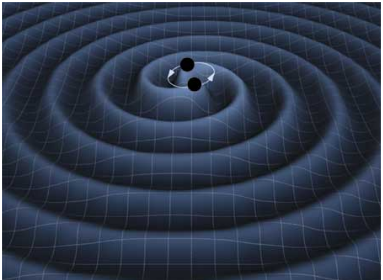
Black Holes and Wave Mechanics (IV)

Dr. Sam R. Dolan

University College Dublin
Ireland

Matematicos de la Relatividad General 08



Course Content

1. Introduction
   - General Relativity basics
   - Schwarzschild's solution
   - Classical mechanics
2. Scalar field + Schwarzschild Black Hole
   - Klein-Gordon equation
   - Wave-packet scattering
   - Quasi-normal modes
3. Hawking Radiation
   - Key results
   - QFT on curved spacetime
   - Black hole collapse model
4. Scattering theory
   - Perturbation theory
   - Partial wave analysis
   - Glories and diffraction patterns
5. Acoustic Black Holes
   - Navier-Stokes eqn $\rightarrow$ Lorentzian geometry
   - Simple models

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Recap

- Scalar wave:

$$\Phi = \sum_{lm} \int d\omega \frac{u_l(r)}{r} Y_{lm}(\theta, \phi) e^{-i\omega t}$$

- Ordinary differential equation:

$$\frac{d^2 u_l}{dr_*^2} + \left[\omega^2 - V_l(r)\right] u_l = 0,$$

- Effective potential

$$V_l(r) = \left(1 - \frac{2M}{r}\right)\left(\frac{l(l+1)}{r^2} + \frac{2M}{r^3} + m^2\right).$$

Current

- Near the horizon, two independent solutions:

$$u_l \sim e^{\pm i\omega r_*} \quad \text{as} \quad r \rightarrow 2M$$

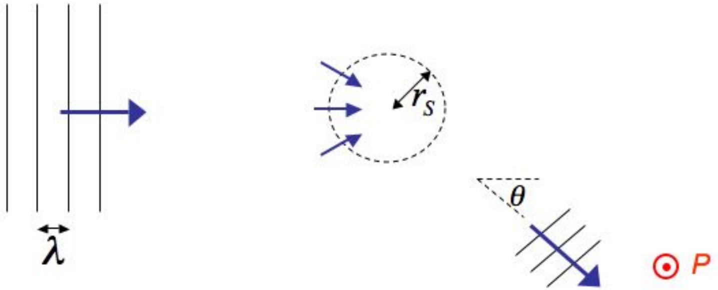
- Infalling observer $\dot{x}^\mu = [(1-2M/r)^{-1}, -\sqrt{2M/r}, 0, 0]$ measures a time-like component

$$\dot{x}^\mu J_\mu \sim \omega \left(1 \mp \sqrt{\frac{2M}{r}}\right)^{-1} \quad \text{as} \quad r \rightarrow 2M$$

- At $r = 2M$, this is regular for ingoing $e^{-i\omega r_*}$ solution ...
- ... but divergent for outgoing solution $e^{i\omega r_*}$.
- $\Rightarrow$ Ingoing boundary condition at $r = 2M$.
- Horizon acts like a one-way membrane.

Time-Independent Scattering

Let a long-lasting monochromatic plane wave impinge upon an isolated black hole:



Time-Independent Plane Wave Scattering

Dimensionless Parameter:

- Coupling strength : $GM\omega/c^3 \sim M\omega \sim \pi r_s/\lambda$

Physical Observables:

- σ_a : absorption cross section.
- $\frac{d\sigma}{d\Omega}$: differential scattering cross section.
- $0 < P < 1$: partial polarisation.

Weak-Field Approximations: $\lambda \gg r_s$

In the long-wavelength limit (low coupling $M\omega \ll 1$), can use perturbation theory to show:

	scattering cross section: $\frac{1}{M^2}\frac{d\sigma}{d\Omega}$	
scalar	$\frac{1}{\sin^4(\theta/2)}$	
neutrino	$\frac{\cos^2(\theta/2)}{\sin^4(\theta/2)}$	
photon	$\frac{\cos^4(\theta/2)}{\sin^4(\theta/2)}$	
gravitational	$\frac{\cos^8(\theta/2)+\sin^8(\theta/2)}{\sin^4(\theta/2)}$	: extra term!

General rule:

$$\lim_{\lambda\to\infty}\left(\frac{1}{M^2}\frac{d\sigma}{d\Omega}\right) \sim \frac{\cos^{4s}(\theta/2)}{\sin^4(\theta/2)} + \delta_{s,2}\sin^4(\theta/2)$$

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Perturbation Theory

- In limit $M\omega \ll 1$, may treat scattering perturbatively.
- Treat BH as interaction potential on a *flat* background.
- Expand in Born series :

$$\frac{d\sigma}{d\Omega} = \left(\frac{GM}{c^2}\right)^2 \left[a_0(v,\theta) + (M\omega)a_1(v,\theta) + (M\omega)^2 a_2(v,\theta) + \ldots\right],$$

  where a_i are dimensionless functions.

- Many coordinate systems ...
- ... but need gauge-invariant result.

Perturbation Method for Scalar Field (I)

- Assume KG equation can be written:

$$\partial^{\mu}\partial_{\mu}\Phi + m^2\Phi + \hat{B}\Phi = 0$$

- where interaction term

$$\hat{B}\Phi = (-g)^{-1/2}\partial_{\mu}\left[(-g)^{1/2}\left(g^{\mu\nu} - \eta^{\mu\nu}\right)\partial_{\nu}\Phi\right]$$

  is in some sense small (warning: not true at origin!)
- Assume time-dependence $e^{-i\omega t}$, so that $\hat{B}$ is a function of r only.

Perturbation Method for Scalar Field (II)

- The propagator (Green's function) Δ_G is defined by

$$\left[(\partial_\mu \partial^\mu)_{x_2} + m^2 + \hat{B}(x_2)\right] \Delta_G(x_2, x_1) = \delta^4(x_2 - x_1)\,.$$

and appropriate boundary conditions

- The propagator may be expanded in a perturbation series,

$$\Delta_G(x_f, x_i) = \Delta_F(x_f, x_i) - \int d^4x_1 \Delta_F(x_f, x_1)\hat{B}(x_1)\Delta_F(x_1, x_i) +$$
$$\int\int d^4x_1 d^4x_2 \Delta_F(x_f, x_1)\hat{B}(x_1)\Delta_F(x_1, x_2)\hat{B}(x_2)\Delta_F(x_2, x_i) + \ldots$$

where Δ_F is the flat-space Feynman propagator.

Perturbation Method for Scalar Field (III)

- The Feynman propagator is simplest in momentum space:

$$\Delta_F(x_2, x_1) = \int \frac{d^4k}{(2\pi)^4} \Delta_F(k) e^{-ik\cdot(x_2 - x_1)}, \quad \Delta_F(k) = \frac{1}{k^2 - m^2}.$$

- To find $\Delta_F(x_2, x_1)$, construct contour with correct causal behaviour. Easiest to do calculation in momentum space.
- Scattering amplitude:

$$\mathcal{M} = B(\boldsymbol{p}_f, \boldsymbol{p}_i) + \int \frac{d^3k}{(2\pi)^3} B(\boldsymbol{p}_f, \boldsymbol{k}) \frac{1}{k^2 - m^2} B(\boldsymbol{k}, \boldsymbol{p}_i) + \ldots$$

- where $B(\boldsymbol{p}_f, \boldsymbol{p}_i)$ is the Fourier transform of the interaction term,

$$B(\boldsymbol{p}_2, \boldsymbol{p}_1) = \int d^3x \, e^{ip_2\cdot x} \hat{B}(x) e^{-ip_1\cdot x} .$$

Perturbation Method for Scalar Field (IV)

- Scattering amplitude:

$$\mathcal{M} = B(\boldsymbol{p}_f, \boldsymbol{p}_i) + \int \frac{d^3k}{(2\pi)^3} B(\boldsymbol{p}_f, \boldsymbol{k}) \frac{1}{k^2 - m^2} B(\boldsymbol{k}, \boldsymbol{p}_i) + \dots$$

- Differential scattering cross section:

$$\frac{d\sigma}{d\Omega} = \frac{1}{4} \frac{|\mathcal{M}|^2}{(2\pi)^2} \,.$$

- Attempt 1st order calculation in AEF and PG coordinates.

Amplitude in AEF coords (I)

- Difference between $g^{\mu\nu}$ and $\eta^{\mu\nu}$ in *tt*, *tr*, *rt* and *rr* components:

$$g^{\mu\nu} - \eta^{\mu\nu} = \frac{2M}{r} \begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix}.$$

- Three terms: $B_{tt}, B_{tr+rt}, B_{rr}$
- Fourier transform of B_{tt} term:

$$B_{tt}(\boldsymbol{p}_f, \boldsymbol{p}_i) = -\omega^2 \int d^3x \left(\frac{2M}{r}\right) e^{-i\boldsymbol{q}\cdot\boldsymbol{x}} = \frac{-8\pi\omega^2 M}{|\boldsymbol{q}|^2},$$

where $\boldsymbol{q} = \boldsymbol{p}_f - \boldsymbol{p}_i$.

Amplitude in AEF coords (II)

- The B_{rr} term may be simplified using integration by parts,

$$B_{rr}(\boldsymbol{p}_f, \boldsymbol{p}_i) = -\int d^3x \frac{\partial}{\partial r}\left(e^{-i\boldsymbol{p}_f\cdot\boldsymbol{x}}\right)\left(\frac{2M}{r}\right)\frac{\partial}{\partial r}\left(e^{i\boldsymbol{p}_i\cdot\boldsymbol{x}}\right)$$
$$= 2M\int (i\boldsymbol{p}_f\cdot\boldsymbol{x})(i\boldsymbol{p}_i\cdot\boldsymbol{x})\frac{e^{-i\boldsymbol{q}\cdot\boldsymbol{x}}}{r^3}.$$

- Integral may be found with sum and difference vectors

$$\boldsymbol{R} = \tfrac{1}{2}\left(\boldsymbol{p}_f + \boldsymbol{p}_i\right), \quad \mathbf{Q} = \tfrac{1}{2}\left(\boldsymbol{p}_f - \boldsymbol{p}_i\right).$$

- Hence

$$B_{rr}(\boldsymbol{p}_f, \boldsymbol{p}_i) = -2M\int d^3x \left[(\boldsymbol{R}\cdot\boldsymbol{x})^2 - (\mathbf{Q}\cdot\boldsymbol{x})^2\right]\frac{e^{-i\boldsymbol{q}\cdot\boldsymbol{x}}}{r^3}.$$

Amplitude in AEF coords (III)

- Align the z-axis with $\mathbf{Q}$ and the x-axis with $\boldsymbol{R}$:

$$\int d^3x \frac{(\boldsymbol{R}\cdot\boldsymbol{x})^2 e^{-i\boldsymbol{q}\cdot\boldsymbol{x}}}{r^3} = \frac{4\pi|\boldsymbol{R}|^2}{|\boldsymbol{q}|^2}$$

and

$$\int d^3x \frac{(\mathbf{Q}\cdot\boldsymbol{x})^2 e^{-i\boldsymbol{q}\cdot\boldsymbol{x}}}{r^3} = -\frac{4\pi|\mathbf{Q}|^2}{|\boldsymbol{q}|^2}.$$

- results in

$$B_{rr}(\boldsymbol{p}_f,\boldsymbol{p}_i) = -\frac{8\pi M|\boldsymbol{p}|^2}{|\boldsymbol{q}|^2},$$

- B_{rt+tr} does not contribute at first order [Exercise!]

Cross Section in AEF coords

- $B_{tt} + B_{rr} \Rightarrow$

$$\mathcal{M}_1 = \frac{-8\pi M\left(\omega^2 + |\boldsymbol{p}|^2\right)}{|\boldsymbol{p}_f - \boldsymbol{p}_i|^2}.$$

- Scattering cross section:

$$\boxed{\frac{d\sigma}{d\Omega} = \left(\frac{GM}{c^2}\right)^2 \frac{\left(1+v^2\right)^2}{4v^4 \sin^4(\theta/2)}.}$$

- Now try same calculation in Painlevé-Gullstrand coordinates ...

Amplitude in PG coords (I)

- Difference between $g^{\mu\nu}$ and $\eta^{\mu\nu}$ in *tt*, *tr*, *rt* and *rr* components:

$$g^{\mu\nu} - \eta^{\mu\nu} = \begin{pmatrix} 0 & -\sqrt{\frac{2M}{r}} \\ -\sqrt{\frac{2M}{r}} & \frac{2M}{r} \end{pmatrix}.$$

- B_{rr} term is identical to AEF calculation
- B_{rt+tr} scales with square root $\sqrt{M}$

$$[B_{tr} + B_{rt}](\boldsymbol{p}_2, \boldsymbol{p}_1) = 2\sqrt{2M} i\omega \int d^3x e^{-i\boldsymbol{p}_2\cdot\boldsymbol{x}} \frac{1}{r^{1/2}} \left(\frac{\partial}{\partial r} + \frac{3}{4r} \right) e^{i\boldsymbol{p}_i\cdot\boldsymbol{x}}$$
$$= 6i\omega\sqrt{M}\pi^{3/2} \frac{\boldsymbol{p}_2{}^2 - \boldsymbol{p}_1{}^2}{|\boldsymbol{p}_2 - \boldsymbol{p}_1|^{7/2}}. \qquad (1)$$

Zero at first order because $\boldsymbol{p}_f^2 = \boldsymbol{p}_i^2$.

Amplitude in PG coords (II)

- To find the extra term that scales with M, go to second order:

$$\left(6i\omega\sqrt{M}\pi^{3/2}\right)^2 I, \quad I = \int \frac{d^3k}{(2\pi)^3} \frac{(\boldsymbol{p}^2 - \boldsymbol{k}^2)}{|\boldsymbol{p}_f - \boldsymbol{k}|^{7/2}} \left(\frac{1}{k^2 - m^2}\right) \frac{(\boldsymbol{k}^2 - \boldsymbol{p}^2)}{|\boldsymbol{k} - \boldsymbol{p}_i|^{7/2}}.$$

- Integral can be found with in centre-of-mass frame using spheroidal coordinates

$$I = \frac{2}{9} \frac{1}{(2\pi)^2 \mathbf{Q}^2}. \qquad (2)$$

- Hence the contribution from $tr + rt$ terms is just

$$-\frac{8\pi M\omega^2}{|\boldsymbol{q}|^2}.$$

- The PG amplitude is identical to the AEF amplitude at first order

Partial Wave Analysis

- Perturbation theory is only appropriate when coupling is small, $M\omega \ll 1$.
- If the wavelength is similar to the horizon size, partial wave approach is best.
- A plane wave can be decomposed into partial wave modes:

$$\Phi_{\text{plane}} = e^{ipz} \sim \frac{1}{2ipr} \sum_{l=0}^{\infty} (2l+1) P_l(\cos\theta) \left[e^{ipr} + (-1)^{l+1} e^{-ipr} \right].$$

- Idea: asymptotically, as $r \to \infty$, solution looks like plane wave + outgoing scattered wave.
- Not quite possible because gravitational force is long-ranged $1/r$ like Coulomb force
- Solutions $u_l \sim e^{\pm ipr_*}$.

Partial Wave Analysis

- Distorted plane wave:

$$\Phi^{\text{dist.}}_{\text{plane}} = \frac{1}{2ipr} \sum_{l=0}^{\infty} (2l+1) P_l(\cos\theta) \left[e^{ipr_*} + (-1)^{l+1} e^{-ipr_*} \right].$$

- $e^{ipr_*} \sim r^{2iMp} e^{ipr}$: extra phase factor which bends wavefronts even in far field.
- Construct partial wave solution:

$$\Phi \sim \Phi^{\text{dist.}}_{\text{plane}} + \frac{f(\theta)}{r} e^{ipr_*} = \frac{1}{r} \sum_{l=0}^{\infty} a_l (2l+1) P_l(\cos\theta) \phi^{(\text{in})}_l (r_*).$$

- Find a_l by matching to ingoing part of plane wave.

Phase shifts δ_l

- Scattering amplitude f is a partial wave series:

$$f(\theta) = \frac{1}{2i\omega} \sum_{l=0}^{\infty} (2l+1) \left(e^{2i\delta_l} - 1 \right) P_l(\cos\theta).$$

  where the phase shifts $e^{2i\delta_l}$ are defined by

$$e^{2i\delta_l} = (-1)^{l+1} \frac{A_{\text{out}}}{A_{\text{in}}}$$

- The phase shifts encode all information about the scattering.

Scattering and Absorption

- The differential scattering cross section is

$$\frac{d\sigma}{d\Omega} = |f(\theta)|^2$$

- The absorption cross section is

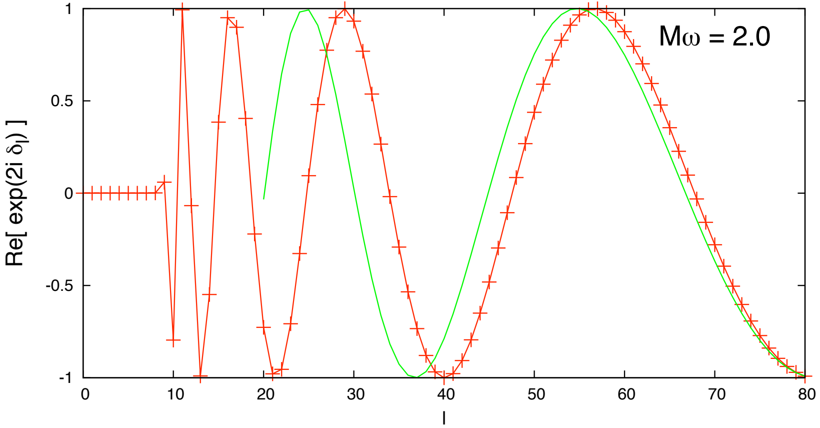
$$\sigma_{\text{abs}} = \frac{\pi}{\omega^2} \sum_{l=0}^{\infty} (2l+1)\mathbb{T}_l$$

where the transmission factors $\mathbb{T}_l$ are

$$\mathbb{T}_l = 1 - \left| e^{2i\delta_l} \right|^2$$

Phase shifts

- Phase shifts can be approximated analytically (in regimes $M\omega \ll 1$ or $M\omega \gg 1$) ...
- ... or computed numerically:



Absorption Cross Sections

- $\sigma_a = \sum \frac{\pi}{\omega^2} \sum_{l=0}^{\infty} (2l+1) \mathbb{T}_l$:

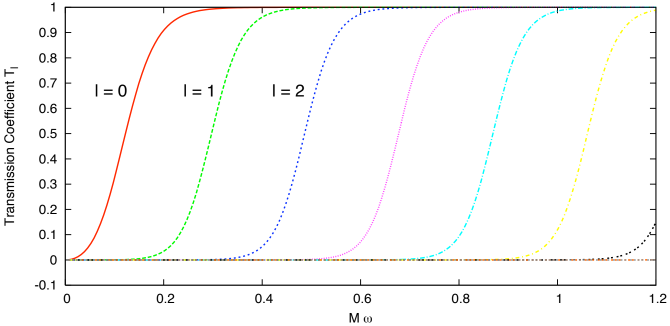


Figure: *Modal transmission factors for the massless scalar wave absorbed by a Schwarzschild black hole.*

Absorption Cross Sections (II)

- Total cross section from sum over modes:

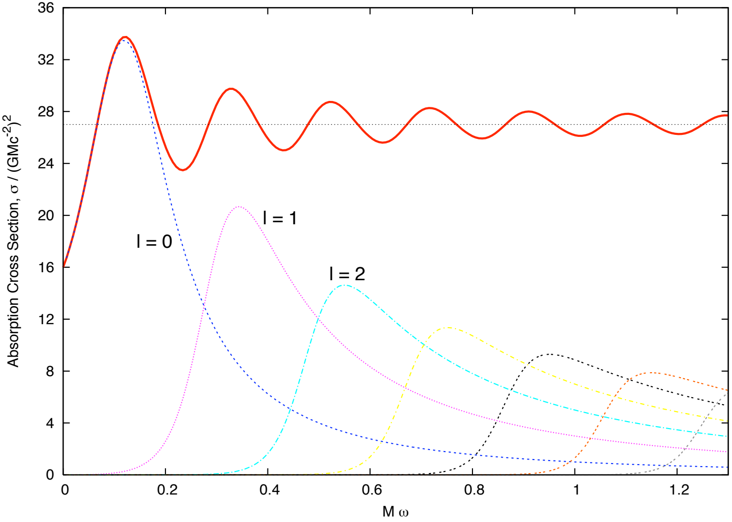


Figure: *Absorption cross section for a massless scalar wave*

Absorption Cross Sections (III)

- Two interesting limits:

$$\sigma_a \sim \begin{cases} 16\pi M^2 & \lambda \gg r_s 0 \\ 27\pi M^2 & \lambda \ll r_s \to \infty \end{cases}$$

- Short-wavelength limit $\Leftrightarrow$ Geometric optics πb_c^2 .
- Long-wavelength limit depends on spin of field :

$$\lim_{M\omega \to 0} \sigma_a = \begin{cases} 16\pi M^2 & s = 0 \\ 2\pi M^2 & s = 1/2 \\ 0 & s = 1 \text{ or } 2 \end{cases}$$

- Das + Gibbons: Scalar cross section approaches horizon area as $M\omega \to 0$ for all black holes.
- Approximations by Starobinskii & Churilov or Unruh (1976).

Absorption Cross Sections (IV)

- South America-UK collaboration:

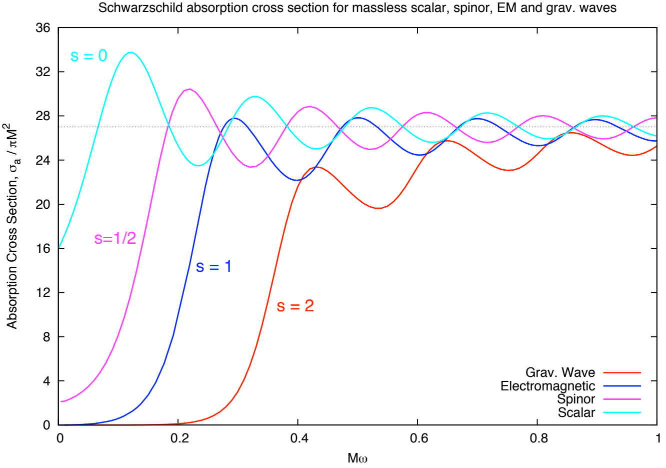
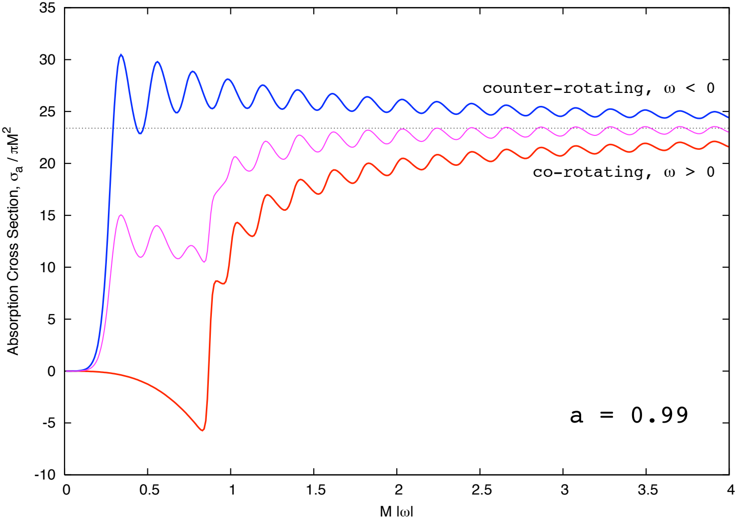


Figure: *σ_a for massless scalar, neutrino, electromagnetic and gravitational waves.*

Absorption Cross Sections (V)

- If black hole is rotating, spin-rotation coupling.



Scattering Cross Sections (I)

- Scattering amplitude

$$f(\theta) = \frac{1}{2i\omega} \sum_{l=0}^{\infty} (2l+1) \left(e^{2i\delta_l} - 1 \right) P_l(\cos\theta).$$

- Series is poorly convergent.
- Why? Amplitude is divergent as θ^{-2} at small angles $\Leftrightarrow$ long-ranged interaction $\Leftrightarrow$ poorly-convergent infinite series.
- Same problem for Coulomb scattering.

$$e^{2i\delta_l^N} = \frac{\Gamma(l+1-i\beta)}{\Gamma(l+1+i\beta)} \sim \beta \ln(l) \text{ as } l \to \infty$$

where $\beta = Z\alpha m/p \Leftrightarrow GM\omega/c^3$.

Scattering Cross Sections (II)

- But Coulomb case has exact solution

$$f_N(\theta) \equiv \frac{1}{2ip}\sum_{l=0}\frac{\Gamma(l+1-i\beta)}{\Gamma(l+1+i\beta)}(2l+1)P_l(\cos\theta) \quad (3)$$

$$= \frac{\beta}{2p}\frac{\Gamma(1-i\beta)}{\Gamma(1+i\beta)}\left[\sin(\theta/2)\right]^{-2+2i\beta}. \quad (4)$$

- Cross section :

$$\frac{d\sigma}{d\Omega} = \frac{\beta^2}{4p^2\sin^4(\theta/2)} = \frac{M^2(1+v^2)^2}{4v^4\sin^4(\theta/2)}. \quad (5)$$

- Idea: use this result to remove the long-range effect of Newtonian potential, leaving a convergent series.

Scattering Cross Sections (III)

- This method works well for scalar wave, but not for waves of higher spin.
- Alternative method : Series reduction method (1950s).
- Given a Legendre polynomial series

$$f(\theta) = \sum_{l=0} a_l^{(0)} P_l(\cos\theta)$$

divergent at $\theta = 0$, define the mth reduced series,

$$(1 - \cos\theta)^m f(\theta) = \sum_{l=0} a_l^{(m)} P_l(\cos\theta).$$

- The reduced series is obviously less divergent at $\theta = 0 \Leftrightarrow$ better convergence.
- Iterative formulae:

$$a_l^{(i+1)} = a_l^{(i)} - \frac{l+1}{2l+3} a_{l+1}^{(i)} - \frac{l}{2l-1} a_{l-1}^{(i)}.$$

Scattering Cross Sections: Results

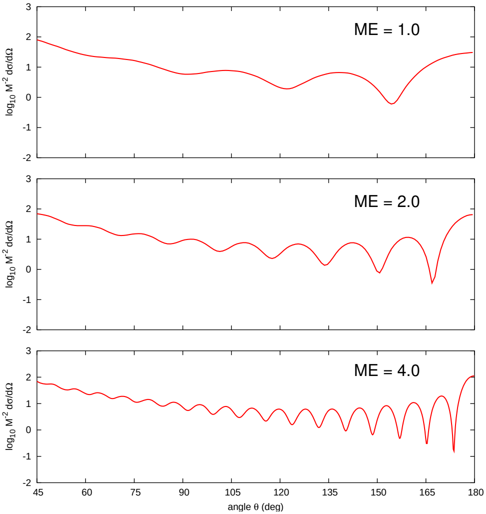


Figure: *Scalar scattering cross sections*. Log scale.

Scattering Cross Sections: Results

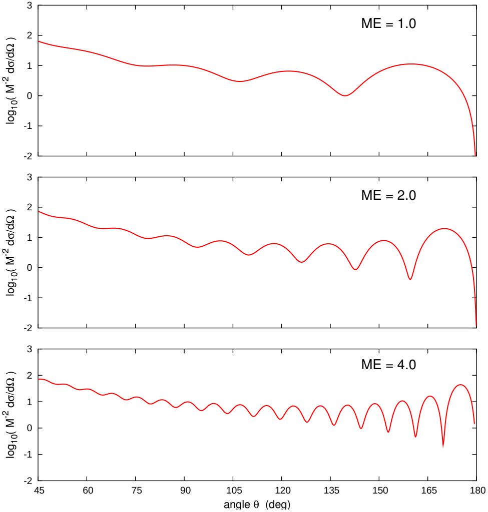


Figure: *Spinor scattering cross sections*. Shows the massless spin-half cross section for various couplings $ME \equiv GM\omega/c^3$.

Scattering Cross Sections: Results

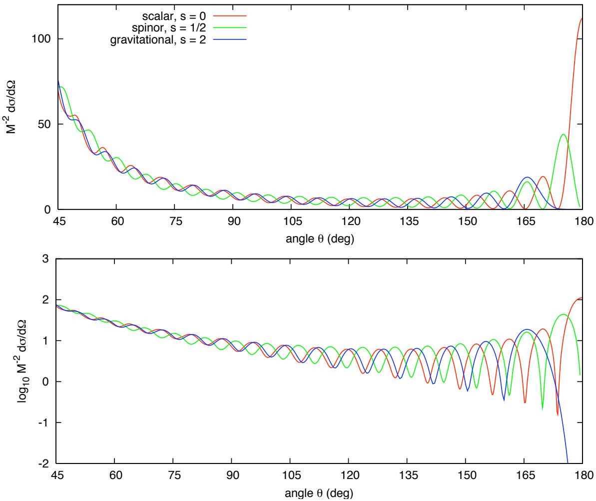
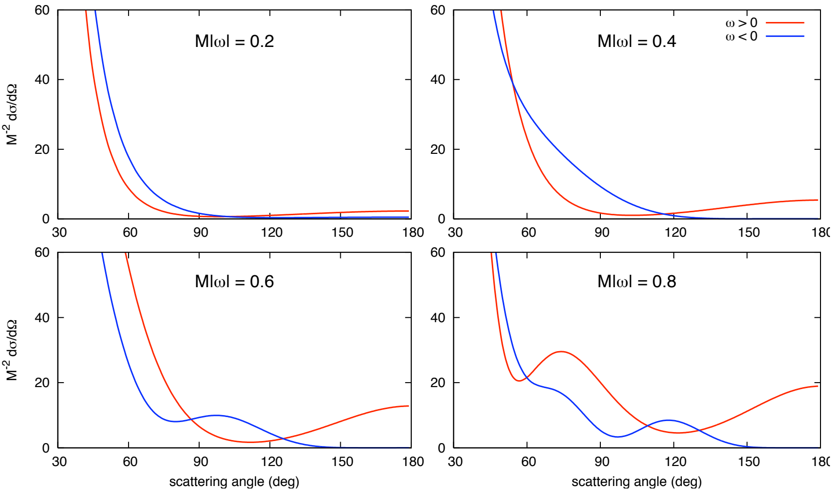


Figure: *Various spins*. The $s = 1$ curve is *missing*: but Luis and Ednilton will fix this!

Scattering Cross Sections: Rotating Case

$a = 0.99M$, lower couplings



Scattering Cross Sections: Rotating Case

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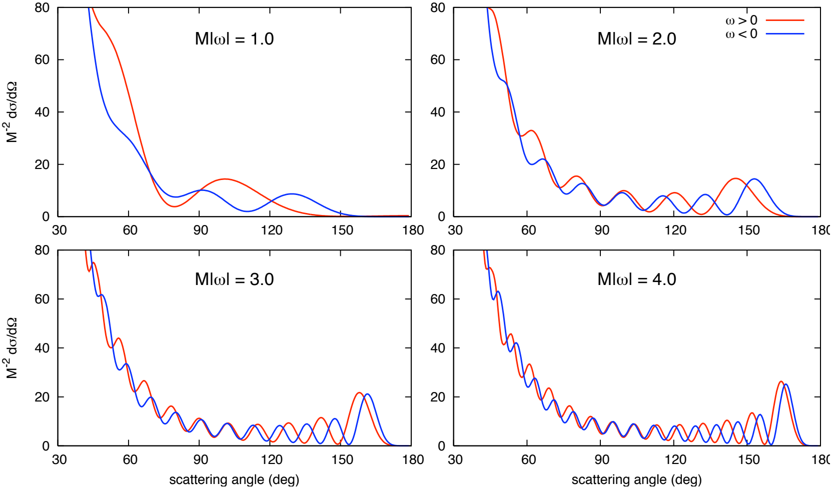
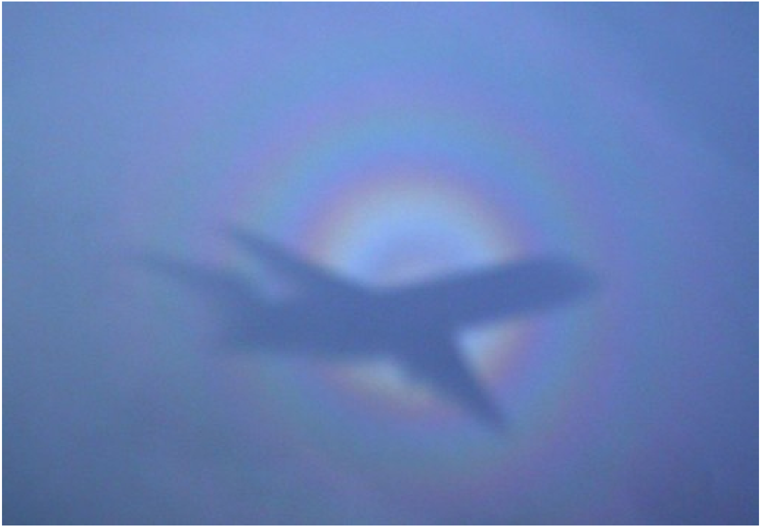


Glory Scattering in Optics (I)



Glory Scattering in Optics (II)

Glory Scattering from Black Holes

- Interference between rays passing around the hole in opposite senses.
- Glory approximation (Matzner *et al* 1985)

$$\frac{d\sigma}{d\Omega} \approx 2\pi\omega b_g^2 \left| \frac{db}{d\theta} \right|_{\theta=\pi} \left[J_{2s}(b_g \omega \sin\theta) \right]^2$$

- Darwin approximation (1959) for rays passing close to photon orbit:

$$\theta(b) \approx -\ln\left(\frac{b - b_c}{0.6702\, b_c} \right)$$

- where $b_c = \sqrt{27}M$. Hence :

$$r_h{}^{-2} \frac{d\sigma}{d\Omega} \approx 3.3772\,\omega r_h \left[J_{2s}(2.67325\,\omega r_h \sin\theta) \right]^2$$

Summary

- Black hole absorbs and scatters incident flux
- Photon orbit $r = 3M \Leftrightarrow$ interference patterns, glories
- Rapid, regular oscillations in amplitude: signature of black holes?
- Tomorrow: Acoustic holes: Wave scattering patterns in the laboratory?