Black Holes and Wave Mechanics (III)

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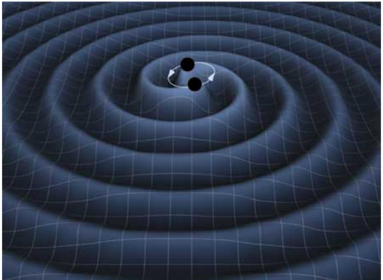
Matematicos de la Relatividad General 08



Course Content

1. Introduction
    - General Relativity basics
    - Schwarzschild's solution
    - Classical mechanics
2. Scalar field + Schwarzschild Black Hole
    - Klein-Gordon equation
    - Wave-packet scattering
    - Quasi-normal modes
3. Hawking Radiation
    - Key results
    - QFT on curved spacetime
    - Black hole collapse model
4. Scattering theory
    - Perturbation theory
    - Partial wave analysis
    - Glories and diffraction patterns
5. Acoustic Black Holes
    - Navier-Stokes eqn $\longrightarrow$ Lorentzian geometry
    - Simple models

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Black Hole Emission

- Black holes are not completely black
- Classical GR + Quantum Field $\Rightarrow$ Hawking radiation (1974)

- Black-body spectrum with Planckian spectrum,

$$\frac{d^2E}{d\omega dt} \sim \frac{\hbar\omega}{\exp(\hbar\omega/k_BT) \pm 1}$$

- Temperature $T_H = \hbar\kappa/(2\pi c) = \hbar c/(4\pi r_s)$
- $T_H \sim 1.2 \times 10^{23}\mathrm{K}(1\mathrm{kg}/M)$
- For solar mass BH, $T_H \sim 6 \times 10^{-8}\mathrm{K} \Rightarrow$ negligible
- Luminosity $L \sim \sigma A T^4 \sim 1/{r_S}^2$

Black Hole Mechanics

GR $\Rightarrow$ Laws of BH mechanics $\Leftrightarrow$ Laws of thermodynamics

- 1st : $dM = \frac{\kappa}{8\pi} dA + \Omega dJ + \Phi dQ$
  $\Leftrightarrow \quad dU = TdS - pdV + \mu dN$
- 2nd : Horizon area always increases, $dA \geq 0 \quad \Leftrightarrow$ entropy always increases $S \geq 0$.
- 3rd : Impossible to form a black hole with zero surface gravity $\kappa \quad \Leftrightarrow \quad$ impossibility of absolute zero $T = 0$.

QFT $\Rightarrow$ Hawking radiation (1970s):

$$k_B T_H = \frac{\hbar \kappa}{2\pi c}, \qquad \text{where surface gravity :} \quad \kappa = \frac{c^4}{4GM}$$

Black hole temperature T_H and entropy $S = A/4$.

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Black Hole Emission

- Small $\Leftrightarrow$ hot!
- Negative heat capacity.
- Temperature $T_H \sim M^{-1}$
- Luminosity $L \sim M^{-2}$
- Lifetime $\tau \sim M^3$
  - $\tau > t_{Hubble}$ if $M > 10^{12}$ kg.
  - $\tau \sim 10^{66}$ years for $M \sim M_\circ$.

Grey-body Spectrum

- Back-scattering of emitted radiation modifies spectrum. Grey body.
- Total luminosity from sum over modes, integral over frequency

$$L = \frac{1}{2\pi} \sum_{l=0}^{\infty} \int_0^{\infty} \frac{\omega \mathbb{T}_{\omega l}}{e^{8\pi M\omega} \mp 1}$$

- Grey-body factor $\Leftrightarrow$ transmission factor

$$\mathbb{T}_{l\omega} = 1 - \left| \frac{A^{\text{out}}_{l\omega}}{A^{\text{in}}_{l\omega}} \right|^2$$

- Schwarzschild: $l = 0$ mode dominant.
- Kerr: $l = m$ modes dominate, loss of angular momentum (spin-down).

Grey-body Spectrum

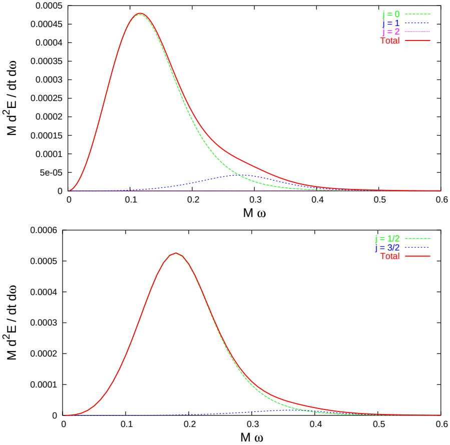


Figure: *Emission spectra of Schwarzschild black hole*. Top: Scalar emission spectrum. Bottom: Fermion emission spectrum.

Particle Creation in Electromagnetic Field

- QFT $\Rightarrow$ Pairs of virtual quanta created in vacuum
- Strong field $\Rightarrow$ creation of real electron-positron ($e^- - e^+$) pairs. Schwinger process (1951).
- Probability $P(d)$ of virtual pair separated by distance d is

$$P(d) \sim \exp(-d/\lambda_c)$$

where Compton wavelength is $\lambda_c = \hbar/mc$

- Field strength E , work done $W = eEd$

Particle Creation in Electromagnetic Field

- Pair creation if $W \geq 2mc^2$ (rest mass energy)
  $\Rightarrow d \sim 2mc^2/Ee$
- Probability $P \sim \exp(-2m^2c^3/\hbar Ee)$
- Barrier tunnelling problem.
- Schwinger process ("rate of particle production by uniform electric field per unit volume per unit time"). If $E \ll E_{crit} = \pi(mc^2)^2/e\hbar c$ then

  $$\frac{dN}{dtdV} \sim (eE)^2 \exp(-\pi m^2c^3/eE\hbar)$$

  where E is the electric field strength
- Boltzmann distribution.

Particle Creation by Gravitational Field (??)

- Naive application of same arguments.
- 'Charge' $e = m$
- $P \sim A \exp(-mc^2/\theta)$ where $\theta = \hbar\Gamma/\pi c$
- c.f. Boltzmann distribution $e^{-E/k_B T}$
- Temperature $T \sim \hbar\Gamma/\pi c k_B$
- Field strength $\Gamma \Leftrightarrow$ surface gravity $\kappa \Rightarrow T \sim T_H$

Heuristic explanation

Pair creation near the horizon.

- Virtual pair creation, tunneling to $r_1 = 2M + d$, $r_2 = 2M - d$
- $g_{00}^{(1)} \approx d/2M$ and $g_{00}^{(2)} \approx -d/2M$
- Killing vector $K^\mu = [1, 0, 0, 0]$
- Virtual $\Rightarrow$ Real particles
- Energy conservation: $K^\mu u_\mu^{(1)} + K^\mu u_\mu^{(2)} = 0$
- Possible because g_{00} changes sign.
- Killing vector becomes spacelike inside the horizon.

Beware : This local ‘explanation’ bears little/no resemblance to QFT analysis!

Quantum Field Theory on Flat Spacetime

- Action

$$S = \int \mathcal{L}(x) d^n x$$

- Field $\phi(x)$, Lagrangian density

$$\mathcal{L}(x) = \frac{1}{2}\eta^{\alpha\beta}\phi_{,\alpha}\Phi_{,\beta}$$

- Wave equation

$$\Box\phi = 0$$

- Modes

$$u_k(t, \mathbf{x}) \propto e^{i\mathbf{k}\cdot\mathbf{x} - i\omega t}$$

- Positive-frequency: $\frac{\partial u}{\partial t} = -i\omega u_k$ where $\omega > 0$.

Quantum Field Theory on Flat Spacetime

- Scalar product

$$(\phi_1, \phi_2) = -i \int [\phi_1(x)\partial_t \phi_2^*(x) - \phi_2^*(x)\partial_t \phi_1(x)] d^3x$$

- Normalised modes:

$$(u_{\mathbf{k}}, u_{\mathbf{k}'}) = \delta^{n-1}(\mathbf{k} - \mathbf{k}') \quad (1)$$

$$(u_{\mathbf{k}}, u_{\mathbf{k}'}^*) = 0 \quad (2)$$

$$(u_{\mathbf{k}}^*, u_{\mathbf{k}'}^*) = -\delta^{n-1}(\mathbf{k} - \mathbf{k}') \quad (3)$$

- Momentum

$$\pi = \frac{\partial \mathcal{L}}{\partial(\partial_t \phi)} = \partial_t \phi$$

Quantization

- Quantization: promote to operators $\phi \to \hat{\phi}$, $\pi \to \hat{\pi}$
- Equal-time commutation relations:

$$[\hat{\phi}, \hat{\phi}] = [\hat{\pi}, \hat{\pi}] = 0$$

$$[\hat{\phi}(t, \mathbf{x}), \hat{\pi}(t, \mathbf{x}')] = i\delta^{n-1}(\mathbf{x} - bx')$$

- Field decomposition

$$\hat{\phi}(t, \mathbf{x}) = \sum_{\mathbf{k}} \hat{a}_{\mathbf{k}} u_{\mathbf{k}}(t, \mathbf{x}) + \hat{a}^{\dagger}_{\mathbf{k}} u^{*}_{\mathbf{k}}(t, \mathbf{x})$$

- Creation $\hat{a}^{\dagger}_{\mathbf{k}}$ and annihilation operators $\hat{a}_{\mathbf{k}}$
- Commutation relations

$$[\hat{a}_{\mathbf{k}}, \hat{a}_{\mathbf{k}'}] = 0, \qquad [\hat{a}^{\dagger}_{\mathbf{k}}, \hat{a}^{\dagger}_{\mathbf{k}'}] = 0,$$

$$[\hat{a}_{\mathbf{k}}, \hat{a}^{\dagger}_{\mathbf{k}'}] = \delta_{\mathbf{k}\mathbf{k}'}$$

Creation and Annihilation

- Vacuum: $\hat{a}_{\mathbf{k}} |0\rangle = 0, \quad \forall\, \mathbf{k}, \quad \langle 0|\, 0\rangle = 1.$
- Particle creation $\hat{a}^\dagger_{\mathbf{k}}$ and annihilation operators $\hat{a}_{\mathbf{k}}$
- Hamiltonian

$$\hat{H} = \sum_{\mathbf{k}} \omega(\hat{N}_{\mathbf{k}} + 1/2)$$

- Number operator

$$\hat{N}_{\mathbf{k}} = \hat{a}^\dagger_{\mathbf{k}} \hat{a}_{\mathbf{k}}$$

- Zero particles in vacuum: $\langle 0|\, \hat{N}_k \,|0\rangle = 0$
- Definition of vacuum depends on choice of states $u_{\mathbf{k}}$
- Under Lorentz transformation: $\hat{a}_j \to \alpha_{ji} \hat{a}_i$, but no mixing of creation and annihilation.
- Zero particles in one inertial frame $\Rightarrow$ zero particles in all inertial frames.

Quantum Field Theory on Curved Spacetime

- Action

$$S = \int \mathcal{L}(x) d^n x$$

- Field $\phi(x)$, Lagrangian density

$$\mathcal{L}(x) = \frac{1}{2}\sqrt{-g}\left(g^{\alpha\beta}\phi_{,\alpha}\phi_{,\beta} - \xi R\phi^2\right)$$

- Wave equation

$$\frac{1}{\sqrt{-g}}\partial_\mu\left(\sqrt{-g}g^{\mu\nu}\partial_\nu\phi\right) + \xi R\phi = 0$$

Quantum Field Theory on Curved Spacetime

- Scalar product on hypersurface

$$(\phi_1, \phi_2) = -i \int_\Sigma \phi_1(x) \overleftrightarrow{\partial}_\mu \phi_2(x) \sqrt{-g_\Sigma} n^\mu d\Sigma$$

- Define Cauchy surface Σ with constraint $f(x^\mu) = 0$. Normal vector $n_\mu = \partial_\mu f$.
- Orthonormal modes u_i:

$$(u_i, u_{i'}) = \delta_{ii'} \quad (4)$$
$$(u_i, u^*_{i'}) = 0 \quad (5)$$
$$(u^*_i, u^*_{i'}) = -\delta_{ii'} \quad (6)$$

- Problem: positive frequency modes?
- If there is a timelike Killing vector X^μ then

$$X^\mu \partial_\mu u_i = -i\omega u_i, \qquad \omega > 0$$

- But general spacetime has no Killing vectors $\Rightarrow$ Ambiguity.

Quantum Field Theory on Curved Spacetime

- In general, no preferred modes. Observer-dependent.
- Alternative set of orthonormal modes $\bar{u}_j$
- Alternative creation $\bar{a}_j^\dagger$ and annihilation operators $\bar{a}_j$.
- Alternative vacuum: $\bar{a}_j \left|\bar{0}\right\rangle = 0$
- According to original observer, alternative vacuum may contain particles

$$\left\langle\bar{0}\right| \hat{a}_i^\dagger \hat{a}_i \left|\bar{0}\right\rangle \neq 0$$

$$\left\langle\bar{0}\right| 0\rangle \neq 0$$

Bogolubov Coefficients

- Express old modes as linear combination of new modes

$$\bar{u}_j = \sum_i \alpha_{ji} u_i + \beta_{ji} u_i^* \quad (7)$$

$$\bar{u}_i = \sum_j \alpha_{ji}^* \bar{u}_j - \beta_{ji} \bar{u}_j^* \quad (8)$$

- α and β are called Bogolubov coefficients

$$\alpha_{ji} = (u_i, \bar{u}_j) \quad (9)$$

$$\beta_{ji} = -(\bar{u}_j, u_i^*) \quad (10)$$

- Linear combinations for creation and annihilation operators

$$a_i = \sum_j \alpha_{ji} \bar{a}_j + \beta_{ji}^* \bar{a}_j^\dagger \quad (11)$$

$$\bar{a}_j = \sum_i \alpha_{ji}^* a_i - \beta_{ji}^* a_i^\dagger \quad (12)$$

Particle Creation

- Particles from vacuum:

$$\langle\bar{0}|\,\hat{a}_i^\dagger\hat{a}_i\,|\bar{0}\rangle = \sum_j |\beta_{ji}|^2 \qquad (13)$$

$$\langle 0|\,\bar{a}_j^\dagger\bar{a}_j\,|0\rangle = \sum_i |\beta_{ji}|^2 \qquad (14)$$

- If both sets of modes are positive-frequency w.r.t timelike Killing vector then $\beta_{ji} = 0 \Rightarrow$ no particle creation.
- Otherwise, vacuum for observer 1 is not vacuum for observer 2.

Unruh Effect

- Unruh effect : uniformly-accelerated observer in flat spacetime sees thermal bath of particles.

$$\langle 0_{Mink}|\, \bar{a}_j^\dagger \bar{a}_j \,|0_{Mink}\rangle = \frac{1}{e^{2\pi c\omega/a} - 1} \quad (15)$$

- Planckian ("black-body") spectrum.
- Temperature $k_B T_{Unruh} = \frac{\hbar a}{2\pi c}$ where a is the acceleration.
- Compare with $k_B T_H = \frac{\hbar\kappa}{2\pi c}$ where κ is the surface gravity.
- The surface gravity κ is, in some sense, the acceleration at the horizon.

Conformal Transformations

- Conformal transformation ($\mathcal{M} \to \mathcal{M}'$)

$$g_{\mu\nu} \to \bar{g}_{\mu\nu}(x) = \Omega^2(x) g_{\mu\nu}(x)$$

- Field $\phi \to \bar{\phi}$, curvature $R \to \bar{R}$, etc.
- Conformal $\Leftrightarrow$ angle-preserving.
- If we use conformal-coupling factor ...

$$\xi = \frac{n-2}{4(n-1)} = 1/6$$

- ... then the wave equation transforms in nice way:

$$\left(\bar{\Box} + \xi \bar{R}\right) \bar{\phi} = \Omega^{-(n+2)/2} \left(\Box + \xi R\right) \phi = 0$$

where

$$\bar{\phi}(x) = \Omega^{(2-n)/2}(x) \phi(x)$$

Compactification

- Represent spacetime structure with Penrose-Carter diagram.
- Compactification: coordinate transform + conformal transformation.
- Example: 2D Minkowski $ds^2 = du dv$ where

$$u = t - r, \qquad v = t + r$$

- Coordinate transformation:

$$u' = 2 \arctan u, \quad v' = 2 \arctan v$$

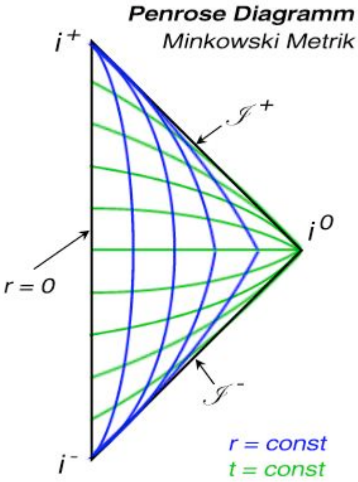
- New coordinate range $-\pi \leq u', v' \leq \pi$ and line element $ds^2 = h(u', v') du' dv'$ where

$$h(u', v') = \frac{1}{4} \sec^2(u'/2) \sec^2(v'/2)$$

- Apply conformal transformation with $\Omega^2 = h^{-1}$ to get conformally-related line element:

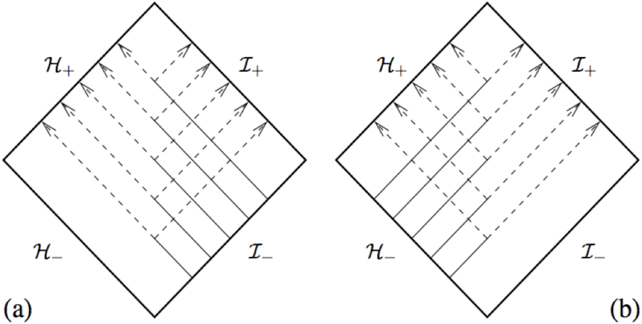
$$d\bar{s}^2 = du' dv'$$

Penrose Diagrams: Minkowski



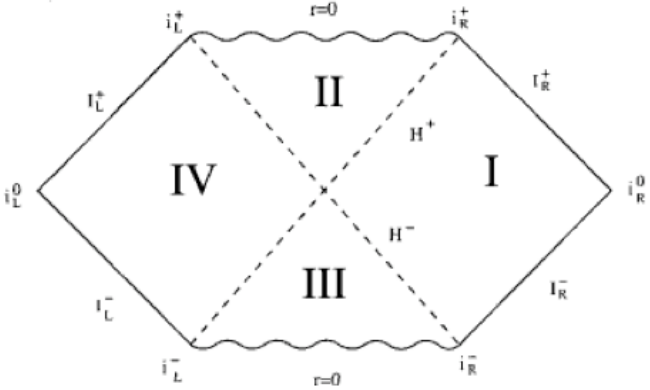
Penrose Diagrams: Schwarzschild

Horizon exterior:



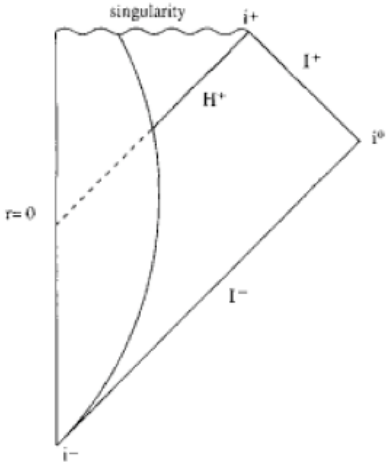
Penrose Diagrams: Schwarzschild (Eternal)

Maximally-extended (Kruskal coordinates):



Penrose Diagrams: Schwarzschild (Collapse)

Black hole formed from collapse:



Vaidya spacetime

- Simplest model of black hole collapse
- Radially-infalling radiation:

$$ds^2 = \left(1 - \frac{2M(v)}{r}\right) dv^2 - 2dvdr - r^2 d\Omega^2$$

- $v = t + r$. Radial infall $\Rightarrow$ constant v.
- Exact solution to field equations, with

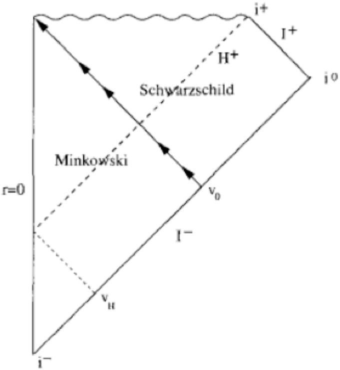
$$T_{vv} = \frac{1}{4\pi r^2} L(v), \qquad L(v) = \frac{dM}{dv}$$

- Simplest assumption: shock wave of mass M propagating along $v = v_0$.

$$M(v) = M\,\Theta(v - v_0)$$

- Timelike singularity forms at $r = 0$
- Define last ingoing ray to avoid singularity $v_H = 0$.

Vaidya spacetime



- Define $v_H = 0 \Rightarrow v_0 = 4M$.
- Spacetime divides into two regions: Mink. and Schw.

Vaidya spacetime

- Minkowski : $v < v_0$

$$ds^2 = dv\, du_{in} - r_{in}^2 d\Omega^2$$

$u_{in} = t_{in} - r_{in}$, $v = t_{in} + r_{in}$.

- Schwarzschild : $v > v_0$

$$ds^2 = (1 - 2M/r) dv\, du_{out} - r_{out}^2 d\Omega^2$$

$u_{out} = t_{out} - r_{out}^*$, $v = t + r_{out}^*$,
$r_{out}^* = r_{out} + 2M \ln(r_{out}/2M - 1)$

- Matching condition: $r_{in} = r_{out}$ along $v = v_0$.

Redshift in Vaidya spacetime

- Idea : use condition $r_{in} = r_{out}$ along $v = v_0$ to relate u_{in} to u_{out}:

$$u_{out} = u_{in} - 4M \ln\left(\frac{-v - u_{in} - 4M}{4M}\right)$$

- On horizon, $u_{in} = -4M$ and $u_{out} = \infty$.
- Just before horizon,

$$u_{out} \approx -4M \ln\left(\frac{-v}{4M}\right)$$

- Infinite redshifting by the horizon.

Hawking radiation: Sketch (I)

- Simplest case: consider only $l = 0$ modes
- On $\mathcal{I}_-$, positive-frequency modes are

$$\phi_{in} \propto e^{-i\omega v}$$

- On $\mathcal{I}_+$, positive-frequency modes are

$$\phi_{out} \propto e^{-i\omega u_{out}} \approx \left(\frac{-v}{4M}\right)^{4Mi\omega}$$

- Also need to consider modes on $\mathcal{H}_+$
- We are interested in flux late times $u_{out} \to \infty$.

Hawking radiation: Sketch (II)

- Late times description: introduce wavepackets localised around $v \leq 0$.
- Calculate Bogolubov coefficients $\alpha_{\omega\omega'}$, $\beta_{\omega\omega'}$ on $\mathcal{I}_+$ using inner product.
- Method: show that

$$|\alpha_{\omega\omega'}|^2 = e^{4\pi M\omega}|\beta_{\omega\omega'}|^2$$

and use identity

$$\sum_{\omega'} |\alpha_{\omega\omega'}|^2 - |\beta_{\omega\omega'}|^2 = 1$$

to obtain Planckian spectrum:

$$N_\omega = \sum_{\omega'} |\beta_{\omega\omega'}|^2 \approx (e^{4\pi M\omega} - 1)^{-1}$$

Hawking Radiation: Key points

- Exponential redshift of late-time modes reaching infinity. $u_{out} \approx -4M \ln\left(\frac{-v}{4M}\right)$
- Planckian spectrum arises from exponential redshift near horizon.
- Temperature proportional to surface gravity at horizon i.e. $\kappa = 1/4M$.
- Hawking spectrum at late times depends only on spacetime structure near horizon.
- Can rederive using ‘eternal’ black hole picture and appropriate Green’s functions.