Black Holes and Wave Mechanics

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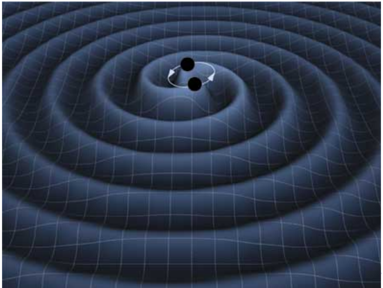
Matematicos de la Relatividad General



Course Content

1. Introduction
   - General Relativity basics
   - Schwarzschild's solution
   - Classical mechanics
2. Scalar field + Schwarzschild Black Hole
   - Klein-Gordon equation
   - Wave-packet scattering
   - Quasi-normal modes
3. Scattering theory
   - Perturbation theory
   - Partial wave analysis
   - Glories and diffraction patterns
4. Radiation Reaction and Black Holes
   - Self-force in curved spacetime
   - Green's functions
5. Acoustic Black Holes
   - Navier-Stokes eqn $\longrightarrow$ Lorentzian geometry
   - Simple models

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Fields in Physics

Fields have spin s and, maybe, rest mass m.

- $s = 0$. Scalar field. Klein-Gordon eqn. Pion π^0.
- $s = \frac{1}{2}$. Spinor field. Dirac eqn. Neutrino ν, electron e^-.
- $s = 1$. Vector field . Maxwell's eqns. Photon γ.
- $s = 2$. Tensor field. Gravitational waves (linearized). Graviton (?).

Treat as classical or quantum fields.

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Black Hole Solutions

"Black holes have no hair". In classical GR, black holes are described by just three parameters.

- Mass M
- Charge Q
- Angular Momentum J.

4D Classification:

- Schwarzschild ($Q = 0$, $J = 0$).
- Reissner-Nordström ($Q \neq 0$, $J = 0$).
- Kerr ($Q = 0$, $J \neq 0$).
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Black Hole Mechanics

GR $\Rightarrow$ Laws of BH mechanics $\Leftrightarrow$ Laws of thermodynamics

- 1st : $dM = \frac{\kappa}{8\pi} dA + \Omega dJ + \Phi dQ$
  $\Leftrightarrow \quad dU = TdS - pdV + \mu dN$
- 2nd : Horizon area always increases, $dA \geq 0 \quad \Leftrightarrow$ entropy always increases $S \geq 0$.
- 3rd : Impossible to form a black hole with zero surface gravity $\kappa \quad \Leftrightarrow \quad$ impossibility of absolute zero $T = 0$.

QFT $\Rightarrow$ Hawking radiation (1970s):

$$k_B T_H = \frac{\hbar\kappa}{2\pi c}, \qquad \text{where surface gravity :} \quad \kappa = \frac{c^4}{4GM}$$

Black hole temperature T_H and entropy $S = A/4$.

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Gravitational Waves are a key prediction of General Relativity

- Very weak ($h \sim 10^{-21}$). Yet to be detected!
- Weakly-interacting, coupled only to bulk motion of matter.

GWs will carry strong signals from black holes in process of:

- Formation: gravitational collapse and supernovae.
- Merger: Binary black holes in galaxy.
- Inspiral. Solar-mass BHs in orbit around supermassive BHs ("radiation reaction" problem).

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- Are black holes stable to small perturbations?
- Precise modelling of BH signals requires full non-linear numerical solutions to Einstein's field equations,
  but ...
- A surprising level of accuracy can be obtained in the linearized approximation,
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- A surprising amount can be learned by just studying a 'toy model': the massless scalar field.

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Motivations (II): "Quantum Gravity"

- Classically, black holes absorb and scatter radiation.
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- Thermal emission spectrum $T_H = (\hbar/2\pi k_B c)\kappa \Rightarrow$ BH entropy $S \sim A/4$.
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Motivations (III): Speculations

- Acoustic ("dumb") holes created in laboratory?
- "Higher-dimensional" black objects (BHs, strings, branes). Experimental signature at LHC?

Newtonian Mechanics

World view : time is absolute and universal.

- Observer-independent t coordinate.
- 3D world-line: $x^i(t) \equiv [x^1(t), x^2(t), x^3(t)]$.
- Newton's Laws $\Rightarrow$ differential equations for $x^i(t)$
- Action principle: $S = \int dt[T(\dot{x}^i(t)) - V(x^i(t))]$.
- e.g. $T = \frac{1}{2}m|\dot{x}|^2$ and $V = \frac{e}{2}[\Phi(x)]^2$
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Special Relativity

World view :

- Time depends on observer.
- Inertial observers are special.

Concept of unified space-time:

- Events in space-time labelled with four coodinates $x^\mu = [x^0, x^1, x^2, x^3]$.
- Set of coordinates systems corresponding to lengths and times measured by inertial observers.
- Inertial observers in constant relative motion.
- Coordinate distances measured by different inertial observers are related by Lorentz transformation.

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Special Relativity: Lorentz Transformation

- Two inertial observers measure ‘coordinate distances’ $\Delta x^{\mu} = [c\Delta t, \Delta x, \Delta y, \Delta z]$ and $\Delta x^{\mu\prime} = [c\Delta t', \Delta x', \Delta y', \Delta z']$.
- If the 2nd observer is moving at speed v in the $+x$ direction relative to the first observer, then

$$c\Delta t' = \gamma\left(c\Delta t - v\Delta x/c\right), \qquad \Delta y' = \Delta y$$
$$\Delta x' = \gamma\left(\Delta x - v\Delta t\right), \qquad \Delta z' = \Delta z,$$

where

$$\gamma = \left(1 - v^2/c^2\right)^{-1/2}.$$

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The Interval

There is one universal quantity on which inertial observers agree: the space-time interval,

$$(\Delta s)^2 = (c\Delta t)^2 - (\Delta x)^2 - (\Delta y)^2 - (\Delta z)^2.$$
$$= \sum_{\mu\nu} \eta_{\mu\nu} \Delta x^\mu \Delta x^\nu.$$

The interval may be positive, negative or zero:

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- Uniform gravitational field $\Leftrightarrow$ Uniformly-accelerating frame. (Principle of Equivalence).
- Gravity $\Rightarrow$ tidal forces : parallel paths are pushed together or pulled apart.
- Locally, space-time still looks flat (Lorentzian) ...
- .. but globally space-time may be curved. *'Over there'* not the same as *'over here'*.
- No global inertial frame.
- Define and compare local quantities.

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General Relativity: The Metric

- Space-time interval in differential (local) form:

$$ds^2 = \sum_{\mu=0}^{3} \sum_{\nu=0}^{3} g_{\mu\nu}(x) dx^\mu dx^\nu \qquad (1)$$

$$\equiv g_{\mu\nu} dx^\mu dx^\nu \qquad (2)$$

- $g_{\mu\nu}$ is a symmetric tensor called the metric.
- Summation convention is used ('one up, one down').
- Metric inverse $g^{\mu\nu}$ is defined by

$$g_{\mu\nu} g^{\nu\lambda} = \delta_\mu^\lambda$$

- The metric (metric inverse) raises (lowers) indices, i.e.

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Coordinate Transformations

- Many different coordinate systems describe the same space-time.
- Under general coordinate transformation,

$$x \mapsto x' = x^{\mu'}(x)$$

'up' and 'down' indices transform in opposite ways:

$$a^{\mu'} = \frac{\partial x^{\mu'}}{\partial x^{\mu}} a^{\mu}, \qquad a_{\mu'} = \frac{\partial x^{\mu}}{\partial x^{\mu'}} a_{\mu}.$$

- i.e. transform like dx^{μ} or like $\frac{\partial}{\partial x^{\mu}}$.
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Geodesics

- Particles follow world-lines in space-time : $x^\mu = x^\mu(\lambda)$ ...
- Free particles follow privileged world-lines called geodesics.
- Geodesics are the generalisation of the Euclidean idea of a straight line.
- Straight line: shortest distance between two points.
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World-lines

- Free particles follow geodesics $\Rightarrow$ action principle

$$S = \int ds = \int d\lambda \, L(x^\mu, \dot{x}^\mu; t) \qquad \text{where } L = \sqrt{g_{\mu\nu}\dot{x}^\mu \dot{x}^\nu}$$

  and $x^\mu(\lambda)$ and $\dot{x}^\mu \equiv \frac{dx^\mu}{d\lambda}$.

- Euler-Lagrange equations:

$$\frac{\partial L}{\partial x^\mu} = \frac{d}{d\lambda}\left(\frac{\partial L}{\partial \dot{x}^\mu}\right)$$

- For time-like paths, set $d\lambda = ds = c d\tau$, where τ is the proper time experienced by the particle.

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The Schwarzschild Space-time

- Unique asymptotically flat space-time exterior to a spherically-symmetric grav. source (e.g. our Sun).
- In Schwarzschild coordinates

$$ds^2 = (1-2M/r)dt^2-(1-2M/r)^{-1}dr^2-r^2(d\theta^2+\sin^2\theta d\phi^2).$$

- Units: $G = c = 1$, so $M \equiv GM/c^2$.
- Event horizon at $r = 2M$.
- Compact objects that lie entirely within their horizon are black holes.
- Many other coordinate systems may be used.

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- Metric is independent of t and $\phi \Rightarrow$ conserved quantities
- In equatorial plane ($\theta = \pi/2$, $\dot{\theta} = 0$):

$$(1 - 2M/r)\dot{t} = k,$$
$$r^2\dot{\phi} = h.$$

- 'Energy' k and 'Angular momentum' h.
- To find an equation for $\dot{r}$, use

$$g_{\mu\nu}\dot{x}^\mu\dot{x}^\nu = \epsilon^2 \equiv \begin{cases} 0 & \text{null} \\ 1 & \text{time-like} \end{cases}$$

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- Insert constants of motion to get energy equation :

$$\dot{r}^2 + V_{\mathrm{eff}}(r) = k^2 - \epsilon^2,$$

with an effective potential

$$V_{\mathrm{eff}}(r) = -\frac{2M\epsilon^2}{r} + \frac{h^2}{r^2}\left(1 - \frac{2M}{r}\right).$$

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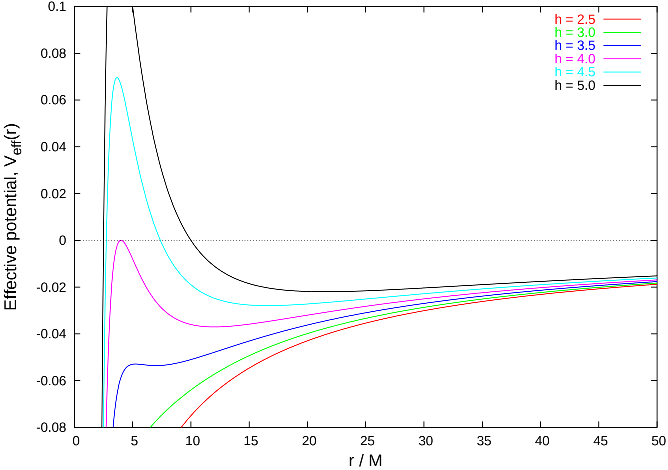
Effective Potential

This plot shows the effective potential for timelike geodesics with a range of angular momenta $h = r^2\dot{\phi}$.



Radial infall

Consider a particle falling radially inwards:

- $(1 - 2M/r)\dot{t} = k$, $h = 0$, and $\dot{r} = k^2 - 1 + 2M/r$
- If it starts from rest at infinity $\Rightarrow k = 1$
- Integrating, we find

$$\tau - \tau_0 = \frac{2}{3(2M)^{1/2}}\left(r_0^{3/2} - r^{3/2}\right)$$

- Passes through horizon smoothly in finite τ.
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$$\tau - \tau_0 = \frac{2}{3(2M)^{1/2}}\left(r_0^{3/2} - r^{3/2}\right)$$

- Passes through horizon smoothly in finite τ.
- But $\dot{t} \to \infty$ as $r \to 2M \Rightarrow$ coordinate singularity.
- Coordinate time t diverges as horizon is approached

Circular Orbits

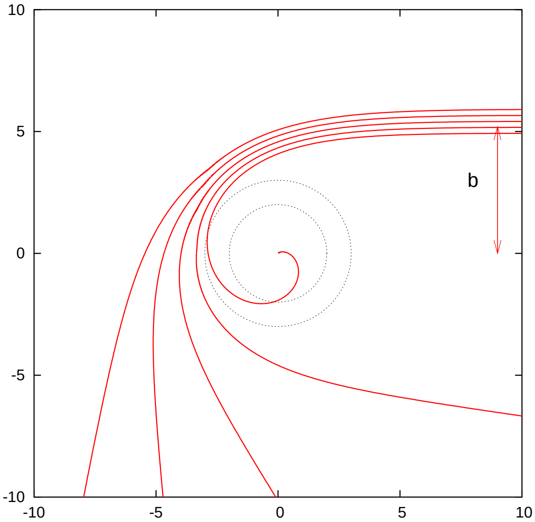
Circular orbits occur at points where $\frac{dV_{eff}}{dr} = 0$.
Orbit is stable if $\frac{d^2 V_{eff}}{dr^2} > 0$

Exercises :

1. Show that the unstable photon (i.e. null) orbit is at $r = 3M$.
2. Show that the stable time-like orbit is at $r = (h^2/2M)\left(1 + \sqrt{1 - 12M^2/h^2}\right)$.
3. Show that the innermost stable time-like orbit is at $r = 6M$.

Scattering and Absorption (I)

Photon geodesics around a Schwarzschild black hole



Scattering (II)

- Divide energy equation by $\dot{\phi}^2$ to get orbit equation

$$\left(\frac{du}{d\phi}\right)^2 + u^2 = \frac{k^2 - \epsilon^2}{h^2} + \frac{2M\epsilon^2}{h^2}u + 2Mu^3$$

where $u = 1/r$.

- Differentiate to get GR version of Binet's equation

$$\frac{d^2u}{d\phi^2} + u = \frac{M\epsilon}{h^2} + 3Mu^2.$$

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Deflection-angle Approximations:

- Weak-field deflection:

$$\Delta\theta \approx 4M/b$$

$$\Rightarrow \lim_{\theta\to 0} \frac{d\sigma}{d\Omega} \sim 1/\theta^4$$

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Absorption

- Critical impact parameter $b_c = 3\sqrt{3}M$ (massless)
  - $b > b_c$: scattered.
  - $b < b_c$: absorbed.

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$$\sigma_a = \pi {b_c}^2 = 27\pi M^2$$

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Alternative Coordinate Systems

- Problem: for ingoing geodesics, $t \to +\infty$ as $r \to 2M$.
- t is the time measured by observer at infinity.
- Solution: to continue geodesics across the horizon, use a horizon-penetrating coordinate system.
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Advanced Eddington-Finkelstein Coordinates

- Define new time coordinate $\bar{t}$:

$$\bar{t} = t + 2M \ln(r - 2M) \quad \Rightarrow \quad d\bar{t} = dt + \frac{2M}{r - 2M} dr$$

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$$ds^2 = (1 - 2M/r)d\bar{t}^2 - (4M/r)drd\bar{t} - (1 + 2M/r)dr^2 - r^2 d\Omega^2$$

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$$\dot{\bar{t}} = -\dot{r}.$$

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Progress so far

- Geodesics on Schwarzschild spacetime
- Interval $\Rightarrow$ Action principle $\Rightarrow$ E-L equation $\Rightarrow$ dynamics
- Skipped differential geometry!
- Now : Recap important concepts in GR:
  - Tensors
  - Covariant differentiation
  - Parallel transport
  - Geodesic equation
  - Connections and metric-compatibility
  - Killing vectors

Tensors

- Under coordinate transform $x \mapsto x' = x^{\mu'}(x)$,

$$\text{contravariant :} \quad a^{\mu'}(x') = \frac{\partial x^{\mu'}}{\partial x^{\mu}} a^{\mu}(x), \tag{3}$$

$$\text{covariant :} \quad b_{\mu'}(x') = \frac{\partial x^{\mu}}{\partial x^{\mu'}} b_{\mu}(x). \tag{4}$$

- Contraction $\Rightarrow$ coordinate-independent scalar $a^{\mu} b_{\mu} = a^{\mu'} b_{\mu'}$
- Tensors :

$$T^{\alpha'\beta'\ldots}{}_{\gamma'\delta'\ldots} = \left(\frac{\partial x^{\alpha'}}{\partial x^{\alpha}} \frac{\partial x^{\beta'}}{\partial x^{\beta}} \cdots \right) \left(\frac{\partial x^{\gamma}}{\partial x^{\gamma'}} \frac{\partial x^{\delta}}{\partial x^{\delta'}} \cdots \right) T^{\alpha\beta\ldots}{}_{\gamma\delta\ldots}$$

Covariant derivative (I)

- Construct a derivative of a vector field a^μ that behaves like a tensor
- Try $\partial_\nu a^\mu$ … no good!

$$\partial_{\mu'} a^{\nu'} = \frac{\partial x^\mu}{\partial x^{\mu'}} \frac{\partial}{\partial x^\mu} \left(\frac{\partial x^{\nu'}}{\partial x^\nu} a^\nu \right) = \frac{\partial x^\mu}{\partial x^{\mu'}} \frac{\partial x^{\nu'}}{\partial x^\nu} \partial_\mu a^\nu + \frac{\partial x^\mu}{\partial x^{\mu'}} \frac{\partial^2 x^{\nu'}}{\partial x^\mu \partial x^\nu} a^\nu$$

- Define covariant derivative ∇_μ

$$\nabla_\mu a^\nu = \partial_\mu a^\nu + \Gamma^\nu{}_{\mu\lambda} a^\lambda$$

where Γ is called a connection (or Christoffel symbol)

Covariant derivative (II)

- Connection is not a tensor. It transforms as

$$\Gamma^{\alpha\prime}{}_{\beta\prime\gamma\prime} = \frac{\partial x^{\alpha\prime}}{\partial x^{\alpha}} \left(\frac{\partial^2 x^{\alpha}}{\partial x^{\beta\prime} \partial x^{\gamma\prime}} + \frac{\partial x^{\beta}}{\partial x^{\beta\prime}} \frac{\partial x^{\gamma}}{\partial x^{\gamma\prime}} \Gamma^{\alpha}{}_{\beta\gamma} \right)$$

- so that $\nabla_{\mu} a^{\nu}$ transforms as a tensor

$$\nabla_{\mu\prime} a^{\nu\prime} = \frac{\partial x^{\mu}}{\partial x^{\mu\prime}} \frac{\partial x^{\nu\prime}}{\partial x^{\nu}} \nabla_{\mu} a^{\nu}$$

- Comma and semicolon notation :

$$a^{\mu}{}_{,\nu} \equiv \partial_{\nu} a^{\mu} \qquad a^{\mu}{}_{;\nu} \equiv \nabla_{\nu} a^{\mu}$$

Parallel Transport

- Transport a vector a^ν along a world-line $x^\mu(\lambda)$
- Tangent vector to world-line $u^\mu = \frac{dx^\mu}{d\lambda}$
- Covariant derivative operator: $\frac{D}{D\lambda} = u^\mu \nabla_\mu$
- Parallel-transport condition

$$\frac{Da^\nu}{D\lambda} = u^\mu \nabla_\mu a^\nu = 0.$$

Geodesics

- Geodesic: 'Straight line in curved spacetime'
- Parallel transport tangent vector $u^\mu = \frac{dx^\mu}{d\tau}$ using $u^\nu \nabla_\nu u^\mu = 0$ to construct geodesic $x^\mu(\tau)$
- Geodesic equation:

$$\frac{Du^\mu}{D\tau} \equiv \frac{du^\mu}{d\tau} + \Gamma^\mu{}_{\nu\lambda} u^\nu u^\lambda = 0$$

- τ is affine parameter.
- Two alternative definitions for geodesics (action principle vs parallel transport). Compatible?

Metric compatibility

- Compatible definitions if connection is symmetric $\Gamma^{\mu}{}_{\nu\lambda} = \Gamma^{\mu}{}_{\lambda\nu}$ (torsion-free) and

$$\nabla_{\mu} g_{\nu\lambda} = 0$$

- $\Rightarrow$ Affine connection (or *Levi-Civita connection*) related to metric by

$$\Gamma^{\mu}{}_{\nu\lambda} = \frac{1}{2} g^{\mu\sigma} \left(\partial_{\sigma} g_{\nu\lambda} - \partial_{\nu} g_{\sigma\lambda} - \partial_{\lambda} g_{\nu\sigma} \right)$$

- Metric compatibility $\Rightarrow$ parallel-transport preserves scalar product.

$$u^{\mu} \nabla_{\mu} (g_{\alpha\beta} a^{\alpha} b^{\beta}) = 0$$

Killing vectors

- Spacetime symmetries (isometries) $\Rightarrow$ constants of motion
- Killing vectors of spacetime X^μ satisfy Killing's equation

$$X_{\mu;\nu} + X_{\nu;\mu} = 0$$

- Killing vectors are generators of infinitessimal isometries
- Contraction of Killing vector and tangent vector $\Rightarrow$ constant of motion

$$u^\nu \nabla_\nu (u^\mu X_\mu) = u^\nu u^\mu X_{\mu;\nu} = \frac{1}{2} u^\nu u^\mu (X_{\mu;\nu} - X_{\nu;\mu}) = 0$$

- Killing vector $\Leftrightarrow$ coordinate system where metric independent of coordinate.
- eg. Schwarzschild coords independent of $t, \phi \Rightarrow k, h$.

Fields on BH space-times

Next time:

- Klein-Gordon equation on Schwarzschild spacetime.
- Assume weak (no back-reaction), minimally-coupled and classical field.
- Scalar field Φ : ‘toy model’ for gravitational radiation.
- Define a field current. Causality $\Rightarrow$ Boundary conditions at horizon, infinity and origin.
- Field dynamics $\Rightarrow$ Quasi-normal modes.