Black Holes and Wave Mechanics (V)

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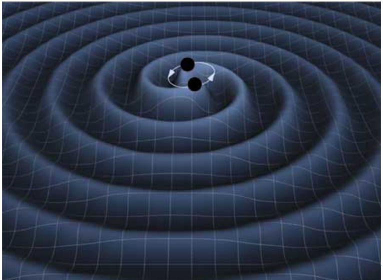
Matematicos de la Relatividad General 08



Course Content

1. Introduction
    - General Relativity basics
    - Schwarzschild's solution
    - Classical mechanics
2. Scalar field + Schwarzschild Black Hole
    - Klein-Gordon equation
    - Wave-packet scattering
    - Quasi-normal modes
3. Hawking Radiation
    - Key results
    - QFT on curved spacetime
    - Black hole collapse model
4. Scattering theory
    - Perturbation theory
    - Partial wave analysis
    - Glories and diffraction patterns
5. Acoustic Black Holes
    - Navier-Stokes eqn $\longrightarrow$ Lorentzian geometry
    - Simple models

Final Lecture

Why study Black Holes and Wave Mechanics?

- Gravitational wave astronomy
- Stability of black hole solutions
- Hawking radiation
- Higher-dimensional black holes at LHC?

Further motivations:

- (I) Scattering patterns from black holes
- (II) Quasi-bound states (?)
- (III) Analogue black holes
- (IV) Self-Force and Radiation Reaction

Quasi-Bound States

- Toy model: Primordial black hole $M \sim 10^{14}$kg + neutrino ν
- Neutrino has small mass $\mu \lesssim 0.01$eV
- Dimensionless mass coupling:

$$\alpha_G = \frac{GM\mu}{\hbar c} = \pi \frac{r_S}{\lambda_C}$$

- Effective potential V_l has local minimum, near the radius of circular orbit.
- Define resonances called Quasi-Bound States (QBSs).

Quasi-Bound States

- QBS satisfy a pair of boundary conditions,

$$u_l \sim \begin{cases} e^{-i\omega r_*}, & r_* \to -\infty \\ Ae^{-qr_*}, & r_* \to +\infty \end{cases}$$

where $q = \sqrt{\mu^2 - \omega^2}$, $-\mathrm{Re}(q) < 0$

- Ingoing at the horizon
- Exponentially decaying at infinity
- In horizon-penetrating coordinates (AEF; PG) the wavefunctions are normalisable.

QBS of Klein-Gordon Equation

- Claim: In the limit $\alpha_G \ll l$, the QBS spectrum is hydrogenic
- In Painlevé-Gullstrand coordinates, the Klein-Gordon equation reads [exercise!] :

$$\left(\partial_t - \sqrt{\frac{2M}{r}}\,\partial_r\right)^2 \Phi - \frac{3}{2r}\sqrt{\frac{2M}{r}}\left(\partial_t - \sqrt{\frac{2M}{r}}\,\partial_r\right)\Phi - \boldsymbol{\nabla}^2\Phi + \mu^2\Phi = 0$$

- Here $\boldsymbol{\nabla}^2 = \partial_i \partial_i$ is 3D Laplacian
- Ingoing solutions are regular at horizon.
- All solutions go as $\Phi \sim r^{-3/4}$ at origin $\Rightarrow r^2|\Phi|^2 \to 0$

Non-relativistic Spectrum

- Split the field Φ into two components χ_1 and χ_2,

$$\chi_1 = \tfrac{1}{2}\left(\Phi + \frac{i}{\mu}\left(\partial_t - \sqrt{\frac{2M}{r}}\,\partial_r\right)\Phi\right), \qquad (1)$$

$$\chi_2 = \tfrac{1}{2}\left(\Phi - \frac{i}{\mu}\left(\partial_t - \sqrt{\frac{2M}{r}}\,\partial_r\right)\Phi\right), \qquad (2)$$

so

$$\chi_1 + \chi_2 = \Phi \qquad \text{and} \qquad \chi_1 - \chi_2 = \frac{i}{\mu}\left(\partial_t - \sqrt{\frac{2M}{r}}\,\partial_r\right)\Phi.$$

- Pair of coupled equations,

$$(i\partial_t - \mu)\,\chi_1 = -\frac{\nabla^2}{2\mu}(\chi_1 + \chi_2) + i\sqrt{\frac{2M}{r}}\,\partial_r\chi_1 + \frac{3i}{4r}\sqrt{\frac{2M}{r}}(\chi_1 - \chi_2)$$

$$(i\partial_t + \mu)\,\chi_2 = +\frac{\nabla^2}{2\mu}(\chi_1 + \chi_2) + i\sqrt{\frac{2M}{r}}\,\partial_r\chi_2 + \frac{3i}{4r}\sqrt{\frac{2M}{r}}(\chi_2 - \chi_1)$$

Non-relativistic Spectrum

- Non-relativistic spectrum: assume $\hbar\omega \sim \mu c^2$ and define $E_{NR} = \hbar\omega - \mu c^2$. Then

$$E_{NR}\chi_1 = -\frac{1}{2\mu}\nabla^2\chi_1 + i\sqrt{\frac{2M}{r}}\left(\partial_r + \frac{3}{4r}\right)\chi_1$$

- Make substitution $\chi_1 = \psi \exp(i\mu\sqrt{8Mr})$ to find Schrödinger equation

$$E_{NR}\,\psi = -\frac{1}{2\mu}\nabla^2\psi - \frac{M\mu}{r}\psi.$$

- Energy levels with gravitational fine-structure constant α_G:

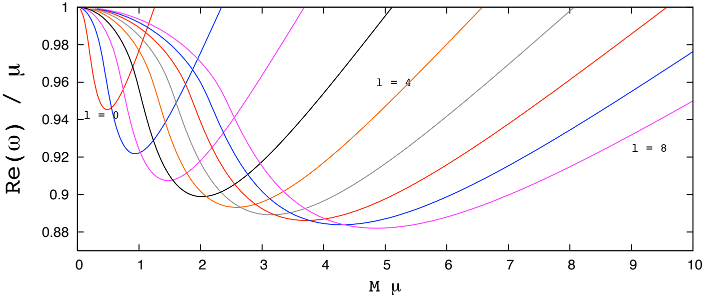
$$\hbar\omega_n \approx \left(1 - \frac{\alpha_G^2}{2n^2}\right)\mu c^2,$$

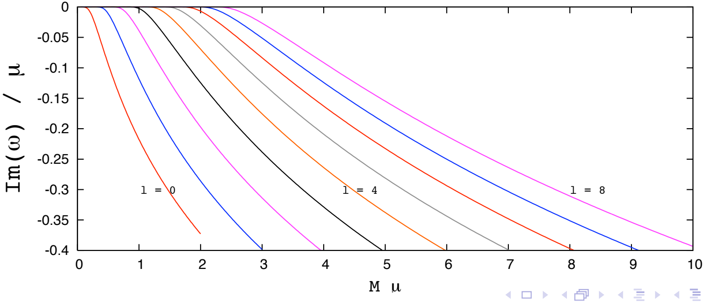
- Hydrogenic wavefunctions.

Relativistic Spectrum

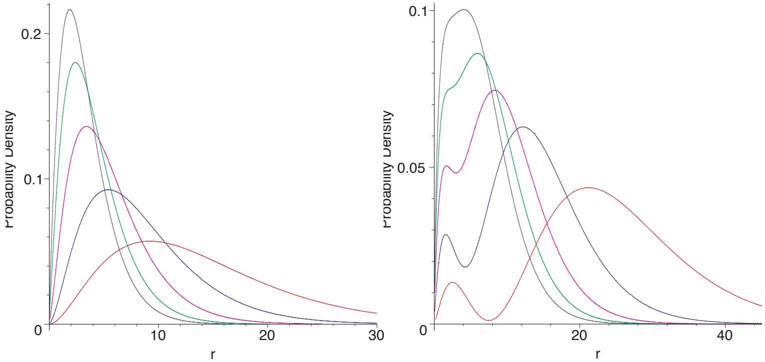
- QBS frequencies are complex, because flux is absorbed at the origin
- $\omega = \pm\omega_r - i\omega_i$
  - Frequency ω_r
  - Decay rate ω_i
- Decay suppressed for $\alpha_G \ll l$, but becomes dominant for $\alpha_G \sim l$.
- Compute spectrum numerically (continued fraction method; direct integration; etc.)
- Need: Existence proofs; bounds; limits; functional analysis etc.

QBS spectrum





QBS Wavefunctions



S-state wavefunctions. Left: 1S wavefunctions in the range $0.1 \leq \alpha \leq 0.3$. Right: 2S wavefunctions in the range $\alpha = 0.2$ – 0.6.

Acoustic Holes

- Under certain assumptions, perturbations to background flow satisfy a 'curved-space' Klein-Gordon equation!

$$\Box\psi \equiv \frac{1}{\sqrt{-g}}\partial_\mu\left(\sqrt{-g}g^{\mu\nu}\partial_\nu\psi\right) = 0$$

  where flow velocity is $\mathbf{v} = \mathbf{v}_0 - \boldsymbol{\nabla}\psi$

- Dumb hole: region of fluid from which no sound may escape (Unruh 1981).
- Horizon: Surface on which speed of sound in fluid = normal bulk flow velocity.
- Ergosphere: region of supersonic flow

Fluid Flow (I)

- Fluid mechanics:
    - (i) continuity equation

$$\partial_t \rho + \nabla \cdot (\rho \mathbf{v}) = 0,$$

    - (ii) Euler's equation

$$\rho \frac{D\mathbf{v}}{Dt} = \rho \left[\partial_t \mathbf{v} + (\mathbf{v} \cdot \nabla) \mathbf{v} \right] = \mathbf{f}$$

- Assume inviscid and 'potential' forces only:

$$\mathbf{f} = -\nabla P - \rho \nabla \Phi$$

- Vector triple product:

$$\mathbf{v} \times (\nabla \times \mathbf{v}) = \frac{1}{2} \nabla v^2 - (\mathbf{v} \cdot \nabla) \mathbf{v}$$

- Assume irrotational : $\nabla \times \mathbf{v} = 0$.

Fluid Flow (II)

- No vorticity $\Rightarrow$ Potential flow: $\mathbf{v} = -\boldsymbol{\nabla}\psi$
- Assume barotropic: density depends on pressure *only* $\rho = \rho(P)$.
- Define enthalpy h

$$h(P) = \int_0^P \frac{dP'}{\rho(P')}$$

so that $h = \frac{1}{\rho}\boldsymbol{\nabla} P$

- Euler's equation $\Rightarrow$ Bernoulli's equation:

$$-\partial_t\psi + h + \frac{1}{2}(\nabla\psi)^2 + \Phi = 0.$$

Perturbations

- Assume background flow + perturbations: $\psi = \psi_0 + \epsilon\psi_1$

$$\Rightarrow P = P_0 + \epsilon P_1 + \mathcal{O}(\epsilon^2), \qquad \rho = \rho_0 + \epsilon P_1 + \mathcal{O}(\epsilon^2),$$

- Linearize :
  - (i) $\partial_t \rho_1 + \nabla \cdot (\rho_1 \mathbf{v}_0 - \rho_0 \nabla\psi_1) = 0$
  - (ii) $\rho_1 = \frac{d\rho}{dP} P_1 = \frac{\rho_0}{c^2}(\partial_t \psi_1 + \mathbf{v}_0 \cdot \nabla \psi_1)$

  where the speed of sound is $c = \left(\frac{d\rho}{dP}\right)^{-1/2}$
- Combine (i) & (ii) in wave equation:

$$-\partial_t \left(\frac{\rho_0}{c^2}(\partial_t\psi_1 + \mathbf{v}_0 \cdot \nabla\psi_1)\right) +$$
$$\nabla \cdot \left(\rho_0 \nabla\psi_1 - \frac{\rho_0}{c^2}\mathbf{v}_0(\partial_t\psi_1 + \mathbf{v}_0 \cdot \nabla\psi_1)\right) = 0.$$

Lorentzian Geometry

- Rewrite perturbation equation in suggestive form:

$$\partial_\mu (f^{\mu\nu} \partial_\nu \psi_1) = 0$$

where

$$f^{\mu\nu}(t, \mathbf{x}) = \frac{\rho_0}{c^2} \left[\begin{array}{c|c} -1 & -v_0^j \\ \hline -v_0^i & (c^2 \delta^{ij} - v_0^i v_0^j) \end{array} \right] .$$

- Let $f^{\mu\nu} = \sqrt{-g}\, g^{\mu\nu}$ where $\det(f^{\mu\nu}) = \det(g_{\mu\nu}) = -\rho_0^4 / c^2$

Effective Metric

- $\Rightarrow$ Perturbation equation

$$\Box\psi \equiv \frac{1}{\sqrt{-g}}\partial_\mu\left(\sqrt{-g}g^{\mu\nu}\partial_\nu\psi\right) = 0$$

- where the acoustic metric is

$$g_{\mu\nu}(t,\mathbf{x}) = \frac{\rho_0}{c}\left[\begin{array}{ccc} -(c^2 - v_0^2) & \vdots & -v_0^j \\ \cdots\cdots & \cdot & \cdots\cdots\cdots\cdots\cdots \\ -v_0^i & \vdots & \delta_{ij} \end{array}\right].$$

- and the line element is

$$ds^2 = \frac{\rho_0}{c}\left[-c^2dt^2 + (dx^i - v_0^i dt)\delta_{ij}(dx^j - v_0^j dt)\right].$$

Comments

- Flat spacetime $\eta_{\mu\nu}$, but perturbations couple to effective metric $g_{\mu\nu}$.
- Laboratory frame.
- Newtonian time coordinate t
- For fluids, $c_{sound} \ll c_{light}$
- Horizon: normal flow speed = speed of sound.
- Quantize perturbations $\Rightarrow$ phonons $\Rightarrow$ Hawking radiation

  . . .
- but no equivalent "Laws of Acoustic Hole Mechanics".

Example I: Canonical Acoustic Hole

- Spherically-symmetric flow in 3D.
- Source (+) or sink (-) at origin $r = 0$
- Conservation of fluid $\Rightarrow v_r = \pm c r_h^2 / r^2$ where r_h is a constant
- Acoustic line element

$$ds^2 = -c^2 dt^2 + \left(dr^2 \pm \frac{r_h^2}{r^2} c dt \right)^2 + r^2 \left(d\theta^2 + \sin^2\theta d\phi^2 \right).$$

- Horizon at $r = r_h$
- Like Painlevé-Gullstrand BH metric $\left(\frac{r_h^2}{r^2} \leftrightarrow \sqrt{\frac{2M}{r}} \right)$

Example I: Canonical Acoustic Hole

- Change to diagonal metric with new (*non-physical*) time coordinate

$$c d\bar{t} = c dt \pm \frac{r_h^2}{r^2}(1 - r_h^4/r^4)^{-1} dr$$

- Schwarzschild-like line element:

$$ds^2 = -c^2 f(r) d\bar{t}^2 + f^{-1}(r) dr^2 + r^2(d\theta^2 + \sin^2\theta d\phi^2)$$

where $f(r) = 1 - r_h^4/r^4$.

- Transformation is singular at horizon

Hawking radiation

- Quantized perturbations ⇒ phonons ⇒ Hawking radiation
- Hawking temperature:

$$k_B T_H = \frac{\hbar g_h}{2\pi c}$$

- Speed of sound c not light
- Surface gravity

$$g_H = \tfrac{1}{2} \frac{\partial(c^2 - v_\perp^2)}{\partial n}$$

- Temperature of canonical acoustic hole:

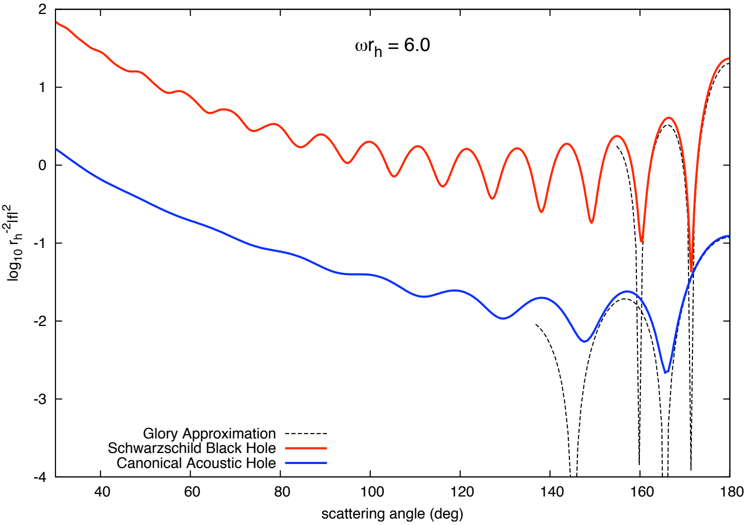
$$T_H \approx 1.2 \times 10^{-9}\,\mathrm{K\,m}\,(c/1000\mathrm{ms}^{-1}) \left(c^{-1} \frac{dv_\perp}{dn} \right)$$

- Cold!

Analogue Models

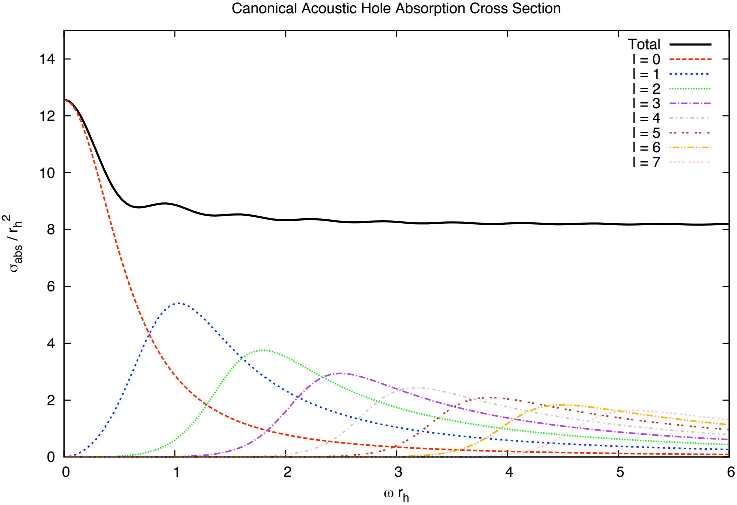
- A range of other black hole analogues have been proposed
  - Bose-Einstein condensates
  - Superfluid Helium
  - Electromagnetic waveguides
  - Optical fibres
- Hawking radiation not yet measured.
- e.g. Bose-Einstein condensate:
  - $T_H \sim 10\text{nK}$
  - $T_{background} \sim 100\text{nK}$
- Classical wave phenomena should be easier to observe.

Wave Scattering by an Acoustic Hole



[Dolan, Crispino and Oliveira 2008]

Wave Absorption by an Acoustic Hole



[Crispino, Oliveira and Matsas 2008]

Example II: Draining Bathtub Model

- Simple circulating fluid flow:

$$\mathbf{v}_0 = \frac{A\hat{r} + B\hat{\theta}}{r}$$

- Constants of flow A, B
- Irrotational ($\nabla \times \mathbf{v} = 0$) except at the vortex core
- Potential is discontinuous on passing through 2π

$$\psi_0(r, \theta) = A \ln(r/a) + B\theta$$

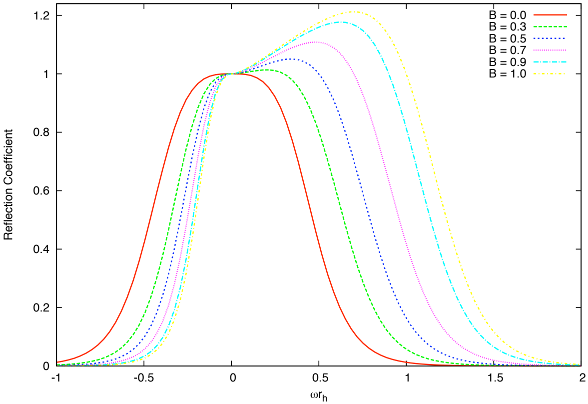
- Line element in cylindrical polars r, θ, z:

$$ds^2 = -c^2 dt^2 + \left(dr - \frac{A}{r} dt\right)^2 + \left(r d\theta - \frac{B}{r} dt\right)^2$$

Example II: Draining Bathtub Model

- Horizon $r_h = |A|/c$
- Ergosphere $r_{erg} = \sqrt{A^2 + B^2}/c$
- Inside ergosphere, all perturbations are co-rotating with circulating fluid.
- Ergosphere $\Rightarrow$ Superradiance: stimulated emission of radiation.
- Perturbation + Superradiance: outgoing flux can exceed ingoing flux
- Rotational energy extracted from the background flow.
- Superradiance also occurs for a rotating (Kerr) black hole
- Superradiance + bound states $\Rightarrow$ Stability?

Superradiance



Superradiance in the first co-rotating mode $n = 1$ as a function of perturbation frequency ωr_h and rotation rate $B = 0 \ldots 1.0$

Conclusion

Black holes + wave mechanics = a range of topics for study

- GR + Geodesics : Gravitational Lensing
- GR + field (classical) : Gravitational Wave Astronomy
    - Black hole mergers
    - QNM signatures
    - Diffraction scattering patterns
- GR + field (quantum) :
    - Particle creation
    - Hawking radiation
    - Higher-dimensional black holes at LHC?
    - Quasi-Bound States?
- Non-Commutative GR : all of the above and more!
- Acoustic holes: black hole analogues in the lab?

Questions, comments, corrections: sam.dolan@ucd.ie