Black Holes and Wave Mechanics (II)

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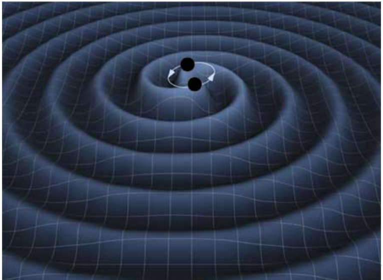
Matematicos de la Relatividad General 08



Course Content

1. Introduction
   - General Relativity basics
   - Schwarzschild's solution
   - Classical mechanics
2. Scalar field + Schwarzschild Black Hole
   - Klein-Gordon equation
   - Wave-packet scattering
   - Quasi-normal modes
3. Scattering theory
   - Perturbation theory
   - Partial wave analysis
   - Glories and diffraction patterns
4. Radiation Reaction and Black Holes
   - Self-force in curved spacetime
   - Green's functions
5. Acoustic Black Holes
   - Navier-Stokes eqn $\longrightarrow$ Lorentzian geometry
   - Simple models

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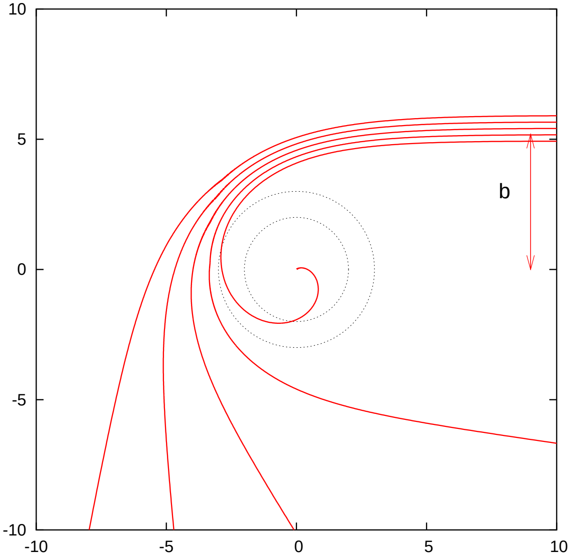
Geodesics on the Schwarzschild spacetime



The Light Cone is Self-Intersecting

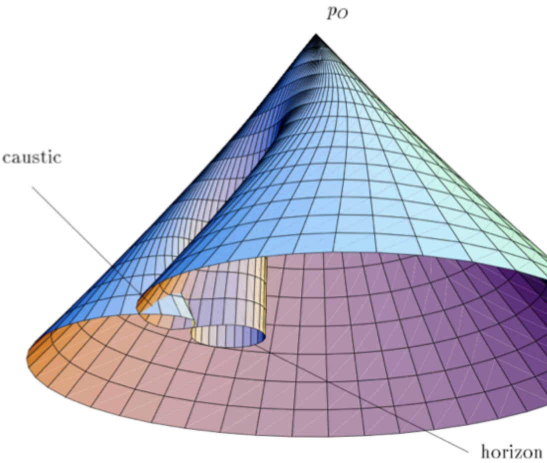


Figure: *Light cone structure*. Reproduced from V. Perlick.

- Caustics: focal points or lines where light cone intersects.

The Scalar Field

- Scalar field action

$$S = \int d^4x \, \mathcal{L}(\Phi, \partial_\mu \Phi; g_{\mu\nu})$$

- where

$$\mathcal{L}(x) = \tfrac{1}{2}\sqrt{-g}\left(g^{\mu\nu}\partial_\mu\Phi\partial_\nu\Phi - (m^2 + \xi R(x))\Phi^2\right)$$

- $R(x)$ is Ricci scalar.
- ξ is a numerical factor
  - minimal coupling : $\xi = 0$
  - conformal coupling : $\xi = (n-2)/4(n-1) = 1/6$
- g is the metric determinant

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Scalar Wave Equation

- Euler-Lagrange equations

$$\frac{\partial \mathcal{L}}{\partial \Phi} = \frac{d}{dx^\mu}\left(\frac{\partial \mathcal{L}}{\partial(\partial_\mu \Phi)}\right).$$

- Wave equation:

$$\frac{1}{\sqrt{-g}}\frac{\partial}{\partial x^\nu}\left(\sqrt{-g}\, g^{\mu\nu}\frac{\partial \Phi}{\partial x^\nu}\right) + (m^2 + \xi R(x))\Phi = 0.$$

- Linearity $\Rightarrow$ Superposition of modes $\Phi = \sum_{lm} a_{lm}\Phi_{lm}$:

$$\Phi_{lm}(t, r, \theta, \phi) = \frac{u_l(t, r)}{r} Y_{lm}(\theta, \phi),$$

- (1+1) partial differential equation :

$$\left(\frac{\partial^2}{\partial t^2} - \left(1 - \frac{2M}{r}\right)\frac{\partial}{\partial r}\left[\left(1 - \frac{2M}{r}\right)\frac{\partial}{\partial r}\right] + V_l(r)\right) u_l(t, r) = 0,$$

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Tortoise Coordinate

- Introduce a tortoise coordinate r_*,

$$\frac{dr_*}{dr} = \left(1 - \frac{2M}{r}\right) \Rightarrow r_* = r + 2M \ln |r/2M - 1|$$

- Moves the horizon to $r_* = -\infty$.

Effective Potential

- (1+1) partial differential equation:

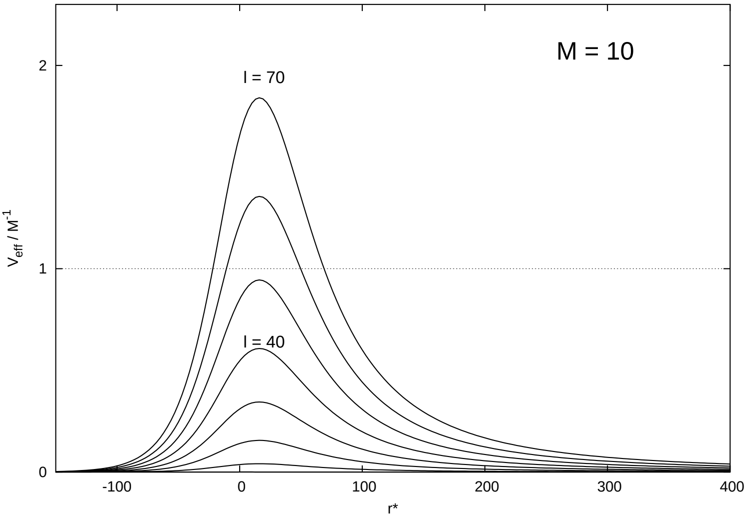
$$\left[\frac{d^2}{dr_*^2} - \frac{\partial^2}{\partial r_*^2} + V_l(r)\right] u_l(t,r) = 0,$$

- with effective potential

$$V_l(r) = \left(1 - \frac{2M}{r}\right)\left(\frac{l(l+1)}{r^2} + \frac{2M}{r^3} + m^2\right).$$

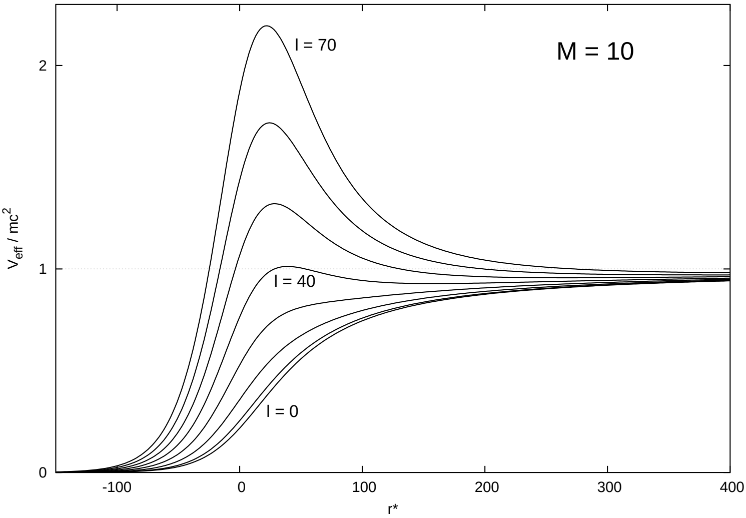
Effective Potential

Massless waves ($m = 0$)



Effective Potential

Massive waves ($m \neq 0$)



Wavepacket Scattering

- What happens when a black hole is perturbed slightly?
- Try firing a massless Gaussian wavepacket at a black hole [Vishveshwara, Nature, 1970].
  - Pick a specific l mode
  - Numerically solve 1+1 PDE wave equation for $u_l(t, r_*)$:
  - Initial condition $\partial_t \phi_l(0, r_*) = -v\phi_l(0, r_*)$.
  - Use finite difference method (e.g. Leapfrog method).
  - Apply ingoing boundary condition as $r_* \to -\infty$.
- Try various wavepacket widths and speeds

Wavepacket Scattering (II)

"Halfway through the defence of my Ph.D, the examiner from the mathematics department asked the question: why should one both to prove the stability of an object that was impossible to observe and was of doubtful existence in the first place? My thesis advisor did not like the question in the least, and the rest of the examination ended up as a verbal battle between the two which I watched with great satisfaction. But the question remained: how do you observe a solitary black hole? To me the answer seemed obvious. It had to be through the scattering of radiation, as the black hole left its fingerprint on the scattered wave."

From *On the black hole trail: a personal journey* by C. V. Vishveshwara (1996).

Wavepacket Scattering (III)

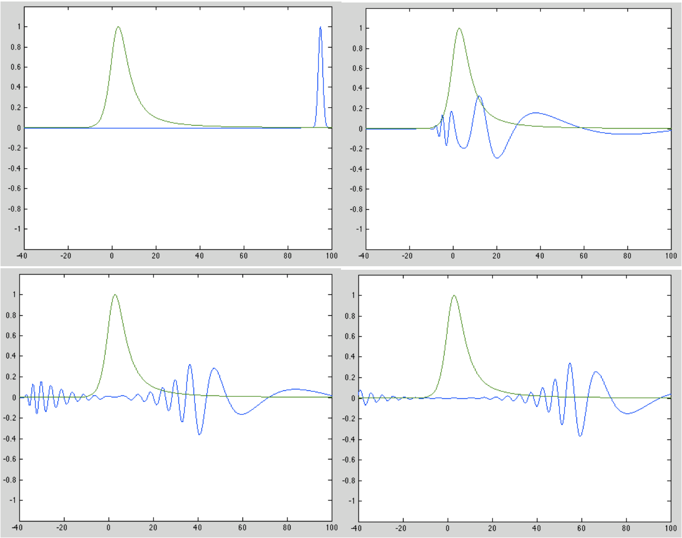


Figure: *Vishveshwara's scattering simulation*

Wavepacket Scattering (IV)

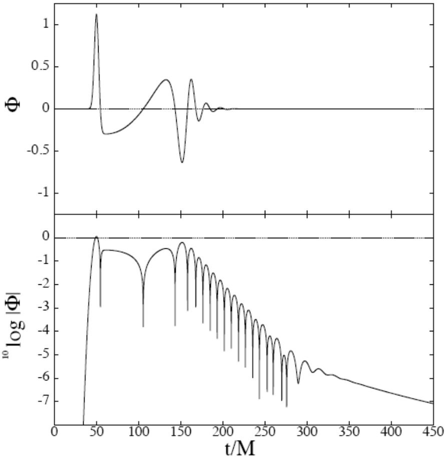


Figure: *The time-dependent response to wavepacket scattering at fixed position far from the hole.* This figure is taken from Andersson & Jensen (2001) [gr-qc/0011025].

Black Hole Response

Response of the black hole undergoes three distinct stages:

- Initial back-scattering
- 'Quasi-normal mode' ringing
- Power-law decay

Observable implications:

- Ringing and decay depend on black hole parameters not initial perturbation
- Quasi-normal modes (QNMs) $\Leftrightarrow$ unstable circular orbit $\Leftrightarrow$ peak in effective potential.
- QNMs $\Rightarrow$ black holes not neutron stars.
- QNM frequencies and decay rates are distinctive BH signature.

Gravitational Waves

- Scalar field Φ is a toy model for gravitational perturbations $h_{\mu\nu}$
- Signal decays as $h_{\mu\nu} \sim 1/r$ rather than $I \sim 1/r^2$.
- Gravitational waves more like sound than light.
  - Long wavelength $\lambda \sim$ source size.
  - Emission for bulk dynamics rather than thermodynamics
  - Coherent emission ($h \sim 1/r$)
  - Two polarizations $\Rightarrow$ stereophonic
  - Detectors cannot focus on small patch of sky

Fourier Decomposition

- Decompose $u_l(t, r)$ into Fourier modes

$$u_l(t, r) = \int_{-\infty+ic}^{\infty+ic} e^{-i\omega t} u_{l\omega}(r) d\omega$$

- Ordinary differential equation

$$\left[\frac{d^2}{dr_*^2} + \omega^2 - V_l(r) \right] u_{l\omega}(r) = 0$$

- Behaviour at the horizon

$$u_{l\omega}(r) \sim e^{\pm i\omega r_*}$$

- Impose ingoing boundary condition $\Rightarrow \; e^{-i\omega r_*}$

Field Current

- Wave equation $\Rightarrow$

$$\Phi^* g^{\mu\nu} \Phi_{;\mu\nu} - \Phi g^{\mu\nu} \Phi^*_{;\mu\nu} = 0.$$

- Define a conserved current J_μ :

$$J^\mu{}_{;\mu} = (-g)^{-1/2} \partial_\mu \left[(-g)^{1/2} g^{\mu\nu} J_\nu \right] = 0,$$

- To find the probability density as perceived by a specific observer, we take the contraction of J_μ and the observer's world line $\dot{x}^\mu$:

$$\rho c^2 = \dot{x}^\mu J_\mu$$

Probability Density

- An observer at fixed $r, \theta, \phi \Rightarrow \dot{x}^\mu = [(1 - 2m/r)^{-1/2}, 0, 0, 0]$
- Measures time-like component

$$\rho c^2 = (1 - 2M/r)^{-1/2}\omega,$$

Diverges as $r \to 2M$.

- Infalling observer $\Rightarrow \dot{x}^\mu = [(1 - 2M/r)^{-1}, -\sqrt{2M/r}, 0, 0]$
- Measures

$$\rho c^2 \sim (1 - 2M/r)^{-1}\omega\left(1 \pm \sqrt{\frac{2M}{r}}\right) \sim \omega\left(1 \mp \sqrt{\frac{2M}{r}}\right)^{-1}$$

- One regular, one divergent solution.

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Flux through the horizon

- Curved-space version of Gauss's Law :

$$\int_V d^4x \sqrt{-g}\, J^{\mu}{}_{;\mu} = \int_{\partial V} d^3x \sqrt{-h}\, J^{\mu} \hat{n}_{\mu}. \tag{1}$$

- 'Flux through 4D surface' = 0
- V is a four-volume with a 3-surface boundary ∂V,
- $\hat{n}$ is the unit normal to an element of ∂V,
- h is the determinant of the induced metric $h_{\mu\nu} = g_{\mu\nu} - \hat{n}_{\mu}\hat{n}_{\nu}$

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Flux through the horizon

- Apply Gauss Law to thin-sandwich $t \to t + dt$ (Schw. coords) to get:

$$\frac{\partial}{\partial t}\left[\int d\Omega \int_{r_0}^{r_1} dr\, r^2 \,(1 - 2M/r)^{-1}\, J_t\right] = -\int d\Omega \left[r^2(1 - 2M/r)J_r\right]_{r_0}^{r_1} \quad (2)$$

- Probability density integral diverges as the horizon is approached
- Infinite number of oscillations in Φ as $r \to 2M$.
- Coordinate singularity as $r = 2M$.
- Try horizon-penetrating coordinate system instead.

Alternative Coordinate Systems

- AEF and PG coordinates $\Rightarrow$ new time coordinate

$$t = \tilde{t} + \alpha(r),$$

- New radial function related to Schw. radial function :

$$\tilde{u}_l(r) = e^{-i\omega\alpha(r)} u_l(r).$$

- The ingoing $e^{-i\omega r_*}$ solution becomes regular:

$$\alpha_{(\mathrm{AEF})} \sim \alpha_{(\mathrm{PG})} \sim -r_* \sim -2M \ln(r/2M - 1) \qquad \text{as } r \to 2M$$

Flux across the horizon

Exercise : Show that, in AEF and PG coordinates, the probability density integral and radial current is well defined as $r \longrightarrow 2M$ if we use the ingoing boundary condition.

Quasi-Normal Modes

What are QNMs and how do we find their frequencies?

Defined by boundary conditions:

- Ingoing at horizon : $u_{l\omega}(r) \sim e^{-i\omega r_*}$ as $r_* \to -\infty$.
- Outgoing at infinity : $u_{l\omega}(r) \sim A_{out}(\omega) e^{+i\omega r_*}$ as $r_* \to +\infty$.

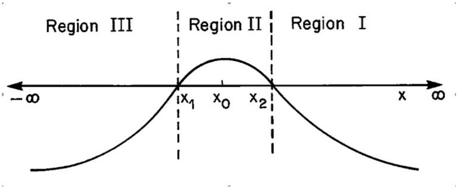
- Two boundary conditions $\Rightarrow$ Discrete spectrum $\omega_q = \omega_{ln}$.
- Complex ω:
  - Frequency : $Re(\omega) \sim 1/t_{\text{period}}$
  - Decay rate : $-Im(\omega) \sim 1/t_{\text{lifetime}}$

QNMs: Chandrasekhar's perspective

"Any initial perturbation will, during its last stages, decay in a manner characteristic of the black hole and independent of the original cause. In other words, during the last stages, the black hole will emit gravitational waves with frequencies and rates of damping characteristic of itself, in the manner of a bell sounding its last dying pure notes."

From *The Mathematical Theory of Black Holes* by S. Chandrasekhar (1983).

Estimating QNMs: A WKB method



- Match across potential barrier.
- On either side of the barrier,

$$\Phi_I(r_*) \approx Q^{-1/4}(r_*) \exp\left(+i\int_{x_2}^{r_*} Q^{1/2}(x)dx\right) \qquad (3)$$

$$\Phi_{III}(r_*) \approx Q^{-1/4}(r_*) \exp\left(-i\int_{r_*}^{x_1} Q^{1/2}(x)dx\right) \qquad (4)$$

where $Q(r_*) = \omega^2 - V(r)$.

- In region II, approximate Q by an inverted parabola, $Q = Q_0 - \frac{1}{2}Q_0''(x-x_0)^2$.

QNMs and the WKB method

- Write radial equation in a standard form,

$$\frac{d^2\Phi_{II}}{dt^2} + \left(\nu + \tfrac{1}{2} - \tfrac{1}{4}t^2\right)\Phi = 0,$$

with definitions

$$t = (4k)^{1/4} e^{i\pi/4}(x - x_0), \quad k = Q_0''/2, \quad \nu + \tfrac{1}{2} = -iQ_0/(2Q_0'')^{1/2}.$$

- Solutions are parabolic cylinder functions

$$\Phi_{II} = AD_\nu(t) + BD_{-\nu-1}(it)$$

- Take asymptotic forms as $r \to \pm\infty$ and match with Φ_I and Φ_{III}.
- Condition: $\Gamma(-\nu) = -\infty \Rightarrow \nu = n$ (integer).

QNMs: WKB result

- Discrete spectrum with frequencies

$$\omega_{ln}{}^2 = V_l(r_0) - i(n + \tfrac{1}{2}) \left(-2 \left. \frac{d^2 V}{dr_*^2} \right|_{r=r_0} \right)^{1/2}$$

- $n = 0, 1, 2, \ldots$ is the overtone number
- Frequency and decay rate related to the height and width of the effective potential barrier.
- Infinite number of modes
- Least-damped QNMs will dominate the signal.

QNMs : Continued Fraction Method

- Make substitution with correct boundary conditions at horizon and infinity:

$$u_{l\omega}(r) = (r-2m)^{\rho} r^{-2\rho} e^{-\rho(r-2M)/2M} \sum_{n=0}^{\infty} a_n \left(\frac{r-2m}{r}\right)^n$$

where $\rho = 2iM\omega$.

- Sub into radial eq. $\Rightarrow$ three term recurrence relation for coefficients a_n

$$\alpha_n a_{n+1} + \beta_n a_n + \gamma_n a_{n-1} = 0,$$

where

$$\begin{aligned}\alpha_n &= n^2 + 2n(\rho+1) + 2\rho + 1\\ \beta_n &= -[2n^2 + 2n(4\rho+1) + 8\rho^2 + 4\rho + l(l+1) + 1]\\ \gamma_n &= n^2 + 4n\rho + 4\rho^2\end{aligned}$$

QNMs : Continued Fraction Method

- Frequency is a QNM if and only if the series $\sum a_n$ converges.
- $\Rightarrow$ continued fraction condition:

$$\beta_0 - \frac{\alpha_0\gamma_1}{\beta_1 -}\frac{\alpha_1\gamma_2}{\beta_2 -}\frac{\alpha_2\gamma_3}{\beta_3 -}\ldots = 0.$$

- Non-linear equation implicitly determining ω_q.
- Locate QNMs with numerical root finder.
- To find the *n*th root, solve *n*th inversion.

QNM Frequencies

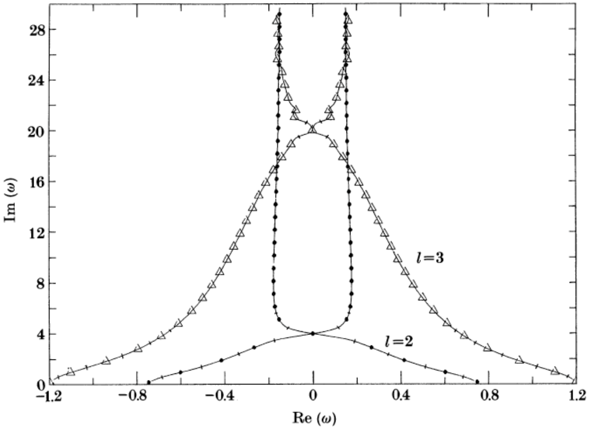


Figure: *The Quasi-Normal Mode Spectrum*. Plot taken from Fig. 1, E.W. Leaver, *Proc. R. Soc. Lond. A* **402**, 285–298 (1985). It shows the QNM frequencies of the grav. field of the Schw. BH., for $l = 2$ and $l = 3$ modes. With our conventions, the y-axis should read $-\mathrm{Im}(\omega)$.

QNM Frequencies (II)

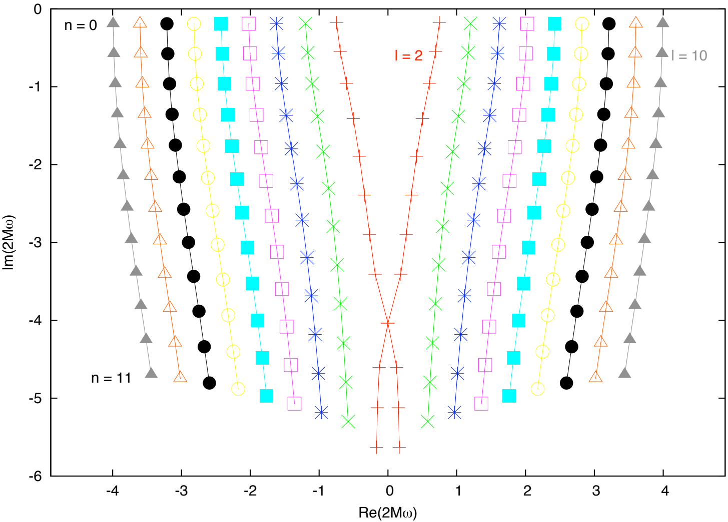


Figure: *The Quasi-Normal Mode Spectrum (II).* (SD)

Green's Function Analysis

- How do QNMs appear in the scattered signal?
- Consider time evolution of field as initial value problem
- Field expressed in terms of a Green's function $G_l(r_*, y, t)$

$$u_l(r_*, t) = \int G_l(r_*, y, t)\partial_t u_l(y, 0)dy + \int \partial_t G_l(r_*, y, t)u_l(y, 0)dy.$$

- The retarded Green's function is defined by

$$\left[\frac{\partial^2}{\partial t^2} - \frac{\partial^2}{\partial r_*^2} + V(r)\right] G_l(r_*, y, t) = \delta(t)\delta(r_* - y),$$

and the condition $G_l(r_*, y, t) = 0$ for $t \leq 0$.

Green's Function Analysis

- Take Fourier transform of G:

$$G(r_*, y, t) = \frac{1}{2\pi}\int_{-\infty+ic}^{\infty+ic} G(r_*, y, \omega)e^{-i\omega t}d\omega$$

- G is ingoing at horizon:

$$\frac{\partial G}{\partial r_*} + i\omega G = 0, \qquad r \to 2M$$

- and outgoing at infinity:

$$\frac{\partial G}{\partial r_*} - i\omega G = 0, \qquad r \to \infty.$$

Green's function

- The Green's function can be found from two linearly-independent solutions of wave equation.
- Use $u_{l\omega}^{(\mathrm{up})}$ and $u_{l\omega}^{(\mathrm{in})}$, with boundary conditions

$$u_{l\omega}^{(\mathrm{in})}(r) \sim \begin{cases} e^{-i\omega r_*}, & r_* \to -\infty, \\ A_{l\omega}^{\mathrm{in}} e^{-i\omega r_*} + A_{l\omega}^{\mathrm{out}} e^{+i\omega r_*}, & r_* \to +\infty, \end{cases}$$

- and

$$u_{l\omega}^{(\mathrm{up})}(r) \sim \begin{cases} B_{l\omega}^{\mathrm{in}} e^{-i\omega r_*} + B_{l\omega}^{\mathrm{out}} e^{+i\omega r_*}, & r_* \to -\infty, \\ e^{+i\omega r_*}, & r_* \to +\infty. \end{cases}$$

where $A_{l\omega}^{\mathrm{in}}$, $A_{l\omega}^{\mathrm{out}}$, $B_{l\omega}^{\mathrm{in}}$ and $B_{l\omega}^{\mathrm{out}}$ are complex constants.

Boundary Conditions

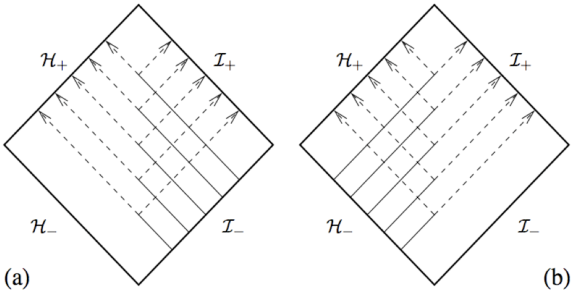


Figure: *Penrose-Carter conformal diagrams showing causal structure of 'eternal' Schwarzschild black hole*.

- Ingoing and outgoing light rays at 45°.
- Left (a): IN boundary conditions $u_{l\omega}^{(\text{in})}$.
- Right (b): UP boundary conditions $u_{l\omega}^{(\text{up})}$.

Green's function

- The Green's function is

$$\hat{G}(r_*, y, \omega) = -\frac{1}{W}\begin{cases} u^{(\text{in})}_{l\omega}(r_*)u^{(\text{up})}_{l\omega}(y), & r_* < y, \\ u^{(\text{up})}_{l\omega}(r_*)u^{(\text{in})}_{l\omega}(y), & r_* > y \end{cases}$$

- Properties of $u^{(\text{in})}_{l\omega}$ and $u^{(\text{up})}_{l\omega}$ take care of boundary conditions.
- W is the Wronskian

$$W = u^{(\text{in})}_{l\omega}\frac{du^{(\text{up})}_{l\omega}}{dr_*} - u^{(\text{up})}_{l\omega}\frac{du^{(\text{in})}_{l\omega}}{dr_*} = 2i\omega A^{\text{in}}_{l\omega}.$$

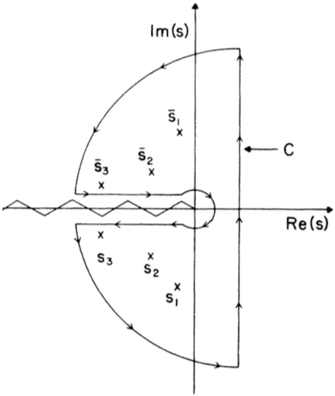
- For QNM frequencies, $A^{\text{in}}_{l\omega} = 0$.
- QNM frequencies correspond to poles in the Green's function.

Green's function

- Deform contour integral in complex plane

$$G(r_*, y, t) = \frac{1}{2\pi} \int_{-\infty+ic}^{\infty+ic} G(r_*, y, \omega) e^{-i\omega t} d\omega$$

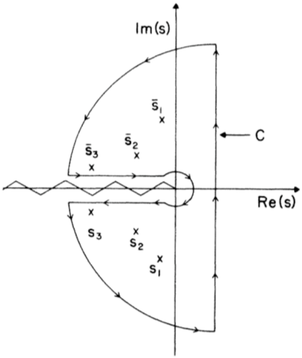
- Branch point at $\omega = 0$ and branch cut along -ve imag. axis.



Black Hole Response

Three stages, corresponding to three parts of contour integral.

- Initial back-scattering (high frequency arc)
- Damped ringing (poles = QNMs)
- Power-law decay (branch cut integral)



Summary

- Scalar field toy model for gravitational radiation
- Perturbed black hole radiates in Quasi-Normal Modes
- QNMs are complex frequencies corresponding to poles of the Green's function
- Tomorrow : sum over l to see interesting diffraction patterns