Swimming in Curved Space

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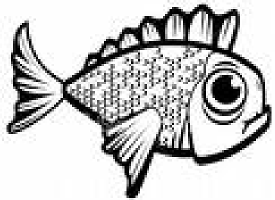
Outline

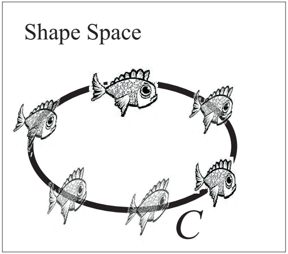
Geometric Swimming

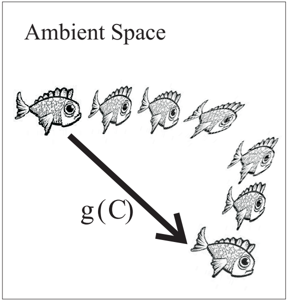
Swimming

The ability of a deformable body to alter its location or orientation relative to an ambient space by controlling variations in its shape.



Geometric Swimming

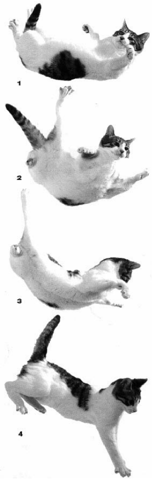




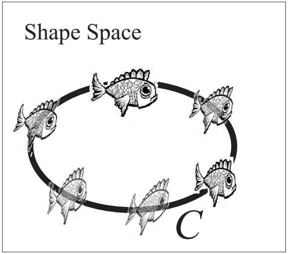
Geometric Swimming

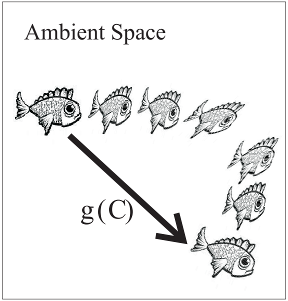
- Location/Orientation of object is only dependent on the sequence of shape changes (i.e., independent of time, external forces, etc...)
- After a complete cycle C of shape changes, object undergoes a net action $g(C)$ of a symmetry group of the ambient space (e.g., Euclidean group)

Some Examples



Geometric Swimming





Two properties:

- Scallop Theorem: Swimming cycle requires enclosing net areas in shape space
- Helix Theorem: Swimming cycle will generally involve both rotations and translations

Naive Treatment

Configuration Description

- "The Body in Space": N mass points in manifold $\mathcal{M}$
- Configuration Space: $\mathcal{Q} = \mathcal{M}^N \equiv \mathcal{M} \times \mathcal{M} \times \ldots \mathcal{M}$

Configuration Variable Split

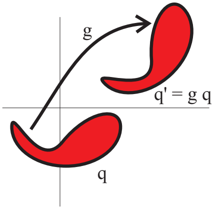
$q \in \mathcal{Q} = (q_{int}, q_{ext})$

- q_{ext}: "External Variables (e.g., Center of Mass Position, Euler Angles)
- q_{int}: "Internal Variables (e.g., relative positions, etc.)

Constraints

$F_i(q_{int}, q_{ext}, \dot{q}_{int}, \dot{q}_{ext}, t) = 0$ yield $q_{ext}(t)$ in terms of history of the $q_{int}(t)$.

Gauge Treatment



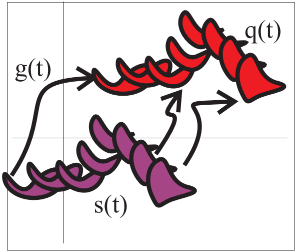
Symmetry group

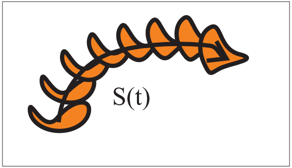
- G: a symmetry group of $\mathcal{M}$
  - e.g. $E(d)$ for $\mathbb{R}^d$
- Action: $G\mathcal{Q} = \mathcal{G}^N\mathcal{M}^N \equiv G\mathcal{M} \times G\mathcal{M} \times \ldots G\mathcal{M}$

Shapes

- "Same Shape" ER: $q \sim q'$ if $q' = gq$ for some $g \in G$
- "Shape Space": $\mathcal{S} \equiv \mathcal{Q}/\sim$

Gauge Treatment





Motion Described by

- A curve in Shape Space $S(t) \in \mathcal{S}$
- An associated curve $s(t) \in \mathcal{Q}$ of representative configurations: $S(t) = [s(t)]$
- A curve $g(t) \in G$ linked to $s(t)$ though non-holonomic constraints
- $q(t) = g(t)s(t)$.

Gauge Freedom

- For $s' = h(S)s$, with $h \in G$
- $q(t) = g'(t)s'(t)$ with $g' = gh^{-1}$

Gauge Treatment

Fiber Bundle Description

- Replace Q by Principal Bundle $P = P(\mathcal{S}, G)$
- Constraints define connection on TP
- $s(t) = \sigma(S(t))$ with σ a local section of P

Gauge Treatment

The Connection one-form

For given σ, constraints define connection one-forms $\mathbf{A}$ on $T^*\mathcal{S}$ valued on the Lie-Algebra $\mathfrak{g}$ of G.

- $g(t)$ Satisfies the equation

$$g^{-1}\frac{dg}{dt} = \langle \mathbf{A}, \frac{d}{dt} \rangle$$

- If $q(0) = s(0)$, integrates on a curve C in shape space to

$$g_C = \overline{P} \exp\left(\int_C \mathbf{A}\right)$$

($\overline{P}$: reverse path ordering)

- Under change of section $s' = h(S)s$,

$$\mathbf{A}' = h\mathbf{A}h^{-1} - \mathbf{d}hh^{-1}$$

Gauge Treatment

Field Strength Two-Form

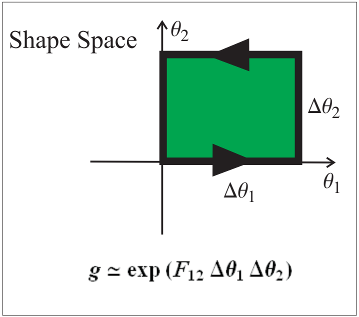
From $\mathbf{A}$ one defines the curvature two-form $\mathbf{F}$

$$\mathbf{F} = \mathbf{DA} = \mathbf{dA} + \mathbf{A} \wedge \mathbf{A}$$

- Under change of section $\mathbf{F}' = h\mathbf{F}h^{-1}$
- For a small, *closed* loop $C = \partial S$ in Shape Space

$$g_C = \overline{P} \exp \left(\oint_C \mathbf{A} \right) \simeq \exp \left(\int_S \mathbf{F} \right)$$

Gauge Treatment



Swimming

- Swimming is possible only if $\mathbf{F} \neq 0$.

$$\mathbf{F} = \sum_{i<j} F_{ij}\,\mathbf{d}\theta^i \wedge \mathbf{d}\theta^j$$

Gauge Treatment

Translational vs. Rotational Swimming

In many cases, $\mathfrak{g} = \mathfrak{t} \oplus \mathfrak{r}$ where

- $\mathfrak{t}$ generates "translations"
- $\mathfrak{r}$ generates "rotations"

$$[\mathfrak{r}, \mathfrak{r}] \subset \mathfrak{r} \quad [\mathfrak{r}, \mathfrak{t}] \subset \mathfrak{t} \quad [\mathfrak{t}, \mathfrak{t}] \subset \mathfrak{r}$$

Thus, $\mathbf{A}$ and $\mathbf{F}$ split as

$$\begin{aligned} \mathbf{A} &= \mathbf{A}^{\mathbf{Trans}} + \mathbf{A}^{\mathbf{Rot}} \\ \mathbf{F} &= \mathbf{F}^{\mathbf{Trans}} + \mathbf{F}^{\mathbf{Rot}} \end{aligned}$$

with

$$\begin{aligned} \mathbf{F}^{\mathbf{Trans}} &= \mathbf{dA}^{\mathbf{Trans}} + \mathbf{2A}^{\mathbf{Rot}} \wedge \mathbf{A}^{\mathbf{Trans}} \\ \mathbf{F}^{\mathbf{Rot}} &= \mathbf{dA}^{\mathbf{Rot}} + \mathbf{A}^{\mathbf{Rot}} \wedge \mathbf{A}^{\mathbf{Rot}} + \mathbf{A}^{\mathbf{Trans}} \wedge \mathbf{A}^{\mathbf{Trans}} \end{aligned}$$

Example: Free Rotational Swimming

Free Fixed Body in Euclidean 3-Space

A free body in Euclidean space conserves angular momentum with respect to CM

$$\vec{L} = \sum_n m_n \vec{x}_n \times \dot{\vec{x}}_n\,, \quad \text{with} \ \sum_n m_n \vec{x}_n = 0$$

Gauge Description

- Configuration Space: $\mathcal{Q} = (\mathbb{R}_3)^N / T$
- Gauge group: $G = SO(3)$
- Define Section: $s(t) = (\vec{z}_1, \vec{z}_2, \ldots, \vec{z}_N)$ with $s(0) = q(0)$

$$\vec{x}_n(t) = R(t)\vec{z}_n(\theta(t))$$

Example: Free Rotational Swimming

Connection (solely rotational)

$\mathbf{A} \equiv \vec{\mathbf{A}} \cdot \vec{M}$, with M_i standard generator matrices of SO(3)

To obtain connection:

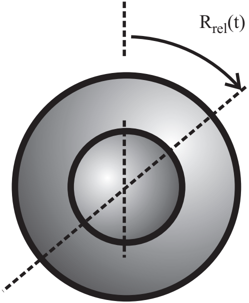
- $R(t)$ satisfies $R^{-1}\dot{R}(t) = \vec{\omega} \cdot \vec{M}$
- When $\vec{L} = 0$,

$$\sum_n m_n \vec{x}_n \times \dot{\vec{x}}_n = 0 \quad \Rightarrow \quad \vec{\omega} = -\mathbb{I}^{-1} \sum_n m_n \vec{z}_n \times \dot{\vec{z}}_n$$

- Hence

$$\vec{\mathbf{A}} = -\mathbb{I}^{-1} \sum_n m_n \vec{z}_n \times \mathbf{d}\vec{z}_n$$

Example: Free Rotational Swimming



Two Concentric Spheres

- Shape Space is $SO(3)$ for relative orientation
- $\mathbf{A}_{rel} = (\mathbf{d}R_{rel})R_{rel}^{-1}$
- $\mathbf{A} = -\frac{I_2}{I_1+I_2}\mathbf{A}_{rel}$
- $\mathbf{F} = -\frac{I_1 I_2}{I_1+I_2}\mathbf{A}_{rel} \wedge \mathbf{A}_{rel} \neq 0$
- Net orientation change of the body is possible by a sequence of relative rotations.

Killing Fields and Symmetric Spaces

Killing Fields

- A symmetric space $\mathcal{M}$ admits a set of $k \leq d(d+1)/2$ Killing vector fields $\overline{K}^{(a)}$ such that

$$\pounds_{\overline{K}^{(a)}}\mathbf{g} = 0$$

- "Geometry is the same along integral curves of $\overline{K}^{(a)}$"

Killing equation

$$K^{(a)}_{\mu;\nu} + K^{(a)}_{\nu;\mu} = 0$$

Killing Fields and Symmetric Spaces

Symmetry

- The Killing fields span the Lie Algebra of $\mathrm{I}so(\mathcal{M})$, the isometry group of $\mathcal{M}$.
- Maximally symmetric spaces: $k = d(d+1)/2$ independent Killing fields

Conservation Laws

Invariance

- Let $q = (x_1, x_2, \ldots x_N)$
- Let the Lagrangian of the body be of the form

$$L(q, \dot{q}) = \frac{1}{2} \sum_n g_{\mu\nu} \dot{x}_n^\mu \dot{x}_n^\nu + V(x_1, \ldots x_N)$$

- Suppose that L is invariant under

$$q \to exp(s\overline{K})q \qquad \forall \overline{K} \in \mathrm{Iso}(\mathcal{M})$$

Conservation Laws

Then the quantities

$$P^{(a)} \equiv \sum_n m_n K_\mu^{(a)} \dot{x}_n^\mu,$$

are constants of the motion.

Gauge description

- Symmetry Group: $G = \text{Iso}(\mathcal{M})$
- Shape Space: $\mathcal{S} = \mathcal{M}^N/G$, with coordinates θ^i
- Section: $x_n = g\, z_n(\theta) \Rightarrow s = (z_1 \ldots z_N)$
- $\mathbf{A} = \mathbf{A}^{(a)} \otimes \overline{K}^{(a)}$

Swimming connection

Using conserved quantities $P^{(a)} = 0$,

$$P^{(a)} = 0 \quad \Rightarrow \quad \sum_n m_n K^{(a)}_\mu(z_n) \left[\frac{d}{dt} + g^{-1}\dot{g} \right] z^\mu_n = 0\,.$$

Hence,

$$\mathbf{A}^{(b)} \sum m_n K^{(a)}_\mu(z_n) \overline{K}^{(b)}(z^\mu) + \sum_n m_n K^{(a)}_\mu(z_n) \mathbf{d} z^\mu_n = 0$$

Swimming connection

Putting Everything together we obtain

$$\mathbf{A}^{(a)} = -(M^{-1})_{ab} \sum_n m_n K_\mu^{(a)}(z_n) \mathbf{d} z_n^\mu$$

Interpret as "Body-Averaged Killing field" where "Inertia Matrix"

$$M_{ab} \equiv \sum_n m_n \left(\mathbf{K}^{(a)} \cdot \mathbf{K}^{(b)} \right) (z_n)$$

Example: Euclidean Space

Killing Vectors

- Translation: $\overline{T}_i = \frac{\partial}{\partial x_i}$
- Rotation: $\overline{M}_i = \epsilon_{ijk} x^j \frac{\partial}{\partial x_k}$

Generate $E(3)$

Gauge

- Choose gauge so that $\sum_n m_n z_n^i = 0$
- Split $\mathbf{A} = \mathbf{A}^{Trans} + \mathbf{A}^{Rot}$

Example: Euclidean Space

- Inertia Matrix

$$\mathbb{M} = \begin{pmatrix} m\mathbf{1} & 0 \\ 0 & \mathbb{I}_{CM} \end{pmatrix} \qquad m = \sum_n m_n$$

- $\sum_n m_n T^{(i)}_\mu \mathbf{d} z^\mu_n = \mathbf{d}(\sum_n m_n z^i) = 0$
- $\sum_n m_n M^{(i)}_\mu \mathbf{d} z^\mu_n = \epsilon_{ijk} \sum_n m_n z^j \mathbf{d} z^j$

Example: Euclidean Space

Connections

- $\mathbf{A}^{Trans} = 0$
- $\vec{\mathbf{A}}^{Rot} = -\mathbb{I}_{CM}^{-1} \sum_n m_n \vec{z}_n \times \mathbf{d}\vec{z}_n$

Swimming

- $\mathbf{F}^{Trans} = 0$
- $\vec{\mathbf{F}}^{Rot} = \mathbf{d}\vec{\mathbf{A}}^{Rot} + \frac{1}{2}\vec{\mathbf{A}}^{Rot} \times \vec{\mathbf{A}}^{Rot}$ ("Falling cat")

Maximally Symmetric 3-Spaces

Constant Curvature Spaces

	$\mathcal{M}$	$Iso(\mathcal{M})$
$K > 0$	S^3	$SO(4)$
$K < 0$	$\mathrm{Hyp}(3)$	$SO(3,1)$
$K = 0$	$\mathbb{R}^3$	$E(3)$

- Riemann Tensor:

$$R_{\mu\nu\lambda\sigma} = K(g_{\mu\lambda}g_{\nu\sigma} - g_{\mu\sigma}g_{\nu\lambda})$$

Maximally Symmetric 3-Spaces

Quasi-Euclidean Coordinates $\vec{x}$

- Locally: $ds^2 = d\vec{x} \cdot d\vec{x} + \frac{K}{1-Kr^2}(\vec{x} \cdot d\vec{x})^2$

  (Euclidean dot product, $r^2 = \vec{x} \cdot \vec{x}$)

Killing Vectors

- Translation: $\overline{T}_i = \sqrt{1 - Kr^2}\frac{\partial}{\partial x_i}$
- Rotation: $\overline{M}_i = \epsilon_{ijk} x^j \frac{\partial}{\partial x_k}$

Maximally Symmetric 3-Spaces

Gauge

- Choose gauge so that

$$\sum_n m_n \sqrt{1 - K r_n^2}\; z_n^i = 0$$

- Inertia Matrix

$$\mathbb{M} = \begin{pmatrix} m\mathbf{1} - K\mathbb{I}_{CM} & 0 \\ 0 & \mathbb{I}_{CM} \end{pmatrix}$$

- $\sum_n m_n T_\mu^{(i)} \mathbf{d} z_n^\mu = \sum_n m_n \sqrt{1 - K r_n^2}\; \mathbf{d} z^i$
- $\sum_n m_n M_\mu^{(i)} \mathbf{d} z_n^\mu = \epsilon_{ijk} \sum_n m_n z^j \mathbf{d} z^j$

Maximally Symmetric 3-Spaces

Connections

- $\vec{\mathbf{A}}^{Trans} = -(m + K\mathbb{I}_{CM})^{-1} \sum_n m_n \sqrt{1 - Kr_n^2}\ \mathbf{d}\vec{z}$
- $\vec{\mathbf{A}}^{Rot} = -\mathbb{I}_{CM}^{-1} \sum_n m_n \vec{z}_n \times \mathbf{d}\vec{z}_n$ (same as Euclidean)

Swimming

- $\vec{\mathbf{F}}^{Trans} = \mathbf{d}\vec{\mathbf{A}}^{Trans} + \vec{\mathbf{A}}^{Rot} \times \vec{\mathbf{A}}^{Trans}$
- $\vec{\mathbf{F}}^{Rot} = \mathbf{d}\vec{\mathbf{A}}^{Rot} + \frac{1}{2}\vec{\mathbf{A}}^{Rot} \times \vec{\mathbf{A}}^{Rot} + \frac{K}{2}\vec{\mathbf{A}}^{Trans} \times \vec{\mathbf{A}}^{Trans}$

Small Bodies

"Small"

$$|z|\sqrt{|K|} \ll 1$$

Translation Swimming

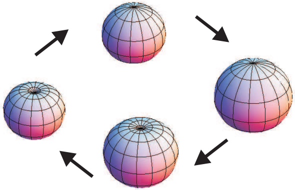
- Connection:

$$\vec{\mathbf{A}}^{Trans} = \frac{K}{2} \sum_n \frac{m_n}{m} \; r_n^2 \; \mathbf{d}\vec{z}_n + O(K^2)$$

- Field Strength:

$$\vec{\mathbf{F}}^{Trans} = \frac{K}{2} \sum_n \frac{m_n}{m} \; \left(\mathbf{d}(r_n^2) + r_n^2 \vec{\mathbf{A}}^{Rot} \times \right) \wedge \mathbf{d}\vec{z}_n$$

Small Deformation of a Uniform Spherical Membrane



$s + p$-wave Deformation

- $\vec{z} = (r_o + \theta_1)\hat{\mathbf{n}} + \theta_2 \left(\hat{\mathbf{k}} - 3(\hat{\mathbf{n}} \cdot \hat{\mathbf{k}})\hat{\mathbf{n}}\right)$
- Satisfies $\vec{\mathbf{A}}^{Rot} = 0$

Translation Swimming

$$\vec{\mathbf{F}}^{Trans} = \frac{2Kr_o}{3} \left(\mathbf{d}\theta_1 \wedge \mathbf{d}\theta_2\right) \hat{\mathbf{k}}$$

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