THE IMPACT OF THE COSMOLOGICAL CONSTANT

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This is what happens if the Dark Energy is not represented by the Cosmological Constant

1 The accelerated Universe explained by a positive cosmological constant Λ
a reminder of the state of art

2 The cosmological aspect
Universes with Λ: critical value set by Ω_{m0}
$\longrightarrow$ avoidance of the initial singularity

3 Scales of Λ
density: $\rho_\Lambda, \ \rho_{\rm vac} = \Lambda/8\pi G_N$
distance: $r_\Lambda = 1/\sqrt{\Lambda}$
small mass: $m_\Lambda = \sqrt{\Lambda}$
big mass: $M_\Lambda = 1/G_N\sqrt{\Lambda}$
connection to Newtonian limit and coincidence problems

4 Aspects of the Schwarzschild-de Sitter metric: r_Λ
combination of Schwarzschild radius and the cosmological constant

5 The semi-classical aspect: m_Λ and M_Λ
Generalized uncertainty relation and Λ; evaporation of a Schwarzschild-de Sitter black hole

6 Propagation of gravitational waves: r_Λ
When is the 'wavy' character lost due to $\Lambda \longrightarrow \mathcal{L}_{\rm crit}$

7 The astrophysical aspect: ρ_{vac}

equilibrium concepts necessary since Λ is a source of an external repulsive force

8 Conclusions

more than Cosmology.....

L. D. Landau: “Cosmologists are often in error, but never in doubt”

$$\hbar = c = k_B = 1$$

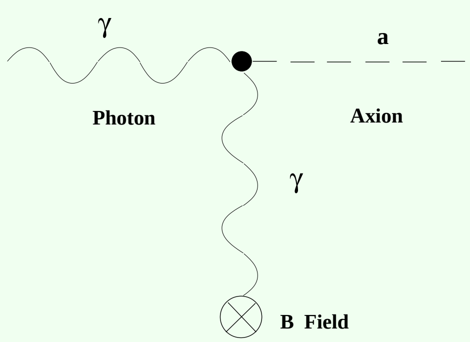
D. Adams in 'The Hitch-hiker's guide to the Galaxy':

"There is a theory which states that if ever anyone discovers exactly what the Universe is for and why it is there, it will instantly disappear and be replaced by something even more bizarre and inexplicable.
There is another theory which states that this has already happened."

This fits the theory of the cosmological constant: many people knew it to be exactly zero.....and so it happened that now nobody can explain why it is non-zero.

1. The accelerated Universe explained by Λ

Observation of standard candles like type I Supernova led us to conclude that the expansion of the Universe as compared to the standard Friedman model is accelerated (the Supernovae are dimer than expected in a standard Friedman model, hence they have to be further away than expected, hence the Universe must have expanded faster than expected). Of course there could be other explanations also, like dust and dimming due to photon-axion conversion in the presence of magnetic fields....



However, the plot seems to thicken now. For example, the age of globular clusters is too big as compared to the age of the Universe (this is actually an old problem) unless we assume e.g. a positive cosmological constant Λ. ΛCDM (Cold Dark Matter) models have been also successful...

General Evidence based on:
Supernovae survey
WMAP (CMB)
Cluster formation (LambdaCDM)
Baryon acoustic oscillations (BAO)
Age of the Universe
Weak Lensing
All evidence in agreement with a positive cosmological constant: equation of state

$$p = p(\rho_{\rm vac}) = w\rho_{\rm vac}, \;\; w = -1$$

Supernova 1994

All what we need to understand the acceleration is Einstein's equations leading to the equations governing the Universe:

$$G_{\mu\nu} = \kappa T_{\mu\nu}, \;\; \kappa = 8\pi G_N$$

$$G_{\mu\nu} = R_{\mu\nu}\frac{1}{2}g_{\mu\nu} + \Lambda g_{\mu\nu}$$

Note that Λ enters the Einstein tensor and not the energy-momentum tensor. Therefore Λ will affect, in principle, any aspect of local physics where gravity plays a role

COSMOLOGY:

$$ds^2 = -dt^2 + a^2(t)R_0^2\left[\frac{dr^2}{1-kr^2} + r^2 d\Omega\right]$$

$$H^2 \equiv \left(\frac{\dot{a}}{a}\right)^2 = \frac{8\pi G_N}{3}\rho + \frac{\Lambda}{3} - \frac{k}{a^2 R_0}, \;\; k = \pm 1,\, 0$$

$$\frac{\ddot{a}}{a} = -\frac{4\pi G_N}{3}(\rho + 3p(\rho)) + \frac{\Lambda}{3}$$

$$\dot{\rho} = -3(\rho + p(\rho))\frac{\dot{a}}{a}$$

- acceleration of the universe (expansion): $\ddot{a} > 0$ BUT

$$\Lambda = 0 \Rightarrow \ddot{a} < 0 \quad (p = 0,\ k = 0)$$

- $$\rho \sim \rho_{\rm crit} \Rightarrow \rho_{\rm vac} > 0.5\rho_{\rm crit}$$

$$\rho_{\rm vac}(\text{today}) \simeq 0.7\rho_{\rm crit}$$

T. H. Huxley: "The great tragedy of science - the slaying of a beautiful hypothesis by an ugly fact."

Example of a flat Universe with $k = 0$

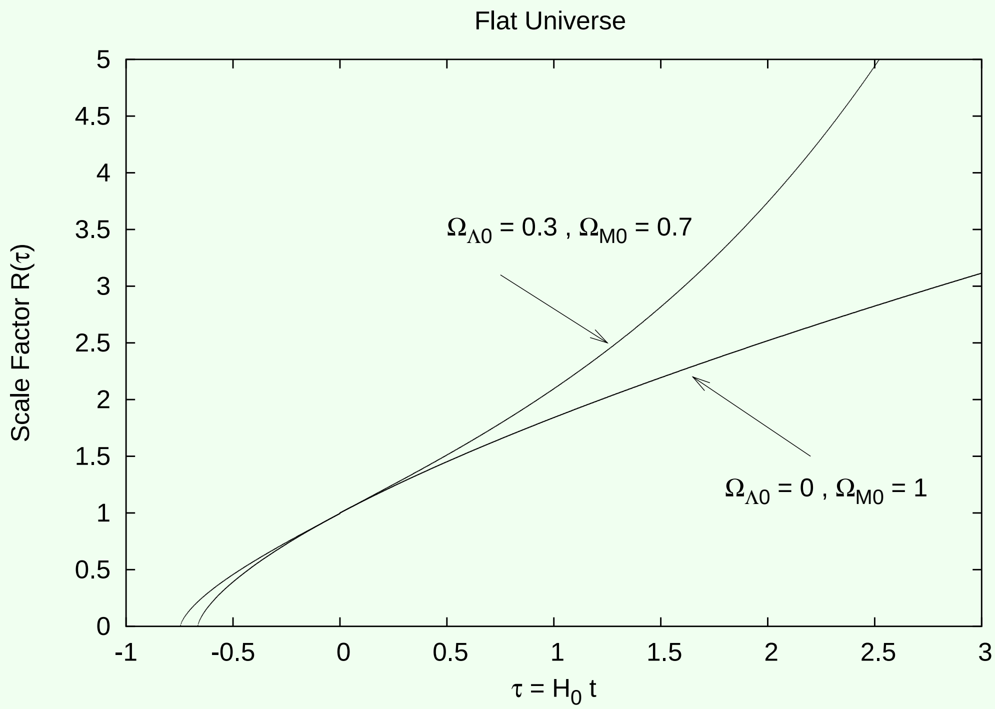


Figure 1: Two fates of matter-filled Universes. The present epoch is at $\tau = 0$.

Friedmann equation at the present epoch

$$\Omega_{m0} + \Omega_{\Lambda 0} - \Omega_{k0} = 1 \rightarrow k = \mathrm{sgn}(\Omega_{m0} + \Omega_{\Lambda 0} - 1)$$

$$\Omega_{m0} = \frac{\rho_0}{\rho_{\rm crit}}, \quad \rho_{\rm crit} = \frac{3H_0^2}{8\pi G_N}$$

$$\Omega_k = \frac{k}{R_0^2 H_0^2}$$

$$\Omega_{\Lambda 0} = \frac{\rho_{\rm vac}}{\rho_{\rm crit}}, \quad \Lambda = 8\pi G_N \rho_{\rm vac}$$

An intuitive understanding of the accelerated expansion is given by the Newtonian limit (1). For a spherically symmetric object with mass M we can use the Schwarzschild metric

$$ds^2 = -e^{\nu(r)}dt^2 + e^{-\nu(r)}dr^2 + r^2 d\theta^2 + r^2 \sin^2\theta d\phi^2$$

$$g_{00} = e^{\nu(r)} = 1 - \frac{2r_s}{r} - \frac{r^2}{3(r_\Lambda)^2}, \quad r_s = G_N M, \quad r_\Lambda = \frac{1}{\sqrt{\Lambda}}$$

and its connection to the gravitational potential

$$g_{00} \simeq -(1 + 2\Phi)$$

to get

$$\Phi(r) = -\frac{r_s}{r} - \frac{1}{6}\frac{r^2}{(r_\Lambda)^2}$$

The last term plays the role of a repulsive external force! The Galilean spacetime gets replaced by Newton-Hooke spacetime where each two space points go apart due to the cosmological constant (this is the part of the cosmological expansion which survives the Newtonian limit).

Taking account:

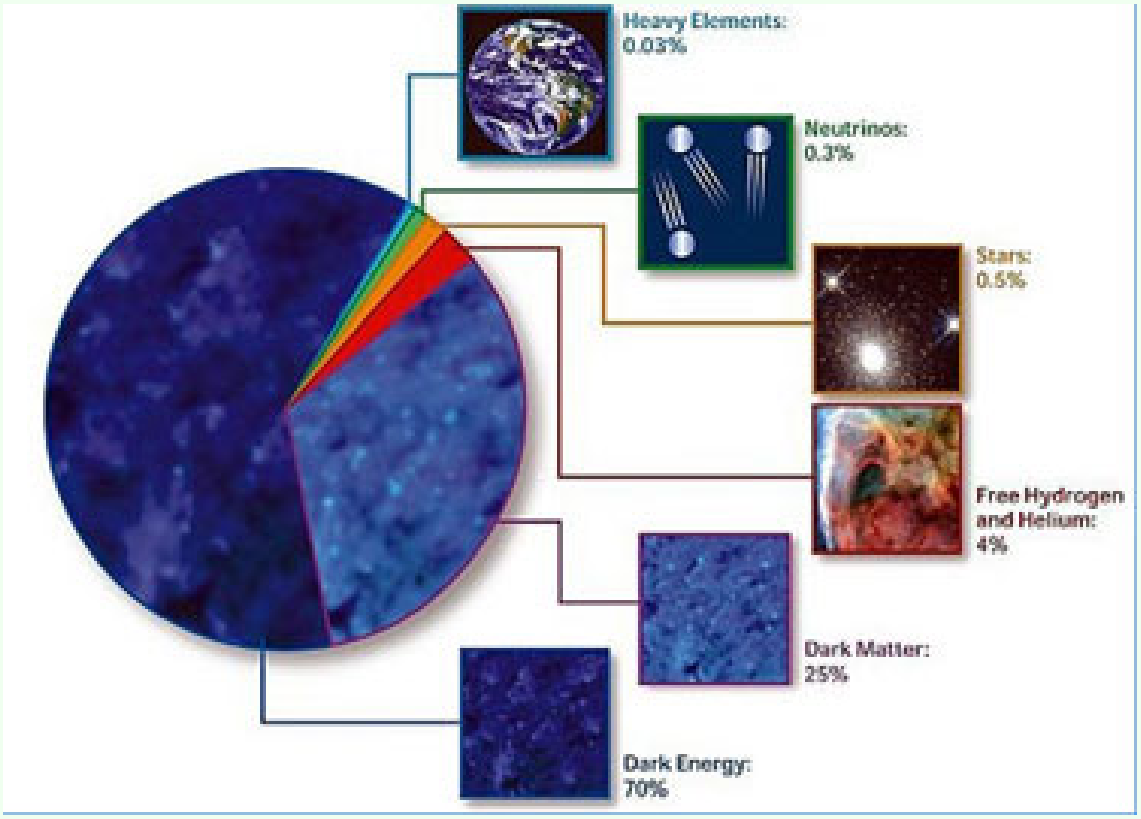


Figure 2: The balance of our Universe.

Conclusions: we are just a dust in the wind.

2. The cosmological aspect:
Universes with Λ: critical value set by Ω_{m0}

J. E. Felten and R.Isaacsman, Rev. Mod. Phys. 58 (1986) 689 We know already that $\Lambda > 0$ leads to acceleration. Are there other new aspect of Λ-Universes?

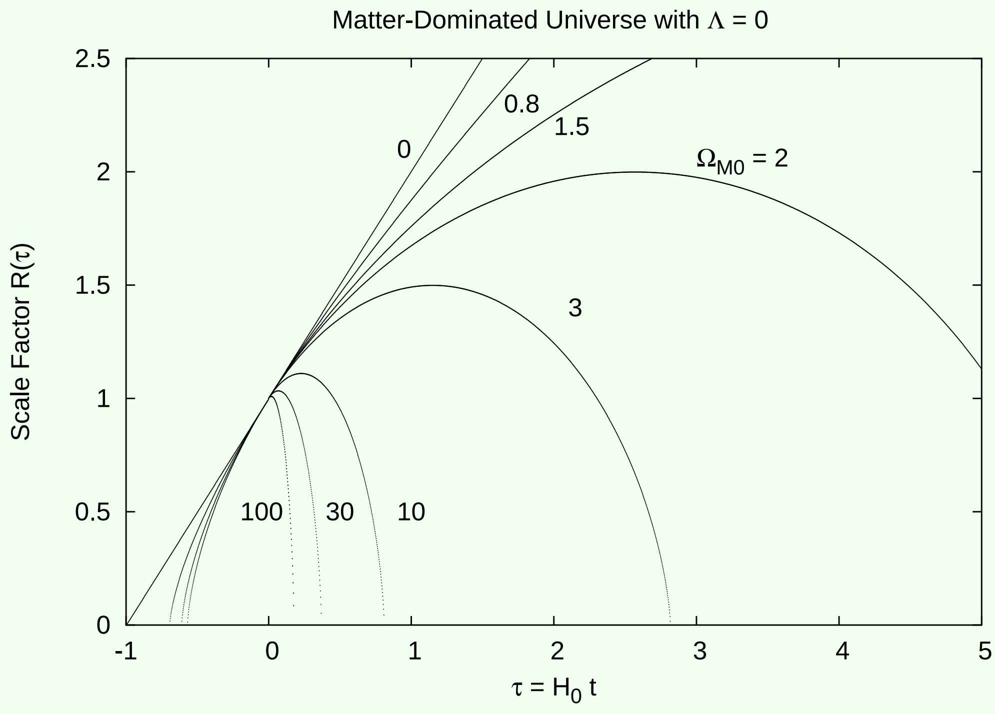
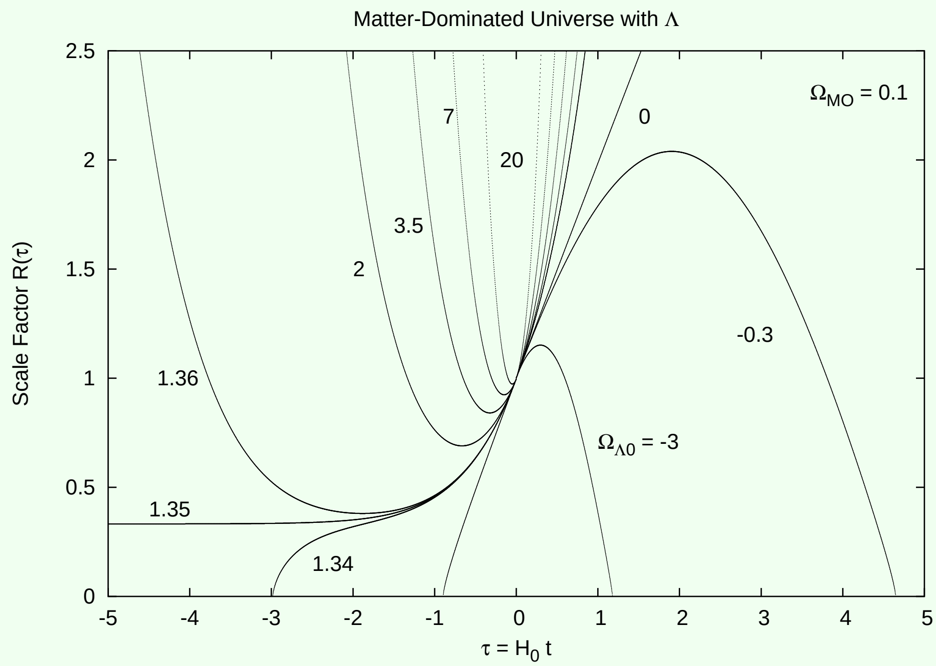
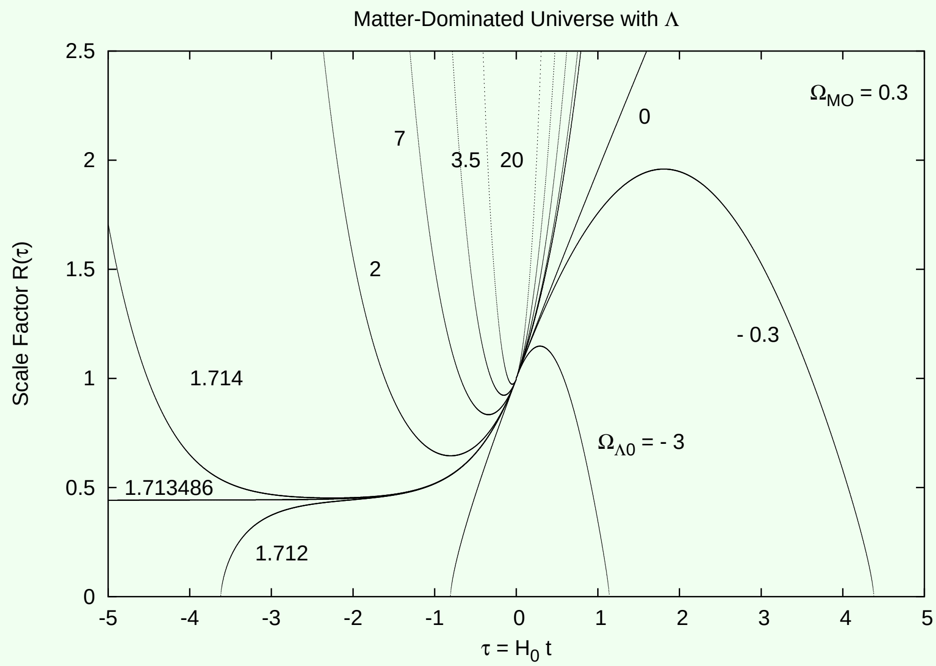


Figure 3: $\Lambda = 0$ Universes.



Matter-Dominated Universe with Λ
Scale Factor R(τ)
τ = H0 t
ΩMO = 0.1
ΩΛ0 = -3
1.36
1.35
1.34
2
3.5
7
20
0
-0.3



Matter-Dominated Universe with Λ
Scale Factor R(τ)
τ = H0 t
ΩMO = 0.3
ΩΛ0 = - 3
1.714
1.713486
1.712
2
7
3.5
20
0
- 0.3

New features appear:
(i) $\Lambda > \Lambda_{\rm crit}$: NO Big Bang
(ii) $\Lambda \sim \Lambda_{\rm crit}$: 'semi-static' coasting Universes

$$\Lambda_{\rm crit}/H_0^2 = 12\Omega_{m0}x^3$$

x solution of

$$x^3 - \frac{3}{4}x + (\Omega_{m0} - 1)/\Omega_{m0} = 0$$

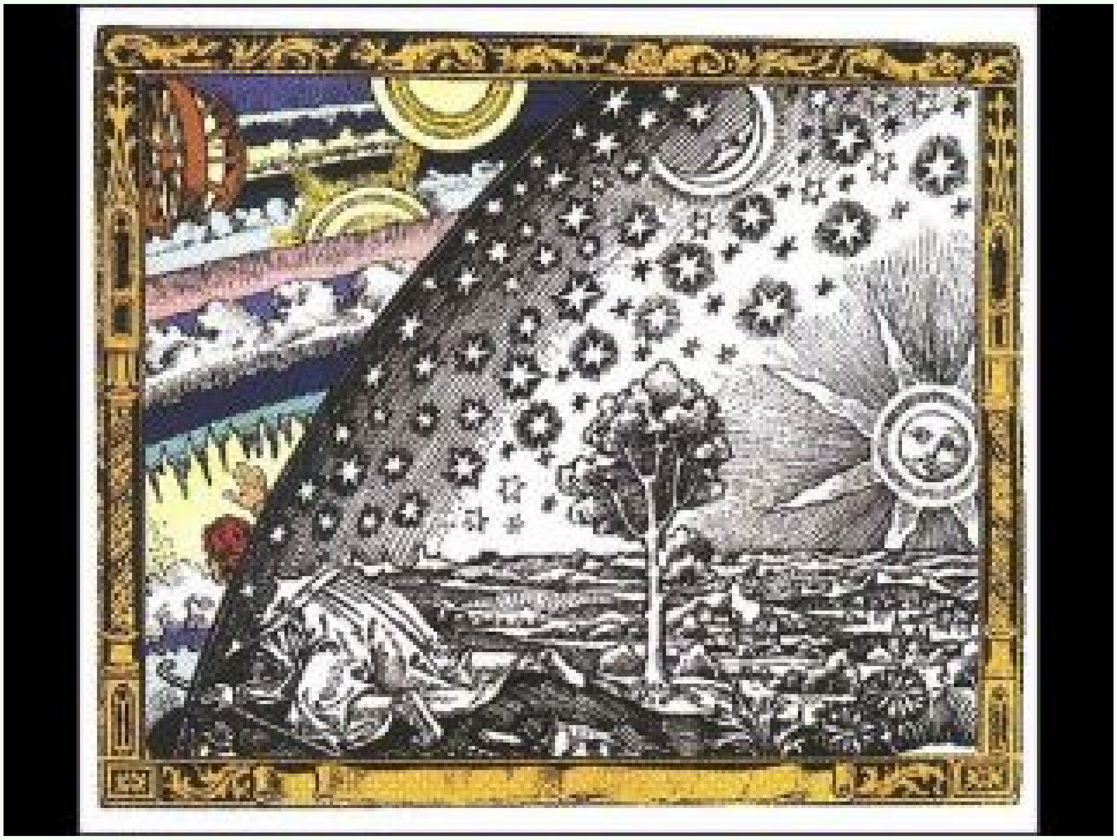
Solutions:

$$0 < \Omega_{m0} \leq \frac{1}{2} : \quad x = \cosh\left[\frac{1}{3}\cosh^{-1}\frac{1-\Omega_{m0}}{\Omega_{m0}}\right]$$

$$\frac{1}{2} \leq \Omega_{m0} \leq 1 : \quad x = \cos\left[\frac{1}{3}\cos^{-1}\frac{1-\Omega_{m0}}{\Omega_{m0}}\right]$$

$$\Omega_{m0} > 1 : \quad x = \cos\left[\frac{1}{3}\cos^{-1}\frac{1-\Omega_{m0}}{\Omega_{m0}} + \frac{3}{4}\pi\right]$$

Uni-verse $\rightarrow$ Multi-verse (anthropic principle):
there must exits universes in the Multi-verse which avoid the initial singularity



Uni-verse or Multi-verse? In a Multi-verse scenario it makes sense to consider different values of the cosmological constant.

4. Scales of Λ

Cosmological coincidences and other curiosities

$$\Lambda = 3\left(\frac{\rho_{\rm vac}}{\rho_{\rm crit}}\right) H_0^2$$

From this we can get different scales

R. Feynman: "Physics is a number", A. Anonymous, "More physics is a plot"

- Density:

$$\rho_{\rm vac}^{\rm obs} \simeq 0.7\rho_{\rm crit} \sim \rho_{\rm crit} \sim 2h_0^2 \times 10^{-29}{\rm gcm}^{-3}$$

$$\rho_{\rm pl} = G_N^{-2} \sim 5 \times 10^{93}{\rm gcm}^{-3}$$
$$\rho_{\rm vac}/\rho_{\rm pl} \sim 10^{122}$$

- Length:

$$r_\Lambda = \frac{1}{\sqrt{\Lambda}} = \frac{1}{\sqrt{3}} \left(\frac{\rho_{\rm vac}}{\rho_{\rm crit}}\right)^{-1/2} H_0^{-1} \sim H_0^{-1} \sim 10^{28} h_0^{-1} {\rm cm}$$

$$r_{\rm pl} = G_N^{1/2} \sim 1.5 \times 10^{-33} {\rm cm}$$
$$r_\Lambda / r_{\rm pl} \sim 10^{61}$$

- large Mass:

$$M_\Lambda = \frac{1}{G_N \sqrt{\Lambda}} = 3.6 \times 10^{22} h_0^{-1} \left(\frac{\rho_{\rm vac}}{\rho_{\rm crit}}\right)^{-1/2} M_\odot$$

with $H_0 = 70 h_{70} {\rm kms}^{-1} {\rm Mpc}^{-1} = 100 h_0 {\rm kms}^{-1} {\rm Mpc}^{-1}$
and $h_{70} = 1.0 \pm 0.15$

$$m_{\rm pl} = G_N^{-1/2} \sim 10^{19} {\rm GeV} \sim 2 \times 10^{-5} {\rm g}$$

$$M_\Lambda / m_{\rm pl} \sim 10^{60}$$

- small mass:

$$m_\Lambda = \sqrt{\Lambda} \sim 3 \times 10^{-42}\text{GeV}, \quad \frac{m_p}{m_\Lambda} \sim 10^{41} \sim \text{Dirac}$$

Dirac's large number $\frac{1}{G_N m_p m_e} \sim 10^{41}$

$$m_{\rm pl}/m_\Lambda \sim 10^{60}$$

$$M_\Lambda \gg m_{\rm pl} \gg m_\Lambda$$

- Time:

$$T_\Lambda = \frac{1}{\sqrt{\Lambda}} \sim T_{H_0}$$

Coincidences:

- Dirac's large number also with the cosmological constant.

- Cosmological coincidence no.1: $\rho_{\rm vac} \sim \rho_{\rm crit}$
- Cosmological coincidence no.2: r_Λ ca. extension of the visible Universe !
- Cosmological coincidence no.3: M_Λ ca. mass of the visible Universe !

The coincidence is that we are living right now in a Universe whose mass, length and density scales are dominated by a constant Λ. After all we could have been living in a different Universe or at a earlier/later epoch in the same Universe. Ω_Λ which is now of order one, was different in the past and will grow away from one in the future. The transitions in both directions are quite steep...

The Λ-scales appear as limiting values in two completely different (non-cosmological) settings, namely in the Newtonian Limit and Hydrostatic equilibrium.

Newtonian Limit (2): condition for the approximation to be valid

$$|\Phi(r)| \ll 1 \rightarrow r_s \ll d(r) \equiv r - \frac{1}{6}\frac{r^3}{(r_\Lambda)^2}$$

$d(r)$ has a local maximum at $r_+ = \sqrt{2} r_\Lambda$. Hence:

$$M_{\max} = \frac{2\sqrt{2}}{3} M_\Lambda \gg M$$

Solving

$$r_s = d(r)$$

(Remark: This is the 2nd time we have to solve a cubic equation.), one gets

$$R_{\max} = \sqrt{6} r_\Lambda \gg r \gg r_s = R_{\min}$$

The Newtonian Limit for non-spherically symmetric objects is given by

$$\nabla^2 \Phi = 4\pi G_N \rho - \Lambda$$

We have a problem to put the Dirichlet boundary condition at infinity...

Hydrostatic equilibrium: existence of a global solution (general relativistic spherically symmetric object in hydrostatic equilibrium) requires (Buchdahl inequalities)

$$3r_s \leq \frac{2}{3}R + R\sqrt{\frac{4}{9} - \frac{1}{3}\frac{R^2}{(r_\Lambda)^2}}$$

We have to satisfy

$$R \leq \sqrt{\frac{4}{3}}r_\Lambda \sim R_{\max}$$

Hence we get also

$$M_{\max} \sim \frac{2}{3}\sqrt{\frac{4}{9}}M_\Lambda \geq M$$

Combination of scales:

very often the relevant physical scale emerge as a combination of two different scales:

$$\rho_{\rm vac} = \frac{1}{8\pi}\sqrt{\rho_\Lambda \rho_{\rm pl}} \sim \sqrt{\rm small \times large}$$

$$\rho_\Lambda = m_\Lambda / r_\Lambda^3, \;\; \rho_{\rm pl} = G_M^{-2} = m_{\rm pl}^4$$

and for the Schwarzschild radius

$$r_s = 2\left(\frac{r_{\rm pl}}{r_M}\right) r_{\rm pl}, \;\; r_M = \frac{1}{M}$$

We have seen already

$$M_{\rm Univ.} \sim \frac{1}{G_N\sqrt{\Lambda}} = \left(\frac{m_{\rm pl}}{m_\Lambda}\right) m_{\rm pl}$$

Does the combination

$$\sqrt{m_{\rm pl} m_\Lambda} \simeq 3 \times 10^{-3} \text{ eV} \simeq 1\text{K}$$

appear anywhere in physics?

Λ on the MOND: the acceleration connection

The acceleration crisis:

- **inflation: $R + \alpha R^2$ gravity etc.**
- **recent Universe: Λ**
- **Pioneer10 anomalous acceleration:**

$$a_0 \sim 10^{-8}\mathrm{cm\ s}^{-2} \sim (\Lambda/3)^{1/2}$$

which is another COINCIDENCE!

- **(Other) Flyby anomalies of spacecrafts (GALILEO, ROSETTA, CASSINI etc.) at low acceleration**
- **MOND (Modified Newtonian Dynamics) of Milgrom (with same value of a_0 as above) versus DARK MATTER:**

$$a^2/a_0 = MG_N r^{-2}, \;\; a_0 \gg a$$

$$a = a_N = MG_N r^{-2}, \;\; a \gg a_0$$

interpolating function μ

$$x = a/a_0, \;\; \mu(x)a = a_N$$

for example

$$\mu(x) = x/(1+x)$$

from which it follows

$$a = G_N M/2r^2 + \sqrt{G_N^2 M^2/4r^4 + a_0 G_N M/r^2}$$

or due to the coincidence

$$a = G_N M/2r^2 \sqrt{r_s^2/4r^4 + (r_s/r_\Lambda)1/r^2}$$

which for

$$r \gg \sqrt{r_s r_\Lambda}$$

becomes

$$a \simeq \frac{\sqrt{r_s/r_\Lambda}}{r} = v^2/r \to v(r) \simeq \mathrm{const}$$

4. Aspects the Schwarzschild-de Sitter metric: length r_Λ

To obtain astronomically relevant length scales from Λ we have to combine r_Λ with another smaller length scale which is r_s.

Consider the motion of test particles in spherically symmetric and static space-time with a cosmological constant. The Schwarzschild metric takes the form

$$ds^2 = -e^{\nu(r)}dt^2 + e^{-\nu(r)}dr^2 + r^2 d\theta^2 + r^2 \sin^2\theta d\phi^2$$

$$e^{\nu(r)} = 1 - \frac{2r_s}{r} - \frac{r^2}{3(r_\Lambda)^2}$$

One would suspect that the inclusion of Λ is irrelevant in this setting. Note, however, that there are now indeed two scales involved, r_s and r_Λ. The combination of the two can lead to new results. Indeed, the equation of motion for a massive particle with proper time τ in the Schwarzschild metric is given by

$$\frac{1}{2}\left(\frac{dr}{d\tau}\right)^2 + U_{\text{eff}} = \frac{1}{2}\left(\mathcal{E}^2 + \frac{L^2\Lambda}{3} - 1\right) \equiv C = \text{constant}$$

where $\mathcal{E}$ (which should not be confused with the energy) and L are conserved quantities defined by

$$\mathcal{E} = e^{\nu(r)}\frac{dt}{d\tau}, \;\; L = r^2\frac{d\Phi}{d\tau}$$

where Φ is the azimuthal angle and $U_{\rm eff}$ is defined by

$$U_{\rm eff}(r) = -\frac{r_s}{r} - \frac{1}{6}\frac{r^2}{(r_\Lambda)^2} + \frac{L^2}{2r^2} - \frac{r_s L^2}{2r^3}$$

is the analog of effective potential potential in classical mechanics.

We now consider radial motion ($L = 0$). From the definition of C we obtain the inequality

$$C > -\frac{1}{2}$$

which will play a crucial role later in the derivation. For the limiting value $C = -\frac{1}{2}$, we have $\mathcal{E} = 0$ which signals an artifact of the Schwarzschild coordinates. This means that there exist some $r = r_\star$ which satisfies the cubic equation

$$y^3 - 3y + 6x = 0$$

$$y = \frac{r_\star}{r_\Lambda}$$

$$x = \frac{r_s}{r_\Lambda} = 1.94 \times 10^{-23} \left(\frac{M}{M_\odot}\right) h_{70} \left(\frac{\rho_{\rm vac}}{\rho_{\rm crit}}\right)^{1/2} \ll 1$$

(Remark: this is the third time we arrive at a cubic equation.) The two positive roots are

$$r_\star^{(1)} = \sqrt{3} r_\Lambda - r_s, \;\; r_\star^{(2)} = 2 r_s \left(1 - \frac{1}{6}\left(\frac{r_s}{r_\Lambda}\right)^2\right)$$

In other words, the condition $C = -1/2$ is satisfied at the Schwarzschild radius and at the edge of the universe. With the same limiting value, we have $|U_{\rm eff}(r_\star)| = \frac{1}{2}$. Motion with $|U_{\rm eff}(r)| \geq \frac{1}{2}$ becomes unphysical since it corresponds to allowing the motion of test particles inside the Schwarzschild radius and beyond the observed universe. The latter is a result of the coincidence in the sense that r_Λ sets the scale of the horizon of the universe. Hence, the particles are allowed to be at some r such that

$$R_{\rm min} \sim r_s < r < \sqrt{3} r_\Lambda \sim R_{\rm max}$$

with

$$|C| < |U_{\rm eff}(r)| < \frac{1}{2}$$

(for negative C and $U_{\rm eff}$). It is clear that at certain distance, the terms $-r_s/r$ and $r^2/(r_\Lambda)^2$ will become comparable leading to a local maximum located at

$$r_{\rm max} = \left(3r_s r_\Lambda^2\right)^{1/3} \simeq 10^{-4}\left(\frac{M}{M_\odot}\right)^{1/3}\left(\frac{\rho_{\rm crit}}{\rho_{\rm vac}}\right)^{1/3} h_{70}^{-2/3}{\rm Mpc}$$

$$U_{\rm eff}(r_{\rm max}) = -7.51\times 10^{-16}\left(\frac{M}{M_\odot}\right)^{2/3}\left(\frac{\rho_{\rm vac}}{\rho_{\rm crit}}\right)^{1/3} h_{70}^{2/3}$$

Beyond $r_{\rm max}$, $U_{\rm eff}$ is a continuously decreasing function. This implies that $r_{\rm max}$ is the maximum value within which we can find bound solutions for the orbit of a test body.

Consider now the following chain of matter conglomeration of astrophysical objects: the smallest are star clusters (globular and open) with stars as members ($M = M_\odot$) and a mass of $10^6 M_\odot$. We proceed to galaxies and galactic clusters. Within this chain, we find for $r_{\rm max}$ the following values as a function of mass:

$M/M_\odot$	$r_{\rm max}/\alpha$ (pc)
1	75
10^6	7.5×10^3
10^{11}	3.5×10^5
10^{13}	1.6×10^6

with $\alpha = h_0^{-2/3}\left(\rho_{\rm vac}/\rho_{\rm crit}\right)^{-1/3}$.

- The value at the first line is of the order of magnitude of the tidal radius of globular clusters.

- The second line agrees with the extension of an average galaxy. Unlike the other three ones, where the argument M of $r_{\rm max}$ has been taken to be the mass of the average members of the astrophysical object, it might appear unjustified to take the mass of the globular cluster to obtain the extension of the galaxy. However, in view of the fact that globular clusters are very old objects and are thought to be of importance in the formation of the galaxy, this choice seems justified. Indeed, with $\Lambda > 0$, our result strengthens the belief that globular clusters are relics of the formation of the galaxy. For instance, $r_{\rm max}$ for open star clusters with a mass $M = 250 M_\odot$ is only 0.5 kpc.

- The next two values are about the size of a galaxy cluster. The value $10^{13} M_\odot$ corresponds to a giant elliptic galaxy encountered often at the center of the clusters.

Conclusions:

Hence, r_Λ in combination with r_s gives us surprisingly accurate and natural astrophysical scales. The combination $r_{\rm max} = (3 r_s (r_\Lambda)^2)^{1/3}$ from which these scales where calculated is not an arbitrary combination with length dimension, but it is the distance beyond which we cannot find bound orbits. Therefore we would expect that $r_{\rm max}$ sets a relevant astrophysical scale. Of course, we are talking here about scales neglecting dynamical aspects of many body interactions, but no doubt $r_{\rm max}$ is roughly the scale to be set for bound systems. Indeed, the agreement of the result in the table with values encountered in nature is striking.

What happens in the case of $r_l \equiv L \neq 0$?. Look for a saddle point i.e.

$$\frac{dU_{\rm eff}}{dr} = \frac{d^2 U_{\rm eff}}{dt^2} = 0$$

The two conditions lead to the position of the saddle point and a condition on one parameter, say $x_l \equiv \frac{r_l^2}{(r_\Lambda)^2}$

$$x_l^4 - \left(\frac{3r_s}{4r_\Lambda}\right)^4 x_l - 12\left(\frac{3r_s}{4r_\Lambda}\right)^6 = 0$$

To solve this fourth order we have to solve the associated third order (and this is the fourth time we deal with third order polynomial equations). After handling hyperbolic functions and their inverses, going through complex numbers and their roots, one gets a simple expression

$$r_l^{\rm max} = r_l^{\rm crit} = 0.9(r_s^2 r_\Lambda)^{1/3}$$

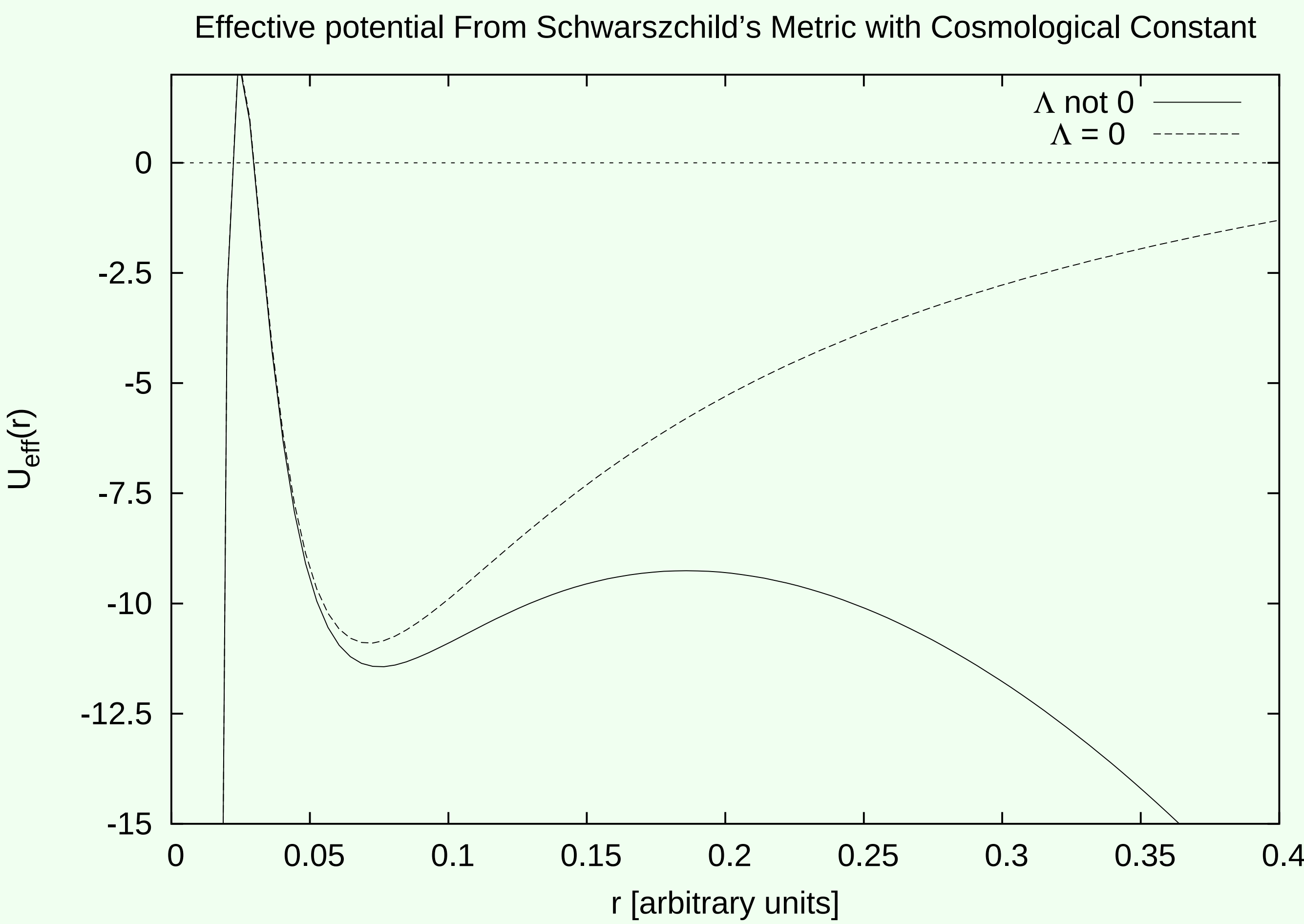
provided $r_s/r_\Lambda \ll 1$. For $r_l \geq r_l^{\text{crit}}$ the minimum and maximum fall together and there are no more bound orbits! Taking now from non-relativistic mechanics the expression for the order of magnitude of a bound orbit we get

$$R_{\text{orbit}} \sim \frac{r_l^2}{r_s} \to R_{\text{orbit}}^{\max} \sim 0.55 r_{\max}$$

which is a very satisfying result as it does not change the order of magnitude of the estimate with zero angular momentum!

Note: to ensure the existence of the first maximum and minimum of U_{eff} one has to require

$$r_l^{\min} = 2\sqrt{3} r_s$$



Effective potential From Schwarszchild’s Metric with Cosmological Constant
Λ not 0
Λ = 0
0
-2.5
-5
-7.5
-10
-12.5
-15
0
0.05
0.1
0.15
0.2
0.25
0.3
0.35
0.4
$U_{eff}(r)$
r [arbitrary units]

5. The semi-classical aspect: m_Λ and M_Λ

J. von Neumann: "There is no sense in being precise when you don't know what your are talking about"

Evaporation of Schwarzschild- de Sitter black hole via the Generalized Uncertainty Principle **GUP**).

Keywords:

- black hole remnant (minimum mass) $\leftrightarrow$ maximum temperature
  deformation of the standard (Hawking) dispersion relation $T(M)$ near the horizon $2r_s$

- effect of Λ:
  maximum possible mass $\leftrightarrow$ minimum temperature
  deformation of the dispersion relation near the second horizon.

SIMILARITIES

Early Quantum	Quantum Gravity today
Uncertainty relation for $\Delta x \ \Delta p$ not yet derived from Schwarz ineq. in Hilbert space	Generalized Uncertainty principle (exists !)
Ubiquitous Black Body Radiation 3 Nobel Prizes so far	(Hawking's) evaporation of **Black Holes** is perfect **Black Body** radiation
Bohr	String Theory, Loop gravity
Schroedinger	Wheeler - de Witt eq.
Interpretation	Many worlds interpretation
Quantum Mechanics = Hilbert space + interpretation	??

DIFFERENCES

Early Quantum	Quantum Gravity today
many experiments relatively easy to invent/perform	NO experiments (so far) Why? Because quantum gravity requires <u>extreme situations!</u> Planck length (tiny) Planck time (tiny) Planck energy (huge) Planck density (huge)
discovery of Hilbert space as a tool	**Construction** of Hilbert space (or beyond)

REMARKS

- Right now we are often estimating orders of magnitude.

- The expectation that something happens at the Planck's values (or that the Planck values are limiting values of physics has been based on purely dimensional grounds. Is there a way to establish the above expectations in a more quantitative way?
  $\Longrightarrow$ **GUP**

- By the same token (as the Newton's constant sets the scale of Planck's length $\sim \sqrt{G_N}$ as the possible smallest length in physics): is $r_\Lambda = 1/\sqrt{\Lambda}$ also a special scale in (quantum) gravity?
  The largest length scale?
  Too scary to contemplate?
  Possible answers:
  $\Longrightarrow$ **GUP**

GUP FOR $\Lambda = 0$ case:
Heuristic derivation

Let $E = p$ be the photon's energy, then

$$a_G = \frac{G_N E}{r^2}$$

As an order of magnitude estimate, we can write

$$\Delta x_G \simeq \frac{G_N E}{r^2} L^2 \simeq G_N E = G_N p$$

where we used $r \sim L$. Using $\Delta p \sim p$. we arrive at GUP

$$\Delta x \geq \frac{1}{2\Delta p} + \frac{G_N \Delta p}{2}$$

Identifying now

$$\Delta x \sim 2r_s = 2G_N M$$

$$\Delta p \sim E \sim T$$

and using GUP we obtain

$$2G_N M = 2\frac{M}{m_{\rm pl}} = \frac{1}{2T} + \frac{T}{m_{\rm pl}}$$

Solving this equation for $T = T(M)$ and introducing a calibration factor (in reality the GUP relation turns out to be valid for surface gravity) $(2\pi)^{-1}$ gives

$$T = \frac{1}{2\pi}\left(M - \sqrt{M^2 - m_{\rm pl}^2/2}\right)$$

Conclusions:

- This reduces to Hawking's formula for large M i.e. $T(M) = \frac{m_{\rm pl}^2}{8\pi M}$
- BH remnant:

$$M > \mathcal{M}_{\rm min} = \frac{m_{\rm pl}}{2}$$

  as expected from dimensional analysis. This corresponds to $T_{\rm max} = m_{\rm pl}/4\pi$

- $$T = \frac{1}{\pi}\frac{r_s}{l_{\rm pl}^2}\left(1 - \sqrt{1 - \frac{1}{4}\left(\frac{l_{\rm pl}}{r_s}\right)^2}\right)$$

  Hence $L_{\rm min} = l_{\rm pl}/2$ as expected.

Black body radiation as a check:

$$0 < g_{00} = 1 - \frac{2G_N M}{R} = 1 - (8\pi/3)G_N \rho R^2$$

where we use later $M = E$ (energy of photons)
Hence

$$\rho < \frac{3}{8\pi}\frac{1}{G_N R^2}$$

Using Stefan-Boltzmann law $\rho = \sigma T^4$ one gets

$$T^4 < \frac{3}{8\pi}\frac{1}{\sigma G_N R^2}$$

Using the quantum mechanical result $R > 1/T$ one gets

$$T < T'_{\rm max} = \sqrt{\frac{45}{8\pi^2}} m_{\rm pl}$$

Inserted into Hawking's formula yields

$$M'_{\rm min} = \left(\frac{2}{5}\frac{1}{8\pi}\right) m_{\rm pl}$$

Sakharov's result as a second check:

- In 1966 Andrei Sakharov derived the result for Black Body radiation

$$T < T_{\rm max}^{\rm Sakharov} \sim m_{\rm pl} \sim 10^{32}\ K$$

independently of all the results above. Obviously, it is of the same order of magnitude as the two previous results and corresponds, via the Hawking formula, to the existence of a black hole remnant of the order of Planck mass Obviously also, only now in the context of Hawking's radiation and semi-classical quantum gravity such a results makes sense.

Repeating the above steps with $\Lambda \neq 0$:

- Let us stay first with the Black Body radiation...
- Schwarzschild-de Sitter metric:

$$0 < g_{00} = 1 - \frac{2r_s}{R} - \frac{R^2}{3{r_\Lambda}^2}$$

$$0 < \rho < \frac{3}{8\pi}\frac{m_{\rm pl}^2}{R^2} - \frac{1}{3}\frac{3}{8\pi}m_{\rm pl}^2{m_\Lambda}^2$$

Finally, using again $R > 1/T$ we obtain

$$\frac{1}{\sqrt{3}}m_\Lambda = T'_{\rm min} < T < T'_{\rm max}$$

- **The task is now to confirm the existence of the minimal temperature and the maximal mass which, via the Hawking formula, must be of the order**

$$\mathcal{M}'_{\rm max} \sim \left(\frac{m_{\rm pl}}{m_\Lambda}\right) m_{\rm pl} \Longrightarrow \mathbf{GUP} - \Lambda$$

GUP FOR $\Lambda \neq 0$ CASE:

Heuristic derivation along the same lines as above

We have seen that the Generalized Uncertainty Principle (**GUP**) can be obtained easily from the gravitational force. To repeat the steps leading to GUP from the previous section we need the gravitational potential Φ for a spherically symmetric mass distribution with blue3Λ

$$\Phi = -\frac{r_s}{r} - \frac{1}{6}\frac{r^2}{r_\Lambda^2}$$

Then following the arguments from above the gravitational force per mass attributed to Λ is

$$\frac{|\vec{F}_\Lambda|}{m} = \frac{1}{3}\Lambda L$$

where L is again a typical length scale in the problem under consideration. The corresponding displacement is

$$\Delta x_\Lambda \sim \frac{1}{3} m_\Lambda^2 L^3$$

We use now the additional assumption $L \sim \frac{1}{\Delta p}$.This assumption is equivalent to say that the precision of the momentum is inversely proportional to the typical length scale and can be found e.g. in textbooks in connection with wave packets. It is analog to similar assumptions like $\Delta t \sim E^{-1}$ in the context of estimating the pion mass in Yukawa's theory or $\Delta x \sim p^{-1}$ in case we want to estimate the precision of the position. Therefore we can write

$$\Delta x_\Lambda \sim \frac{1}{3} \frac{m_\Lambda^2}{\Delta p^3}$$

such that the proposed relation for GUP with the inclusion of the cosmological constant is

$$\Delta x \gtrsim \frac{1}{2\Delta p} + \frac{\Delta p}{2 m_{\rm pl}^2} - \Delta x_\Lambda$$

$$\Delta x \gtrsim \frac{1}{2\Delta p} + \frac{\Delta p}{2m_{\rm pl}^2} - \frac{\gamma}{3}\frac{m_\Lambda^2}{\Delta p^3}$$

where we have taken into account the relative sign difference between the cosmological constant contribution and the standard Newtonian part. We also include a factor $\gamma \sim \mathcal{O}(1)$ which accounts for the fact that we are dealing with orders of magnitudes estimates. In comparing the results from **GUP** with standard results for small masses this factor should come out of the order of 1. If this is not the case, something would be wrong with the uncertainty relation

As before in the context of Hawking radiation the uncertainty in position is associated with the event horizon. Then the Generalized Uncertainty applied to black hole evaporation gives the equation

$$\frac{2M}{m_{\rm pl}^2} = \frac{1}{2T_*} + \frac{T_*}{2m_{\rm pl}^2} - \frac{\gamma}{3}\frac{m_\Lambda^2}{T_*^3}$$

It is worth noting that for high temperatures, the previous results for $\Lambda = 0$ are recovered. Therefore,

$T_{\max}$ in conjunction with $M_{\min}$ also follows from the above equation. For small temperatures, GUP can be approximated to

$$\frac{2M}{m_{\mathrm{pl}}^2} \approx \frac{1}{2T_*} - \frac{\gamma}{3}\frac{m_\Lambda^2}{T_*^3}$$

which amounts to solve a third order polynomial of the form

$$T_*^3 - \left(\frac{m_{\mathrm{pl}}^2}{4M}\right) T_*^2 + \frac{\gamma}{6}\frac{m_\Lambda^2 m_{\mathrm{pl}}^2}{M} = 0$$

SOLUTION

Global aspects:

- Crucial parameters (and their signs)

$$p = -\frac{m_{pl}^4}{48M^2} < 0$$

$$q = \frac{m_{\rm pl}^4}{M}\left(-\frac{1}{864}\frac{m_{\rm pl}^2}{M^2} + \frac{\gamma}{6}\frac{m_\Lambda^2}{m_{\rm pl}^2}\right)$$

$$D = \frac{1}{4}\frac{m_{\rm pl}^6 m_\Lambda^2}{M^2}\left(\frac{\gamma^2}{36}\left(\frac{m_\Lambda^2}{m_{\rm pl}^2}\right) - \frac{\gamma}{3(864)}\left(\frac{m_{\rm pl}^2}{M^2}\right)\right)$$

- It can be demonstrated that for $D > 0$ there are no physical solutions and only $D < 0$ is of interest for us. A limit on the value of M is set by putting $D = 0$. We find from $D = 0$, $M^*_{\rm max}$. A simple algebraic manipulation yields now

$$M^*_{\rm max} = \frac{1}{6\sqrt{2\gamma}}\frac{m_{\rm pl}^2}{m_\Lambda} \sim \mathcal{M}'_{\rm max}(black - body)$$

for $\gamma = 5/9$ as we will show later.

- $q > 0$ $q < 0$ is a branch point and has to be treated separately. We introduce

$$M = \frac{M^*_{\rm max}}{\zeta}$$

where $\zeta = 1$ corresponds to $M^*_{\max}$.

Explicit solution for the branch $q > 0$ ($1 < \zeta < \sqrt{2}$) i. e. large masses :
We find:

$$T(\zeta) = -\frac{\sqrt{2\gamma} m_\Lambda}{2\pi\zeta} \left(\cos\left(\frac{1}{3}\left(\cos^{-1}\left(-1 + \frac{2}{\zeta^2}\right) + 2\pi\right)\right) - \frac{1}{2}\right)$$

which is a monotonically decreasing function of ζ. This means

$$T(1) = T_{\min} = \frac{\sqrt{2\gamma}}{2\pi} m_\Lambda \sim T'_{\min}(black - body)$$

Explicit solution for the branch $q < 0$ ($\zeta > \sqrt{2}$) i.e. smaller masses:
We find:

$$T(\zeta) = \frac{\sqrt{2\gamma} m_\Lambda}{2\pi} \zeta \left(\cos\left(\frac{1}{3}\cos^{-1}\left(1 - \frac{2}{\zeta^2}\right)\right) + \frac{1}{2}\right)$$

The matching condition;
For $M << M_{\rm max}$, we can expand:

$$T(M) \approx \frac{1}{8\pi}\frac{m_{\rm pl}^2}{M} - \frac{9}{10\pi}\gamma\left(\frac{m_\Lambda}{m_{\rm pl}}\right)^2 M$$

and compare it with the standard result obtained via surface gravity:

$$T(M) = \frac{m_{pl}^2}{8\pi M} - \frac{1}{2\pi}\frac{m_\Lambda^2}{m_{\rm pl}^2}M$$

This gives

$$\gamma = \frac{5}{9}$$

Checks and Conclusions of **GUP** with Λ:

- $\gamma \sim \mathcal{O}(1)$ as it should be
- For relative moderate masses the **GUP** result goes over to the standard one. The standard dispersion relation $T(M)$ gets modified near $2r_s$ and $\sim r_\Lambda$.

- Existence of

$$T_{\rm min} \sim m_\Lambda, \leftrightarrow M_{\rm max} \sim \frac{m_{\rm pl}^2}{m_\Lambda} \sim M_\Lambda$$

$$T_{\rm max} \sim m_{\rm pl} \leftrightarrow M_{\rm min} \sim m_{\rm pl}$$

- These results are confirmed in different contexts: Sakharov's result and black body radiation (in Schwarzschild-de Sitter).

Digression: Classical Standard Result

Via the surface gravity κ and the relation

$$T = \frac{\kappa}{2\pi}$$

as well as

$$\kappa = Va$$

where all quantities are evaluated at the horizon r_c.Here a is the invariant scalar acceleration and V is the red-shift factor.

$$V(r) = \sqrt{1 - \frac{2r_s}{r} - \frac{1}{3}\frac{r^2}{r_\Lambda^2}}$$

$$a(r) = \frac{\frac{r_s}{r^2} - \frac{1}{3}\frac{r}{r_\Lambda^2}}{\sqrt{1 - \frac{2r_s}{r} - \frac{1}{3}\frac{r^2}{r_\Lambda^2}}}$$

Hence the surface gravity takes the following simple expression

$$\kappa(r_c) = \left|\frac{r_s}{r^2} - \frac{1}{3}\frac{r}{r_\Lambda^2}\right|_{r=r_c}$$

The standard $T - M$ relation reads

$$T(M) = \frac{m_{pl}^2}{8\pi M} - \frac{1}{2\pi}\frac{m_\Lambda^2}{m_{\rm pl}^2}M$$

A psychedelic black hole. Its surrounding is a superposition of X-ray and optical information (unfortunately no Hawking radiation)

6. Propagation of gravitational waves: length r_Λ

I. Asimov: “The most exciting phrase to hear in science, the one that heralds new discoveries, is not Eureka! (I found it!) but rather, hmm. thats funny.

In testing Einstein’s theory of gravity, its modifications and ramifications, two important sub-areas (among other) of research can be explored and explained in more detail. The first one has to do with cosmology and goes back to the discovery of dark energy ten years ago which drives the acceleration of the universe The second one is the possibility to detect gravitational waves directly by already operating (e.g. LIGO) or forthcoming gravitational wave detectors (e.g. LISA).

Will Λ affect the propagation of a garvitationa wave? The question does not addresses the cosmological aspect of Λ (the fact that the cosmological constant is part of the Friedmann equations) where the wave is interpreted as a ripple on cosmological background. Since Λ is part of the Einstein tensor, it will also play a role in the local production of gravitational waves.

Answer in six steps. Step 1: The Linearized Einstein's equation with Λ

The metric is

$$g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}$$

where $\eta_{\mu\nu}$. Einstein's equations in frist order (usage of Minkowski metric):

$$R^{(1)}_{\mu\nu} = -8\pi G S_{\mu\nu} - \Lambda\eta_{\mu\nu}$$

where we have used the trace-reversed part of the energy-momentum tensor

$$S_{\mu\nu} \equiv T_{\mu\nu} - \frac{1}{2}\eta_{\mu\nu}T$$

The linearized expression of the Ricci tensor is easily obtained to be

$$R^{(1)}_{\mu\nu} \equiv \frac{1}{2}(\Box h_{\mu\nu} - \partial^\lambda \partial_\mu h_{\lambda\nu} - \partial^\lambda \partial_\nu h_{\lambda\mu} + \partial_\mu \partial_\nu h)$$

which gives us the linearized equations

$$\Box h_{\mu\nu} - \partial^\lambda \partial_\mu h_{\lambda\nu} - \partial^\lambda \partial_\nu h_{\lambda\mu} + \partial_\mu \partial_\nu h = -16\pi G S_{\mu\nu} - 2\Lambda \eta_{\mu\nu}$$

Which without the cosmological constant is the Fierz-Pauli equation for a masless spin-2 object ($h_{\mu\nu}$) . With it only a part of a Fierz-Pauli equation for a massive spin-2 object i.e. Λ is not a mass!

This equation is clearly covariant under the local gauge transformation

$$h_{\mu\nu} \to h_{\mu\nu} + \partial_\mu \epsilon_\nu + \partial_\nu \epsilon_\mu$$

as imposed by the general diffeomorphic covariance of the Einstein's equations with Λ. Any attempt to make the cosmological constant more dynamical by replacing

$$\Lambda \eta_{\mu\nu} \to \Lambda g_{\mu\nu}$$

This gauge freedom allows us to fix the gauge which we choose to be the de Donder condition:

$$\partial^\mu h_{\mu\nu} = \frac{1}{2}\partial_\nu h$$

The equation to be solved becomes a wave equation with two kinds of inhomogeneities; one the standard source $S_{\mu\nu}(x)$, the other one a constant term proportional to the cosmological constant:

$$\Box h_{\mu\nu} = -16 G S_{\mu\nu} - 2\Lambda\eta_{\mu\nu}$$

Step 2: Check via the Veltmann Lagrangian
Veltmann while deriving Feynman rules for graviton scattering derived the Lagrangian

$$\mathcal{L}_h = -2\Lambda\left(1+\frac{1}{2}h\right) - \frac{1}{4}\partial_\nu h_{\alpha\beta}\partial^\nu h^{\alpha\beta}$$

$$+\frac{1}{4}\partial_\mu h\partial^\mu h - \frac{1}{2}\partial_\beta h\partial_\mu h^{\beta\mu} + \frac{1}{2}\partial_\alpha h_{\nu\beta}\partial^\nu h^{\alpha\beta}$$

invariant under the gauge transformations. The Euler-Lagrange equations give the same linearized equation as above.

Step 3: Solution

Since the linearized equation is linear we can split its solution in two parts

$$h_{\mu\nu} = \gamma_{\mu\nu} + \xi_{\mu\nu}$$

where

$$\gamma_{\mu\nu} = e_{\mu\nu}(\mathbf{r}, \omega) e^{ik_\alpha x^\alpha} + \text{c.c.}$$

is thestandard retarded solution (written here for a monochromatic source at a distance far away from the source and $\xi_{\mu\nu}$ solves $\Box \xi_{\mu\nu} = -2\Lambda \eta_{\mu\nu}$. The latter, should satisfy the de Donder gauge and, in addition, we demand that up to a diffeomorphism its asymptotic form is of the de Sitter metric.

$$\xi_{00} = -\Lambda t^2,\ \xi_{0i} = \frac{2}{3}\Lambda t x_i,\ \xi_{ij} = \Lambda t^2 \delta_{ij} + \frac{1}{3}\Lambda \epsilon_{ij}$$

where $\epsilon_{ij} = x_i x_j$ for $i \neq j$ and 0 otherwise. These solutions are to be used in the energy momentum pseudo-tensor $\hat{t}_{\mu\nu}$ for gravitational waves.

Step 4: The energy momentum pseudo-tensor (leadind to gravitational Poynting vector)
In the absence of the cosmological constant the latter is defined as

$$(G_{\mu\nu} - G^{(1)}_{\mu\nu})/8\pi G$$

where , again, the index 1 indicates that we expand the tensor in the order $\mathcal{O}(\mathbf{h})$. Taking into account that $G_{\mu\nu}$ is now modified, the very same procedure can be adopted for theories with Λ leading to

$$\hat{t}_{\mu\nu} = t_{\mu\nu} - \frac{1}{8\pi G}\Lambda h_{\mu\nu}$$

where $t_{\mu\nu}$ is the part defined by

$$t_{\mu\nu} = \frac{1}{8\pi G}\biggl(-\frac{1}{2}h_{\mu\nu}R^{(1)} + \frac{1}{2}\eta_{\mu\nu}h^{\sigma\rho}R^{(1)}_{\sigma\rho}$$

$$+R^{(2)}_{\mu\nu} - \frac{1}{2}\eta_{\mu\nu}\eta^{\sigma\rho}R^{(2)}_{\sigma\rho}\biggr) + \mathcal{O}(\mathbf{h}^3)$$

It remains to calculate the gravitational averaged Poynting vector

$$< \hat{t}^{03} >_{\rm wave} = < t^{03} >_{\rm wave} = \frac{\omega^2 \hat{h}^2}{8\pi G}, \;\; < \hat{t}^{03} >_{\Lambda} = -\frac{1}{8\pi G}\frac{5}{18}\frac{1}{r_{\Lambda}^4}L^2$$

where $\hat{h}$ is either $|e_{11}|$ or $|e_{12}|$. Note that due to Λ, the power

$$\frac{dP}{d\Omega} = r^2 \frac{x_i}{r} < \hat{t}^{0i} >$$

receives a negative contribution. The power is only well defined i.e. positive definite below a certain critical distance $\mathcal{L}_{\rm crit}$ where the oscillatory character of the solution dominates. To calculate this critical distance it suffices to compare the magnitudes of the two contributions to $< \hat{t}^{03} >$. The result is

$$\mathcal{L}_{\rm crit} = \frac{6\sqrt{2}\pi f \hat{h}}{\sqrt{5}} r_{\Lambda}^2$$

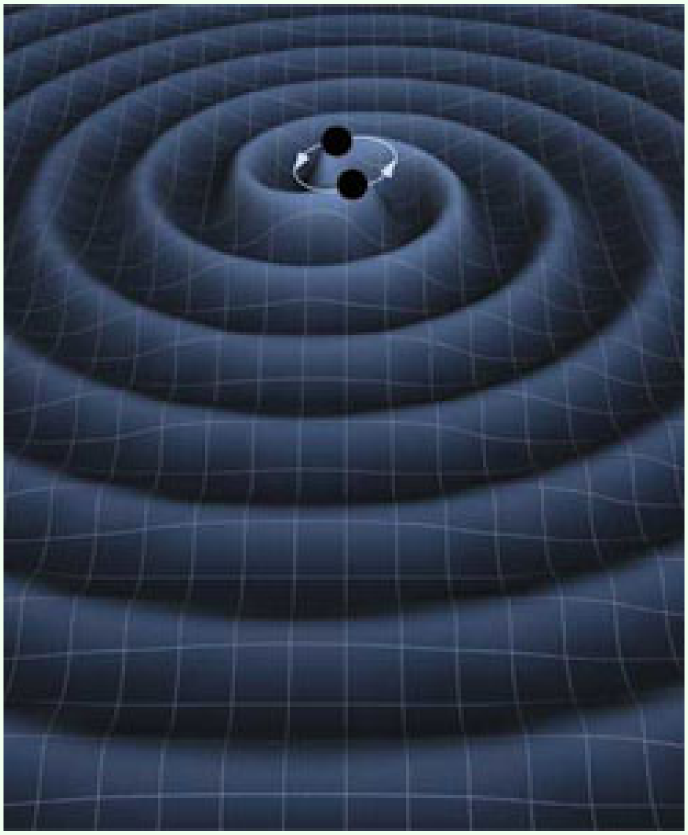
Notice that what we are really comparing is the averaged solution proportional Λ with the averaged wave component of the solution. We then say that the wave character of the solution is lost when both are comparable.

Step 5: Phenomenology

Table 1: Sources of gravitational waves for LIGO. AIC means accretion induced collapse. For P we have used geometrized units $G = c = 1$.

System	f [Hz]	$\hat{h}$ 10^{-23}	Dist. 10^8 pc	$\mathcal{L}_{\rm crit}$ 10^8 pc	$dP/d\Omega$ $\Lambda = 0$	$dP/d\Omega$ $\Lambda \neq 0$
NS/NS binary	100	1	10	0.13	$-$	$-$
BH/BH binary	100	10	2	1.3	$-$	$-$
Collapse & explosion of Supernova	20	4.1	0.1	0.11	1.1×10^{-12}	1.2×10^{-13}
NS formed from AIC	450	8	1	4.6	2.2×10^{-7}	2.1×10^{-7}
NS/NS binary	1000	1000	0.23	1280	8.8×10^{-4}	8.8×10^{-4}
Stell. collap. Centrifugal hang up	100	10	0.15	1.3	3.8×10^{-10}	3.7×10^{-10}

Step 6: Waiting for ex experimental signal (so far only an indirect evidence through Hulse-Taylor pulsars)



Gravitational waves due to a binary system as ripples on spacetime

7. The astrophysical aspect: density $\rho_{\rm vac}$

M. Geller: " There is certainly not a lack a chutzpah in extragalactic astronomy."

If the cosmological constant alone sets only cosmological scales, how does it happen that it can also set relevant astrophysical scales? In case of the density this happens for non-spherical objects with at least two length scales l_1 and l_2. Then in equilibria concepts we often get $(l_1/l_2)^n$.
Gravitational equilibrium via virial equations
The standard non-relativistic virial theorem reads

$$\frac{d^2 I_{jk}}{dt^2} = 4K_{jk} + 2W_{jk}$$

where I_{jk} is the inertial tensor, K_{jk} the kinetic and W_{jk} the gravitational potential tensor. If an external force is exerted on the object, we have to add to the right hand side of the equation the term

$$V_{jk} = -\frac{1}{2}\int \rho\left(x_k\frac{\partial\Phi_{\rm ext}}{\partial x_j} + x_j\frac{\partial\Phi_{\rm ext}}{\partial x_k}\right)d^3x$$

where $\Phi_{\rm ext}$ is the external potential and the case of a cosmological constant corresponds to

$$\Phi_{\rm ext} = -\frac{1}{6}\Lambda r^2$$

Therefore the new virial theorem which accounts for the cosmological constant takes the form

$$\frac{d^2 I_{jk}}{dt^2} = 4K_{jk} + 2W_{jk} + \frac{2}{3}I_{jk}\Lambda$$

It is very often more convenient to consider a less demanding task by simply noting that the trace W of W_{jk} is negative whereas the trace K of K_{jk} is positive definite. Then the gravitational equilibrium i.e. $d^2 I_{jk}/dt^2 = 0$ yields the inequality

$$-\frac{1}{3}\Lambda I + |W| \geq 0$$

where I denotes the trace of the inertial tensor I_{jk}. To appreciate the meaning of this inequality we specialize to the case of constant density. It is then easy to show that

$$8\pi G_N \rho \geq A\Lambda, \ \ \rho \geq A\rho_{\rm vac}$$

where the quantity A depends only on the geometry of the object under consideration.

$$A = \frac{16\pi}{3} \frac{\int r^2 d^3x}{\int \frac{|\Phi_{\rm N}|}{\rho} d^3x}$$

where $\Phi_{\rm N}$ is the Newtonian part of the non-relativistic gravitational potential. For spherically symmetric objects one easily calculates $A = 2$ and therefore the virial inequality is simply

$$4\pi G_N \rho \geq \Lambda, \ \ \rho \geq 2\rho_{\rm vac}$$

Where is the ratio of two length scales to a power n? In principle it is in A if we consider non-spherical objects. Instead of the inequality we can also calculate from the virial theorem the mean velocity of the objects

$$\langle v^2 \rangle = \frac{|W|}{M} - \frac{8\pi}{3}\frac{\rho_{\rm vac}}{M} I$$

To appreciate the effect of Λ let us assume a constant density and the shape of the astrophysical object to be an ellipsoid. The mean velocity can be now written as

$$\langle v^2 \rangle_{\rm ellipsoid} = \frac{32\pi}{45M} \rho \rho_{\rm vac} a_1 a_2 a_3 \left(a_1^2 + a_2^2 + a_3^2\right) \times$$

$$\left(\frac{3}{4}\frac{\rho}{\rho_{vac}}\Gamma_{\rm ellipsoid} - 1\right)$$

The prolate case ($a_1 = a_2 < a_3$, $e = \sqrt{1 - a_1^2/a_3^2}$) gives

$$\Gamma_{\rm prolate} = \frac{\left(\frac{a_1}{a_3}\right)^3}{1 + 2\left(\frac{a_1}{a_3}\right)^2} \frac{\ln\left(\frac{1+e}{1-e}\right)}{e}$$

Note now that for a flattened prolate ellipsoid we can approximate

$$\Gamma_{\rm prolate} \simeq \left(\frac{a_1}{a_3}\right)^3 \ln\left(\frac{1+e}{1-e}\right)$$

Since the nowadays preferred value of $\rho_{\rm vac}$ is $(0.7-0.8)\rho_{\rm crit}$ we can say that if the constant $\rho/\rho_{\rm crit}$ is, say, 10^3, it suffices for the ellipsoid to have the ratio $a_1/a_3 \sim 10^{-1}$ in order that the mean velocity of its components approaches zero. This is valid always under the assumption that the object is in gravitational equilibrium. This effect is due to the relatively large cosmological constant. In general we can say that in flattened astrophysical systems in gravitational equilibrium, the mean velocity gets affected by the cosmological constant. The denser the system, the bigger should be the deviation from spherical symmetry to have a sizeable effect. It is interesting if such an effect can be observed in reality which would confirm the existence of Λ. One can paraphrase this also by looking at the results from a different perspective. If we are certain that a given astrophysical object is in gravitational equilibrium, then the above equations would put a stringent bound on Λ in the case of strong deviation from spherical symmetry. Hydrostatic equilibrium for spherically symmetric objects:

Is the hydrostatic equilibrium related to the gravitational one? Note that in the virial equations used above no pressure appeared (indeed, with pressure the virial equations change), but the hydrostatic equilibrium is

$$\nabla P = -\rho \nabla \Phi, \;\; \nabla^2 \Phi = 4\pi G_N \rho - \Lambda$$

This condition leads to

$$P'(r) = -r\rho(r)\left(G_N \frac{m(r)}{r^3} - \frac{\Lambda}{3}\right)$$

which is sometimes called the “fundamental equation of Newtonian astrophysics”. The mass function is as usual defined by

$$m(r) = \int_0^r 4\pi \rho(s) s^2 ds$$

Furthermore let the mean density be defined by

$$\bar{\rho}(r) = \frac{3}{4\pi}\frac{m(r)}{r^3}$$

Then

$$P'(r) = -r\frac{\rho(r)}{3}\Big(4\pi\bar{\rho}(r) - \Lambda\Big)$$

For any physically reasonable astrophysical object, the pressure and density must be monotonically decreasing functions of the object's radius. Hence negativity of the derivative of the pressure implies (1)

$$\Lambda < 4\pi G_N \bar{\rho}_b$$

This equation we also obtained via the virial condition (2)! Indeed, one can arrive at it also via the general relativistic Tolman-Oppenheimer-Volkoff equation (3)

$$P'(r) = -r\frac{\rho(r)}{3}\Big(1+\frac{P(r)}{\rho(r)}\Big)\left(\frac{12\pi P(r) + 4\pi G_N\bar{\rho}(r) - \Lambda}{1 - \frac{8\pi}{3}\bar{\rho}(r)r^2 - \frac{\Lambda}{3}r^2}\right)$$

(via a positive definite denominator and the boundary condition $P_{\mathrm{boundary}} = 0$) and by demanding stability of circular orbits (4). This calls for two things: include pressure in the virial equations and generalize the hydrostatic equilibrium concept to non spherically symmetric objects.

Let me quote at the end of this section from a book **"The measure of the Universe" by North** which describes the history of cosmology in general and the work of some authors on the cosmological constant: **"The essential difficulty with a relativistic theory in which λ [the Cosmological Constant] is positive is that of accounting for the formation and condensation in terms of gravitational instability; for, to use the *'force'* metaphor, the present expansion indicates that the force of cosmic repulsion exceeds those of gravitational attraction. This is not likely to disturb the stability of systems (such as the galaxy) of high average density, but it is likely to prevent new condensation in regions of low density."**

8. Conclusions

- Due to Λ there can exist uni-verses in the multi-verse which do not have the initial singularity (no Big Bang).

- Combinations of large scales and small scales e.g.

$$r_{\rm max} \sim (r_s r_\Lambda^2)^{(1/3)}, \;\; r_0 = (r_s r_\Lambda)^{(1/2)}$$

  are meanngful quanities and result in astrophysical orders of magnitude
  $\Rightarrow$ Local effects of the cosmlogical constant.

- **DUALITY:**

$$R_{\rm min} \sim 2r_s << r << R_{\rm max} \sim r_\Lambda, \;\; (Newtonian{-}limit)$$

$$L_{\rm min} \sim r_s << L << L_{\rm max} \sim (r_s^2 r_\Lambda)^{(1/3)}, \;\; (Ang.mom.)$$

$$T_{\rm max} \sim m_{\rm pl} \leftrightarrow T_{\rm min} \sim m_\Lambda$$

$$M_{\min} \sim m_{\rm pl} \leftrightarrow M_{\max} \sim m_{\rm pl}^2/m_\Lambda$$

- Gravitational waves:

$$r < \mathcal{L}_{\rm crit}(\Lambda)$$

in order for the 'wavy' character to dominate.

- Gravitational equilibrium:

$$\rho > A\rho_{\rm vac}$$

A can be large if the shape of the object deviates sizeably from spahrical symmetry.

Third attempt:
The fearful sphere or how centuries influence each other

- 500 B.C.Xenophanes of Colophon:
  GOD IS AN ETERNAL SPHERE [enters spherical symmetry]

- 40 years later Parmenides:
  THE DIVINE BEING IS LIKE THE MASS OF A WELL-ROUNDED SPHERE WHOSE FORCE IS CONSTANT FROM THE CENTRE IN ANY DIRECTION

- 300 B.C. Hermes Trismegistus in Corpus Hermeticum, rediscovered by Alain de Lille in 1300 :
  GOD IS AN INTELLIGIBLE SPHERE, WHOSE CENTRE IS EVERYWHERE AND WHOSE CIRCUMFERENCE IS NOWHERE.

- 1580 Giordano Bruno:
  WE CAN ASSERT WITH CERTAINTY THAT

THE UNIVERSE IS ALL CENTRE, OR THAT
THE CENTRE OF THE UNIVERSE IS EVERYWHERE
AND ITS CIRCUMFERENCE NOWHERE.

- 1650 Blaise Pascal:
  NATURE IS AN INFINITE SPHERE WHOSE
  CENTRE IS EVERYWHERE, WHOSE CIRCUMFERENC
  IS NOWHERE.

- today :
  THE UNIVERSE HAS A SPHERICAL SYMMETRY.
  THERE IS NO CENTRE.