# Geometry of Families of Elliptic Complexes, Duality and Anomalies


Alexander Cardona

*Laboratoire de Mathématiques Appliquées*
*Université Blaise Pascal (Clermont II)*
*Complexe Universitaire des Cézeaux*
*63177 Aubière Cedex, France.*
cardona@ucfma.univ-bpclermont.fr

*a la memoria de Julio Alba*
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**Abstract**

This thesis is devoted to the study of the relation between (weighted) trace anomalies and:

- anomalies in Quantum Field Theory –illustrated by Chern-Simons models– on one hand,
- duality in antisymmetric field theories with a discussion of the factorization of the geometry of determinant line bundles associated to families of elliptic complexes, on the other hand.

It is shown that the anomaly coming from a phase ambiguity for $\zeta$-regularized determinants, modelling partition functions in Chern-Simons theory, can be seen as a tracial anomaly. These arise from the fact that weighted traces fail to commute with exterior differentiation. Since they can be expressed in terms of Wodzicki residues, it follows that they have a local feature.
Following Schwarz's Ansatz for partition functions in antisymmetric field theories, duality can be interpreted as a factorization of the analytic torsion, which can be seen as a metric on the determinant line associated to a de Rham acyclic complex. We extend this to a "factorization" of the geometry of the determinant line bundle associated to a family of elliptic complexes, showing how the curvature of the Bismut-Freed connection decomposes as a sum of two forms which, as a consequence of the locality of trace anomalies, carry the same locality feature as the Bismut-Freed curvature.
The thesis also presents a Fresnel integral approach to path integrals underlying formal computations used by physicist to establish duality between partition functions in antisymmetric field theories.

**Résumé**

Cette thèse est consacrée à l'étude des relations entre des anomalies traciales et:

- des anomalies en théorie des champs quantiques —illustrés par le cas de modèles de Chern-Simons— d'une part,
- la dualité en théorie des champs antisymétriques, avec l'étude de la factorisation de la géométrie des fibrés déterminants associés à des familles de complexes elliptiques, d'autre part.

On montre que l'anomalie provenant d'une ambiguité de phase pour les déterminants $\zeta$-regularisés, qui décrivent des fonctions de partition de théories de Chern-Simons, peut être interprétée comme une anomalie traciale. Ceci s'explique par le fait que les traces régularisées ne commutent pas avec la différentiation extérieure. Pouvant s'exprimer en termes de résidus de Wodzicki, ces anomalies traciales héritent d'une propriété de localité.

En appliquant l'Ansatz de Schwarz pour des fonctions de partition en théorie des champs antisymétriques, la dualité peut s'interpréter comme factorisation de la torsion analytique, qui peut être vue comme métrique sur l'espace déterminant associé à un complexe de de Rham acyclique. On étend ceci à une "factorisation" de la géométrie du fibré déterminant associé à une famille de complexes elliptiques en montrant que la courbure de la connexion de Bismut-Freed se décompose en une somme de deux formes qui, en consequence de la localité des anomalies traciales, sont elles-mêmes porteuses de cette propriété de localité.

Cette thèse présente de plus une approche utilisant l'outil des intégrales de Fresnel pour donner un sens aux intégrales de chemin qui sous-tendent les calculs formels utilisés par les physiciens pour établir une dualité entre des fonctions de partition de théorie des champs antisymétriques.

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# Contents

## Introduction and Statement of Results

In order to describe physical systems at very short distances, Quantum Field theorists use "path integrals", which are objects without a rigorous mathematical definition due mainly to the infinite-dimensional character of the spaces of fields (vector valued functions, or sections of fibrations on a Riemannian or Mikowskian *space-time manifold* $M$) on which these objects must be defined. These "integrals" model probability amplitudes, and formal manipulations –which rely on classical facts and properties of finite-dimensional integrals (Gaussian integrals, change of variable formulae, Fourier transforms,...)– lead to numerical data which are in extraordinary accordance with experiments. This phenomenological success has encouraged mathematical physicists to search for a mathematical theory of path integrals, but at present this goal has not yet been reached.

The general form of a path integral is

$$Z(F) = \frac{1}{Z_o} \int_\Phi F(\phi) \exp\{-S(\phi)\} \; [\mathcal{D}\phi],$$

where $\Phi$ is the space of configurations of the fields $\phi$, $F(\phi)$ a functional on $\Phi$, $[\mathcal{D}\phi]$ a *formal* Lebesgue-type measure on $\Phi$ and $S : \Phi \to \mathbb{R}$ (or $\mathbb{C}$) the classical action of the theory under consideration. Here $Z_o$ denotes the *partition function* of the theory, given by the integral at the right hand side of this equation when $F(\phi) = 1$, a normalization factor. $\Phi$ is typically an infinite-dimensional manifold, so that the formal Lebesgue-type measure $[\mathcal{D}\phi]$ is generally ill-defined.

Roughly speaking, there are mainly two approaches to path integrals. The first one tries to describe path integrals as properly defined integrals, through the study of measure theory on functional spaces. This approach, known as the "constructive approach", uses in an essential way the fact that Gaussian measures (unlike Lebesgue measures) on infinite-dimensional spaces do exist. Another approach, known as the "non-perturbative approach", uses heuristic manipulations of path integrals and their "semiclassical" limits. Important examples of this second kind of approach are the so-called *Topological Quantum Field Theories*, born from pioneering work of A. Schwarz and E. Witten in the late 70's and the 80's. The search for a "regularized" definition of the partition function for some particular models led them to set up links with topological invariants of combinatorial type, such as the *analytic torsion* defined by D. Ray and I.M. Singer in the early 70's.

The basic idea is to interpret a partition function as a "regularized" determinant, imitating the Gaussian integral identity

$$\int_V e^{-\frac{1}{2}S(v)}\, dv = (\det T_s)^{-\frac{1}{2}},$$

valid for Lebesgue-type integrals on a finite-dimensional euclidean vector space $V$, where $S(v) = \langle T_s v, v\rangle$ is a symmetric and positive quadratic form defined on $V$, and $dv$ denotes the "Lebesgue measure".

## $\zeta$-Regularization of Determinants and Traces

In the case of action functionals defined by elliptic differential operators with positive order acting on infinite-dimensional spaces of sections, the ordinary determinant on the right hand side of the previous equality is replaced by the *regularized determinant*, defined by Ray and Singer through $\zeta$-function regularization [RS71]. Combining this with the Faddeev-Popov procedure led A. Schwarz to a definition for the partition function associated with a degenerate action functional. The latter mimicks Milnor's definition of the Reidemeister torsion of a complex of vector spaces, and therefore yields a relation with the Ray-Singer torsion, a secondary topological invariant used to classify topological spaces with the same homotopy type. Using this approach in his study of three-dimensional Chern-Simons theories, E. Witten showed that in this context the corresponding partition function must contain (a phase given by) another secondary invariant: the $\eta$-invariant defined by Atiyah, Patodi and Singer in order to state an index theorem for manifolds with boundary, which corresponds to a $\zeta$-function regularization of the "signature" of an operator.

Regularization techniques used to define determinants of elliptic positive-order differential operators are also used to regularize other ill-defined extensions of finite dimensional concepts, such as traces (see Section 1.1). The classical identity $\det(AB) = \det(A)\det(B)$, that holds for finite-dimensional determinants, breaks down for $\zeta$-regularized determinants, giving rise to the so-called "multiplicative anomalies" for $\zeta$-determinants. This is closely related to the fact that the fundamental tracial identity $\mathrm{tr}(AB) = \mathrm{tr}(BA)$ which holds for ordinary matrices breaks down for $\zeta$-regularized traces. As a matter of fact, it is well-known that the only trace on the algebra $\mathcal{C}l(E)$ of classical pseudo-differential operators acting on sections of the vector bundle $E$ over a closed connected manifold $M$ of dimension $> 1$ is the Wodzicki residue which, for $A \in \mathcal{C}l(E)$, is defined by

$$\mathrm{res}(A) = q\, \mathrm{Res}_{z=0}\left(\mathrm{tr}(AQ^{-z})\right),$$

where $Q$ is any invertible admissible pseudo-differential operator and $q$ denotes the order of $Q$. An important feature of the Wodzicki residue of a classical pseudo-differential operator $A$ is its *locality*, i.e. it can be described as an integral of local density on $M$, namely

$$\mathrm{res}(A) = \frac{1}{(2\pi)^n} \int_M \mathrm{res}_x(A) d\mu_M(x)$$

where $n$ is the dimension of $M$, $\mu_M$ the volume measure on $M$. A drawback of the Wodzicki residue is that it vanishes on finite-rank operators and hence it is not an extension of the usual trace. We therefore consider instead other linear functionals that extend the finite-dimensional trace.

### Weighted Trace Anomalies

*Weighted traces* of classical pseudo-differential operators are linear functionals on the algebra of such operators, which were investigated in [P][CDMP] and implicitly used, both in theoretical physics and mathematics, under the name of $\zeta$-regularized traces. By weighted trace of a classical admissible pseudo-differential operator $A$ we mean the complex number given by

$$\mathrm{tr}^Q(A) := \mathrm{f.p.}|_{z=0} \mathrm{tr}(AQ^{-z}),$$

where f.p. refers to the finite part, the weight $Q$ being an admissible invertible elliptic operator of positive order. It follows from the definition that weighted traces extend usual finite-dimensional traces, i.e. $\mathrm{tr}^Q(A) = \mathrm{tr}(A)$, whenever $A$ is a finite rank operator. Taking the finite part and leaving out the divergences leads to discrepancies, which we refer to as *weighted trace anomalies* or *tracial anomalies* [CDMP][CDP]. These give rise to Wodzicki residues, and hence have some locality features (see Section 1.1.1).

One of the purposes of this work is to relate logarithmic variations of regularized determinants of certain families of admissible operators with tracial anomalies, thus giving an a priori explanation for the locality of these variations. For families of self-adjoint elliptic operators, such as Dirac operators in odd dimensions, $\zeta$-determinant functions can be defined using the Atiyah-Patodi-Singer eta invariant [APSI] (see Section 3.1). For other types of elliptic operators, such as chiral Dirac operators, one works instead with determinant sections, namely sections of determinant line bundles.

Let us first turn to the self-adjoint case. Consider a family $\{A_x\}_{x\in[0,1]}$ of elliptic self-adjoint positive order operators parametrized by $[0,1]$. Then, the $\eta$-invariant $\eta(A_x) = \eta_{A_x}(0)$ varies smoothly in $x$ modulo integers, i.e. except for jumps coming from eigenvalues of $A_x$ "crossing zero", and we prove in Section 3.1 the following

**Theorem 7** [CDP] *Let $A_0$ and $A_1$ be two elliptic invertible self-adjoint operators and $\{A_x\}_{x\in[0,1]}$ a smooth family of elliptic self-adjoint operators interpolating them, then*

$$\eta(A_1) - \eta(A_0) = 2\Phi(\{A_x\}) + \int_0^1 \dot{\mathrm{tr}}^{A_x}(\mathrm{sign}(A_x))\, dx,$$

*where $\Phi(\{A_x\})$ denotes the spectral flow of the family and $\dot{\mathrm{tr}}^{A_x} = [\frac{d}{dx}, \mathrm{tr}^{A_x}]$ is the variation of the weighted trace $\mathrm{tr}^{A_x}$.*

The local term given by the Wodzicki residue coming from the weighted trace anomaly $\int_0^1 \dot{\mathrm{tr}}^{A_t}(\mathrm{sign}(A_t))\, dt$ corresponds to the local term in the Atiyah-Patodi-Singer index theorem. Furthermore, since the $\zeta$-determinant of a (non necessarily positive) self-adjoint elliptic operator is given by

$$\det_\zeta A = \det_\zeta |A| e^{\frac{\pi}{2}\left(\eta_A(0) - \zeta_{|A|}(0)\right)},$$

this leads to

**Corollary 1** *Let $\{A_x\}_{x\in[0,1]}$ be a smooth family of self-adjoint elliptic operators with vanishing spectral flow and such that $A_0$ and $A_1$ are invertible. Then, if $\det_\zeta |A_x|$ and $\zeta_{|A_x|}(0)$ are constant,*

$$\begin{aligned}
\log \frac{\det_\zeta A_1}{\det_\zeta A_0} &= \frac{\pi}{2} \int_0^1 \dot{\mathrm{tr}}^{A_x}(\mathrm{sign}(A_x))\, dx \\
&= -\frac{\pi}{2a} \int_0^1 \mathrm{res}\left[|A_x|^{-1} \frac{d}{dx} A_x\right] dx.
\end{aligned}$$

Thus, under the above assumptions, the logarithmic variation of the $\zeta$-determinant is expressed as a weighted trace anomaly and is therefore local. Although these assumptions seem strong, they are fulfilled in the case of families of signature operators (Section 3.2).

*Families of Signature Operators.* Let $M$ be a Riemannian manifold of odd dimension $n = 2k+1$, and $W$ a Hermitian vector bundle over $M$ with flat connection. Given a smooth family of connections $\{\nabla_t^W, t \in [0,1]\}$ on the exterior bundle $W$, there is an associated family of operators $\{*d_t, t \in [0,1]\}$, where $d_t$ is the exterior differential on $M$ coupled with the connection $\nabla_t^W$ and $*$ the Hodge star operator. Let $*d_{k,t}$ denote their restriction to $k$-forms. If $n = 2k+1$, for $k$ odd, $*d_{k,t}$ is self-adjoint, elliptic and $\det_\zeta *d''_{k,t}$ is well-defined, where $*d''_{k,t} = *d_{k,t}|_{\ker(*d_{k,t})^\perp}$ (see Section 3.2.2). Since the signature of the manifold $M \times [0,1]$ is zero, then (using the results of [APSI, APSIII]) the index of the operator $A = *d''_{k,t} \otimes \frac{d}{dt}$ is zero and hence the spectral flow of the family $\{*d''_{k,t}\}$ vanishes. Theorem 7 shows that

$$\eta(*d''_{k,1}) - \eta(*d''_{k,0}) = \int_0^1 \dot{\mathrm{tr}}^{*d''_{k,t}}(\mathrm{sign}(*d''_{k,t}))\, dt,$$

so that the difference of the eta invariants is given by a tracial anomaly, and hence is local. Furthermore, in the case $k = 1$ ($n = 3$), if the family $\{*d''_{k,t}\}$ is build from a family $\{g_t\}_{t\in[0,1]}$ of Riemannian metrics, the modulus of the $\zeta$-determinant $\det_\zeta |*d_{1,t}|$ is independent of $t$ on the grounds of the topological invariance of the analytic torsion. Therefore, in view of the above Corollary,

$$\begin{aligned} \log \frac{\det_\zeta(*d''_{1,1})}{\det_\zeta(*d''_{1,0})} &= \frac{\pi}{2}\left\{\eta(*d''_{1,1}) - \eta(*d''_{1,0})\right\} \\ &= -\frac{\pi}{2}\int_0^1 \mathrm{res}\left[|*d''_{1,t}|^{-1}\frac{d}{dt}*d''_{1,t}\right] dt, \end{aligned}$$

being an integrated tracial anomaly, and hence a Wodzicki residue, is the integral of a local term on the base manifold. This example plays a fundamental role in phase anomaly computations in Chern-Simons theory, as we explain in the sequel.

## Weighted Trace Anomalies and Phase Anomalies in Chern-Simons theory

Using the results previously stated, in Chapter 5 we relate phase anomalies in odd dimensions –coming from logarithmic variations of $\zeta$-determinants of Dirac operators– to weighted trace anomalies, thus giving an apriori explanation for the locality we expect from these anomalies. In QFT an *anomaly* occurs when a transformation in the fields, leaving invariant the action functional, changes the corresponding path integral. In particular, when the classical action is quadratic, $S(\phi) = \langle T\phi, \phi\rangle$, transformations in the path integral can be read off the transformations of the regularized determinant associated to the corresponding partition function $Z = (\det_\zeta T)^{-\frac{1}{2}}$. The "anomaly" is defined to be the logarithmic variation of such partition function and hence of the corresponding regularized determinants. Thus, the difference of logarithms of $\zeta$-determinants ($\log\det_\zeta T_1 - \log\det_\zeta T_0$) in Corollary 1, seen as an anomaly of partition functions, can be seen as an "integrated tracial anomaly" ($\int_0^1 \dot{\mathrm{tr}}^{T_x}(\mathrm{sign}(T_x))\, dx$), under the assumptions of the Corollary, in which case the anomaly term comes from the phase of the determinant.

The variation of the partition function of the Chern-Simons model in dimension 3 –under a change of metric– gives rise to an "anomaly", which can be written as an integrated tracial anomaly. Indeed, the Chern-Simons theory in dimension $n = 2k+1$ (see [S79][W89]) is modelled by the metric invariant action functional $S_k^{CS}(\omega_k) = \langle \omega_k, *d_k\omega_k\rangle$, which is degenerate on the space $\Omega^k$ of $W$-valued $k$-forms. Applying Schwarz's Ansatz to $S_k^{CS}$ yields the corresponding partition function

$$Z_k^{CS}(*d_k'') = \left[\prod_{l=0}^{k-1}\left(\det{}_\zeta(\Delta_l'')\right)^{(-1)^{k-l+1}}\right]^{\frac{1}{2}} \det{}_\zeta\left(*d_k''\right)^{-\frac{1}{2}},$$

which is well defined since in $n = 2k+1$ dimensions, for $k$ odd, the operator $*d_k''$ is self-adjoint and hence has a well-defined $\zeta$-determinant. We show in Section 5.1 that

$$Z_k^{CS}(*d_k'') = (T(M))^{\frac{(-1)^{k+1}}{2}}\, e^{-i\frac{\pi}{4}\eta(*d_k'')},$$

where $\eta(*d_k'')$ denotes the eta invariant of the operator $*d_k''$ and $T(M)$ the analytic torsion of the manifold $M$. Note that the classical action functional $S_k^{CS}$ is metric independent, but its associated partition function has a phase which depends on the metric on $M$, i.e. there is a *phase anomaly.* Specializing to the case $k=1$, $n=3$ the logarithmic variation of the partition function under such transformation reads

$$\log\frac{Z_1^{CS}(*d_{1,1}'')}{Z_1^{CS}(*d_{1,0}'')} = -\frac{\pi}{2}\int_0^1 \mathrm{res}\left[|*d_{1,t}''|^{-1}\frac{d}{dt}*d_{1,t}''\right]dt,$$

so that the *anomaly* $\log\frac{Z_1^{CS}(*d_{1,1}'')}{Z_1^{CS}(*d_{1,0}'')}$ corresponds to an integrated weighted trace anomaly. This leads (in Section 5.2) to the following

**Theorem 12** *The Chern-Simons phase anomaly between two Riemannian metrics $g_0$ and $g_1$ is an integrated weighted trace anomaly, i.e.*

$$\begin{array}{ccc} \text{phase anomaly} & = & \text{integrated weighted trace anomaly} \\ \downarrow & & \downarrow \\ \log\dfrac{Z_k(*d_{k,1}'')}{Z_k(*d_{k,0}'')} & = & -i\dfrac{\pi}{4}\displaystyle\int_0^1 \dot{\mathrm{tr}}^{*d_{k,t}''}(\mathrm{sign}(*d_{k,t}''))\,dt. \end{array}$$

*Using the APS index theorem* [APSI], *this anomaly is given by the Chern-Simons term $i\frac{32}{\pi^2}\int_M tr(A\wedge dA + \frac{2}{3}A\wedge A\wedge A)$.*

Factorization of the Geometry of Determinant Bundles and Topological Field Theory

In [S79] and [ST84] the relation between the partition function of degenerate action functionals and the analytic torsion of the underlying space-time manifold $M$ is considered in the cases of Chern-Simons theories and antisymmetric field theories. In [ST84] (see also [C01]) the Ansatz of Schwarz is used to study *duality* in antisymmetric quantum field theories, i.e. the equivalence between two a priori different antisymmetric field theories. We riview these facts in Section 6.2.

Let us consider two antisymmetric field theories defined by the action functionals

$$S(\omega_{k-1}) = \langle d_{k-1}\omega_{k-1}, d_{k-1}\omega_{k-1}\rangle$$

and

$$S(\omega_{n-k+1}) = \langle d_{n-k+1}\omega_{n-k+1}, d_{n-k+1}\omega_{n-k+1}\rangle,$$

where $d_k$ denotes the restriction to $W$-valued $k$-forms of the exterior differential coupled to the connection $\nabla^W$. Hodge star duality implies the equivalence between the action functionals $S(\omega_{n-k+1})$ and $S^*(\omega_{k+1}) = \langle d_k^*\omega_{k+1}, d_k^*\omega_{k+1}\rangle$. The partition functions $Z_k(M)$ for $S(\omega_{k-1})$ and $Z_k^*(M)$ for $S^*(\omega_{k+1})$, are defined using Schwarz Ansatz's here again. $Z_k(M)$ and $Z_k^*(M)$ combine to give back the analytic torsion.

**Proposition 17** [S79] *For any $k \in \{0, 1, \ldots, n\}$,*

$$Z_k(M) \cdot Z_k^*(M)^{-1} = T(M)^{(-1)^k}.$$

Thus, the analytic torsion of the underlying space-time manifold $M$ factorizes as a product of the partition functions associated to degenerate "dual" actions. In the even dimensional case this yields $Z_k(M) = Z_k^*(M)$, for all $k$, and hence an identification of the two dual partition functions.

Proposition 17 can also be interpretated as follows. $T(M)$ is the Quillen metric $\|\cdot\|_Q$ on the determinant line associated to the acyclic elliptic de Rham complex

$$0 \to \Omega^0 \xrightarrow{d_0} \cdots \to \Omega^{k-1} \xrightarrow{d_{k-1}} \Omega^k \xrightarrow{d_k} \Omega^{k+1} \to \cdots \xrightarrow{d_{n-1}} \Omega^n \to 0,$$

$Z_k(M)$ the Quillen metric $|\cdot|_{(k)}$ on the determinant line associated to the elliptic resolvent

$$0 \to \Omega^0 \xrightarrow{d_0} \cdots \to \Omega^{k-2} \xrightarrow{d_{k-2}} \Omega'_{k-1} \xrightarrow{d_{k-1}^* d_{k-1}} 0,$$

$Z_k^*(M)$ the Quillen metric $|\cdot|_{(k)*}$ on the determinant line associated to "dual" elliptic resolvent

$$0 \to \Omega^n \stackrel{d^*_{n-1}}{\longrightarrow} \cdots \to \Omega^{k+2} \stackrel{d^*_{k-1}}{\longrightarrow} \Omega''_{k-1} \stackrel{d_k d^*_k}{\longrightarrow} 0.$$

Proposition 17 can be read as a factorization of the metric $\|\cdot\|_Q$ in terms of the metrics of the two dual resolvents. Our next goal is to extend this factorization to families of acyclic elliptic complexes, thus working on determinant line bundles.

*Geometry of Determinant Line Bundles.* Let $I\!\!M \stackrel{\pi_M}{\to} X$ be a smooth locally trivial fibration of manifolds, where $X$ is a smooth manifold of finite dimension and the fibre $M_x = \pi_M^{-1}(x)$ a closed Riemannian manifold. To a Hermitian vector bundle $E \to I\!\!M$ we associate the infinite-rank vector bundle $\mathcal{E} \to X$ whose fibre above $x \in X$ is the space of smooth sections $\mathcal{E}_x = \Gamma(M_x, E_x)$, where $E_x \to M_x$ denotes the restriction to $M_x$ of $E$. Following [Q86] and [BF88], in section 4.2 we consider the determinant line bundle $\mathrm{Det}\mathbf{T}$ associated to a family $\{T_x\}_{x\in X}$ of positive-order elliptic differential operators $T_x : \mathcal{E}_x \to \mathcal{E}_x$. In particular, we recall the construction — by $\zeta$-function regularization— of the Quillen metric on $\mathrm{Det}\mathbf{T}$ and, assuming the existence of a connection on $\mathcal{E}$, that of the Bismut-Freed connection $\nabla^{BF}$, which is unitary for the Quillen metric. In Theorem 10, along the lines of [PR], we prove that the curvature $\Omega^{BF}$ of the Bismut-Freed connection is local, i.e. it can be written as the integral of a local density on the fibre $M/X$. This is a consequence of the fact that the Bismut-Freed connection is built point-wise from a family of unitary connections on the Hermitian bundles $\{E_x\}_{x\in X}$.

Consider an acyclic elliptic complex $(I\!\!E_\bullet, T_\bullet)$ of positive-order differential operators acting on sections of Hermitian vector bundles over the manifold $I\!\!M$,

$$0 \to E_0 \stackrel{T_0}{\longrightarrow} \cdots \to E_{k-1} \stackrel{T_{k-1}}{\longrightarrow} E_k \stackrel{T_k}{\longrightarrow} E_{k+1} \to \cdots \stackrel{T_{n-1}}{\longrightarrow} E_n \to 0.$$

For $0 \le k \le n$, let $\mathcal{E}_k \to X$ be the infinite-rank vector bundle associated to $E_k$. The acyclic elliptic complex $(I\!\!E_\bullet, T_\bullet)$ gives rise to an acyclic elliptic complex $(\mathcal{E}_\bullet, \mathbf{T}_\bullet)$ of positive-order differential elliptic bundle maps on infinite-rank vector bundles over $X$, namely

$$0 \to \mathcal{E}_0 \stackrel{\mathbf{T}_0}{\longrightarrow} \cdots \to \mathcal{E}_{k-1} \stackrel{\mathbf{T}_{k-1}}{\longrightarrow} \mathcal{E}_k \stackrel{\mathbf{T}_k}{\longrightarrow} \mathcal{E}_{k+1} \to \cdots \stackrel{\mathbf{T}_{n-1}}{\longrightarrow} \mathcal{E}_n \to 0,$$

where each map $\mathbf{T}_k$ corresponds to a family $\{T_{k,x}\}_{x\in X}$ of elliptic positive-order differential operators, parametrized by the manifold $X$. Quillen's construction associates to each positive-order differential elliptic bundle map $\mathbf{T}_k$ a determinant line bundle $\mathrm{Det}\mathbf{T}_k \to X$ with smooth Quillen metric and, assuming the existence of a unitary connection on $\mathcal{E}_k$, a Bismut-Freed connection unitary for the Quillen metric. From the determinant line bundles $\mathrm{Det}\mathbf{T}_k$, for each $k$, we define in Section 6.4 the determinant line bundle of the acyclic complex $(\mathcal{E}_\bullet, \mathbf{T}_\bullet)$ by

$$\mathcal{L}_{\mathbf{T}} = \bigotimes_{k=0}^{n} (\mathrm{Det}\mathbf{T}_k)^{(-1)^{k+1}}.$$

Let $\|\cdot\|_{Q,k}$ denote, for $0 \leq k \leq n$, the Quillen metric on the line bundle $\mathrm{Det}\mathbf{T}_k \to X$. Then, the natural metric on $\mathcal{L}_{\mathbf{T}}$, $\|\cdot\|_{\mathcal{L}_{\mathbf{T}}} = \otimes_{k=0}^{n} \|\cdot\|_{Q,k}^{(-1)^{k+1}}$, is the *analytic torsion*. Let us denote by $\nabla^{BF}_{(k)}$, for each $0 \leq k \leq n$, the Bismut-Freed connection on $\mathrm{Det}\mathbf{T}_k$, whose curvature $\Omega^{BF}_{(k)}$ is local. This implies that the curvature of the connection $\nabla^{\mathcal{L}_{\mathbf{T}}}$, induced by the unitary connections $\{\nabla^{BF}_{(k)}\}_{0\leq k\leq n}$, also has a local curvature, denoted by $\Omega^{\mathcal{L}_{\mathbf{T}}}$. Let $(\mathcal{E}^{(k)}_\bullet, \mathbf{T}_\bullet)$ and $(\mathcal{E}^{(k)*}_\bullet, \mathbf{T}^*_\bullet)$ be the acyclic elliptic complexes given by

$$(\mathcal{E}^{(k)}_\bullet) \qquad 0 \to \mathcal{E}_0 \xrightarrow{\mathbf{T}_0} \cdots \longrightarrow \mathcal{E}_{k-1} \xrightarrow{\mathbf{T}_{k-1}} \mathbf{T}_{k-1}\mathcal{E}_{k-1} \to 0,$$

and

$$(\mathcal{E}^{(k)*}_\bullet) \qquad 0 \leftarrow \mathbf{T}^*_k\mathcal{E}_{k+1} \xleftarrow{\mathbf{T}^*_k} \mathcal{E}_{k+1} \longleftarrow \cdots \xleftarrow{\mathbf{T}^*_{n-1}} \mathcal{E}_n \leftarrow 0,$$

respectively. In Section 6.4 we show that the splitting of the geometry of the determinant line bundle associated to the complex $(\mathcal{E}_\bullet, T_\bullet)$ holds as in the finite-dimensional case (considered in Section 6.3), and the locality property of the curvature is conserved.

**Theorem 13** *Let $\mathcal{L}_{\mathbf{T}} \to X$ be the determinant line bundle associated to the family $\{\mathcal{E}_{\bullet,x}, T_{\bullet,x}\}_{x\in X}$ of acyclic elliptic complexes. Then,*

1. *The Quillen metric factorizes according to* (6.19)*, in terms of the metrics of the determinant line bundles associated to the complexes $\mathcal{E}^{(k)}_\bullet$ and $\mathcal{E}^{(k)*}_\bullet$, as*
$$\|\cdot\|_Q = \|\cdot\|_{(k)} \|\cdot\|_{(k)*}^{(-1)^{k+1}},$$
*where $\|\cdot\|_{(k)}$ and $\|\cdot\|_{(k)*}$ denote the curvature of the determinant line bundles associated to the complexes $\mathcal{E}^{(k)}_\bullet$ and $\mathcal{E}^{(k)*}_\bullet$, respectively.*

2. *The curvature splits*
$$\Omega^{\mathcal{L}}_{\mathbf{T}} = \Omega_{(k)} \oplus (-1)^{k+1}\Omega^*_{(k)}, \tag{1}$$
*where $\Omega_{(k)}$ and $\Omega^*_{(k)}$ denote the curvature of the determinant line bundles associated to the complexes $\mathcal{E}^{(k)}_\bullet$ and $\mathcal{E}^{(k)*}_\bullet$, respectively.*

3. *This splitting respects the locality properties of the curvature given by* Theorem 10.

The text is organized in three parts; although the heart of the thesis is contained in the second and third part, Part I is essential to set up the framework used in Parts II and III.

In Part I we introduce the mathematical tools used throughout the text (Chapter 1) and some physical prerequisites (Chapter 2) aimed to give a self-contained exposition of the physical applications.
In Chapter 1, Section 1.1, we define *weighted traces*, and we show how *weighted trace anomalies* give rise to Wodzicki residues of pseudo-differential operators, and hence are "local". We also consider variations of $\zeta$-regularized determinants of invertible admissible pseudo-differential operators and eta invariants. *Elliptic complexes* and the *analytic torsion* are considered in Section 1.2. The first part of this section is devoted to the study of the three types of algebraic torsions of a chain complex of finite-dimensional vector spaces (namely, the Reidemeister torsion, the Torsion and the Analytic Torsion), their properties and the relation between them. The analytic torsion for general elliptic complexes is defined in Section 1.2.3, where its topological invariance is proven, as well as the main features of the Ray-Singer analytic torsion of a Riemannian manifold. Finally, in Section 1.3, we recall the definition of Dirac operators on Clifford bundles and two index theorems for Dirac operators we use in the sequel.
In Chapter 2, Section 2.1, we introduce the Fresnel integral approach to path integrals along the lines of [AlH76], in which a mathematically rigorous definition can be given of heuristic infinite-dimensional integrals arising in quantum physics. This approach to path integrals is used later (in Section 6.1) to give a measure theoretical interpretation of duality in antisymmetric tensor fields. In Section 2.2 we describe Schwarz's Ansatz to define the partition function of a degenerate action functional. This heuristic treatment of partition functions underlies the geometric approach we follow in the rest of the work. The Ansatz used to define anomalies in quantum fields is discussed in Section 2.3.

In Part II weighted trace anomalies are used as a geometrical tool. In Chapter 3, Section 3.1, we prove Theorem 7 on logarithmic variations of regularized determinants and tracial anomalies, and discuss in Section 3.2 its application to the case of families of signature operators –relevant in the analysis of phase anomalies in Chern-Simons models.
Chapter 4 is devoted to the study of the geometry of determinant line bundles through $\zeta$-regularization tools. In Section 4.1 the geometry of the determinant line bundle in finite dimensions is reviewed, as background material for the ex-

tension in Section 4.2 to the infinite-dimensional case. We carry out there the construction of the determinant line bundle associated to a family of elliptic positive-order differential operators and, following Quillen [Q86] and Bismut and Freed [BF88], we study its geometry. Finally, by means of weighted trace anomalies, we prove Theorem 10 [PR] concerning the *locality* of the curvature of the Bismut-Freed connection on the determinant line bundle.

In Part III we describe the physically relevant applications of the mathematical theory developed in the two previous parts, namely the study of *phase anomalies* in Chern-Simons theories (Chapter 5), on one hand, and the *splitting* of the geometry of determinant line bundles associated to families of acyclic complexes (Chapter 6), on the other hand. Chern-Simons models, the relation between the partition function –defined through Schwarz's Ansatz– and the analytic torsion are considered in Section 5.1, and Theorem 12 is proven in Section 5.2.
In the last chapter, after a brief introduction to the heuristic manipulations involved in path integral interpretations of duality, we give in Section 6.1 a measure theoretical interpretation of this fact in terms of Fresnel integrals. In Section 6.2 we state Proposition 17 and interpret it as a splitting in the metric of a determinant line. Section 6.3 is devoted to the study of the splitting of determinant line bundles associated to complexes of finite-rank vector bundles, and the situation in infinite dimensions is considered in Section 6.4, where we prove Theorem 13.

Appendices A and B cover some basic background on pseudo-differential operators and the path integral approach in quantum physics, respectively, frequently used in the text.

# Part I

# Mathematical Tools and Physical Prerequisites

# Chapter 1

# Mathematical Tools

In this chapter we define *weighted traces*, *weighted trace anomalies* and we show how they give rise to Wodzicki residues of pseudo-differential operators, and hence are given by local terms. We also consider variations of $\zeta$-regularized determinants and eta invariants of invertible admissible pseudo-differential operators, Dirac operators on Clifford Bundles and the Analytic Torsion of an elliptic complex of bundles over a Riemannian manifold.

## 1.1 Weighted Traces of Pseudo-differential Operators, Regularized Determinants and Tracial Anomalies

### 1.1.1 The Wodzicki Residue, Weighted Traces and Tracial Anomalies

Let $E$ be a vector bundle above a smooth $n$-dimensional closed Riemannian manifold $M$, and let $\mathcal{C}l(E)$ denote the algebra of classical pseudo-differential operators acting on smooth sections of $E$. Let as before $Ell(E)$, $Ell^*(E)$ and $Ell^*_{ord>0}(E)$ denote the set of elliptic, invertible elliptic and invertible elliptic with positive order operators acting on sections of $E$, respectively, and $\mathcal{A}d(E)$ the subset of $Ell^*_{ord>0}(E)$ containing the invertible admissible elliptic classical pseudo-differential operators which have positive order.

For $Q \in \mathcal{A}d(E)$ and $A \in \mathcal{C}l(E)$, the map $z \mapsto \mathrm{tr}(AQ^{-z})$ is meromorphic with a simple pole at zero [KV]. Given $Q \in \mathcal{A}d(E)$, the *Wodzicki residue* of $A \in \mathcal{C}l(E)$ is defined by

$$\mathrm{res}(A) = q\,\mathrm{Res}_{z=0}\left(\mathrm{tr}(AQ^{-z})\right), \tag{1.1}$$

where $q$ denotes the order of $Q$. The definition of $\mathrm{res}(A)$ is independent of the choice of $Q$. Among the many remarkable properties of the Wodzicki residue (for a review see [K89]), let us point out some which will be relevant in this work:

1. *Traciality.* If $M$ is connected, the Wodzicki residue is (up to a constant) the only trace on the algebra $\mathcal{C}l(E)$, i.e.

$$\operatorname{res}([A,B]) = 0, \tag{1.2}$$

   for any $A, B \in \mathcal{C}l(E)$.

2. *Locality.* The Wodzicki residue of a classical pseudo-differential operator $A$ can be described as an integral of local expressions involving the symbol of the operator [W69]

$$\operatorname{res}(A) = \frac{1}{(2\pi)^n} \int_M \int_{|\xi|=1} \operatorname{tr}_x \left(\sigma_{-n}(x,\xi)\right) d\xi d\mu_M(x), \tag{1.3}$$

   where $n$ is the dimension of $M$, $\mu_M$ the volume measure on $M$, $\operatorname{tr}_x$ the trace on the fibre above $x$ and $\sigma_{-n}$ the homogeneous component of order $-n$ of the symbol of $A$.

3. *Triviality in finite dimensions.* If $A$ is of finite rank, or if its order is less than $-n$, then

$$\operatorname{res}(A) = 0. \tag{1.4}$$

Even though property (1.2) is mathematically satisfactory, property (1.4) is not what we want for our purposes. We should keep in mind –as recalled in the introduction– that the heuristic objects considered by physicists in the quantum description of field theories are built extending and imitating finite-dimensional objects and relations between them; namely measures, integrals, change of variable formulae, etc. But the Wodzicki residue "hides" the finite-dimensional objects; it is *not* an extension of the finite-dimensional trace, and extensions to the algebra $\mathcal{C}l(E)$ of the ordinary trace on trace class operators do not exist. Instead, we shall define an object –called weighted trace– that, even if is no longer tracial on $\mathcal{C}l(E)$, extends the usual trace on finite-rank operators (matrices), allowing us to regard our work as an extension of the finite-dimensional theory in the same line of thought as [P]. Nevertheless, Wodzicki residues and their properties, in particular traciality and locality, will play a very important role in the study of weighted traces.

In what follows by a *weight* we shall mean an element of $\mathcal{A}d(E)$, often denoted by $Q$, and by $q$ we shall denote its order. We shall very often take complex powers $Q^{-z}$ of operators $Q \in \mathcal{A}d(E)$, which involves a choice of spectral cut for the operator $Q$. However, in order to simplify notations, we shall drop the explicit mention of the spectral cut. In the case when $Q$ is a positive operator, any ray in $\mathbb{C}$ different from the positive real half line serves as a spectral cut ray of the leading symbol.

Recall that the finite part at $z = p$ of a meromorphic function $\varphi$ with a simple pole at $z = p$ is given by

$$\text{f.p.}|_{z=p}\,\varphi = \lim_{z\to p}\left(\varphi(z) - \frac{\mathrm{Res}_{z=p}\varphi(p)}{(z-p)}\right).$$

For $A \in \mathcal{C}l(E)$ and $Q \in \mathcal{A}d(E)$ the map $z \mapsto \mathrm{tr}(AQ^{-z})$ is a meromorphic function with a simple pole at $z = 0$ and we can set

**Definition 1** [CDMP] *Let $Q$ be a weight and $A$ in $\mathcal{C}l(E)$. We call $Q$-*weighted trace of $A$ *the expression*

$$\mathrm{tr}^Q(A) = \text{f.p.}|_{z=0}\left(\mathrm{tr}(AQ^{-z})\right). \tag{1.5}$$

This definition can be extended to $\mathcal{C}l(E)$-valued forms on $M$ by

$$\mathrm{tr}^Q(\omega) = \mathrm{tr}^Q(\alpha\otimes A) = \alpha\mathrm{tr}^Q(A), \tag{1.6}$$

where $\alpha$ denotes a form on $M$ and the $\mathcal{C}l(E)$-valued form $\omega$ on $M$ is given by $\omega = \alpha\otimes A$. It follows from the definition that weighted traces extend usual finite-dimensional traces, i.e.

$$\mathrm{tr}^Q(A) = \mathrm{tr}(A) \tag{1.7}$$

whenever $A$ is a finite rank operator.

### Weighted Trace Anomalies

Unlike Wodzicki residues, weighted traces are not tracial and depend on the weight $Q$. As a matter of fact both $\mathrm{tr}^Q([A, B])$ and $\mathrm{tr}^{Q_1}(A) - \mathrm{tr}^{Q_2}(A)$, for $Q, Q_1, Q_2 \in \mathcal{A}d(E)$ and $A, B \in \mathcal{C}l(E)$, can be expressed in terms of Wodzicki residues (see Proposition 1 below). This is the price we pay for having left out divergences when taking the finite part of otherwise diverging expressions, and we call these obstructions *weighted trace anomalies.* Weighted trace anomalies play an important role in Chapters 3, 4 and 5, where we shall use them in the study of the geometry of the determinant line bundle, and we shall show how they relate to phase anomalies in Chern-Simons theories.

Recall that although the logarithm of a classical pseudo-differential operator is not classical, the bracket $[\log Q, A]$ and the difference $\frac{\log Q_1}{q_1} - \frac{\log Q_2}{q_2}$ of two such logarithms lie in $\mathcal{C}l(E)$ (see Appendix A).

**Definition 2** *1. For $A, B \in \mathcal{C}l(E)$ and $Q \in \mathcal{A}d(E)$, we define the* coboundary anomaly *by*

$$\partial\mathrm{tr}^Q(A, B) = \mathrm{tr}^Q([A, B]), \tag{1.8}$$

*where $\partial tr^Q$ denotes the coboundary of the linear functional $tr^Q$ on the Lie algebra $\mathcal{C}l(M,E)$ in the Hochschild cohomology*[1].

2. *We define the* weight anomaly, *which expresses how weighted traces depend on the choice of the weight, by*

$$\triangle_{Q_2}^{Q_1}(A) = \mathrm{tr}^{Q_1}(A) - \mathrm{tr}^{Q_2}(A), \tag{1.9}$$

*where $Q_1, Q_2 \in \mathcal{A}d(E)$.*

The following proposition shows how these tracial anomalies can be written in terms of Wodzicki residues.

**Proposition 1** 1. *Given $A, B \in \mathcal{C}l(E)$, $Q \in \mathcal{A}d(E)$ with positive order $q$, we have* [CDMP][MN]

$$\partial \mathrm{tr}^Q(A,B) = -\frac{1}{q}\mathrm{res}\left(A[\log Q, B]\right). \tag{1.10}$$

2. *For $Q_1, Q_2 \in \mathcal{A}d(E)$ with positive orders $q_1, q_2$ we have* [CDMP]

$$\triangle_{Q_2}^{Q_1}(A) = -\mathrm{res}\left(A\left(\frac{\log Q_1}{q_1} - \frac{\log Q_2}{q_2}\right)\right). \tag{1.11}$$

*Proof.* It follows from the properties of the canonical trace established by Kontsevich and Vishik (see [KV], Proposition 3.4) applied to the holomorphic families $\frac{1}{z}[Q^{-z}, A]B$ and $A\left(\frac{1}{z}(Q_1^{-z} - Q_2^{-z})\right)$, which yields (1.10) and (1.11), respectively. □

We can extend these results to variations of traces of one parameter families of operators, which gives rise to another tracial anomaly. Let $\{Q_x\}_{x\in X} \subset \mathcal{A}d(E)$ be a smooth family of weights, with constant positive order $q$ and common spectral cut, parametrized by a smooth manifold $X$. We define for a *fixed* operator $A \in \mathcal{C}l(E)$

$$(d\,\mathrm{tr}^Q)(A) = d(\mathrm{tr}^Q(A)). \tag{1.12}$$

**Proposition 2** *Let $\{Q_x\}_{x\in X}$ be a smooth family in $\mathcal{A}d(E)$ with constant order $q$, parametrized by a smooth manifold $X$. Then, for a* fixed *$A \in \mathcal{C}l(E)$ we have* [CDMP] [P]

$$d\,\mathrm{tr}^Q(A) = -\frac{1}{q}\mathrm{res}\left(A\,d\log Q\right). \tag{1.13}$$

*Proof.* It follows from the fundamental property of the canonical trace of Kontsevich and Vishik [KV] applied to the family $\frac{1}{t}A\left(\frac{1}{z}(Q_x^{-z} - Q_{\gamma_t(x)}^{-z})\right)$, where $\gamma_t(x)$ is a 1-parameter curve starting at $x$, $t \geq 0$ generated by a tangent

[1] This coboundary extends the Radul cocycle in the physics literature [R][M][MN][CDMP].

vector $h$. $\square$

Thus, if we consider a smooth family $\{A_x\}_{x\in X}$ in $\mathcal{C}l(E)$, it follows that

$$\begin{aligned} d\mathrm{tr}^Q(A) &= (d\mathrm{tr}^Q)(A) + \mathrm{tr}^Q\,(dA) \\ &= -\frac{1}{q}\mathrm{res}\,(A\,d\log Q) + \mathrm{tr}^Q\,(dA)\,. \end{aligned} \tag{1.14}$$

This extends to $\mathcal{C}l(E)$-valued $k$-forms on $M$ by

$$d\,\mathrm{tr}^Q(\omega) = \frac{(-1)^{k+1}}{q}\mathrm{res}\,(\omega\,d\log Q) + \mathrm{tr}^Q\,(d\omega)\,. \tag{1.15}$$

An important observation in view of what follows is that all these weighted trace anomalies being Wodzicki residues of some operator, can be expressed in terms of integrals on the underlying manifold $M$ of local expressions involving the symbols of that operator.

**Remarks.**

1. For $C \in \mathcal{C}l(E)$ invertible, $A \in \mathcal{C}l(E)$ and $Q$ any weight [CDMP]

$$\mathrm{tr}^{C^{-1}QC}(A) = \mathrm{tr}^Q(CAC^{-1}). \tag{1.16}$$

   We shall refer to this property as the *covariance* property of weighted traces.

2. When $Q$ has positive leading symbol, we can recover the $\zeta$-regularized trace (1.5) using a heat-kernel expansion. Let us first recall some results about the Mellin transform of a smooth $C^\infty$ function on the positive real line (here we follow [BGV92]). Let $f \in C^\infty(\mathbb{R}^+)$ decaying exponentially at infinity, then the *Mellin Transform* of $f$ is the function defined by

$$\mathrm{M}[f](z) = \frac{1}{\Gamma(z)}\int_0^\infty f(t)t^{z-1}\,dt. \tag{1.17}$$

   Integration by parts shows that

$$\mathrm{M}[t\,f'](z) = -z\,\mathrm{M}[f](z). \tag{1.18}$$

   If $f$ has an asymptotic expansion for small $t$ of the form

$$f(t) \sim \sum_{k\geq -n} f_k t^{\frac{k}{q}} + c\log t, \tag{1.19}$$

   then its Mellin transform $\mathrm{M}[f]$ is a meromorphic function with poles contained in the set $\frac{n}{q} - \frac{\mathbb{N}}{q}$, and with a Laurent series around zero of the form $-c\,s^{-1} + (f_0 - \gamma c) + O(s)$, where $\gamma$ is the Euler constant.

For $A \in \mathcal{C}l(E)$ let $f(t) = \mathrm{tr}(Ae^{-tQ})$, then $f(t)$ behaves as in (1.19), where $q = \mathrm{ord}Q$ and (1.17) yields

$$\mathrm{f.p.}|_{z=0}\left(\mathrm{tr}(AQ^{-z})\right) = \mathrm{f.p.}|_{z=0}\mathrm{M}[f](z) = \mathrm{f.p.}|_{t=0}\left(\mathrm{tr}(Ae^{-tQ})\right) - \gamma \cdot \mathrm{res}(A)$$

where $\gamma$ is the Euler constant. Thus, if $\mathrm{res}(A) = 0$,

$$\mathrm{tr}^Q(A) = \mathrm{f.p.}|_{t=0}\left(\mathrm{tr}(Ae^{-tQ})\right). \tag{1.20}$$

When $Q$ is a *differential* operator and $A = I$ we have, as $t \to 0$, $\mathrm{tr}\left(e^{-tQ}\right) \sim \sum_{k \geq -\frac{\dim M}{q}} a_k t^{\frac{k}{q}}$ (see e.g. [G95]), so $\mathrm{tr}^Q(I) = \mathrm{f.p.}|_{t=0}\left(\mathrm{tr}(e^{-tQ})\right)$.

3. The notion of weighted trace can be extended to the case when $Q$ is a non injective self-adjoint elliptic with positive order. Being elliptic, such an operator has a finite dimensional kernel and the orthogonal projection $P_Q$ onto this kernel is a pseudodifferential operator of finite rank. Since $Q$ is an elliptic operator so is the operator $\bar{Q} = Q + P_Q$, for the ellipticity is a condition on the leading symbol which remains unchanged when adding $P_Q$. Moreover, $Q$ being self-adjoint the range of $Q$ is given by $R(Q) = (\ker Q^*)^\perp = (\ker Q)^\perp$ so that $\bar{Q}$ is onto. $\bar{Q}$ being injective and onto is invertible and being self-adjoint, then $\bar{Q} \in \mathcal{A}d(E)$ (it has the same order as $Q$). We set

$$\mathrm{tr}^Q(A) = \mathrm{f.p.}|_{z=0}\left(\mathrm{tr}(A(\bar{Q})^{-z})\right). \tag{1.21}$$

### 1.1.2 Weighted Traces and $\zeta$-Determinants of invertible admissible operators

Given $A, Q \in \mathcal{A}d(E)$ with common spectral cut, the map $z \mapsto \mathrm{tr}\left((\log A)Q^{-z}\right)$ is meromorphic on the complex plane with a simple pole at the origin [KV]. In order to define determinants in infinite dimensions we now extend weighted traces to logarithms of pseudo-differential operators.

**Definition 3** *Given $A, Q \in \mathcal{A}d(E)$ we set*

$$\mathrm{tr}^Q(\log A) := \mathrm{f.p.}|_{z=0}\left(\mathrm{tr}(\log A Q^{-z})\right).$$

As before, $Q$ is referred to as the weight and $\mathrm{tr}^Q(\log A)$ as the $Q$-weighted trace of $\log A$. We shall not make explicit mention in the notation of the determination of the logarithm underlying this definition. Extending (1.11) to logarithms we set $\triangle_{Q_2}^{Q_1}(\log A) = \mathrm{tr}^{Q_1}(\log A) - \mathrm{tr}^{Q_2}(\log A)$.

**Theorem 1** [OI], (see also [D]) *For $Q_1, Q_2, A \in \mathcal{A}d(E)$ with positive orders $q_1, q_2$ and $a$ respectively,*

$$\begin{aligned}\triangle_{Q_2}^{Q_1}(\log A) &= \mathrm{res}\left(\left(\log A - \frac{a}{q_1}\log Q_1\right)\left(\frac{\log Q_2}{q_2} - \frac{\log Q_1}{q_1}\right)\right) \\ &- \frac{a}{2}\mathrm{res}\left(\left(\frac{\log Q_2}{q_2} - \frac{\log Q_1}{q_1}\right)^2\right).\end{aligned}$$

Let us recall here the definition and some basic properties of $\zeta$-determinants of admissible operators. Let $A \in \mathcal{A}d(E)$, then the spectrum of $A$ is discrete and it is entirely contained in the real line, $\{\lambda_k\}_{k\in\mathbb{Z}} \subset \mathbb{R}$. The $\zeta$-function of $A$ is defined as the Mellin transform of its heat kernel or, which is equivalent, the trace of the operator $A^{-z}$, i.e.

$$\begin{aligned}\zeta_A(z) &= \mathrm{M}[\mathrm{tr}\left(e^{-tA}\right)](z) = \frac{1}{\Gamma(z)}\int_0^\infty t^{z-1}\mathrm{tr}\left(e^{-tA}\right)\,dt \\ &= \mathrm{tr}\left(A^{-z}\right) = \sum_{\lambda\in\mathrm{spec}A}\lambda^{-z},\end{aligned} \tag{1.22}$$

which is an analytic function for $z \in \mathbb{C}$ with $\Re(z) >> 0$, and extends by analytical continuation to a meromorphic function on $\mathbb{C}$, regular at $z = 0$, as can seen from the previous remarks about the Mellin transform.

**Definition 4** [RS71] *The* $\zeta$-determinant *of $A$, denoted by* $\det_\zeta A$, *is the complex number given by*

$$\det{}_\zeta A = \exp\left\{-\zeta'_A(0)\right\} = \exp\mathrm{tr}^A(\log A). \tag{1.23}$$

**Remarks.**

1. Given $Q \in \mathcal{A}d(E)$ and $C \in \mathcal{C}l(E)$ invertible, then $\log CAC^{-1} = \log A$ and from (1.16) it follows that

$$\mathrm{tr}^{CQC^{-1}}(C\log AC^{-1}) = \mathrm{tr}^Q(\log A).$$

   Thus,

$$\det{}_\zeta(CAC^{-1}) = \det{}_\zeta(A), \tag{1.24}$$

   so the $\zeta$-determinant is invariant under inner automorphisms of $\mathcal{C}l(E)$.

2. Note that the operator $A$ is used as a weight to define $\det_\zeta A$. This is a source of anomaly. In particular, $\zeta$-determinants are not multiplicative. In fact the *multiplicative anomaly* [KV], defined by the expression $F_\zeta(A,B) = \frac{\det_\zeta(AB)}{\det_\zeta(A)\det_\zeta(B)}$ which generally differs from one, is given in terms of Wodzicki residues, namely (see [Wo] for the case $[A,B] = 0$, [D] for the general case)

$$\begin{aligned}\log F_\zeta(A,B) &= \frac{1}{2a}\mathrm{res}\left(\left(\log A - \frac{a}{a+b}\log(AB)\right)^2\right) \\ &+ \frac{1}{2b}\mathrm{res}\left(\left(\log B - \frac{b}{a+b}\log(AB)\right)^2\right) \\ &+ \mathrm{tr}^{AB}\left(\log(AB) - \log A - \log B\right)\end{aligned}$$

for any two operators $A, B \in \mathcal{A}d(E)$ of order $a$ and $b$, respectively. Specializing to $B = A^*$, the formal adjoint of $A$ for the $L^2$-structure induced by a Riemannian metric on $M$ and a hermitian one on $E$, in general we have $F_\zeta(A, A^*) \neq 0$ and hence

$$\det_\zeta(A^*A) \neq |\det_\zeta(A)|^2.$$

However, if $A$ is self-adjoint this multiplicative anomaly vanishes, i.e. $\det_\zeta(A^2) = \det_\zeta(A^*A) = |\det_\zeta(A)|^2$.

3. The $\zeta$-determinants extends of the usual determinant in finite-dimensional vector spaces, i.e. if $A$ is a finite-rank operator (a matrix) then

$$\det_\zeta(A) = \prod_{i=1}^{N} \lambda_i, \tag{1.25}$$

where $\lambda_i$, $1 \leq i \leq N$, are the eigenvalues of $A$.

**Lemma 1** [RS71] *Let $\{A_x\}_{x\in X} \subset \mathcal{A}d(E)$ be a one parameter smooth family of admissible operators with constant order and common spectral cut, parametrized by a manifold $X$. Then*

$$d \log \det_\zeta A_x = \mathrm{tr}^{A_x}\left(A_x^{-1} dA_x\right). \tag{1.26}$$

*Proof.* Follows from Theorem 1, using the covariance property and the commutativity of $A_x$ with any power of itself (see [CDP]). □

### 1.1.3 Variations of determinants of invertible self-adjoint operators

Let $A \in Ell^*_{ord>0}(E)$ be a self-adjoint elliptic (classical) pseudo-differential operator. If $A$ is not positive, its spectrum contains negative eigenvalues but its $\zeta$-function can still be defined by (1.22), taking now $\lambda_k^{-z}$ to be $|\lambda_k|^{-z}e^{-i\pi z}$ if $\lambda_k$ is negative. In [APS73] Atiyah, Patodi and Singer define, for large $\Re(z)$, the $\eta$-function of $A$ as the trace of the operator $A|A|^{-z-1}$, i.e.

$$\eta_A(z) = \sum_{k\in\mathbb{Z}} (\mathrm{sign}\lambda_k)\lambda_k^{-z}. \tag{1.27}$$

They showed that this function extends meromorphically to the whole $z$-plane and, moreover, that $\eta_A(z)$ is finite at $z = 0$. Its value at $z = 0$ measures the *asymmetry* of the spectrum of A. Following [APS73] we define the *$\eta$-invariant* of $A$ by

$$\eta_A(0) = \mathrm{f.p.}|_{z=0}\, \mathrm{tr}\left(\mathrm{sign}A|A|^{-z}\right) = \mathrm{tr}^{|A|}(\mathrm{sign}(A)), \tag{1.28}$$

where the sign of $A$ is the classical pseudo-differential operator defined by $\mathrm{sign}(A) = A|A|^{-1}$.

The following well-known result (see e.g. [Si] [AS95]) shows that the phase of the $\zeta$-determinant of a self-adjoint operator can be expressed in terms of its $\eta$-invariant.

**Proposition 3** *Let $A \in Ell^*_{ord>0}(E)$ be any self-adjoint elliptic pseudo-differential operator. Then*

$$\det_\zeta A = \exp \mathrm{tr}^{|A|} (\log A) = \det_\zeta |A| \cdot \exp\{\frac{i\pi}{2}(\eta_A(0) - \zeta_{|A|}(0))\}. \qquad (1.29)$$

*Proof.* Let us give a proof here in the language of weighted traces. Let $A$ be an elliptic self-adjoint operator and let $a$ be the order of $A$, then [APSIII]

$$\mathrm{res}(U_A) = 0,$$

where $U_A = \mathrm{sign}(A) = A|A|^{-1}$ denote the sign of $A$. Using the polar decomposition $A = |A|U_A = U_A|A|$ it follows that

$$\log A = \log |A| + \log U_A$$

since $[|A|, U_A] = 0$. Applying the results of Theorem 1, and using the fact that $U_A = \exp\left(\frac{i\pi}{2}(U_A - I)\right)$, we get

$$\begin{aligned} \mathrm{tr}^A(\log A) - \mathrm{tr}^{|A|}(\log A) &= -\frac{a}{2}\mathrm{res}\left((\log U_A)^2\right) \\ &= a\frac{\pi^2}{8}\mathrm{res}((U_A - I)^2). \end{aligned}$$

But $U_A^2 = I$ for $A$ self-adjoint, so

$$\mathrm{tr}^A(\log A) - \mathrm{tr}^{|A|}(\log A) = a\frac{\pi^2}{4}\mathrm{res}(I - U_A) = -a\frac{\pi^2}{4}\mathrm{res}(U_A) = 0.$$

Thus,

$$\det_\zeta(A) = \exp \mathrm{tr}^A(\log A) = \exp \mathrm{tr}^{|A|}(\log A) = \det_\zeta |A| e^{i\phi(A)},$$

where $\phi(A) = -i\mathrm{tr}^{|A|} \log_{(\frac{\pi}{2})} U_A = \frac{\pi}{2}\left(\eta_A(0) - \zeta_{|A|}(0)\right)$ is the "phase" of the zeta determinant of $A$. The equality $\eta_A(0) = tr^{|A|}(U_A)$ yields (1.29).
□

As a consequence of this, using the fact that $\zeta_{|A|}(0) = 0$ when $A$ is a differential operator acting on sections of some vector bundle based on an odd-dimensional closed manifold [S][Si], it follows that

$$\det_\zeta(A) = \det_\zeta |A| \cdot e^{\frac{i\pi}{2}\mathrm{tr}^{|A|}(U_A)} = \det_\zeta |A| \cdot e^{\frac{i\pi}{2}(\eta_A(0))}, \qquad (1.30)$$

whenever $n$ is odd.

**Example 1** [BBW] *Consider the operator*

$$A_t = i\frac{d}{dx} + t \tag{1.31}$$

*on* $C^\infty(S^1)$*, where* $S^1$ *is identified with* $\mathbb{R}/[0, 2\pi]$ *and* $0 \leq t \leq 1$*. Then, if* $f \in C^\infty(S^1)$*,* $A_t f(x) = \lambda f(x)$ *if and only if* $f(x) = \exp\{-i(\lambda - t)x\}$ *which, under the given boundary condition, implies that* $\lambda = n + t$ *where* $n \in \mathbb{Z}$*, or* $\lambda = \pm n + t$ *where* $n \in \mathbb{Z}^+$*. Thus, the spectrum of* $A_t$ *is the set* $\{\pm n + t\}_{n\in\mathbb{Z}^+}$*, so that*

$$\zeta_{A_t}(z) = \sum_{n\in\mathbb{Z}^+} (n+t)^{-z} + \sum_{n\in\mathbb{Z}^+} (n-t)^{-z} e^{-i\pi z}$$

*and*

$$\eta_{A_t}(z) = \sum_{n\in\mathbb{Z}^+} (n+t)^{-z} - \sum_{n\in\mathbb{Z}^+} (n-t)^{-z}.$$

*In terms of the Riemann-Hurwitz zeta function* $\zeta(z,t) = \sum_{n=0}^{\infty}(n+t)^{-z}$ *these equations can be written as*

$$\zeta_{A_t}(z) = \zeta(z,t) - e^{-i\pi z}\zeta(z,-t)$$

*and*

$$\eta_{A_t}(z) = \zeta(z,t) - \zeta(z,-t).$$

*Hence the* $\zeta$*-determinant of* $A_t$ *reads*

$$\det{}_\zeta A_t = \frac{2\pi}{\Gamma(t)\Gamma(-t)} \exp\left\{\frac{i\pi}{2}(1-2t)\right\}, \tag{1.32}$$

*where* $\Gamma(z)$ *denotes the gamma function*

$$\Gamma(z) = \int_0^\infty e^{-t} t^{z-1} dt, \tag{1.33}$$

*and*

$$\eta_{A_t}(0) = 1 - 2t. \tag{1.34}$$

*This shows that when* $t = \frac{1}{2}$ *and the spectrum of* $A_t$ *is symmetric,* $\eta_{A_t}(0) = 0$*.*

*Finally, note that*

$$\zeta_{|A_t|}(z) = \zeta(z,t) + \zeta(z,-t)$$

*so* $\zeta_{|A_t|}(0) = 0$*, which is in accordance with our previous result for the* $\zeta$*-determinant of* $A_t$*.*

**Remark.** The definition of $\eta(A)$ extends to non invertible operators in a similar way as weighted traces do, i.e. replacing $A$ by $\bar{A} = A + P_A$, where $P_A$ denotes the projection on $\ker A$ (see Remark 2 at the end of Section 1.1.1).

## 1.2 Elliptic Complexes and Analytic Torsion

### 1.2.1 Determinant and Torsion of a Chain Complex

In this section we review the theory of determinants and torsions for chain complexes of finite-dimensional vector spaces. Three types of algebraic "torsions" arising in the literature are considered (Reidemeister torsion, Torsion and Analytic torsion) and the relation between them are described.

**Determinants in Finite Dimensions.** Let $E$ be a finite dimensional complex vector space. The *determinant* of $E$ is the one dimensional complex vector space $\det E = \Lambda^m E$, where $m = \dim E$. If there exists a $\mathbb{Z}$-grading on $E$, i.e. if $E = E_0 \oplus E_1 \oplus \cdots \oplus E_n$, $n \in \mathbb{N}$, where $E_k$ is a finite dimensional complex vector space for all $k \in \{0, 1, \ldots, n\}$, its determinant is defined by the tensor product

$$\det E = \bigotimes_{k=0}^{n} (\det E_k)^{(-1)^{k+1}},$$

where we denote by $V^{-1} = V^* = \mathrm{Hom}(V, \mathbb{C})$ the dual vector space to $V$, letting $\det V = (\det V)^{-1} = (\det V)^* = \mathbb{C}$ if $V = 0$.

Recall that given a linear map $T : E \to E$, the determinant of $T$ is the complex number given by the equality

$$(\det T)(e_1 \wedge \cdots \wedge e_n) = Te_1 \wedge \cdots \wedge Te_n,$$

i.e. $\det T = \langle T\mathbf{e} \,|\, \mathbf{e}^* \rangle$, where $\{e_1, \ldots, e_n\}$ is an oriented basis for $E$, $\mathbf{e}^*$ the dual of $\mathbf{e} = e_1 \wedge \cdots \wedge e_n$, $T\mathbf{e} = Te_1 \wedge \cdots \wedge Te_n$ and $\langle \cdot | \cdot \rangle$ denote duality pairing between $\det E$ and its dual vector space. It is equal to the product of the eigenvalues of $T$, independently of the chosen basis. However, if we consider a linear map between different finite-dimensional complex vector spaces (of the same dimension)

$$T : E \to F,$$

the complex number given by

$$(\det T)(f_1 \wedge \cdots \wedge f_n) = Te_1 \wedge \cdots \wedge Te_n, \tag{1.35}$$

where $f_1, \ldots, f_n$ is an oriented basis for $F$, and $f_1 \wedge \cdots \wedge f_n$, $Te_1 \wedge \cdots \wedge Te_n$, are elements of the one-dimensional complex vector space $\det F$, is no longer independent of the chosen basis for $E$ and $F$. Thus, in this case the *determinant* of $T$ must be regarded as an element of the one-dimensional complex vector space $\det E^* \otimes \det F$. A canonical representation of $\det T$ is giving by taking any $\mathbf{x} \in \det E$ such that $T\mathbf{x} \neq 0$, then the element $\mathbf{x}^* \otimes T\mathbf{x}$ of $\det E^* \otimes \det F$ is independent of the $\mathbf{x}$ chosen.

### Algebraic Torsions of a Chain Complex

Consider a *chain complex* $(E_\bullet, T_\bullet)$ of finite-dimensional vector spaces, i.e. a set of finite-dimensional vector spaces $\{E_k\}_{k=0,\dots,n}$ (which we assume equipped with a Hermitian inner product) and linear maps $\{T_k\}_{k=0,\dots,n}$,

$$0 \to E_0 \xrightarrow{T_0} \cdots \to E_{k-1} \xrightarrow{T_{k-1}} E_k \xrightarrow{T_k} E_{k+1} \to \cdots \xrightarrow{T_{n-1}} E_n \to 0, \tag{1.36}$$

such that

$$T_k \circ T_{k-1} = 0. \tag{1.37}$$

Associated to a complex $(E_\bullet, T_\bullet)$ we shall define three objects, its *Reidemeister Torsion*, its *Torsion* and its *Analytic Torsion*, which generalize the idea of determinant of a given linear map.

**Reidemeister Torsion.** The Reidemeister torsion of a complex was introduced by Reidemeister, Franz and de Rham in the $30's$ in order to distinguish topological spaces with the same homotopy type, and measures -in some sense- the volume of the complex. (for a historical reference containing original references see Milnor's *Collected Works* [M95]).

Given a complex $(E_\bullet, T_\bullet)$ consider the vector space $\mathbb{E} = \bigoplus_{k=0}^n E_k$. The standard short exact sequences

$$0 \to Z_k \longrightarrow E_k \xrightarrow{T_k} B_{k+1} \to 0 \tag{1.38}$$

and

$$0 \to B_k \longrightarrow Z_k \longrightarrow H_k \to 0, \tag{1.39}$$

where $B_k = \mathrm{Im} T_{k-1}$, $Z_k = \ker T_k$ and $H_k$ is the $k^{th}$-cohomology space of complex, induce canonical isomorphisms $\det E_k \cong \det Z_k \otimes \det B_{k-1}$ and $\det Z_k \cong \det B_k \otimes \det H_k$. Combining these isomorphisms gives an isomorphism

$$\Phi : \det \mathbb{E} \to \det \mathbb{H},$$

between the *determinant space* of the complex $(E_\bullet, T_\bullet)$, defined as the determinant space of $\mathbb{E}$,

$$\det \mathbb{E} = \bigotimes_{k=0}^{n} (\det E_k)^{(-1)^{k+1}},$$

and the determinant of its cohomology $\det \mathbb{H} = \bigotimes_{k=0}^n (\det H_k)^{(-1)^{k+1}}$. Let $\mathbf{e}$ be an ordered basis for $E_\bullet$, i.e. an ordered basis $\mathbf{e}_k = \{e_k^1, e_k^2, \dots e_k^{n_k}\}$, where $n_k = \dim E_k$, for each $E_k$. Let $\mathbf{h}$ be an ordered basis for $H_\bullet$. Let $[\mathbf{e}] \in \det \mathbb{E}$ and $[\mathbf{h}] \in \det \mathbb{H}$ denote the resulting volume forms.

**Definition 5** [M62] *The* Reidemeister torsion *of the complex* $(E_\bullet, T_\bullet)$ *is the non-zero complex number given by*

$$\tilde{\tau}_R(E_\bullet, \mathbf{e}, \mathbf{h}) = \langle \Phi([\mathbf{e}]) | [\mathbf{h}]^{-1} \rangle,$$

*where the power* $-1$ *stands for dual and* $\langle | \rangle$ *denotes duality pairing in* $\det \mathbb{H}$. *We also define* $\tau_R(E_\bullet, \mathbf{e}) = \tilde{\tau}_R(E_\bullet, \mathbf{e}, 1)$ *if the complex is* acyclic, *i.e.* $H_\bullet = 0$, $\tau_R(E_\bullet, \mathbf{e}) = 0$ *otherwise. Note that we omit in the notation the dependence of the chain maps* $(T_\bullet)$, *although the torsion depends on it.*

The canonical isomorphism $\Phi$ can be described as follows [M62] [T01]. Let $\mathbf{e} = (\mathbf{e}_0, \mathbf{e}_1, \mathbf{e}_2, \dots \mathbf{e}_n)$ be the given ordered basis of $E_\bullet$, and consider, for each $k$, the volume element $\mathrm{v}_{\mathbf{e}_k} = e_k^1 \wedge e_k^2 \wedge \dots \wedge e_k^{n_k} \in \det E_k$ associated to the basis $\mathbf{e}_k$ of $E_k$. Then $[\mathbf{e}] = \mathrm{v}_{\mathbf{e}_0}^{-1} \otimes \mathrm{v}_{\mathbf{e}_1} \otimes \dots \otimes \mathrm{v}_{\mathbf{e}_n}^{(-1)^{n+1}} \in \det \mathbb{E}$, where the $-1$ power denote the dual element, and $[\mathbf{h}] = \mathrm{v}_{\mathbf{h}_0}^{-1} \otimes \mathrm{v}_{\mathbf{h}_1} \otimes \dots \otimes \mathrm{v}_{\mathbf{h}_n}^{(-1)^{n+1}}$ is the corresponding ordered basis $\mathbf{h}$ of $H_\bullet$.

By exactness of the short exact sequences (1.38) and (1.39), there is a canonical isomorphism $E_k \cong B_k \oplus H_k \oplus B_{k+1}$. Let us choose, for each $k$, a subset $\mathbf{b}_{k-1} = \{b_{k-1}^1, \dots, b_{k-1}^{l_k}\}$ of $E_{k-1}$ such that $T_{k-1}\mathbf{b}_{k-1} = \{T_{k-1}b_{k-1}^1, \dots T_{k-1}b_{k-1}^{l_k}\}$ is a basis of $B_k = \mathrm{Im} T_{k-1}$, where $l_k = \dim B_k$. Then, the collection $\tilde{\mathbf{e}}_k = \{T_{k-1}\mathbf{b}_{k-1}, \mathbf{h}_k, \mathbf{b}_k\}$ is a basis of $E_k$. Let us denote by $\widetilde{T}_k$ the transition matrix taking the basis $\mathbf{e}_k$ into the basis $\tilde{\mathbf{e}}_k$ of $E_k$. Then [T01],

$$\Phi([\mathbf{e}]) = (-1)^{\mathrm{n}_E} \prod_{k=0}^{n} \left( \det \widetilde{T}_k \right)^{(-1)^{k+1}} [\mathbf{h}],$$

where $\mathrm{n}_E = \sum_{k=0}^{n} \chi_k^E(\mathbb{E}) \bar{\chi}_k^H(\mathbb{E}) \pmod 2$, with $\chi_k^E(\mathbb{E}) = \sum_{k=0}^{k} \dim E_k \pmod 2$ and $\bar{\chi}_k^H(\mathbb{E}) = \sum_{k=0}^{k} \dim H_k \pmod 2$. The definition is of course independent of the basis $\mathbf{b}_k$ used in the calculations.

In what follows we shall be interested in the acyclic case (so in $\tau_R$ rather than $\tilde{\tau}_R$). Observe that if $H_\bullet = 0$ then $\mathrm{n}_E = 0$, and the previous discussion shows that we can compute the Reidemeister torsion $\tau_R$ in terms of an alternating product of determinants, as given by the following:

**Lemma 2** *If* $H_\bullet = 0$, *the Reidemeister torsion of the chain complex* $(E_\bullet, T_\bullet)$ *is given by*

$$\tau_R(E_\bullet, \mathbf{e}) = \prod_{k=0}^{n-1} (\det \bar{T}_k)^{(-1)^{k+1}}.$$

*where* $\bar{T}_k$ *is the isomorphism taking the subcollection* $\mathbf{b}_k$ *into the subcollection* $\{e_{k+1}^1, \dots, e_{k+1}^{l_{k+1}}\}$ *of the basis* $\mathbf{e}_{k+1}$.

Indeed, acyclicity implies that, for any $0 \leq k \leq n$,

$$0 \to B_k \hookrightarrow E_k \stackrel{T_k}{\to} B_{k+1} \to 0, \tag{1.40}$$

is a short exact sequence. The result follows taking $[\mathbf{h}] = 1$ in our previous discussion.

In the acyclic case a $\mathbb{Z}_2$-grading is defined naturally on the vector spaces appearing in the complex $(E_\bullet, T_\bullet)$, and the torsion can also be computed from the determinant of a map induced canonically for such a grading.

**Lemma 3** [T01] *Any acyclic chain complex $(E_\bullet, T_\bullet)$ of finite-dimensional vector spaces induces a canonical isomorphism*

$$\iota^R_{\tau(E_\bullet,\mathbf{e})} : \mathbb{E}^+ \to \mathbb{E}^-,$$

*where $\mathbb{E}^+ = \bigoplus_{k\,\mathrm{even}} E_k$ and $\mathbb{E}^- = \bigoplus_{k\,\mathrm{odd}} E_k$, such that*

$$\tau_R(E_\bullet, \mathbf{e}) = \det \iota^R_{\tau(E_\bullet,\mathbf{e})}. \tag{1.41}$$

(see also [N01] and references therein).

**The Torsion.** As for the determinant of a linear map between different vector spaces, to the Reidemeister torsion (which depends on the chosen basis $\mathbf{e}$ and $\mathbf{h}$ involved in the computation) we can associate a canonical element of the determinant line $\det \mathbb{E}$. Let $(E_\bullet, T_\bullet)$ be an acyclic complex and let, for each $0 \leq k \leq n-1$, $\mathbf{x}_k \in \Lambda^{l_{k+1}} E_k$ be such that $T_k \mathbf{x}_k \neq 0$. Then, since $\dim E_{k+1} = l_k + l_{k+1}$, $T_k\mathbf{x}_k \wedge \mathbf{x}_{k+1} \in \det E_{k+1}$ is a non-zero element.

**Definition 6** *The* Torsion *of the complex $(E_\bullet, T_\bullet)$ is the canonical element of* $\det \mathbb{E}$ *given by*

$$\tau(E_\bullet, T_\bullet) = \mathbf{x}_0^{-1} \otimes (T_0\mathbf{x}_0 \wedge \mathbf{x}_1) \otimes (T_1\mathbf{x}_1 \wedge \mathbf{x}_2)^{-1} \otimes \cdots \otimes (T_{n-1}\mathbf{x}_{n-1})^{(-1)^{n+1}}. \tag{1.42}$$

Here, as before, canonical means independent of the $\mathbf{x}_0, \mathbf{x}_1, \ldots \mathbf{x}_{n-1}$ chosen. The relation between the torsion and the Reidemeister torsion of $(E_\bullet, T_\bullet)$ is given by a duality pairing. Indeed, observe that taking an ordered basis $\mathbf{e}$ of $E_\bullet$ and the dual of the induced element $[\mathbf{e}]$ in $\det \mathbb{E}$, $[\mathbf{e}]^{-1} = \mathrm{v}_{\mathbf{e}_0} \otimes \mathrm{v}_{\mathbf{e}_1}^{-1} \otimes \cdots \otimes \mathrm{v}_{\mathbf{e}_n}^{(-1)^n} \in \det \mathbb{E}^{-1}$, then

$$\begin{aligned} &\langle \quad \tau(E_\bullet, T_\bullet) \;\mid\; [\mathbf{e}]^{-1} \quad \rangle \\ = \quad &\langle \mathbf{x}_0^{-1} \otimes (T_0\mathbf{x}_0 \wedge \mathbf{x}_1) \otimes \cdots \otimes (T_{n-1}\mathbf{x}_{n-1})^{(-1)^{n+1}} | \mathrm{v}_{\mathbf{e}_0} \otimes \mathrm{v}_{\mathbf{e}_1}^{-1} \otimes \cdots \otimes \mathrm{v}_{\mathbf{e}_n}^{(-1)^n} \rangle \\ = \quad &\langle \mathbf{x}_0^{-1} | \mathrm{v}_{\mathbf{e}_0} \rangle_{E_0} \langle (T_0\mathbf{x}_0 \wedge \mathbf{x}_1) | \mathrm{v}_{\mathbf{e}_1}^{-1} \rangle_{E_1} \cdots \langle (T_{n-1}\mathbf{x}_{n-1})^{(-1)^{n+1}} | \mathrm{v}_{\mathbf{e}_n}^{(-1)^n} \rangle_{E_n} \end{aligned}$$

where $\langle | \rangle_V$ denotes duality pairing between the vector spaces $\det V$ and $(\det V)^{-1}$. Splitting $\mathrm{v}_{\mathbf{e}_k} = \mathrm{v}'_{\mathbf{e}_k} \wedge \mathrm{v}''_{\mathbf{e}_k}$, where $\mathrm{v}'_{\mathbf{e}_k} \in \Lambda^{l_k} E_k$ and $\mathrm{v}''_{\mathbf{e}_k} \in \Lambda^{l_{k+1}} E_k$, this gives

$$\begin{aligned} &\langle \quad \tau(E_\bullet, T_\bullet) \;\mid\; [\mathbf{e}]^{-1} \rangle = \\ &\langle \mathbf{x}_0^{-1} | (\mathrm{v}''_{\mathbf{e}_0}) \rangle_{E_0} \langle (T_0\mathbf{x}_0 \wedge \mathbf{x}_1) | (\mathrm{v}'_{\mathbf{e}_1} \wedge \mathrm{v}''_{\mathbf{e}_1})^{-1} \rangle_{E_1} \cdots \langle (T_{n-1}\mathbf{x}_{n-1})^{(-1)^{n+1}} | (\mathrm{v}'_{\mathbf{e}_n})^{(-1)^n} \rangle_{E_n}. \end{aligned}$$

Using the fact that $\tau(E_\bullet, T_\bullet)$ is independent of the $\mathbf{x}_k$, taking $\mathbf{x}_k = \mathrm{v}''_{\mathbf{e}_k}$ we find that

$$\langle \tau(E_\bullet, T_\bullet) \ | \ [\mathbf{e}]^{-1} \rangle = \langle (T_0 \mathbf{x}_0) | (\mathrm{v}'_{\mathbf{e}_1})^{-1} \rangle \cdots \langle (T_{n-1} \mathbf{x}_{n-1})^{(-1)^{n+1}} | (\mathrm{v}'_{\mathbf{e}_n})^{(-1)^n} \rangle,$$

and comparing with equation (1.35), which can be written $\det T = \langle T\mathbf{e} \,|\, \mathbf{f}^{-1} \rangle$, we recover the alternating product of determinants appearing in the expression for $\tau_R(E_\bullet, T_\bullet, \mathbf{e})$ given by Lemma 2. This leads to

**Proposition 4**

$$\tau_R(E_\bullet, T_\bullet, \mathbf{e}) = \langle \tau(E_\bullet, T_\bullet) \ | \ [\mathbf{e}]^{-1} \rangle.$$

Note that the advantage of the torsion $\tau(E_\bullet, T_\bullet)$ over the Reidemeister torsion $\tau_R(E_\bullet, T_\bullet, \mathbf{e})$ is that the former does not involve the choice of a basis.

**Analytic Torsion.** Given a complex $(E_\bullet, T_\bullet)$, consider the family $(T^*_\bullet)$ of formal adjoints of the family of linear maps $(T_\bullet)$. The map $T^*_k : E_{k+1} \to E_k$ is defined by

$$\langle T_k e_k, e_{k+1} \rangle_{k+1} = \langle e_k, T^*_k e_{k+1} \rangle_k,$$

where $\langle , \rangle_k$ and $\langle , \rangle_{k+1}$ denote the inner products in $E_k$ and $E_{k+1}$, respectively.

**Definition 7** *Let $(E_\bullet, T_\bullet)$ be an acyclic chain complex of finite dimensional vector spaces, and let, for $0 \le k \le n$, $\Delta_k : E_k \to E_k$ be the* Laplacian*, defined by*

$$\Delta_k = T^*_k T_k + T_{k-1} T^*_{k-1}.$$

*The* Analytic Torsion *of the complex $(E_\bullet, T_\bullet)$ is the positive real number given by* [BGS88]

$$\mathcal{T}(E_\bullet, T_\bullet) = \prod_{k=0}^{n} (\det \Delta_k)^{\frac{k}{2}(-1)^{k+1}} \tag{1.43}$$

The relation between the torsion and the analytic torsion of an acyclic chain complex is given by the following:

**Proposition 5** [BGS88] *Let $(E_\bullet, T_\bullet)$ be an acyclic complex of vector spaces, then*

$$\mathcal{T}(E_\bullet, T_\bullet) = |\tau(E_\bullet, T_\bullet)|, \tag{1.44}$$

*where $|\cdot|$ denotes the norm on* $\det \mathbb{E}$ *induced by the norms on the vector spaces $E_k$.*

Note that, through Proposition 4, Lemma 2 gives us an expression of $\tau_R(E_\bullet, T_\bullet, \mathbf{e})$ as a contraction of $\tau(E_\bullet, T_\bullet)$, an element of $\det \mathbb{E} = \bigotimes_{k=0}^{n} (\det E_k)^{(-1)^{k+1}}$, with an element of $(\det \mathbb{E})^{-1}$. In the same way, Lemma 3 can be seen as a contraction of an element of $\left(\det \mathbb{E}^{+}\right)^{*} \otimes \det \mathbb{E}^{-}$, where

$$\det \mathbb{E}^{+} = \bigotimes_{k\,\text{even}} \det E_k \quad , \quad \det \mathbb{E}^{-} = \bigotimes_{k\,\text{odd}} \det E_k,$$

with an element of its dual. In fact, these complex lines (one-dimensional complex vector spaces) are isomorphic, the isomorphism being given by

$$\begin{array}{rcl} \mathbf{i}_{\pm} : \quad \bigotimes_{k=0}^{n} (\det E_k)^{(-1)^{k+1}} & \rightarrow & \left(\bigotimes_{k\,\text{even}} \det E_k\right)^{*} \otimes \left(\bigotimes_{k\,\text{odd}} \det E_k\right) \\ e_0^{-1} \otimes e_1 \otimes e_2^{-1} \otimes \ldots \otimes \epsilon_n^{(-1)^{n+1}} & \mapsto & (e_0 \otimes e_2 \otimes \ldots)^{*} \otimes e_1 \otimes e_3 \otimes \ldots \end{array} \tag{1.45}$$

There is a canonical element in $\det \mathbb{E}$, namely the torsion $\tau(E_\bullet)$, and the determinant of the isomorphism $D^{+}_{(E_\bullet, T_\bullet)} = \oplus_{k=0}^{n}(T_k + T_k^{*}) : \mathbb{E}^{+} \rightarrow \mathbb{E}^{-}$ (which we shall denote simply by $D^{+}$ when no explicit reference to the complex be necessary) gives us a canonical element in $\left(\det \mathbb{E}^{+}\right)^{*} \otimes \det \mathbb{E}^{-}$. However, these canonical elements do not correspond under isomorphism (1.45). As a matter of fact [BGS88]

$$\det D^{+} = \left[\prod_{\text{k even}} \det \Delta_k'\right] \mathbf{i}_{\pm}\left(\tau(E_\bullet, T_\bullet)\right), \tag{1.46}$$

where $\Delta_k' = \Delta_k|_{\mathrm{Im} T_{k-1}}$.

Let $(E_\bullet, T_\bullet)$ be an acyclic complex of finite dimensional vector spaces. Consider the decomposition $E_k = E_k' \oplus E_k''$, where $E_k' = T_{k-1}E_{k-1}$ and $E_k'' = T_k^{*}E_{k+1}$, at each level of the complex. Let $\Delta_k = \Delta_k' \oplus \Delta_k''$ be the corresponding decomposition of the Laplacians, so that

$$\det \Delta_k = \det \Delta_k' \, \det \Delta_k''. \tag{1.47}$$

**Proposition 6**

$$\mathcal{T}(E_\bullet, T_\bullet) = \prod_{k=1}^{n} \left(\det \Delta_k'\right)^{\frac{(-1)^{k+1}}{2}} = \prod_{k=0}^{n-1} \left(\det \Delta_k''\right)^{\frac{(-1)^{k}}{2}}.$$

*Proof.* It follows from the definition of the analytic torsion and equation (1.47). □

### 1.2.2 Elliptic Complexes

A complex of Hilbert spaces $\{E_k\}_{k=0,1,\ldots,n}$ and continuous linear maps $\{T_k\}_{k=0,1,\ldots,n}$,

$$(\mathbb{E}, \mathbf{T}) : \quad 0 \rightarrow E_0 \xrightarrow{T_0} \cdots \longrightarrow E_{k-1} \xrightarrow{T_{k-1}} E_k \longrightarrow \cdots \xrightarrow{T_{n-1}} E_n \rightarrow 0, \tag{1.48}$$

is called *Fredholm* if $\dim H^k < \infty$ for all $k = 1, \ldots, n$, where $H^k = Z^k/B^k$ is the *cohomology* of the complex $(\mathbb{E}, \mathbf{T})$. This is equivalent to all the Laplacians of the complex

$$\Delta_k = T_k^* T_k + T_{k-1} T_{k-1}^*$$

being Fredholm operators in their respective spaces $E_k$. If this is the case, Hodge's theorem says that the dimension of the kernel of $\Delta_k$ coincides with that of $H^k$. The *Euler characteristic* of the complex is defined as

$$\chi(E) = \sum_{i=0}^{n} (-1)^k \dim H^k, \tag{1.49}$$

which, in the case in which $\dim E_k < \infty$, coincides with $\sum_{i=0}^{n}(-1)^k \dim E_k$.

Let us now consider hermitian vector bundles $\{E_k\}_{k=0,1,\ldots,n}$ over a manifold $M$ and, for each $k$, let $D_k : \Gamma(E_k) \to \Gamma(E_{k+1})$ be classical differential operators of the same positive order $l$. Then, if the sequence

$$0 \to \Gamma(E_0) \xrightarrow{D_0} \cdots \longrightarrow \Gamma(E_{k-1}) \xrightarrow{D_{k-1}} \Gamma(E_k) \longrightarrow \cdots \xrightarrow{D_{n-1}} \Gamma(E_n) \to 0, \tag{1.50}$$

is a complex (i.e. $D_k \circ D_{k-1} = 0$ for $0 \leq k \leq n$), it is called an *elliptic complex* whenever the sequence of vector bundles

$$0 \to \pi_o^* E_0 \xrightarrow{\sigma_{D_0}} \cdots \longrightarrow \pi_o^* E_{k-1} \xrightarrow{\sigma_{D_{k-1}}} \pi_o^* E_k \longrightarrow \cdots \xrightarrow{\sigma_{D_{n-1}}} \pi_o^* E_n \to 0, \tag{1.51}$$

is exact, where $\pi_o : T_o^* M \to M$ denotes the cotangent bundle to $M$ without the zero section and the maps

$$\sigma_{D_k} : \pi_o^* E_{k+1} \to \pi_o^* E_k$$

are defined by the principal symbols of the operators $D_k$ between sections of the pull-back bundles over $T_o^* M$ defined by $\pi_o$ (see Appendix A). This means that, for every $(x, \xi) \in T_o^* M$, the sequence of vector spaces

$$0 \to E_0^x \xrightarrow{\sigma_{D_0}(x,\xi)} \cdots \longrightarrow E_{k-1}^x \xrightarrow{\sigma_{D_{k-1}}(x,\xi)} E_k^x \longrightarrow \cdots \xrightarrow{\sigma_{D_{n-1}}(x,\xi)} E_n^x \to 0, \tag{1.52}$$

where $E_k^x$ is the fibre above $x$ in $\pi_o^* E_k$, is exact.

This definition of ellipticity of the complex (1.50) is equivalent to the ellipticity of *all* the laplacians

$$\Delta_k = D_k^* D_k + D_{k-1} D_{k-1}^*,$$

where $D_k^*$ is the pseudo-differential operator formal adjoint to $D_k$, defined with respect to the Riemannian structure on $M$ and the Hermitian structure on each bundle $E_k$.

**Example 2** *Consider a closed $n$-dimensional Riemannian manifold $M$ and let $\rho : \pi_1(M) \to \mathrm{End}(V)$ be a representation of the fundamental group of $M$ on an inner product vector space $V$. The representation $\rho$ defines a hermitian vector bundle $V_\rho$ over $M$, with fibre $V$, by taking pairs $(m, v), (m', v')$ in $M \times V$ to be equivalent iff $\gamma \cdot m = m'$ and $v = \rho(\gamma^{-1}) \cdot v'$ for some $\gamma \in \pi_1(M)$. $V_\rho$ comes with a* flat *connection $\nabla^\rho$ which we couple with exterior differentiation of $k$-forms on $M$ to define a complex of differential forms on $M$ with values in $V_\rho$. Indeed, consider the vector bundle of twisted $k$-forms $\Lambda^k T^* M \otimes V_\rho$ and the operator $d_k^\rho = d_k \otimes 1 \oplus 1 \otimes \nabla^\rho$ acting on the space of sections of this bundle $\Omega^k(M, \rho) = \Gamma\left((\Lambda^k T^* M)^* \otimes V_\rho\right)$; ${d_k^\rho}^2 = 0$ for all $k$, as a consequence of the flatness of $\nabla^\rho$. We say that the representation $\rho$ is* acyclic *if the sequence*

$$0 \longrightarrow \Omega^0 \xrightarrow{d_0^\rho} \cdots \Omega^{k-1} \xrightarrow{d_{k-1}^\rho} \Omega^k \xrightarrow{d_k^\rho} \Omega^{k+1} \xrightarrow{d_{k+1}^\rho} \cdots \Omega^n \xrightarrow{d_n^\rho} 0, \tag{1.53}$$

*is an* acyclic complex, *i.e. all the de Rham cohomology groups of the complex are trivial ($H^k(M, \rho) = \{0\}$, $0 \le k \le n$). The representation of $\pi_1(M)$ will be fixed and no specific reference to it will be given (in the notation) in the sequel (we shall denote $\Omega^k(M, \rho)$ simply by $\Omega^k$, and $d_k^\rho$ by $d_k$, for all $k$).*
*The inner product on $k$-forms defined by the Riemannian metric on $M$ through the Hodge-star map $* : \Omega^k(M) \to \Omega^{n-k}(M)$,*

$$\langle \alpha_k, \beta_k \rangle = \int_M \alpha_k \wedge * \beta_k,$$

*and the Hermitian structure on each fibre of $V_\rho$, also couple to define an inner product on $\Omega^k$. From it we define the formal adjoints to $d_k$, $k = 0, 1, \ldots, n$, by*

$$\langle d_k \omega_k, \eta_{k+1} \rangle = \langle \omega_k, d_k^* \eta_{k+1} \rangle,$$

*from which it follows that $d_k^* = (-1)^{nk+1} * d_{n-k-1} *$. The Laplacians acting on twisted $k$-forms are given by*

$$\Delta_k = d_k^* d_k + d_{k-1} d_{k-1}^*.$$

*The operators $d_k$ are differential operators of order 1 (see* Example 10 *in* Appendix A*), and hence so are the maps $d_k^\rho = d_k \otimes 1 \oplus 1 \otimes \nabla^\rho$ (note that the $V_\rho$ component does not affect the ellipticity of $d_k^\rho$). The sequence*

$$\cdots \to \Lambda^{k-1} T_m^* M \xrightarrow{\xi} \Lambda^k T_m^* M \xrightarrow{\xi} \Lambda^{k+1} T_m^* M \to \cdots,$$

*where the maps are left exterior multiplication by $\xi$, is exact. From* Example 10 *in* Appendix A *we know that $\sigma_{d_k}(m, \xi)$ is left exterior multiplication by $\xi$. On the other hand, from the definition of the $*$-operator (which, at each $m \in M$ is given by $\langle \omega, \omega' \rangle = *(\omega \wedge * \omega')$) it follows that*

$$(-1)^{nk} * \xi * : \Lambda^{k+1} T_m^* M \to \Lambda^k T_m^* M$$

*is the adjoint to* $\Lambda^k T_m^* M \xrightarrow{\xi} \Lambda^{k+1} T_m^* M$. *Thus,* $d_k^*$ *is also a first order elliptic differential operator for* $k = 0, 1, \dots, n$, *and hence the Laplacian* $\Delta_k$ *is elliptic. In fact*

$$\sigma_{\Delta_k}(m, \xi) = |\xi|^2,$$

*which is clearly non-singular for* $(m, \xi) \in T_m^* M - \{0\}$.
*Note that, because of the acyclicity assumption on* $\rho$, *we have at each level of the complex the* Hodge decomposition

$$\Omega^k = \Omega_k' \oplus \Omega_k'' \tag{1.54}$$

*where* $\Omega_k' = \mathrm{Im}\ d_{k-1} = \mathrm{Ker}\ d_k$ *and* $\Omega_k'' = \mathrm{Im}\ d_k^* = \mathrm{Ker}\ d_{k-1}^*$.

### 1.2.3 Analytic Torsion of a Riemannian Manifold

Consider a closed $n$-dimensional Riemannian manifold $M$ and let $(\mathbb{E}_\bullet, T_\bullet)$ denote the complex

$$0 \to \Gamma(E_0) \xrightarrow{D_0} \cdots \to \Gamma(E_k) \xrightarrow{D_k} \Gamma(E_{k+1}) \to \cdots \xrightarrow{D_{n-1}} \Gamma(E_n) \to 0, \tag{1.55}$$

where the maps $D_k$ are fixed positive-order differential operators acting on spaces of sections of the Hermitian vector bundles $E_k$ over $M$. Let $\Delta_k = D_k^* D_k + D_{k-1} D_{k-1}^*$ be the elliptic self-adjoint positive Laplacian operator acting on sections of $E_k$. The formal adjoint operators $D_k^*$ are defined with respect to the Hermitian inner product $h_k$ on the spaces of smooth sections $\Gamma(E_k)$ induced by the Hermitian structure $\langle \cdot, \cdot \rangle_k$ on $E_k$, and the Riemannian metric g on $M$, namely

$$h_k(\sigma_1, \sigma_2) = \int_M \langle \sigma_1(m), \sigma_2(m) \rangle_k \, d\mu_M(m). \tag{1.56}$$

Thus, the Laplacians $\Delta_k$, their spectra and $\zeta$-regularized determinants are functions of the Riemannian structure on $M$ and the Hermitian structure on the vector bundles. In [RS71], Ray and Singer define, when the elliptic complex is acyclic and from a combination of $\zeta$-determinants of the Laplacians, a topological invariant of $M$ – its *Analytic Torsion*, i.e. a quantity independent of the Riemannian structure on $M$. They consider the case in which $E_k = \Gamma\left((\Lambda^k T^* M)^* \otimes V_\rho\right)$, the space of differential $k$-forms with values in the vector bundle $V_\rho$ associated to a representation $\rho$ of the fundamental group of $M$ on an inner product vector space, and $D_k$ is the flat connection (exterior differentiation of forms coupled to the flat connection on $V_\rho$) on sections of this vector bundle (see Example 2). This construction can be extended to the case of general (acyclic) elliptic complexes, as observed by Schwarz in [S79]. In this section we shall review their main features.

**Definition 8** [RS71] *Let $\Delta_k$ denote the Laplacian operator at the level $k$ in the* acyclic *elliptic complex* $(\mathbb{E}_\bullet,\, \mathbb{D}_\bullet)$ *given by* (1.55). *The* Analytic Torsion *of the complex* $(\mathbb{E}_\bullet,\, \mathbb{D}_\bullet)$ *of vector bundles over the manifold $M$ is the complex number given by*

$$T(\mathbb{E}_\bullet,\, \mathbb{D}_\bullet) = \exp\left\{ \frac{1}{2} \sum_{k=0}^{n} (-1)^k k\, \zeta'_{\Delta_k}(0) \right\}. \tag{1.57}$$

Note that, using the definition of the $\zeta$-determinant of an elliptic differential operator (1.23), this expression yields the same equality as in the finite-dimensional case (Definition 7).

**Theorem 2** [S79] *If $\dim M$ is odd, then $T(\mathbb{E}_\bullet,\, \mathbb{D}_\bullet)$ is independent of the Riemannian metric on $M$ and the Hermitian structure on the bundles $E_k$.*

*Proof.* Let us consider a family $\{g_u, \langle\cdot,\cdot\rangle_k^u\}_u$ of Riemannian metrics and Hermitian structures on $E_k$, respectively, parametrized by $u \in [0, \infty)$. Let

$$h_k(u)(\sigma_1, \sigma_2) = \int_M \langle \sigma_1(m), \sigma_2(m)\rangle_k^u \, d\mu_{M_u}(m), \tag{1.58}$$

be the induced Hermitian inner product on sections of $E_k$, where $\mu_{M_u}$ is the Riemannian volume element on $M$ defined by the metric $g_u$. Then, for $u \neq 0$, $h_k(0)(\sigma_1, \sigma_2) = h_k(\sigma_1, \sigma_2)$ and $h_k(u)(\sigma_1, \sigma_2)$ are related by

$$h_k(u)(\sigma_1, \sigma_2) = h_k(A_u \sigma_1, \sigma_2),$$

where $A_u : \Gamma(E_k) \to \Gamma(E_k)$ is a zero order self-adjoint positive operator, uniquely determined by the variation of the metrics. Thus, since $h_{k+1}(D_k\sigma_k, \varphi_{k+1}) = h_k(\sigma_k, ({A_u}^{-1} D_k^* A_u)\varphi_{k+1})$, where $\sigma_k \in \Gamma(E_k)$ and $\varphi_{k+1} \in \Gamma(E_{k+1})$, then

$$D_k^*(u) = {A_u}^{-1} D_k^* A_u$$

and $\Delta_k(u) = {A_u}^{-1} D_k^* A_u D_k + D_{k-1} {A_u}^{-1} D_{k-1}^* A_u$ . Consider now the function

$$f(u,t) = \sum_{k=0}^{n} (-1)^k k \ \mathrm{tr}\left(e^{-t\Delta_k(u)}\right),$$

where $\mathrm{tr}\left(e^{-t\Delta_k(u)}\right)$ is the heat kernel of the Laplacian $\Delta_k(u)$. We are interested in the variation of this function with respect to the parameter $u$. Let $\dot{\Delta}_k(u) = \frac{\partial \Delta_k(u)}{\partial u}$, then

$$\frac{\partial}{\partial u} f(t,u) = -t \sum_{k=0}^{n} (-1)^k k \ \mathrm{tr}\left(e^{-t\Delta_k(u)} \dot{\Delta}_k(u)\right),$$

and

$$\frac{\partial}{\partial u} D_k^*(u) = D_k^*(u)\, X - X\, D_k^*(u),$$

where $X = {A_u}^{-1}\dot{A}_u$. It follows that

$$\dot{\Delta}_k(u) = D_{k-1}D_{k-1}^*(u)\,X - D_{k-1}\,X\,D_{k-1}^*(u) + D_k^*(u)\,X\,D_k - X\,D_k^*(u)D_k,$$

which, through the relations $\Delta_k(u)D_k^*(u)D_k = \Delta_{k+1}(u)D_kD_k^*(u)$, implies

$$\begin{aligned}
\frac{\partial}{\partial u}f(t,u) &= -t\sum_{k=0}^{n}\operatorname{tr}\left(e^{-t\Delta_k(u)}\Delta_k(u)X\right)\\
&= t\frac{\partial}{\partial t}\left\{\sum_{k=0}^{n}(-1)^k\operatorname{tr}\left(e^{-t\Delta_k(u)}X\right)\right\}.
\end{aligned}$$

Thus, letting

$$F(t) = \sum_{k=0}^{n}(-1)^k\operatorname{tr}\left(e^{-t\Delta_k(u)}X\right),$$

for $\Re(z)$ large enough, (1.18) implies that

$$\frac{\partial}{\partial t}\left\{\sum_{k=0}^{n}(-1)^k\,\zeta_{\Delta_k(u)}(z)\right\} = \mathrm{M}[tf'(t)](z) = -z\mathrm{M}[F(t)](z).$$

It follows from (1.20) applied to $Q = \Delta_k(u) + \lambda X$ (since $X$ is a multiplication operator) that

$$\operatorname{tr}\left(e^{-t\Delta_k(u)}X\right) = -\frac{1}{t}\frac{d}{d\lambda}\operatorname{tr}\left(e^{-t(\Delta_k(u)+\lambda X)}\right).$$

Then for $t\to 0$, the function $F(t)$ has an asymptotic expansion of the form

$$F(t) = \sum_{j=0}^{\infty}a_jt^{j-\frac{n}{2}},$$

where $n = \dim M$, so $\mathrm{M}[F(t)]$ is holomorphic at $z = 0$ and $\mathrm{M}[F(t)](0) = a_{\frac{n}{2}}$. On the other hand, since

$$\log\left(T(\mathbb{E}_\bullet,\,\mathbb{D}_\bullet)(u)^2\right) = \sum_{k=0}^{n}(-1)^k\,k\zeta'_{\Delta_k(u)}(0),$$

it follows that

$$\mathrm{M}[F(t)](0) = -\frac{\partial}{\partial u}\log\left(T(\mathbb{E}_\bullet, T_\bullet)(u)^2\right),$$

where $T(\mathbb{E}_\bullet,\,\mathbb{D}_\bullet)(u)$ denotes the analytic torsion defined by the $\zeta$-determinants of the Laplacians $\Delta_k(u)$, which implies that for $n$ odd $T(\mathbb{E}_\bullet,\,\mathbb{D}_\bullet)$ is constant. □

**Ray-Singer Torsion.** Let us now consider the analytic torsion in the context of Ray and Singer, i.e taking the twisted de Rham complex described in Example 2,

$$0 \longrightarrow \Omega^0 \xrightarrow{d_0} \cdots \Omega^{k-1} \xrightarrow{d_{k-1}} \Omega^k \xrightarrow{d_k} \Omega^{k+1} \xrightarrow{d_{k+1}} \cdots \ \Omega^n \xrightarrow{d_n} 0, \tag{1.59}$$

where $\rho$ denote a (fixed) representation of the fundamental group of $M$ on an inner product vector space, $d_k$ is the flat connection (exterior differentiation of forms coupled to the flat connection on $V_\rho$) on sections of the vector bundle $\Omega^k(M,\rho) = \Gamma\left((\Lambda^k T^*M)^* \otimes V_\rho\right)$ (that we shall denote by $\Omega^k$) of differential $k$-forms with values in $V_\rho$. We assume that the complex (1.59) is acyclic. In this case the formal adjoint to the first-order differential operator $d_k$, denoted $d_k^*$ and defined through the Hodge-star map $* : \Omega^k \to \Omega^{n-k}$, acts on $\Omega^{k+1}$ by

$$d_k^* = (-1)^{nk+1} * d_{n-k-1} * . \tag{1.60}$$

The *Ray-Singer Torsion* corresponds to the analytic torsion of the complex (1.53), and we shall denote it by $T_{RS}(M)$. It follows from Theorem 2 that $T_{RS}(M)$ is independent of the Riemannian structure on $M$ when $n$ is odd. Other important properties of the Ray-Singer torsion of $M$ follow from the particular form of the inner product on $\Omega^k$, which is defined in terms of the Hodge star map, and the Hodge decomposition in each level of the complex. Recall that acyclicity of (1.53) implies, for any $1 \leq k \leq n-1$, a *Hodge decomposition* (1.54)

$$\Omega^k = \Omega'_k \oplus \Omega''_k$$

where $\Omega'_k = \text{Im } d_{k-1} = \text{Ker } d_k$ and $\Omega''_k = \text{Im } d_k^* = \text{Ker } d_{k-1}^*$, while $\Omega^0 = \Omega''_0$ and $\Omega^n = \Omega'_n$. Hence, the spaces $\Omega^k$ of $k$-forms are completely determined by their neighbors in the complex

$$\Omega^{k-1} \xrightarrow{d_{k-1}} \Omega^k \xleftarrow{d_k^*} \Omega^{k+1}.$$

By restriction on the respective domains of the maps $d_k$ and $d_k^*$, it follows that

$$d_k|_{\Omega''_k} : \Omega''_k \to \Omega'_{k+1} \quad \text{ and } \quad d_k^*|_{\Omega'_{k+1}} : \Omega'_{k+1} \to \Omega''_k$$

are isomorphisms, giving rise to the bijective maps

$$d_k^* d_k : \Omega''_k \ \to \ \Omega''_k, \quad \text{ and } \quad d_k d_k^* : \Omega'_{k+1} \to \Omega'_{k+1}.$$

Thus, Hodge decomposition yields an isomorphism

$$\Omega^k \cong \Omega''_{k-1} \oplus \Omega'_{k+1}.$$

Let us consider the Laplacian operator on $k$-forms,

$$\Delta_k = d_{k-1} d_{k-1}^* + d_k^* d_k, \tag{1.61}$$

and the restricted Laplacians, w.r.t. the decomposition (1.54),

$$\Delta'_k = \Delta_k|_{\Omega'_k} \quad \text{and} \quad \Delta''_k = \Delta_k|_{\Omega''_k}. \tag{1.62}$$

Note that, since $\Delta'_k = d_{k-1}d^*_{k-1}|_{\Omega'_k}$ and $\Delta''_k = d^*_k d_k|_{\Omega''_k}$, we have

$$\Delta_k = \Delta'_k + \Delta''_k. \tag{1.63}$$

Moreover, $\Omega^0 = \Omega''_0$ and $\Omega^n = \Omega'_n$ imply that $\Delta_0 = \Delta''_0$ and $\Delta_n = \Delta'_n$, so that we have $2n$ positive selfadjoint elliptic operators $\Delta'_1, \ldots, \Delta'_n, \Delta''_0, \ldots, \Delta''_{n-1}$. Finally, the identities

$$\begin{aligned} *d_{k-1}d^*_{k-1} &= d^*_{n-k}d_{n-k} * & (1.64)\\ *d^*_k d_k &= d_{n-k-1}d^*_{n-k-1}*, & (1.65) \end{aligned}$$

imply that

$$\begin{aligned} *\Delta'_k &= \Delta''_{n-k} * \\ *\Delta''_k &= \Delta'_{n-k}*, \end{aligned} \tag{1.66}$$

which yields for the Laplacian on $\Omega^k$ the well-known equality

$$*\Delta_k = \Delta_{n-k} * . \tag{1.67}$$

As consequence of acyclicity, and the corresponding Hodge decomposition, the zeta-regularization techniques used to define the determinant of the Laplacian operators can also be used to define regularized determinants for the restricted Laplacians $\Delta'_k$ and $\Delta''_k$. As the restriction to $\Omega'_k$ and $\Omega''_k$ of a self adjoint elliptic operator on a closed manifold, the operators $\Delta'_k$ and $\Delta''_k$ have purely discrete real spectrum. Indeed, from (1.54) and the nilpotency of the $d^*_k$ and $d_k$ operators, it follows that the set of eigenvalues of the Laplacian $\Delta_k$ is the union of the eigenvalues of $d_{k-1}d^*_{k-1}$ and $d^*_k d_k$. Hence,

$$\zeta_{\Delta_k}(z) = \zeta_{\Delta'_k}(z) + \zeta_{\Delta''_k}(z), \tag{1.68}$$

where $\zeta_{\Delta'_k}(z)$ and $\zeta_{\Delta''_k}(z)$ are the $\zeta$-functions of the restricted Laplacians (1.62), defined by

$$\zeta_{\Delta'_k}(z) = \operatorname{tr}\left(\Delta'^{-z}_k\right) = \sum_{\lambda' \in \mathrm{Spec}\Delta'_k} \lambda'^{-z}$$

and

$$\zeta_{\Delta''_k}(z) = \operatorname{tr}\left(\Delta''^{-z}_k\right) = \sum_{\lambda'' \in \mathrm{Spec}\Delta''_k} \lambda''^{-z},$$

where $\lambda''$ and $\lambda'$ denote non-zero eigenvalues of $\Delta'_k$ and $\Delta''_k$, respectively.

Let us set $\mathcal{E}_k(\lambda) = \text{Ker } (\Delta_k - \lambda)$, then the Hodge decomposition (1.54) induces a decomposition of such eigenspaces $\mathcal{E}_k(\lambda) = \mathcal{E}'_k(\lambda) \oplus \mathcal{E}''_k(\lambda)$, where $\mathcal{E}'_k(\lambda) = \mathcal{E}_k(\lambda) \cap \Omega'_k$ and $\mathcal{E}''_k(\lambda) = \mathcal{E}_k(\lambda) \cap \Omega''_k$. Let $\omega_{k-1} \in \mathcal{E}_{k-1}(\lambda)''$, then $d_{k-1}\omega_{k-1} \in \Omega'_k$ and

$$\Delta_k d_{k-1}\omega_{k-1} = d_{k-1}d^*_{k-1}d_{k-1}\omega_{k-1} = d_{k-1}\Delta_{k-1}\omega_{k-1} = \lambda d_{k-1}\omega_{k-1},$$

so $d_{k-1}$ maps $\mathcal{E}''_{k-1}(\lambda)$ bijectively into $\mathcal{E}'_k(\lambda)$, leading to a bijective correspondence between (non-zero) eigenvalues (and their corresponding eigenvectors) of the operators $d^*_{k-1}d_{k-1}$ and $d_k d^*_k$, which implies

$$\zeta_{\Delta'_k}(z) = \zeta_{\Delta''_{k-1}}(z). \tag{1.69}$$

Using (1.68) and (1.69) we have,

$$\zeta_{\Delta''_k}(z) = \zeta_{\Delta_k}(z) - \zeta_{\Delta_{k-1}}(z) + \zeta_{\Delta_{k-2}}(z) - \cdots + (-1)^k \zeta_{\Delta_0}(z).$$

Hence, from the properties of the zeta-function of the Laplacian, it follows that, for all $k$, $\zeta_{\Delta'_k}$ and $\zeta_{\Delta''_k}$ are well defined and analytic for $z \in \mathbb{C}$ with $\Re(z) >> 0$, and extend by analytic continuation to meromorphic functions on $\mathbb{C}$, regular at the origin. Thus, the operators $\Delta'_k$ and $\Delta''_k$ have honest $\zeta$-determinants.

**Proposition 7**

$$T_{RS}(M) = \prod_{k=0}^{n} (\det{}_\zeta \Delta_k)^{\frac{(-1)^{k+1}k}{2}} = \prod_{k=1}^{n} \left(\det{}_\zeta \Delta'_k\right)^{\frac{(-1)^{k+1}}{2}} = \prod_{k=0}^{n-1} \left(\det{}_\zeta \Delta''_k\right)^{\frac{(-1)^k}{2}}.$$

*Proof.* From (1.68) and the definition of $\zeta$-determinant for $\Delta'_k$ and $\Delta''_k$ it follows that

$$\det{}_\zeta \Delta_k = \det{}_\zeta \Delta'_k \det{}_\zeta \Delta''_k. \tag{1.70}$$

Since $\log \det_\zeta \Delta_k = -\zeta'_{\Delta_k}(0)$, from the definition of $T_{RS}(M)$ we find,

$$\log T_{RS}(M) = \sum_{k=0}^{n} \log \left\{ (\det{}_\zeta \Delta_k)^{\frac{(-1)^{k+1}k}{2}} \right\}$$

so

$$T_{RS}(M) = \prod_{k=0}^{n} (\det{}_\zeta \Delta_k)^{\frac{(-1)^{k+1}k}{2}}.$$

Notice that (1.69) implies that $\det_\zeta \Delta'_k = \det_\zeta \Delta''_{k-1}$ which, combined with (1.70), yields

$$T_{RS}(M) = \prod_{k=1}^{n} \left(\det{}_\zeta \Delta'_k\right)^{\frac{(-1)^{k+1}}{2}} = \prod_{k=0}^{n-1} \left(\det{}_\zeta \Delta''_k\right)^{\frac{(-1)^k}{2}}.$$

□

As a consequence of these relations we find the following

**Lemma 4** [RS71] *If* $\dim M$ *is even, then* $T_{RS}(M) = 1$.

*Proof.* Let $n = 2m$ be the dimension of $M$, Proposition 7 shows that

$$T_{RS}(M) = \prod_{k=1}^{2m} \left(\det{}_\zeta \Delta'_k\right)^{\frac{(-1)^{k+1}}{2}} = \prod_{k=1}^{m} \left(\det{}_\zeta \Delta'_k\right)^{\frac{(-1)^{k+1}}{2}} \prod_{l=m+1}^{2m} \left(\det{}_\zeta \Delta'_l\right)^{\frac{(-1)^{l+1}}{2}}$$

and (1.69) implies that $\det_\zeta \Delta'_k = \det_\zeta \Delta''_{k-1}$ which, by $*$-Hodge duality (1.66), yields

$$\det{}_\zeta \Delta'_k = \det{}_\zeta \Delta'_{n-k+1}.$$

Putting this into the previous equality yields

$$T_{RS}(M) = \prod_{k=1}^{m} \left(\det{}_\zeta \Delta'_{n-k+1}\right)^{\frac{(-1)^{k+1}}{2}} \prod_{l=1}^{m} \left(\det{}_\zeta \Delta'_{m+l}\right)^{\frac{(-1)^{m+l+1}}{2}} = 1.$$

□

## 1.3 Dirac Operators and Index Theorems

### 1.3.1 Dirac operators on Clifford bundles

Let $M$ be a closed Riemannian manifold of dimension $n$ and let $C(M) \to M$ be the bundle of Clifford algebras over $M$, whose fibre above $m \in M$ is the Clifford algebra $C(T^*_m M)$ of the Euclidean space $T^*_m M$. A *Clifford module* over $M$ is a $\mathbb{Z}_2$-graded Hermitian vector bundle $E = E^+ \oplus E^- \to M$ with an odd action

$$\begin{array}{ccc} \Gamma(M, C(M)) \times \Gamma(E) & \to & \Gamma(E) \\ (a, \sigma) & \to & c(a)\sigma \end{array}$$

of the bundle of algebras $C(M)$ on sections of $E$, i.e. $c(a)E^\pm = E^\mp$, such that

$$\langle c(a)\sigma_1, \sigma_2\rangle + \langle \sigma_1, c(a)\sigma_2\rangle = 0,$$

where $\langle\, , \rangle$ denotes the inner product on $\Gamma(E)$ induced by the Hermitian structure on $E$ and the Riemannian metric on $M$, namely

$$\langle \sigma_1, \sigma_2\rangle = \int_M \langle \sigma_1(m), \sigma_2(m)\rangle_m \, d\mu_M(m). \tag{1.71}$$

Let $L^2(E)$ be the completion of $\Gamma(E)$ with respect to the induced norm. We say that the Clifford module $E$ is self-adjoint when $c(a^*) = -c(a)$ for any $a \in \Gamma(M, C(M))$.

Let $\nabla$ be a *Clifford connection* on $E$ i.e. a connection $\nabla$ on $E$ such that, $\forall a \in C(M)$, $\sigma \in \Gamma(E)$ and any $X \in \Gamma(TM)$,

$$[\nabla_X, c(a)]\sigma = \nabla_X(c(a)\sigma) - c(a)\nabla_X\sigma = c(\nabla_X^{L.C.} a)\sigma,$$

where $\nabla^{L.C.}$ denotes the extension of the Levi-Civita connection to the bundle $C(M)$. From a Clifford connection $\nabla$ and the Clifford multiplication, we build an associated *Dirac operator* $D_\nabla$ acting on sections of the Clifford module $E$ by composition,

$$\Gamma(E) \xrightarrow{\nabla} \Gamma(M, T^*M \otimes E) \xrightarrow{c} \Gamma(E).$$

In local coordinates it reads

$$D_\nabla \sigma = \sum_{i=1}^{n} c(dx_i)\nabla_{\frac{\partial}{\partial x_i}}\sigma, \tag{1.72}$$

where $\sigma \in \Gamma(E)$, $\{x_i\}$, and where $\{\frac{\partial}{\partial x_i}\}$ and $\{dx_i\}$ denote local coordinates for $M$, its tangent bundle and its cotangent bundle, respectively. $D_\nabla$ is a (formally) self-adjoint first order differential operator on $\Gamma(E)$.

With respect to the $\mathbb{Z}^2$-grading of the Clifford module, i.e. the direct sum decomposition $E = E^+ \oplus E^-$, the Dirac operator $D_\nabla$ can be written as

$$\begin{array}{rcl} D_\nabla : \Gamma(E) & \rightarrow & \Gamma(E) \\ \sigma & \mapsto & \begin{bmatrix} 0 & D_\nabla^- \\ D_\nabla^+ & 0 \end{bmatrix} \sigma, \end{array}$$

where $D_\nabla^\pm : \Gamma(E^\pm) \to \Gamma(E^\mp)$ denote its corresponding components, with respect to the induced $\mathbb{Z}_2$-graduation on $\Gamma(E)$.

The operator $\Delta_\nabla = D_\nabla^2$ is a generalized Laplacian, in the sense that its leading symbol $\sigma_L(\Delta_\nabla)$ satisfies the relation $\sigma_{\Delta_\nabla}(x, \xi) = |\xi|^2$ for any $(x, \xi) \in T_o^*M$.

**Example 3** *If $M$ is an oriented spin manifold then any Clifford bundle is a twisted spinor bundle $F = S \otimes W \to M$, where $S$ is the spinor bundle on $M$ and $W$ an exterior vector bundle on $M$. The Clifford connection $\nabla$ arises in that case from coupling the connection $\nabla^S$ on $S$ induced by the Levi-Civita connection with a connection $\nabla^W$ on $W$, $\nabla = \nabla^S \otimes 1 \oplus 1 \otimes \nabla^W$. The Dirac operator thus obtained is the* twisted classical Dirac operator. *In the case of an even-dimensional manifold $M$, the $\mathbb{Z}_2$ grading on $E$ is the one induced by the $\mathbb{Z}_2$-grading $S = S^+ \oplus S^-$ of the spinor bundle, the decomposition being orthogonal for the inner product induced from the metric on $M$, the Dirac operator reads* $D = \begin{bmatrix} 0 & D^- \\ D_+ & 0 \end{bmatrix}$.

**Example 4** *In the context of* Example 2 *let $M$ be a closed Riemannian manifold, $\rho$ a representation of the fundamental group of $M$ on an inner product space $V$ and let $V_\rho$ be the vector bundle over $M$ defined by $\rho$. Then, the bundle $\mathbf{E}_\rho = \bigoplus_k \Lambda^k T^*M \otimes V_\rho$ is a Clifford module for the Clifford multiplication given by*

$$\begin{array}{rcl} \Gamma(T^*M) \times \Gamma(\mathbf{E}_\rho) & \to & \Gamma(\mathbf{E}_\rho) \\ (a, \alpha) & \mapsto & \epsilon(a) \wedge \alpha - i(a)\alpha, \end{array}$$

*where $\epsilon(a)$ and $i(a)$ denotes exterior and interior product, respectively. This Clifford bundle is naturally graded by the parity on forms:*

$$\mathbf{E}_\rho = \mathbf{E}_\rho^+ \oplus \mathbf{E}_\rho^- = \left( \bigoplus_{k \text{ even}} \Lambda^k T^*M \otimes V_\rho \right) \oplus \left( \bigoplus_{k \text{ odd}} \Lambda^k T^*M \otimes V_\rho \right).$$

*The bundle $V_\rho$ comes with a flat (self-adjoint) connection $\nabla^\rho$ that couples with the Levi-Civita connection $\nabla^{LC}$ to give a (self-adjoint) connection $\nabla = \nabla^{LC} \otimes 1 \oplus 1 \otimes \nabla^\rho$ on $\mathbf{E}_\rho$ from which we can construct a Dirac operator $D_\nabla$, which we also call the* de Rham operator. *On the other hand, as seen in* Example 2, *exterior differentiation $d$ couples with the connection $\nabla^\rho$ to yield a twisted exterior differential $d_\rho : \Gamma(\mathbf{E}_\rho) \to \Gamma(\mathbf{E}_\rho)$. Identifying $d_\rho$ with $\epsilon \circ \nabla^{LC}$, $d_\rho^*$ identifies to $-i \circ \nabla^{LC}$ from which it easily follows that $d_\rho + d_\rho^* = (\epsilon - i) \circ \nabla = c \circ \nabla^{LC}$, and hence*

$$D_\nabla = d_\rho + d_\rho^*.$$

### 1.3.2 Index Theorems

Let $\mathcal{H}$ and $\mathcal{H}'$ be two Hilbert spaces and $A : \mathcal{H} \to \mathcal{H}'$ a Fredholm operator. Then, the (formal) adjoint operator $A^* : \mathcal{H}' \to \mathcal{H}$ is also Fredholm, so that $\mathrm{Im}A$ and $\mathrm{Im}A^*$ are closed and there are orthogonal splittings

$$\mathcal{H} = \ker A \oplus \mathrm{Im}A^* \qquad \text{and} \qquad \mathcal{H}' = \ker A^* \oplus \mathrm{Im}A.$$

There exists a bounded operator $R : \mathcal{H}' \to \mathcal{H}$, called a *parametrix* of $A$, such that $\pi_A = I_{\mathcal{H}} - RA$ and $\pi_{A^*} = I_{\mathcal{H}'} - AR$ are orthogonal projectors onto $\ker A$ and $\ker A^*$, respectively.

The *analytic index* of $A$ is the integer number given by

$$\mathrm{ind}A = \dim \ker A - \dim \mathrm{coker}A. \tag{1.73}$$

Since $\mathrm{coker}A \cong \ker A^*$, this is equivalent to

$$\begin{array}{rcl} \mathrm{ind}A & = & \dim \ker A - \dim \ker A^* \\ & = & \dim \ker(A^*A) - \dim \ker(AA^*). \end{array}$$

A positive order elliptic differential operator $A : \Gamma(E) \to \Gamma(E)$, where as before $\Gamma(E)$ denotes the space of smooth sections of a vector bundle $E$ on a compact manifold, extends to a Fredholm operator $A : \mathrm{H}^s(E) \to \mathrm{H}^{s-a}(E)$, where $\mathrm{H}^s(E)$ is the completion with respect to the Sobolev $\mathrm{H}^s$-norms of the spaces of smooth sections of $E$ and $a = \mathrm{ord}A$. If there exists a $\mathbb{Z}_2$-graduation of $E$ such that $E = E^+ \oplus E^-$, and $A = \begin{pmatrix} 0 & A^- \\ A^+ & 0 \end{pmatrix}$ is self-adjoint and odd with respect to this grading, then $A^- = (A^+)^*$. In this case we have $\mathrm{ind}A^+ = \dim\ker A^+ - \dim\ker A^-$, and we define the $Q$-weighted *supertrace* of the operator $A$ by

$$\mathrm{tr}_s^Q(A) = \mathrm{tr}^Q(\gamma A), \tag{1.74}$$

where $\gamma = \begin{pmatrix} \mathrm{I} & 0 \\ 0 & -\mathrm{I} \end{pmatrix}$ on $E$ (i.e. $E^\pm$ is the $\pm 1$ eigenspace of $\gamma$) and $Q$ denotes a weight.

**Example 5** *In the context of* Example 2*, note that*

$$D = \sum_{k=0}^{n} (d_k + d_k^*) : \bigoplus_{k=0}^{n} \Omega_k \to \bigoplus_{k=0}^{n} \Omega_k$$

*is a first order elliptic differential operator,* $\Delta_k = D^2|_{\Omega_k}$*, and that the index of the operator* $D^+ = D|_{\Omega^+}$*, where* $\Omega^+ = \bigoplus_{k\,\mathrm{even}} \Omega_k$*, is the Euler characteristic of the complex defined in* (1.49)*.*

The main goal in index theory is to express the analytic index of an elliptic pseudo-differential operator acting on sections of a vector fibration over a Riemannian manifold as a local term, i.e. as an integral on the manifold of characteristic classes associated to the underlying geometry of the fibration. Recall that locality is also a feature of weighted trace anomalies, so it is natural to ask if there is a relation between those anomalies and the index, a question we shall address in Chapter 3.

The index of an elliptic operator can be calculated from its symbol, as was shown by Atiyah, Singer and collaborators in the sixties. The original proofs used methods of algebraic topology and K-theory [AS], but index theorems can also be proven by *heat kernel methods*, which use analytical properties of the asymptotic expansions of (some functions of) the involved operators.

Let us consider a graded Clifford module $E = E^+ \oplus E^-$ over a closed Riemannian manifold $M$. A smoothing operator $A \in \mathcal{C}l(E)$ has smooth kernel $k$ and, $M$ being closed, its trace can be computed from the trace $\mathrm{tr}_x$ on $\mathrm{End}E_x$ of the linear maps $k(x,x) \in E_x^* \otimes E_x \cong \mathrm{Hom}E_x$,

$$\mathrm{tr}A = \int_M \mathrm{tr}_x k(x,x)\, d\mu_M(x). \tag{1.75}$$

Applying this to the smoothing heat kernel operator $e^{-tD^2}$, $t > 0$, where $D$ is the Dirac operator acting on sections of $E$, there is a section $k_t(x,y)$ (parametrized by $t$) of the bundle $E \boxtimes E^*$ over $M \times M$, called the *heat kernel*, such that

$$\operatorname{tr}\left(e^{-tD^2}\right) = \int_M \operatorname{tr}_x k_t(x,x)\, d\mu_M(x). \tag{1.76}$$

On the other hand, the McKean-Singer index formula [MS] show that, for all $t > 0$,

$$\operatorname{ind}(D^+) = \operatorname{tr}_s(e^{-tD^2}) = \operatorname{tr}(e^{-tD^*D}) - \operatorname{tr}(e^{-tDD^*}). \tag{1.77}$$

The Atiyah-Singer index theorem follows from the asymptotic behavior of the heat kernel.

**Proposition 8** [G95] *Let $M$ be a closed Riemannian manifold of dimension $n$ and $E$ a Clifford module on $M$ with associated Dirac operator $D$. Then there exists an asymptotic expansion for the heat kernel $k_t$ of $M$ of the form*

$$k_t(x,y) \sim h_t(x,y)\left[\kappa_0(x,y) + t\kappa_1(x,y) + t^2\kappa_2(x,y) + \cdots\right], \tag{1.78}$$

*where*

$$h_t(x,y) = \frac{1}{(4\pi t)^{\frac{n}{2}}} \exp\{-d(x,y)^2/(4t)\} \tag{1.79}$$

*and the $\kappa_i$ are sections of $E \boxtimes E^*$ whose values $\kappa_i(x,x)$ along the diagonal can be computed by algebraic expressions involving the metrics, connection coefficients and their derivatives.*

Thus, from the asymptotic expansion (1.78) for the heat kernel associated to the smoothing operator $e^{-tD^2}$, it follows that

$$\operatorname{tr}_s(e^{-tD^2}) \sim \frac{1}{(4\pi t)^{\frac{n}{2}}}\left[\int_M \kappa_0(x,x)\, d\mu_M(x) + t\int_M \kappa_1(x,x)\, d\mu_M(x) + \cdots\right].$$

On the other hand, the McKean-Singer index formula (1.77) says that $\operatorname{tr}_s(e^{-tD^2})$ is constant so, if $n$ is even,

$$\operatorname{ind}(D^+) = \frac{1}{(4\pi)^{\frac{n}{2}}} \int_M \kappa_{\frac{n}{2}}(x,x)\, d\mu_M(x), \tag{1.80}$$

where $\kappa_{\frac{n}{2}}(x,x)$ can be expressed in terms of the underlying geometrical data, and $\operatorname{ind}(D^+) = 0$ if $n$ is odd. The Atiyah-Singer theorem [AS] (see [Pal] for a careful exposition) gives an explicit expression of the right hand side of (1.80) in terms of characteristic classes when the Clifford bundle is the given by the spinor bundle on an even-dimensional spin manifold.

**Theorem 3** [AS] *The index of a Dirac operator $D^+$ on a Clifford module $E = S \otimes W$ based on an even dimensional spin manifold $M$, $S$ being the spinor bundle and $W$ an exterior bundle, is given by:*

$$\operatorname{ind}(D^+) = \int_M \hat{A}(\nabla^{LC}) \operatorname{Ch}(\nabla^W), \tag{1.81}$$

*where $\hat{A}(\nabla^{LC}) = \sqrt{\det\left(\frac{\Omega^{LC}/4\pi}{\sinh \Omega^{LC}/4\pi}\right)}$ is the $\hat{A}$ genus of the Levi-Civita connection on $M$ and $\operatorname{Ch}(\nabla^W) = \operatorname{tr}\left(e^{-\Omega^W}\right)$ the Chern character of the Clifford connection on $W$.*

If we consider now a manifold with non-empty boundary, a similar statement can be made about the index of the signature operator, having in mind some precise boundary conditions. In this case a new term appears in the index formula from the boundary, containing the eta invariant of a particular differential operator. This is the *Atiyah-Patodi-Singer theorem.*

**Theorem 4** [APSI] *Let $X$ be an oriented Riemannian manifold of dimension $4l$ with boundary $M$ such that $X$ is isometric to a product near the boundary. Let $\nabla^W$ be a connection on the exterior bundle $W$ based on $X$ and $\nabla^{LC}$ the Levi-Civita connection on $X$. Let $D_\nabla = \oplus_{k=1}^n (d_k^\nabla + d_k^{\nabla *})$ where $d_k^\nabla = d_k \otimes 1 \oplus 1 \otimes \nabla^W$ and $d_k^{\nabla *} = d_k^* \otimes 1 \oplus 1 \otimes \nabla^W$ as in* Example 2*, and let as before $D_\nabla^+$ denote the restriction of $D_\nabla$ to the even forms on $X$. Near the boundary,*

$$D_\nabla^+ = c \circ (\frac{d}{dt} + B^-)$$

*where $B^-$ is the restriction to odd forms on the boundary of the operator defined on $2p$ or $2p+1$ forms by $B_\nabla = \oplus_{k=1}^n (-1)^{k+p+1}(\epsilon * d_k^\nabla - d_k^\nabla *)$, $\epsilon$ denoting the grading operator on forms. Then, the index of the operator $D_\nabla^+$ (restricted to the subspace of smooth sections satisfying the boundary condition $P_+(s|_M) = 0$, $P_+$ being the projector on the space spanned by eigenfunctions with non-negative eigenvalues) is given by*

$$\operatorname{ind} D_\nabla^+ = \int_X L(\nabla^{LC}) \operatorname{Ch}(\nabla^W) + \eta(B^-), \tag{1.82}$$

*where $L(\nabla^{LC}) = \sqrt{\det\left(\frac{\Omega^{LC}/4\pi}{\tanh \Omega^{LC}/4\pi}\right)}$ is the Hirzebruch polynomial of $\nabla^{LC}$ and $\eta(B^-)$ denotes the $\eta$ invariant of $B^-$.*

# Chapter 2

# Physical Prerequisites

In this chapter we consider the Fresnel integral approach to path integrals, which gives a rigorous definition to some of the heuristic integrals considered in quantum physics. We also describe the Ansatz given by Schwarz to define partition functions associated to degenerate action functionals by the use of $\zeta$-regularized determinants, and the Ansatz to define anomalies from the path integral point of view.

## 2.1 Fresnel Integrals

In this section we introduce the framework of Fresnel integrals as defined by Albeverio and Høegh-Krohn [AlH76].

### 2.1.1 Infinite Dimensional Gaussian Integrals

In a finite dimensional vector space *Bochner's theorem* ensures a one-to-one correspondence between characteristic functions [1] (positive definite continuous functions) and measures [Y85]. For example, to the function

$$\begin{array}{rcl} \chi : \mathbb{R}^n & \to & \mathbb{R}^+ \\ \xi & \mapsto & \chi(\xi) = e^{-\frac{1}{2}\langle \xi, \xi \rangle} \end{array}$$

there corresponds a unique Borel measure on $\mathbb{R}^n$, called *Gaussian Measure* and denoted by $\mu$, such that

$$\chi(\xi) = \int_{\mathbb{R}^n} e^{i\langle \xi, \phi \rangle} d\mu(\phi).$$

[1] A *characteristic function* on a topological vector space $E$ is a continuous (on every finite dimensional subspace of $E$) function $\chi$ satisfying

$$\sum_{j,k=1}^{N} \alpha_j \bar{\alpha_k}\, \chi(\xi_j - \xi_k) \geq 0$$

for $\alpha_k \in \mathbb{C}, \xi_j \in E \ \ (j, k = 1, ..., N.)$

In infinite dimensions, starting from a characteristic function $\chi$ on a topological vector space $E$, one typically ends up with a measure with support in a larger space. Even in the case of a Hilbert space, the measure corresponding to a characteristic function on this space lies in some Hilbert-Schmidt extension of it. However, Bochner's theorem holds in the case of continuous characteristic functions on a nuclear Hilbert space (a topological vector space whose topology is defined by a family $\{|| \cdot ||_\alpha\}$ of Hilbertian semi-norms such that $\forall\alpha\ \exists\alpha' : || \cdot ||_\alpha$ is HS with respect to $|| \cdot ||_{\alpha'}$.) [GV64].

Let $\mathcal{H}$ be a Hilbert Space (with inner product $\langle\, ,\rangle_{\mathcal{H}}$) and, for $\alpha > 0$, consider the *characteristic function*

$$\chi_\alpha(\xi) = e^{-\frac{1}{2\alpha}\langle\xi,\xi\rangle_{\mathcal{H}}}.$$

Corresponding to this function there is a infinite dimensional *Gaussian measure* $\mu_\alpha$ which, keeping in mind the path integral heuristic expressions, can be formally written

$$d\mu_\alpha(\phi) = \frac{1}{Z_\alpha}\, e^{-\frac{\alpha}{2}\langle\phi,\phi\rangle_{\mathcal{H}}}\mathcal{D}\phi,$$

the support of which lies in a Hilbert-Schmidt extension of $\mathcal{H}$, say $\mathcal{H}'$. Here $Z_\alpha = \int \mathcal{D}\phi \;\; e^{-\frac{\alpha}{2}\langle\phi,\phi\rangle_{\mathcal{H}}}$. All this can be summarized in the single equation

$$\chi_\alpha(\xi) = \int_{\mathcal{H}} e^{i\langle\xi,\phi\rangle_{\mathcal{H}}} d\mu_\alpha(\phi), \tag{2.1}$$

that generalizes the classical relation

$$e^{-\frac{1}{2\alpha}|\vec{x}|^2} = \int_{\mathbb{R}^n} e^{i\langle\vec{x},\vec{y}\rangle - \frac{\alpha}{2}|\vec{y}|^2} d\vec{y}, \tag{2.2}$$

where $\vec{x}, \vec{y} \in \mathbb{R}^n$ and $\langle\, ,\, \rangle$ denotes the inner product in this space. Equation (2.1) defines the function $\chi_\alpha$ as the *Fourier Transform* of the Gaussian measure $\mu_\alpha$, so we shall denote it as $\widehat{\mu}_\alpha$.

### 2.1.2 Fresnel Integrals

Let $\mathcal{H}$ be a separable Hilbert space with inner product $\langle,\rangle$ and norm $\|\ \|$, and let $\mathcal{M}(\mathcal{H})$ be the commutative Banach algebra of bounded complex Borel measures on $\mathcal{H}$ (with the norm induced by "total variation" and the product given by convolution [DS58]).

**Definition 9** *The class $\mathcal{F}(\mathcal{H})$ of* Fresnel integrable functions *of $\mathcal{H}$ is the set of continuous bounded complex-valued functions on $\mathcal{H}$ which are Fourier transforms of some element of $\mathcal{M}(\mathcal{H})$. Namely, $f \in \mathcal{F}(\mathcal{H})$ if there exists $\mu_f \in \mathcal{M}(\mathcal{H})$ such that*

$$f(x) = \int_{\mathcal{H}} \exp\{i\langle x, y\rangle\}\, d\mu_f(y). \tag{2.3}$$

*The* Fresnel integral *of $f \in \mathcal{F}(\mathcal{H})$ is defined by*

$$\mathcal{F}(f) = \int_{\mathcal{H}} \exp\left\{-\frac{i}{2}\|x\|^2\right\} d\mu_f(x). \tag{2.4}$$

In [AlH76] it is shown that $\mathcal{F}(\mathcal{H})$ is a Banach algebra with identity isometrically isomorphic to $\mathcal{M}(\mathcal{H})$, and we refer to [AlH76] for additional information concerning this algebra and proofs of some of the results which we are going to use.

Note that $\mathcal{F}(f)$ is not properly speaking an integral, but it verifies some properties of integrals that prompt that name (for example a Fubini theorem for Fresnel integrals exist). Remark also that in the very suggestive notation used by Albeverio and Høegh-Krohn, the expression

$$\mathcal{F}(f) = \tilde{\int_{\mathcal{H}}} \exp\left\{\frac{i}{2}\|x\|^2\right\} f(x)\, dx,$$

which comes from the usual integral relation in finite-dimensional Gaussian integration, looks like a generating functional as defined in Appendix B (see also equation 2.2). Actually the work of Albeverio and Høegh-Krohn on oscillatory integrals was aimed to find a mathematically rigorous theory of integration corresponding to the heuristic Feynman path integration in quantum physics. Among the various applications of oscillatory integrals to physics, applications to quantum and statistical mechanics and the theory of quantized fields can be found in [AlH76], and more recently to Chern-Simons field theories in [AlS92] [AlS95]. For more applications and the relation of Fresnel integrals with other approaches to the Feynman integral see also [JL00].

**Theorem 5** [AlH76] *Let $\mathcal{H} = \mathcal{H}_1 \oplus \mathcal{H}_2$ be the orthogonal sum of two subspaces $\mathcal{H}_1$ and $\mathcal{H}_2$. For $f \in \mathcal{F}(\mathcal{H})$ set $f(x_1, x_2) = f(x_1 \oplus x_2)$, $x_i \in \mathcal{H}_i$ $(i = 1, 2)$, then, for fixed $x_2 \in \mathcal{H}_2$, $\tilde{f} : x_1 \mapsto f(x_1, x_2) \in \mathcal{H}_1$ and*

$$g(x_2) = \int_{\mathcal{H}_1} \exp\left\{-\frac{i}{2}\|x_1\|^2\right\} d\mu_{\tilde{f}}(x_1),$$

*belongs to $\mathcal{F}(\mathcal{H}_2)$. Moreover,*

$$\int_{\mathcal{H}_2} \exp\left\{-\frac{i}{2}\|x_2\|^2\right\} d\mu_g(x_2) = \int_{\mathcal{H}} \exp\left\{-\frac{i}{2}\|x\|^2\right\} d\mu_f(x). \tag{2.5}$$

The case when the inner product on $\mathcal{H}$ is defined by a symmetric bilinear form is particularly interesting for us, for in that case we can define an associated Fresnel integral as follows. Consider a densely defined symmetric operator $B$ acting on $\mathcal{H}$, with domain and range equal to $\mathcal{H}$, and with bounded inverse $B^{-1}$.

**Definition 10** *Let*

$$\langle\!\langle x, y \rangle\!\rangle_B = \langle x, By \rangle,$$

*then we define the* Fresnel integral *of a function $f$ in $\mathcal{F}_B(\mathcal{H})$, the Banach algebra of Fresnel integrable functions (with respect to the bilinear form defined by $B$), by*

$$\mathcal{F}_B(f) = \int_{\mathcal{H}} \exp\left\{-\frac{i}{2}\langle x, B^{-1}x\rangle\right\} d\mu_f(x), \tag{2.6}$$

*where $\mu_f$ denotes the bounded complex measure defined by $f$ through*

$$f(x) = \int_{\mathcal{H}} \exp\{i\langle\!\langle x, y\rangle\!\rangle_B\}\, d\mu_f(y). \tag{2.7}$$

Let us now consider the particular situation where $\mathcal{H}_1$ is a closed subspace of $\mathcal{H}$ such that the restriction of $\langle\!\langle x, y\rangle\!\rangle_B$ to $\mathcal{H}_1 \times \mathcal{H}_1$ is non-degenerate. Let $\mathcal{H}_1^{\perp}$ be the orthogonal complement of $\mathcal{H}_1$ in $\mathcal{H}$ and $\mathcal{H}_2 = B^{-1}\mathcal{H}_1^{\perp}$, so that $\mathcal{H} = \mathcal{H}_1 \oplus \mathcal{H}_2$ yields a splitting of $\mathcal{H}$ in closed subspaces such that the restriction of $\langle\!\langle x, y\rangle\!\rangle_B$ to $\mathcal{H}_1 \times \mathcal{H}_2$ is identically zero. Then the restriction of $\langle\!\langle x, y\rangle\!\rangle_B$ to $\mathcal{H}_2 \times \mathcal{H}_2$ is non-degenerate, and Fubini's theorem takes the following form.

**Theorem 6** [AlH76] *For any $f \in \mathcal{F}(\mathcal{H})$,*

$$\begin{aligned}
\mathcal{F}_B(f) &= \int_{\mathcal{H}} \exp\left\{-\frac{i}{2}\langle x, B^{-1}x\rangle\right\} d\mu_f(x) \\
&= \int_{\mathcal{H}_1} \exp\left\{-\frac{i}{2}\langle x_1, B^{-1}x_1\rangle\right\} \left[\int_{\mathcal{H}_2} \exp\left\{-\frac{i}{2}\langle x_2, B^{-1}x_2\rangle\right\} d\mu_f(x_2)\right] d\mu_f(x_1)
\end{aligned}$$

*where $f(x_1, x_2) = f(x_1 \oplus x_2)$ with respect to the orthogonal splitting of $\mathcal{H}$ defined by the inner product induced by $B$, the restrictions of $f$ on the right hand side being indicated by the subscripts.*

**Example 6** [AlS95] *The partition function of abelian Chern-Simons theory can be rigorously defined in the Fresnel integral approach, as shown in* [AlS92] [AlS95]. *Let $M$ be a three dimensional closed Riemannian manifold, $G$ a Lie group, $A$ a* Lie$(G)$*-valued one form on $M$, and consider the Chern-Simons action functional*

$$S(A) = \frac{k}{4\pi}\int_M A \wedge d_1 A, \qquad k \in \mathbb{R}, \tag{2.8}$$

*where $d_1$ denotes the exterior differential on* Lie$(G)$*-valued one forms. Let $D = \sum d_i + d_i^*$ be the Dirac operator associated to the de Rham complex* (1.53)*, where $d_i^*$ denotes the formal adjoint to the exterior differential $d_i$, with respect to the inner product $\langle \alpha, \beta\rangle = \int_M \alpha \wedge *\beta$, $*$ being the Hodge star map. Let $\mathcal{H}$ be the closure of $\Omega^1 \oplus \Omega^3$ with respect to the previously defined scalar product, and consider the operator*

$$B_{CS} = \frac{k}{2\pi}\left(*D\,J + P_D\right), \tag{2.9}$$

*where* $J : \Omega^1 \oplus \Omega^3 \to \Omega^1 \oplus \Omega^3$ *is the operator given by the matrix* $\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ *and* $P_D$ *denotes the projection from* $\mathcal{H}$ *onto* $\ker D$. *Then* $B_{CS}$ *is a self-adjoint surjective operator and its inverse* $B_{CS}^{-1}$ *is compact. This means that the Fresnel integral of a function* $f$ *in the Banach algebra of Fresnel integrable functions with respect to the linear form defined by* $B_{CS}$, *as in* Definition 10, *is well-defined and reads*

$$\mathcal{F}_{B_{CS}}(f) = \int_{\mathcal{H}} \exp\left\{-\frac{i}{2}\langle \mathbf{x}, B_{CS}^{-1}\mathbf{x}\rangle\right\} d\mu_f(\mathbf{x}). \tag{2.10}$$

*Setting* $\mathbf{x} = (A, \omega)$, *from the definition of* $B_{CS}$ *it follows that*

$$\begin{aligned} \langle \mathbf{x}, B_{CS}\mathbf{x}\rangle &= \langle (A,\omega), \begin{pmatrix} *d_1 & d_0* \\ -d_0^* & 0 \end{pmatrix} (A,\omega)\rangle \\ &= \langle A, *d_1 A\rangle + \langle d_0^* A, *\omega\rangle - \langle \omega, d_0^* A\rangle \\ &= \langle A, *d_1 A\rangle \end{aligned}$$

*which shows that this Fresnel integral models the Chern-Simons action functional* (2.8). *We refer to* [AlS92] [AlS95] *for further results concerning the treatment of Wilson loops and more general correlation functions in this setting.*

## 2.2 The Partition Function of a Degenerate Gaussian Action Functional

Consider the partition function[2] (see Appendix B for the relevant prerrequisites)

$$Z_o(S) \text{“}=\text{”} \int_{\Xi} \exp\{-S(\xi)\}\,[\mathcal{D}\xi] \tag{2.11}$$

associated to a non degenerate quadratic action functional $S$, i.e. when $S(\xi) = \langle \xi, T_s \xi\rangle$ and $\ker T_s = \{0\}$, $T_s$ being a self-adjoint elliptic positive-order differential operator on the inner-product Hilbert space $\Xi$ (typically the space of sections of a vector fibration on a manifold). As we pointed out in the Introduction, in the “non perturbative approach” of quantum field theory, there is a natural Ansatz to give a rigorous definition of this object through regularized determinants, namely (as prompted by the equality (B.3) that holds in finite dimensions) to define

$$Z_o(S) \equiv (\det{}_\zeta T_s)^{\frac{1}{2}}. \tag{2.12}$$

[2]The notation “ = ” will be used to distinguish the heuristic statements communly used in physics from mathematical equalities.

**Schwarz's Ansatz for the Partition Function of a Degenerate Gaussian Action Functional.** The partition function associated to a degenerate action functional was first studied by Schwarz [S79], who proposed an Ansatz to associate to it an acyclic elliptic complex (called "resolvent"). Schwarz's Ansatz, inspired in the so-called Fadeev-Popov procedure, then imitates the (combinatorial) definition of Reidemeister torsion [M66] of a chain of linear maps between vector spaces –introduced by topologists in order to classify topological spaces with the same homotopy type– making use of the theory of zeta-determinants introduced by Ray and Singer in [RS71]. This was later refined by Adams and Sen [AS95], who used this Ansatz to test the conjectured behavior of the partition function of Chern-Simons theory (i.e. its behavior for large $k$, in the notations of (2.8)).

Consider the partition function (2.11), but assume that $\ker T_s \neq 0$. Then, given the degeneracy in the action $Z_o(S)$ heuristically diverges,

$$Z_o(S) \text{``}={}\text{''} \mathrm{vol}(\ker T_s) \int_{(\ker T_s)^\perp} \exp\{-S(\xi)\}\,[\mathcal{D}\xi].$$

The formal extension of the Faddeev-Popov procedure led to Schwarz to the following Ansatz to "compute" the divergent part of this formal equality, and hence to define the partition function $Z_o(S)$.

**Definition 11** *An elliptic* resolvent *for $S$ is an acyclic elliptic complex of the form*

$$R(S): \qquad 0 \to \Gamma(E_N) \xrightarrow{T_N} \cdots \to \Gamma(E_1) \xrightarrow{T_1} \Gamma(E_0) \xrightarrow{T_0} \Xi \xrightarrow{T_s} 0, \tag{2.13}$$

*where the $\Gamma(E_i)$ are spaces of sections of Hermitian vector bundles $E_i$ over $M$, and $T_i$, $i = 0, 1, \ldots, N$, are differential operators of the same positive order.*

Recall that ellipticity of the complex is equivalent to ellipticity of the formal Laplacians $\Delta_k = T_k^* T_k + T_{k-1} T_{k-1}^*$, $0 \leq k \leq N$.

If the action is given by $S(\xi) = \langle T_s\xi, \xi\rangle$, $\ker T_s = E \neq \{0\}$, and there exists an elliptic resolvent $R(S)$ associated to it, Schwarz defined the partition function of $S$ in terms of the $\zeta$-determinants of the Laplacian corresponding to the differential operators $T^k$ by[3]

$$Z(T_s) \equiv (\det\nolimits_\zeta T_s)^{-\frac{1}{2}} \prod_{k=0}^{N} (\det\nolimits_\zeta \Delta_k)^{\frac{(-1)^{k+1}k}{2}}. \tag{2.14}$$

Notice that the determinants $\det_\zeta T_s$ and $\det_\zeta \Delta_k$ on the right hand side must be understood as the $\zeta$-determinant of the operators $T_s$ and $\Delta_k$ restricted to the orthogonal of their respective kernels.

[3]Here we follow the definition of $Z(T_s)$ given in [AS95], the original definition of [S79] is given in terms of determinants of the maps $T_k$, which are not defined.

**Example 7** [S79] *Consider the partition function associated to the Chern-Simons action functional* (2.8) *in the previous example. In this case* $T_s = *d_1$*, and the associated resolvent is*

$$0 \to \Omega^0 \xrightarrow{d_0} \Omega_1' \xrightarrow{*d_1} 0.$$

*Then*

$$Z(*d_1) = (\det_\zeta *d_1'')^{-\frac{1}{2}} (\det_\zeta d_0 d_0^*)^{\frac{1}{2}},$$

*where* $*d_1''$ *denotes the restriction of* $*d_1$ *to* $\Omega_1''$*.*

## 2.3 Anomalies

In classical physics, Noether's theorem associates to each symmetry of the classical action (or Lagrangian) of a physical system, a corresponding conservation law or conserved current. This correspondence between symmetry and conservation laws may not be preserved by quantization, in which case we say that the theory under consideration suffers from anomalies. The first historical example of this kind of phenomena is the so-called *chiral anomaly*, which concerns fermionic lagrangians invariant under certain transformations giving rise to currents that have been experimentally observed to fail to vanish. The relevant experimental evidence in this case is the $\pi^o \to \gamma\gamma$ decay, and the corresponding action functional is given by a Dirac type operator (a general introduction containing an extensive list of reference is given in [Ber]).

From a path integral point of view, we say that an *anomaly* occurs when a transformation in the fields, leaving invariant the action functional, changes the corresponding "effective action" $\mathcal{W}$, defined by the path integral

$$e^{-\mathcal{W}} \text{``} = \text{''} \int_\Phi [\mathcal{D}\phi]\, e^{-S_o(\phi)} = Z.$$

The right hand side of this formal expression is the partition function of the theory, so that the variations of the effective action under transformations leaving the classical action invariant will be given by (logarithmic) variations of the partition function. In the case in which the action functional is non degenerate, the corresponding partition function is described by a regularized determinant, so that the variations of the effective action will be given by (logarithmic) variations of regularized determinants. If the fields are interpreted as sections of a vector bundle $E$ over a closed Riemannian manifold $M$, and if the classical action is the quadratic functional $S(\phi) = \langle T\phi, \phi \rangle$, $T$ being a positive order elliptic differential operator, then the variation of the effective action will be given by

$$\delta\mathcal{W} \text{``} = \text{''} - \delta \log \det_\zeta T.$$

Recall that here we work with $\zeta$-regularized determinants, but there are several regularization procedures to define the right hand side of this last equality. The anomaly must be, in principle, independent of the regularization procedure used to define the determinant (a different approach can be found e.g. in [LM], and a discussion on the independence of the regularization chosen in [Ber], section 5.3).

There are several kinds of anomalies, depending on the nature of the transformation defining the symmetry of the classical action and on the way it is performed. Let us say a word about the type of anomaly we shall come across in this work. Recall that we consider Clifford modules over a Riemannian manifold $M$, and Dirac operators defined on it, and that in both examples we consider (the spinor bundle on a spin manifold and the bundle of twisted differential forms on a Riemannian manifold) the connection used to define such Dirac operators is the coupling of the "exterior bundle" connection ($\nabla^W$ in the first example, $\nabla^\rho$ in the second) with a connection defined from the geometry of $M$. Here, unlike in the case of gravitational anomalies, and as is usual in gauge theory, we only consider the (logarithmic) variations of the regularized determinant of the Dirac operator induced by transformations in the "exterior" part of the connection of the Clifford bundle. Also we shall not consider the infinitesimal (logarithmic) variations of the $\zeta$-determinants, but rather (logarithmic) quotients of $\zeta$-determinants of a smooth family of operators under gauge transformations, i.e.

$$\log \frac{Z_1}{Z_0} = \log \frac{\det_\zeta(T_1)}{\det_\zeta(T_0)}.$$

One of our aims is to relate this type of anomaly occurring in physics with the tracial anomalies described in section 3.1. Other types of anomalies in physics can also be analysed using a weighted trace approach. When looking at the geometry of line bundles associated to families of elliptic complexes we shall implicitely be considering (first and second) logarithmic variations of $\zeta$-determinants of degenerate actions (namely, the Bismut-Freed connection and its curvature), which can be seen as manifestations of chiral anomalies. In the case of Dirac operators involved in QFT anomalies, the corresponding local terms can be expressed via *index theorems* as integrals over a compact manifold $M$ of local expressions involving the underlying geometric data. These questions are partially addressed in [CDP].

# Part II

# Tracial Anomalies and Geometry

# Chapter 3

# Tracial Anomalies and Index Theorems

Recall that in general, as shown in Proposition 1, tracial anomalies –which can be expressed as Wodzicki residues– are local. Locality is also the main feature of the index of a geometric operator, so it is natural to ask if there is some relation between them, i.e. if it is possible to identify the local term corresponding to a tracial anomaly in terms of indexes or viceversa. A first relation between the coboundary anomaly $\partial \mathrm{tr}^Q$ (see equation 1.8) and the index of a positive-order elliptic differential operator $A : \Gamma(E) \to \Gamma(F)$, acting as before on spaces of sections of vector fibrations on smooth manifolds (which extend to a Fredholm operator on the corresponding Sobolev completions...), can be seen as follows. Suppose $E = F$, let $I_E$ denote the identity map on $\Gamma(E)$ and $R$ be a parametrix for $A$. Since $RA = I_E - \pi_A$ and $AR = I_E - \pi_{A^*}$, where $A^*$ denotes the formal adjoint of $A$, it follows that

$$\begin{aligned} \mathrm{ind}A &= \dim\ker A - \dim\ker A^* \\ &= \mathrm{tr}(\pi_A) - \mathrm{tr}(\pi_{A^*}) \\ &= \mathrm{tr}^Q(\pi_A - \pi_{A^*}) \\ &= \mathrm{tr}^Q(AR - RA) = \mathrm{tr}^Q([A, R]) = \partial \mathrm{tr}^Q(A, R), \end{aligned}$$

where $Q$ denotes an arbitrary weight, and we use property (1.7) for the trace of the finite-rank operator $\pi_A - \pi_{A^*}$.

We shall see that in the case of some differential operators acting on sections of vector fibrations on smooth manifolds, a similar feature holds for the local term in the Atiyah-Patodi-Singer index theorem, it can be written as a trace anomaly of the type $\int \dot{\mathrm{tr}}^Q(\mathrm{sign}Q)$. These anomalies appear in the study of Chern-Simons theories.

## 3.1 Logarithmic Variations of Determinant and Weighted Trace Anomalies

Our purpose here is to relate logarithmic variations of regularized determinants of certain families of admissible operators with tracial anomalies, thus giving an a priori explanation for the locality of these variations. For families of geometric operators these variations can be expressed, via *index theorems*, as integrals of characteristic forms on the underlying manifold.

Let $\{A_x\}_{x\in X}$ be a smooth family of elliptic self-adjoint operators of constant order $A_x : \Gamma(E) \to \Gamma(E)$, parametrized by a smooth manifold $X$. The eta invariant $\eta(A_x) = \eta_{A_x}(0)$ –which is part of the phase of the $\zeta$-determinant of $A_x$– varies smoothly in $x$ modulo integers, i.e. except for jumps coming from eigenvalues of $A_x$ "crossing zero". Indeed, since $\eta(A_x) = \mathrm{tr}^{|A_x|}(\mathrm{sign}A_x)$, it gives the difference between the number of positive eigenvalues and negative eigenvalues of $A_x$, and if one of the eigenvalues of $A_x$ passes from positive to negative $\eta(A_x)$ jumps by two.

**Example 8** *Consider for instance the family of Example 1,*

$$A_x = i\frac{d}{ds} + x,$$

*on $C^\infty(S^1)$, letting $x \in \mathbb{R}$. The eta function is given by $\eta(A_x) = 1 - 2x$ for $x \in (0,1)$ and $\eta(A_x) = 0$ for $x \in \mathbb{Z}$. Hence, the value of $\eta(A_x)$ jumps by two when $x$ goes from a positive to a negative value, showing that one of the eigenvalues of $A_x$ passes from positive to negative.*

Let $A_1$ and $A_0$ be two invertible self-adjoint elliptic operators, the *spectral flow* of a family of self-adjoint elliptic operators $\{A_x\}_{x\in[0,1]}$ interpolating them, denoted $\Phi(\{A_x\})$, measures the net number of times the spectrum of the family crosses the zero axis, i.e. the net change in the number of negative eigenvalues of $A_x$ as $x$ varies between 0 and 1, it was introduced in [APSIII] in order to study the "non-continuous" part of the $\eta$ invariant. Making this definition precise requires some care since there might well be an infinite number of crossings of the zero axis (here we follow [Me], see also [CDP]).
Let us first observe that there is a partition $x_0 = 0 < x_1 < \cdots < x_N = 1$ of the interval $[0,1]$ and there are real numbers $\lambda_i, i = 1, \cdots, N$, $\lambda_0 = \lambda_{N+1} = 0$ such that the spectrum of $A_x$ avoids $\lambda_i$ for any $x$ in the interval $[x_i, x_{i+1}]$. The spectral flow of the family $\{A_x\}$ is defined by (see [Me] formula (8.134)):

$$\Phi(\{A_x\}) := \sum_{i=0}^{N} \sum_{\lambda\in S_i} \mathrm{sgn}(\lambda_{i+1} - \lambda_i) m(\lambda, x_i), \tag{3.1}$$

where $S_i = \mathrm{Spec}(A_{x_i}) \cap [\lambda_i, \lambda_{i+1}]$, $m(\lambda, x)$ denotes the multiplicity of $\lambda$ in the spectrum $\mathrm{Spec}(A_{x_i})$ of $A_x$ and $\mathrm{sgn}(\alpha)$ is $-1, 0$ or $1$ as $\alpha$ is negative, 0 or positive. This definition is independent of the chosen partition and, if $A_x$ is invertible for any $x \in [0, 1]$, then $\Phi(A_x) = 0$ as expected.

Consider now a parametrized family of elliptic pseudo-differential operators of positive order $\{A_x\}$, where $x$ varies smoothly in a manifold $X$. Let, for $z \in \mathbb{C}$,

$$\eta_{A_x}(z) = \mathrm{tr}(A_x |A_x|^{-z-1}),$$

which we recall is a meromorphic function on the complex plane.

**Proposition 9** [APSIII] *For $\Re(z) \gg 0$,*

1.
$$d\eta_{A_x}(z) = -z \, \mathrm{tr}(d\, A_x |A_x|^{-z-1}). \tag{3.2}$$

2.
$$\mathrm{res}\,[d\,(\mathrm{sign} A_x)] = 0. \tag{3.3}$$

**Lemma 5** *Let $\{A_x\}_{x\in[0,1]}$ be a smooth family of elliptic self-adjoint operators of positive constant order $a$. Then, at points $x \in X$ for which $A_x$ is invertible,*

$$-\frac{1}{a}\mathrm{res}\left[|A_x|^{-1}\frac{d}{dx}A_x\right] = -\frac{1}{a}\mathrm{res}\left[A_x^{-1}\frac{d}{dx}|A_x|\right] = \dot{\mathrm{tr}}^{|A_x|}\,(\mathrm{sign} A_x) = \dot{\mathrm{tr}}^{A_x}\,(\mathrm{sign} A_x) \tag{3.4}$$

*Proof.* Recall that from [APSIII], see Proposition 9 (*2*),

$$\mathrm{res}\left(\frac{d}{dx}U_x\right) = 0,$$

where $U_x = A_x|A_x|^{-1} = |A_x|A_x^{-1} = \mathrm{sign} A_x$. Then

$$\begin{aligned}
\mathrm{res}\left[|A_x|^{-1}\frac{d}{dx}A_x\right] &= \mathrm{res}\left[|A_x|^{-1}\frac{d}{dx}(A_x U_x^2)\right] \\
&= \mathrm{res}\left[|A_x|^{-1}\frac{d}{dx}(|A_x| U_x)\right] \\
&= \mathrm{res}\left[\frac{d}{dx}(U_x) + {A_x}^{-1}\frac{d}{dx}|A_x|\right],
\end{aligned}$$

from which the first equality follows.

On the other hand, from Proposition 2, using the fact that $[U_x, |A_x|] = [A_x, |A_x|] = 0$,

$$\begin{aligned}
\dot{\mathrm{tr}}^{|A_x|}(\mathrm{sign}A_x) - \dot{\mathrm{tr}}^{A_x}(\mathrm{sign}A_x) &= -\frac{1}{a}\mathrm{res}\left[U_x \frac{d}{dx}\log|A_x|\right] + \frac{1}{a}\mathrm{res}\left[U_x \frac{d}{dx}\log A_x\right] \\
&= -\frac{1}{a}\mathrm{res}\left[U_x|A_x|^{-1}\frac{d}{dx}|A_x|\right] + \frac{1}{a}\mathrm{res}\left[U_x A_x^{-1}\frac{d}{dx}A_x\right] \\
&= -\frac{1}{a}\mathrm{res}\left[A_x^{-1}\frac{d}{dx}|A_x|\right] + \frac{1}{a}\mathrm{res}\left[|A_x|^{-1}\frac{d}{dx}A_x\right],
\end{aligned}$$

where $a = \mathrm{ord}(A_x)$, so the last equality in (3.4) follows from the first. $\square$

The following theorem relates the variation of the continuous part of the eta invariant to an integrated tracial anomaly:

**Theorem 7** [CDP] *Let $A_0$ and $A_1$ be two elliptic self-adjoint operators and $\{A_x\}_{x\in[0,1]}$ a smooth family of elliptic self-adjoint operators of constant order interpolating them. Then, at points $x \in X$ for which $A_x$ is invertible,*

$$\eta(A_1) - \eta(A_0) = 2\Phi(\{A_x\}) + \int_0^1 \dot{\mathrm{tr}}^{A_x}(\mathrm{sign}(A_x))\, dx, \tag{3.5}$$

*where $\Phi(\{A_x\})$ denotes the spectral flow of the family and $\dot{\mathrm{tr}}^{A_x} = [\frac{d}{dx}, \mathrm{tr}^{A_x}]$.*

*Proof.* Using the invariance of the difference $\eta(A_1) - \eta(A_0)$ under a shift $A_x \mapsto A_x + \alpha$, $\alpha \in \mathbb{R}$, it can be seen that we can reduce the proof of (3.5) to the case of a family of invertible operators (for details see [CDP], section 3). In order to show (3.5) in that case, let us first to show that

$$\mathrm{tr}^{|A_x|}\left[\frac{d}{dx}\mathrm{sign}A_x\right] = 0. \tag{3.6}$$

By (3.4)

$$\begin{aligned}
\frac{d}{dx}\left[\mathrm{tr}^{|A_x|}(\mathrm{sign}A_x)\right] &= \dot{\mathrm{tr}}^{|A_x|}(\mathrm{sign}A_x) + \mathrm{tr}^{|A_x|}\left[\frac{d}{dx}\mathrm{sign}A_x\right] \\
&= -\frac{1}{a}\mathrm{res}\left[A_x^{-1}\frac{d}{dx}|A_x|\right] + \mathrm{tr}^{|A_x|}\left[\frac{d}{dx}\mathrm{sign}A_x\right]. \qquad (3.7)
\end{aligned}$$

On the other hand,

$$\begin{aligned}
\frac{d}{dx}\left[\mathrm{tr}^{|A_x|}(\mathrm{sign}A_x)\right] &= \frac{d}{dx}\left[\mathrm{f.p.}|_{z=0}\left(\mathrm{tr}\left(\mathrm{sign}A_x|A_x|^{-z}\right)\right)\right] \\
&= \frac{d}{dx}\left[\mathrm{f.p.}|_{z=0}\left(\mathrm{tr}\left(A_x|A_x|^{-z-1}\right)\right)\right].
\end{aligned}$$

But Proposition 9 (i) implies that

$$\frac{d}{dx}\eta_{A_x}(z) = \frac{d}{dx}\mathrm{tr}\left(A_x|A_x|^{-z-1}\right) = -z\mathrm{tr}\left(\frac{dA_x}{dx}|A_x|^{-z-1}\right),$$

hence

$$\begin{aligned}\frac{d}{dx}\left[\mathrm{tr}^{|A_x|}(\mathrm{sign}A_x)\right] &= \mathrm{f.p.}|_{z=0}\left[-z\mathrm{tr}\left(\frac{dA_x}{dx}|A_x|^{-z-1}\right)\right]\\ &= -\mathrm{Res}_{z=0}\left[\mathrm{tr}\left(\frac{d}{dx}A_x|A_x|^{-1-z}\right)\right]\\ &= -\frac{1}{a}\mathrm{res}\left[\frac{d}{dx}A_x|A_x|^{-1}\right],\end{aligned}$$

where $a = \mathrm{ord}A_x$. Combining this with the first equality in (3.4) and (3.7) yields equality (3.6). Now, by definition,

$$\eta(A_1) - \eta(A_0) = \mathrm{tr}^{|A_1|}(\mathrm{sign}A_1) - \mathrm{tr}^{|A_0|}(\mathrm{sign}A_0),$$

so that, whenever $A_x$ is invertible,

$$\eta(A_1) - \eta(A_0) = \int_0^1 \frac{d}{dx}\left[\mathrm{tr}^{|A_x|}(\mathrm{sign}A_x)\right]\,dx. \tag{3.8}$$

Putting (3.6) into the first equality in (3.7) gives

$$\eta(A_1) - \eta(A_0) = \int_0^1 \dot{\mathrm{tr}}^{|A_x|}(\mathrm{sign}A_x)\,dx, \tag{3.9}$$

so, by the last equality in equation (3.4), the result follows. □

**Remark.** Notice that, even if $A_x$ is not invertible, the weighted traces $\mathrm{tr}^{A_x}$ can be defined as in equation (1.21) because $\ker A_x$ has constant dimension on each continuity interval.

It follows from this that variations of $\eta$ (on continuity intervals) are local. In particular, for families of signature operators on a three dimensional Riemannian manifold, a classical result of [APSI] expresses the local term given by the Wodzicki residue coming from the weighted trace anomaly $\int \dot{\mathrm{tr}}^{A_x}(\mathrm{sign}(A_x))\,dx$, as an integral on $M$ in terms of the underlying geometry.

**Corollary 1** *Let $\{A_x\}_{x\in[0,1]}$ be a smooth family of self-adjoint elliptic operators with vanishing spectral flow and constant order $a$, such that $A_0$ and $A_1$ are invertible. Let $\phi(A_x) = \frac{\pi}{2}\left(\eta_{A_x}(0) - \zeta_{|A_x|}(0)\right)$ be the phase of $\det_\zeta A_x$.*

*Then, if $\zeta_{|A_x|}(0)$ is constant, the difference of the phases $\phi(A_1) - \phi(A_0)$ reads*

$$\begin{aligned} \phi(A_1) - \phi(A_0) &= \frac{\pi}{2} \int_0^1 \dot{\mathrm{tr}}^{A_x}(\mathrm{sign}(A_x))\, dx \\ &= -\frac{\pi}{2a} \int_0^1 \mathrm{res}\left[|A_x|^{-1} \frac{d}{dx} A_x\right] dx. \end{aligned} \tag{3.10}$$

*If furthermore* $\det_\zeta |A_x|$ *is constant, then*

$$\begin{aligned} \log \frac{\det_\zeta A_1}{\det_\zeta A_0} &= \phi(A_1) - \phi(A_0) \\ &= -\frac{\pi}{2a} \int_0^1 \mathrm{res}\left[|A_x|^{-1} \frac{d}{dx} A_x\right] dx. \end{aligned} \tag{3.11}$$

*Proof.* From the equality (1.29), relating the $\zeta$-determinant of a (non necessarily positive) self-adjoint elliptic operator and its $\eta$ invariant, it follows that

$$\log \det_\zeta A = \log \det_\zeta |A|\, i\frac{\pi}{2} \left(\eta_A(0) - \zeta_{|A|}(0)\right).$$

Thus, given that

$$\dot{\mathrm{tr}}^{|A_x|}(\mathrm{sign} A_x) = \dot{\mathrm{tr}}^{A_x}(\mathrm{sign} A_x) = -\frac{1}{a} \mathrm{res}\left[|A_x|^{-1} \frac{d}{dx} A_x\right],$$

the result follows. □

Thus, under the above assumptions, the logarithmic variation of the $\zeta$-determinant is expressed as a weighted trace anomaly and is therefore local. Although the assumptions of the previous corollary seem strong, they are fulfilled for the example of signature operators of interest to us here. Indeed, there is a natural family of examples where the equality between the index of an elliptic differential operator and the spectral flow of an associated family of operators can be made explicit (see [BBW] [Woj] [RoSa] and references therein), and to have $\zeta_{|A|}(0) = 0$ it is enough to work on an odd-dimensional manifold. This is ilustrated by an example described in the next section.

## 3.2 The Signature Operator on an odd Dimensional Manifold

In this section we give an application of Corollary 1 to a family of geometric operators that appears in our discussion about phase anomalies in Section 5.2.

Let $M$ be a Riemannian manifold of odd dimension $n = 2k + 1$, and $V_\rho$ the Hermitian vector bundle over $M$ with flat connection $\nabla^\rho$ defined by a representation $\rho$ of the fundamental group of $M$ as in Example 2. Consider the acyclic de Rham complex (1.53)

$$0 \longrightarrow \Omega^0 \xrightarrow{d_0} \cdots \Omega^{k-1} \xrightarrow{d_{k-1}} \Omega^k \xrightarrow{d_k} \Omega^{k+1} \xrightarrow{d_{k+1}} \cdots \Omega^n \xrightarrow{d_n} 0,$$

where $\Omega^k = C^\infty(\Lambda^k T^*M \otimes V_\rho)$ and $d_k$ denotes the restriction to $\Omega^k$ of the exterior differential on twisted forms given by $d_\rho = d \otimes 1 \oplus 1 \otimes \nabla^\rho$. Then the de Rham operator $D_\nabla = \sum_{k=0}^n (d_k + d_k^*)$ is a Dirac operator taking even (odd) forms to odd (even) forms. Let us set $\Delta_\nabla = D_\nabla^2$ and define the Laplacian operator on $\Omega^k$ by $\Delta_k = d_k^* d_k + d_{k-1} d_{k-1}^*$. Acyclicity of the complex (1.53) implies a *Hodge decomposition* (1.54)

$$\Omega^k = \Omega'_k \oplus \Omega''_k$$

where $\Omega'_k = \operatorname{Im} d_{k-1} = \operatorname{Ker} d_k$ and $\Omega''_k = \operatorname{Im} d_k^* = \operatorname{Ker} d_{k-1}^*$

In odd dimensions the square of the Hodge star operator $* : \Omega^p \to \Omega^{n-p}$ is the identity map, and the operator $*d_k : \Omega^k \to \Omega^k$ as a formally self-adjoint elliptic differential operator of order one. Restricting $*d_k$ to $\Omega''_k$ gives us an invertible self-adjoint elliptic differential operator that we shall denote $*d''_k$, which has a well defined $\zeta$-determinant.

Consider a family of operators $\{*d''_t, t \in [0,1]\}$ (which can be built from a smooth family of connections $\{\nabla^\rho_t, t \in [0,1]\}$ on the exterior bundle $V_\rho$, which interpolates two given connections $\nabla^\rho_0$ and $\nabla^\rho_1$, or from a smooth family $\{g_t, t \in [0,1]\}$ of metrics on $M$, which induces a family of Hodge star operators) and let $*d''_{k,t}$ denote their restriction to $k$-forms. We consider now the manifold $M \times [0,1]$, and from the family $*d''_{k,t}$ we construct an elliptic operator acting on sections of the bundle $\Omega^k \times [0,1]$, namely

$$A = *d''_{k,t} + \frac{d}{dt}.$$

It follows from [APSIII] (see also [Woj] [RoSa] and references therein) that

$$\operatorname{ind} A = \Phi(\{*d''_{k,t}\}),$$

but, since the signature of the manifold $M \times [0,1]$ is zero, then $\operatorname{ind} A = 0$ and hence the spectral flow of the family $\{*d''_{k,t}\}$ vanishes. Thus, it follows from Theorem 7 that

$$\eta(*d''_{k,1}) - \eta(*d''_{k,0}) = \int_0^1 \dot{\operatorname{tr}}^{*d''_{k,t}}(\operatorname{sign}(*d''_{k,t}))\, dt, \tag{3.12}$$

so that the difference of the eta invariants is given by a tracial anomaly, which makes it a local quantity. Furthermore, in odd dimensions $\zeta_{|A|}(0) = 0$ for any elliptic self-adjoint differential operator $A$, so that

$$\log\left[\frac{\det_\zeta(*d''_{k,1})}{\det_\zeta(*d''_{k,0})}\frac{\det_\zeta(|*d''_{k,0}|)}{\det_\zeta(|*d''_{k,1}|)}\right] = \frac{\pi}{2}\left\{\eta(*d''_{k,1}) - \eta(*d''_{k,0})\right\}$$

coincides with the tracial anomaly given by the right hand side of (3.12). From Corollary 1 follows the following

**Proposition 10** *Let $M$ be a 3-dimensional closed manifold and consider the family of first-order invertible differential operators $\{*d''_{1,t}, t \in [0,1]\}$ —built from a smooth family of connections $\{\nabla^\rho_t, t \in [0,1]\}$ on the exterior bundle $V_\rho$ or from a smooth family $\{g_t, t \in [0,1]\}$ of metrics on $M$. Then the difference of phases of the $\zeta$-determinants of $*d''_{1,t}$ at $t = 0$ and $t = 1$ is given by a Wodzicki residue coming from a tracial anomaly*

$$\begin{aligned}\phi(*d''_{1,1}) - \phi(*d''_{1,0}) &= \frac{\pi}{2}\int_0^1 \dot{\mathrm{tr}}^{*d''_{1,t}}(\mathrm{sign}(*d''_{1,t}))\,dt \\ &= -\frac{\pi}{2}\int_0^1 \mathrm{res}\left[|*d''_{1,t}|^{-1}\frac{d}{dt}*d''_{1,t}\right]dt. \qquad (3.13)\end{aligned}$$

Finally, note that for $k = 1$ ($n = 3$) the analytic torsion of $M$ is given by

$$T(M) = \det_\zeta \Delta_0 \cdot (\det_\zeta \Delta''_{1,t})^{-1},$$

and the determinant of $\Delta_0$ is constant given the definition of $D_{\nabla t}$. Hence, it is clear that when the family $\{*d''_t\}$ is built from a family of metrics on $M$, topological invariance of $T(M)$ implies that

$$\det_\zeta|*d''_{1,t}| = (\det_\zeta \Delta''_{1,t})^{\frac{1}{2}}$$

is constant. Therefore, in view of Corollary 1,

$$\begin{aligned}\log\frac{\det_\zeta(*d''_{1,1})}{\det_\zeta(*d''_{1,0})} &= \frac{\pi}{2}\int_0^1 \dot{\mathrm{tr}}^{*d''_{k,t}}(\mathrm{sign}(*d''_{k,t}))\,dt, \\ &= -\frac{\pi}{2}\int_0^1 \mathrm{res}\left[|*d''_{k,t}|^{-1}\frac{d}{dt}(*d''_{k,t})\right]dt.\end{aligned}$$

Thus, being an integrated tracial anomaly, and hence a Wodzicki residue, the term $\log\frac{\det_\zeta(*d''_{1,1})}{\det_\zeta(*d''_{1,0})}$ is the integral of a local term on the base manifold. This example plays a fundamental role in phase anomaly computations in Chern-Simons theory, as we explain in Chapter 5.

# Chapter 4

# Geometry of Determinant Line Bundles and Tracial Anomalies

In this chapter we consider Quillen's determinant line bundle associated to a family of elliptic differential operators with constant positive order acting on an infinite-dimensional vector bundle over a closed Riemannian manifold. Using the $\zeta$-regularization method introduced by Ray and Singer [RS71] to define determinants of elliptic operators, we follow Quillen [Q86] to define a smooth metric and Bismut and Freed to define a compatible connection on the determinant line bundle associated to the family. We discuss the locality of its curvature on the basis of [PR]. Following Bismut and Freed [BF88], we specialize to a family of Dirac operators defined by a fibration of spin manifolds, for which this curvature can be expressed as an integral of Chern-Weail forms.

## 4.1 Determinant Line Bundles in Finite Dimensions

Let $M$ be a smooth closed finite-dimensional Riemannian manifold, $E \to M$ a Hermitian complex line bundle over $M$ and $\nabla^E$ a connection on $E$. If in some local frame the connection has the form

$$\nabla^E = d + \theta, \tag{4.1}$$

where $\theta$ denotes a U(n)-valued 1-form, then the curvature of $\nabla^E$, defined by $\Omega^E = (d+\theta)^2$, is locally given by $\Omega^E = d\theta + \theta \wedge \theta$, and hence it satisfies the *Bianchi identity*

$$d\Omega^E = [\Omega^E, \theta]. \tag{4.2}$$

Induced by $\nabla^E$ there is a connection on the bundle $\mathrm{Hom}(E) \cong E^* \otimes E$ given by $\nabla^{\mathrm{Hom}(E)} = \nabla^{E^*} \otimes 1 \oplus 1 \otimes \nabla^E$, which extends to $\mathrm{Hom}(E)$-valued forms on

$M$ and is given locally by

$$\nabla^{\mathrm{Hom}(E)} = d + [\theta,\ ]. \tag{4.3}$$

It follows from (4.2) that

$$\nabla^{\mathrm{Hom}(E)}\Omega^E = 0,$$

and, for any $\mathrm{Hom}(E)$-valued form $\alpha$ on $M$,

$$d\left(\mathrm{tr}(\alpha)\right) = \mathrm{tr}\left([\nabla^E, \alpha]\right),$$

since $\mathrm{tr}\left([\theta, \alpha]\right) = 0$. Bianchi identity shows then that $\mathrm{tr}(\Omega^E)$ is closed. Moreover it is related with the curvature of the connection on the determinant line bundle $\det E \to M$ induced by $\nabla^E$ as expressed in the following well known

**Proposition 11** *Let $\Omega^{\det E}$ denote the curvature of the connection on the determinant bundle associated to $E \to M$ induced by $\nabla^E$. Then*

$$\mathrm{tr}(\Omega^E) = \Omega^{\det E}. \tag{4.4}$$

*Proof.* It follows from the definition of the Hermitian structure on $\det E$ induced by the one on $E$, namely

$$\langle \sigma, \sigma' \rangle = \det[\langle \sigma_i, \sigma'_j \rangle]_{i,j}$$

where $\sigma, \sigma' \in \Gamma(\det E)$ are given by $\sigma = \sigma_1 \wedge \sigma_2 \ldots \wedge \sigma_n$ and $\sigma' = \sigma'_1 \wedge \sigma'_2 \ldots \wedge \sigma'_n$ in local bases of sections of $E$. $\square$

If there is a $\mathbb{Z}_2$-graduation $E = E^+ \oplus E^-$ such that $\mathrm{rank}(E^+) = \mathrm{rank}(E^-)$, a section $T \in \Gamma\left(\mathrm{Hom}(E^+, E^-)\right)$ of the homomorphism bundle $\mathrm{Hom}(E^+, E^-) \to M$ induces a canonical section, denoted $\det T$ and called the *determinant* of $T$, of the line bundle $\mathcal{L} = (\det E^+)^* \otimes \det E^-$ over $M$. This section associates to each $m \in M$ the element $\det T_m \in \mathrm{Hom}(\det E^+_m, \det E^-_m) \cong (\det E^+_m)^* \otimes \det E^-_m$, i.e. the determinant of the map $T_m \in \mathrm{Hom}(E^+_m, E^-_m)$. If $T_m$ is invertible for all $m \in M$ we say that $T$ is invertible, in which case $\det T$ defines a trivialization of the line bundle $\mathcal{L}$.

As before, Hermitian structures and connections on the bundles $E^\pm$, which we denote $|\cdot|_{E^\pm}$ and $\nabla^{E^\pm}$, respectively, induce (by fibrewise operations) Hermitian structures $|\cdot|_{\det E^\pm}$ and $|\cdot|_{\mathrm{Hom}(E^\pm)}$, and connections $\nabla^{\det E^\pm}$ and $\nabla^{\mathrm{Hom}(E^\pm)}$, on the bundles $\det E^\pm$ and $\mathrm{Hom}(E^+, E^-)$, respectively. The connection induced on $\mathrm{Hom}(E^+, E^-)$, is given by $\nabla^{\mathrm{Hom}(E^\pm)} = \nabla^{(E^+)^*} \otimes 1 \oplus 1 \otimes \nabla^{E^-}$. It induces a connection $\nabla^{\mathcal{L}}$ on the determinant bundle $\mathcal{L}$ given, on a neigborhood on which $T$ is invertible, by

$$\nabla^{\mathcal{L}}(\det T) = (\det T)\ \mathrm{tr}_{E^+}(T^{-1}\nabla^{\mathrm{Hom}(E^\pm)}T), \tag{4.5}$$

where $\mathrm{tr}_{E^+}$ is the trace on $\mathrm{Hom}(E^+)$. Notice that this definition of $\nabla^{\mathcal{L}}$ is a generalization of the classical equality

$$(\det A_t)^{-1}\frac{d}{dt}(\det A_t) = \mathrm{tr}(A_t^{-1}\frac{d}{dt}A_t), \tag{4.6}$$

which holds for every smooth family $\{A_t\}$ of invertible finite-dimensional matrices parameterized by $t$.

The curvature of $\nabla^{\mathcal{L}}$ defined by $\Omega^{\mathcal{L}} = (\nabla^{\mathcal{L}})^2 = \mathrm{tr}(\nabla^{\mathrm{Hom}(E^{\pm})})^2$, is equal to

$$\Omega^{\mathcal{L}} = \Omega^{E^{+*}} \otimes 1 \oplus 1 \otimes \Omega^{E^-},$$

(we shall write this in the following $\Omega^{\mathcal{L}} = -\Omega^{E^{+*}} \oplus \Omega^{E^-}$, the tensorization with 1 in each case being understood). All the above mentioned results extend (up to a sign) to the super-vector bundle setting with connection replaced by superconnections, commutators replaced by supercommutators and traces replaced by supertraces (see [BGV92]), in particular $d\,(\mathrm{tr}_s(\alpha)) = \mathrm{tr}_s\left([\nabla^E, \alpha]_s\right)$, and

$$\mathrm{tr}_s(\Omega^E) = -\Omega^{\mathcal{L}}, \tag{4.7}$$

where $\Omega^E$ is the curvature of the superconnection on $E$ defined by the connections $\nabla^{E^{\pm}}$.

## 4.2 Regularized Traces, Regularized Determinants and Quillen's construction

### 4.2.1 Some Geometry of Families of Fibrations

Let $X$ be a smooth manifold of finite dimension, $I\!M$ be a closed finite-dimensional Riemannian manifold and $\pi_M : I\!M \to X$ a smooth locally trivial fibration such that $\pi_M^{-1}(x) = M_x$, the fibre over $x$ (which will be also denoted by $I\!M/X$ or simply $M$ when no reference to the base point be nescesary) be a closed Riemannian manifold. By a smooth family $\{E_x\}_{x\in X}$ of Hermitian vector bundles over the fibration of manifolds $I\!M \to X$, we mean a smooth, $\mathbb{Z}_2$-graded Hermitian vector bundle $\pi_E : E \to I\!M$ so that $E_x$ –the restriction of the bundle $E$ to $M_x$– is a Hermitian vector bundle with connection $\nabla^{E_x}$. Let $\mathcal{E}^{\pm}$ be the infinite-rank bundle over $X$ whose fibre at $x \in X$ is the space of smooth sections $\mathcal{E}_x^{\pm} = \Gamma(M_x, E_x^{\pm})$, where $E_x^{\pm} \to M_x$ is the restriction of the bundle $E = E^+ \oplus E^-$ to $M_x$, and let $\nabla^{E_x^{\pm}}$ be the Hermitian connection on $E_x^{\pm}$.

Let us assume that on the bundle $M/X$ there is a horizontal distribution, i.e. a splitting $T\,I\!M = T_H\,I\!M \oplus T(M/X)$ so that the subbundle $T_H\,I\!M$ is isomorphic to the bundle $\pi_M^* TX$. This gives us a canonical projection on the vertical tangent bundle $T(M/X)$

$$\Pi : T\,I\!M \to T(M/X),$$

with kernel the chosen horizontal tangent subbundle $T_H \mathbb{M}$. This projection allows to lift tangent vectors $\xi \in TX$ to horizontal vector fields along the fibres. Let $\xi_M \in T\mathbb{M}$ to be the vector field on $\mathbb{M}$ defined as a section of $T_H \mathbb{M}$ which projects to $\xi$ under the push-forward $\pi_{M*} : (T_H \mathbb{M})_m \to T_{\pi_M(m)}X$. Then a tangent vector $\xi(x)$, $x \in X$, lifts to a vector field $\xi_M^x$ along the fibre $M_x$.

From the Riemannian metric $\mathrm{g}_{M/X}$ on the fibre $M/X$ and the Hermitian structure on $E_x^\pm$ we define a Hermitian structure on the spaces of smooth sections $\mathcal{E}_x^\pm = \Gamma(M_x, E_x^\pm)$ by

$$\langle \sigma_1, \sigma_2 \rangle_x = \int_{M/X} \langle \sigma_1(x), \sigma_2(x) \rangle_{E_x^\pm} \, d\mu_{M_x}(x), \tag{4.8}$$

where $\sigma_1, \sigma_2 \in \mathcal{E}_x^\pm$ and $\mu_{M_x}$ denotes the volume element defined by the Riemannian metric on each fibre $M_x = \pi_M^{-1}(x)$.

Since the volume form $\mu_{M_x}$ changes from fibre to fibre, the connection $\nabla^{E_x^\pm}$ fails to be unitary for the $L^2$-inner product (4.8). Following [BF88], we modify the connection $\nabla^{E_x^\pm}$ taking[1]

$$\tilde{\nabla}^\pm = \nabla^{E_x^\pm} + \frac{1}{2}\mathrm{div}_{M_x}(m), \tag{4.9}$$

where $\mathrm{div}_{M_x}(m)$ is the divergence of the volume form at $m$ in the base directions.

**Definition 12** [BF88] *Let $\xi$ be a tangent vector to $X$, $\xi_M$ its horizontal lift to a fibre in $\mathbb{M}$ and $\psi$ a section of $\mathcal{E}^\pm$. The* Bismut connection *on the bundles $\mathcal{E}^\pm \to X$ is defined as*

$$\nabla_\xi^{\mathcal{E}^\pm} \psi = \tilde{\nabla}_{\xi_M}^\pm \psi, \tag{4.10}$$

*by point-wise action.*

The choice of $\tilde{\nabla}^\pm$ in (4.9) makes this connection unitary for the inner product (4.8).

**The Bundle $\mathcal{C}l(\mathcal{E})$.** Let $\mathcal{E}$ be the infinite-rank bundle over $X$ modelled on $\Gamma(M, E)$, whose fibre at $x \in X$ is the space of smooth sections $\mathcal{E}_x = \Gamma(M_x, E_x)$. Let $\mathcal{C}l(\mathcal{E}_x)$ be the class of operators $A_x : \mathcal{E}_x \to \mathcal{E}_x$ such that, for any local trivialization of $\mathcal{E}$ around $x$

$$\Psi : \mathcal{E}|_{\mathrm{U}_x} \to \mathrm{U}_{\varphi(x)} \times \Gamma(M, E),$$

[1] This modification of the connection can also be done by introducing a "density bundle" [BGV92], here we follow [BF88].

where $\varphi$ is a local chart of $X$ on an open set $U_x$ containing $x$, the operator

$$A^{\Psi}(x) \equiv \Psi(x) A_x \Psi(x)^{-1}$$

lies in $\mathcal{C}l(E)$. The collection $\{\mathcal{C}l(\mathcal{E}_x)\}_{x\in X}$ defines a vector bundle over $X$, modelled on $\mathcal{C}l(E)$, which we denote $\mathcal{C}l(\mathcal{E})$ [P][CDMP]. Moreover, given that the properties (on the operators and their principal symbols) characterizing the notions of order, ellipticity and admissibility are independent of the choice of the local trivialization $\Psi$, it makes sense to talk about the order of $A_x \in \mathcal{C}l(\mathcal{E}_x)$, its ellipticity or admissibility, and these notions can be extended to sections of $\mathcal{C}l(\mathcal{E})$. In this context a *weight* is defined as section $Q$ of $\mathcal{C}l(\mathcal{E})$ which is locally elliptic and admissible, and has a given (constant) positive order.

Given a section $A \in \Gamma(\mathcal{C}l(\mathcal{E}))$ and a weight $Q$, the covariance property (1.16) implies that the expression $\mathrm{tr}^{Q^{\Psi}}(A^{\Psi})$ is independent of the local trivialization $\Psi$. Hence, the definition of weighted trace can be extended to sections of $\mathcal{C}l(\mathcal{E})$ setting

$$\mathrm{tr}^{Q}(A) = \mathrm{tr}^{Q^{\Psi}}(A^{\Psi}).$$

In the same way, the notion of Wodzicki residue carries out to sections of $\mathcal{C}l(\mathcal{E})$, aswell as weighted trace formulae of Section 1.1. In particular, for $A, B \in \Gamma(\mathcal{C}l(\mathcal{E}))$ and a weight $Q$

$$\mathrm{tr}^{Q}([A,B]) = -\frac{1}{q}\mathrm{res}([\log Q, A]B). \tag{4.11}$$

This extends to $\mathcal{C}l(\mathcal{E})$-valued forms on $X$ in a straightforward way using the equality

$$\mathrm{tr}^{Q}(\alpha \otimes A) = \alpha \mathrm{tr}^{Q}(A), \tag{4.12}$$

for $A \in \Gamma(\mathcal{C}l(\mathcal{E}))$ and $\alpha$ a differential form on $X$. Extending (1.14) we have that

$$d\mathrm{tr}^{Q}(\omega) = \mathrm{tr}^{Q}(d\omega) + \frac{(-1)^{k+1}}{q}\mathrm{res}(\omega d \log Q), \tag{4.13}$$

for a $\mathcal{C}l(\mathcal{E})$-valued $k$-form $\omega$ and a weight $Q$ of order $q$ on $\mathcal{E}$.

**Proposition 12** [P][CDMP] *Let $\mathcal{E}$ be the infinite-rank bundle over $X$ whose fibre at $x \in X$ is the space of smooth sections $\mathcal{E}_x = \Gamma(M_x, E_x)$ and $\nabla^{\mathcal{E}}$ a connection on $\mathcal{E}$ which induces a connection on $\Gamma(\mathcal{C}l(\mathcal{E}))$. Then, for any weight $Q$ with (constant) order $q$ and $\omega \in \Gamma\left(\Lambda^k T^*X \otimes \mathcal{C}l(\mathcal{E})\right)$, a $\mathcal{C}l(\mathcal{E})$-valued $k$-form*

$$[\nabla^{\mathcal{E}}, \mathrm{tr}^{Q}](\omega) = \frac{(-1)^{k+1}}{q}\mathrm{res}\left(\omega[\nabla^{\mathcal{E}}, \log Q]\right). \tag{4.14}$$

*Proof.* By definition

$$[\nabla^{\mathcal{E}}, \mathrm{tr}^Q](\omega) = d\,\mathrm{tr}^Q(\omega) - \mathrm{tr}^Q([\nabla^{\mathcal{E}}, \omega]).$$

Let $(U,\Psi)$ be a local trivialization in which $\nabla^{\mathcal{E}} = d + \Theta^{\mathcal{E}}$, where $\Theta^{\mathcal{E}}$ is a $\mathcal{C}l(\mathcal{E})$-valued 1-form on $U$, then

$$[\nabla^{\mathcal{E}}, \omega] = d\omega + [\Theta^{\mathcal{E}}, \omega]$$

so that, given that $[\nabla^{\mathcal{E}}, \omega] \in \Gamma\left(\mathcal{C}l(\mathcal{E})\right)$ by the assumption on $\nabla^{\mathcal{E}}$, $[\Theta^{\mathcal{E}}, \omega] \in \Gamma\left(\mathcal{C}l(\mathcal{E})\right)$ and it follows from (4.11) that

$$\mathrm{tr}^Q\left([\Theta^{\mathcal{E}}, \omega]\right) = -\frac{1}{q}\mathrm{res}\left([\log Q, \Theta^{\mathcal{E}}]\omega\right),$$

which implies that

$$\mathrm{tr}^Q\left([\tilde{\nabla}^{\mathcal{E}}, \omega]\right) = \mathrm{tr}^Q\left(d\omega\right) - \frac{1}{q}\mathrm{res}\left([\log Q, \Theta^{\mathcal{E}}]\omega\right). \tag{4.15}$$

Thus, from (4.13) it follows that

$$\begin{aligned}
[\nabla^{\mathcal{E}}, \mathrm{tr}^Q](\omega) &= d\mathrm{tr}^Q(\omega) - \mathrm{tr}^Q\left(d\omega\right) + \frac{1}{q}\mathrm{res}\left([\log Q, \Theta^{\mathcal{E}}]\omega\right) \\
&= \frac{(-1)^{k+1}}{q}\mathrm{res}\left(\omega d\log Q\right) + \frac{1}{q}\mathrm{res}\left([\log Q, \Theta^{\mathcal{E}}]\omega\right) \\
&= \frac{(-1)^{k+1}}{q}\mathrm{res}\left(\omega d\log Q\right) + \frac{(-1)^{k}}{q}\mathrm{res}\left(\omega[\log Q, \Theta^{\mathcal{E}}]\right) \\
&= \frac{(-1)^{k+1}}{q}\mathrm{res}\left(\omega d\log Q\right) + \frac{(-1)^{k+1}}{q}\mathrm{res}\left(\omega[\Theta^{\mathcal{E}}, \log Q]\right) \\
&= \frac{(-1)^{k+1}}{q}\mathrm{res}\left(\omega[\nabla^{\mathcal{E}}, \log Q]\right).
\end{aligned}$$

□

### 4.2.2 Tracial Anomalies and the Locality of the Curvature of Determinant Line Bundles

Let $I\!\!M \overset{\pi_M}{\to} X$ be a smooth locally trivial fibration, where $X$ is a smooth manifold of finite dimension, such that $M_x = \pi_M^{-1}(x)$ be a closed Riemannian manifold for every $x \in X$. Let $E$ be a smooth $\mathbb{Z}_2$-graded Hermitian vector bundle over $I\!\!M$. Consider a smooth family of differential elliptic operators of order $d$

$$T_x : \Gamma(M_x, E_x^+) \to \Gamma(M_x, E_x^-),$$

where $E_x^\pm \to M_x$ denotes the restriction to $M_x$ of the Hermitian vector bundle $E^\pm \overset{\pi_\pm}{\to} M$. This family extends to a family of Fredholm operators $T : \mathrm{H}^s(E_x^+) \to \mathrm{H}^{s-d}(E_x^-)$, where $\mathrm{H}^s(E_x^\pm)$ denote the Sobolev space of $s$-differentiable $L^2$-sections on $E_x^{\pm}$. Thus, both $\ker T_x$ and $\mathrm{coker} T_x$ are finite-dimensional vector spaces (of smooth sections), so that $(\det \ker T_x)^* \otimes (\det \mathrm{coker} T_x)$ defines a one-dimensional complex vector space which we call the *determinant line* associated to $T_x$, and which we denote by $\det T_x$. If $T_x^*$ denotes the formal adjoint of $T_x$, defined with respect to the inner product induced by the Riemannian structure on $M$ and the Hermitian structure on $E_x$, then

$$\det T_x = (\det \ker T_x)^* \otimes (\det \ker T_x^*)\,. \tag{4.16}$$

Let $\mathcal{E}^{\pm} \to X$ be the smooth Hermitian infinite-rank Fréchet bundle whose fibre above $x \in X$ is the space of sections $\mathcal{E}_x^{\pm} = \Gamma(M_x, E_x^{\pm})$. We summarize this saying that we have an *elliptic positive-order differential bundle map* of order $d$ between smooth Hermitian infinite-rank Fréchet bundles

$$\mathbf{T} : \mathcal{E}^+ \to \mathcal{E}^-,$$

by which we mean a family $\{T_x\}$, parametrized by the manifold $X$, of elliptic positive-order differential operators $T_x : \mathcal{E}_x^+ \to \mathcal{E}_x^-$ of constant order $d$, taking the Fréchet space $\mathcal{E}_x^+$ into the Fréchet space $\mathcal{E}_x^-$. This bundle map gives rise to a corresponding family of complex lines $\{\det T_x\}_{x\in X}$, given by (4.16). We shall build, following [Q86], a complex line bundle over $X$, denoted

$$\mathrm{Det}\mathbf{T} \to X$$

and called the *determinant bundle associated to the family* $\{T_x\}_{x\in X}$. Using the $\zeta$-regularization method introduced by Ray and Singer [RS71] to define determinants of elliptic operators, we shall define a smooth metric on this determinant line bundle, called the Quillen metric, which can be seen as the regularization of the Hermitian structure induced on it by the Hermitian structure on $\mathcal{E}$. Moreover, assuming the existence of a unitary connection on $\mathcal{E}^{\pm}$, we shall define the Bismut-Freed connection on $\mathrm{Det}\mathbf{T}$, a connection which is unitary for the Quillen metric. Finally, following [PR], we shall show that the curvature of this connection is "local", i.e. can be written as the integral of a density on the fibre $X$. In the next section, following [BF88] and [F90], we shall apply these results to describe the line bundle associated to a family of Dirac operators on spin fibrations.

**Quillen's Determinant Line Bundle**

Consider a family $\{T_x\}_{x\in X}$ of fixed positive-order differential operators acting fibrewise from $\mathcal{E}^+$ to $\mathcal{E}^-$ as before, $T_x$ depending smoothly on $x$ in a manifold $X$. We want to patch up the lines $\det T_x = (\det \ker T_x)^* \otimes (\det \ker T_x^*)$ into a line bundle over $X$.

For each $x \in X$, let us consider the formal Laplacians,

$$\Delta_x^+ = T_x^* T_x \quad \text{and} \quad \Delta_x^- = T_x T_x^*. \tag{4.17}$$

The construction of the determinant line bundle associated to the family $\{T_x\}_{x\in X}$ is based on specific properties of the spectrum of these Laplacian operators. Recall from the theory of elliptic operators acting on compact manifolds, that ellipticity and self-adjointness of the "Laplacians" imply that $\Delta_x^+$ and $\Delta_x^-$ have a discrete real spectrum, with the same set $\{\lambda_x\}$ of non-zero eigenvalues, and that the spaces of smooth sections $\mathcal{E}^\pm$ decompose into direct sums of (finte-dimensional) eigenspaces of $\Delta_x^+$ and $\Delta_x^-$, respectively. This eigenspace decomposition gives us a complete orthogonal basis for the metric $\langle\,,\rangle_{\mathcal{E}^\pm}$, in terms of which –when restricted to a finite number of eigenvalues– we can define determinant spaces just like in the finite-dimensional case, i.e. by taking direct products and sums.

Let us make this more precise. Following [Q86] [F90] (see also [BGV92]), we exhibit a trivialization over a collection of open sets covering $X$. For $a > 0$ such that $a \notin \mathrm{Spec}\Delta_x^+$, let $E_x^{+,a}$ and $E_x^{-,a}$ be the spaces defined by

$$E_x^{+,a} = \bigoplus_{\lambda\in\Lambda_a^+} E_x^+(\lambda) \qquad \text{and} \qquad E_x^{-,a} = \bigoplus_{\lambda\in\Lambda_a^-} E_x^-(\lambda),$$

respectively, where $\Lambda_a^\pm = \{\lambda \in \mathrm{Spec}\Delta_x^\pm \,:\, \lambda < a\}$, $E_x^+(\lambda)$ and $E_x^-(\lambda)$ denote the subspaces (of $\mathcal{E}_x^+$ and $\mathcal{E}_x^-$, respectively) spanned by eigenvectors (of $\Delta_x^+$ and $\Delta_x^-$, respectively) with eigenvalues lower than $a$. Since the spectra of the Laplacians $\Delta_x^\pm$ are discrete and bounded below by zero, $E_x^{+,a}$ and $E_x^{-,a}$ are finite-dimensional spaces, and the set

$$\mathcal{U}_a = \{x \in X : a \notin \mathrm{Spec}\Delta_x^+\} \tag{4.18}$$

is open in $X$. Moreover, since $T_x$ varies smoothly with $x$, the number of eigenvalues of $\Delta_x^+$ less than $a$ is constant in $\mathcal{U}_a$, so that $E_x^{+,a}$ and $E_x^{-,a}$ define vector bundles over $\mathcal{U}_a$, from which it follows that $L_x^a = (\det E_x^{+,a})^* \otimes (\det E_x^{-,a})$ defines a complex line bundle $\mathcal{L}_a$ over $\mathcal{U}_a$.

On the other hand, for each $x \in X$, the sequence

$$0 \to \ker \Delta_x^+ \to E_x^{+,a} \xrightarrow{T_x} E_x^{-,a} \to \ker \Delta_x^- \to 0, \tag{4.19}$$

is exact, and hence, given that $\ker \Delta_x^+ = \ker T_x$ and $\ker \Delta_x^- = \ker T_x^*$, for each $x \in X$, there is a canonical isomorphism

$$L_x^a \cong (\det \ker T_x)^* \otimes (\det \ker T_x^*). \tag{4.20}$$

**Theorem 8** [Q86] *The line bundles $\mathcal{L}_a$ over $\mathcal{U}_a$ defined above patch up to a line bundle over $X$, the* determinant line bundle *associated to the familly $\{T_x\}_{x\in X}$, denoted by* $\mathrm{Det}\mathbf{T} \to X$.

*Outline of the proof.* Consider the bundles $\mathcal{L}_a$ and $\mathcal{L}_b$, defined on a non-empty intersection $\mathcal{U}_a \cap \mathcal{U}_b$, for $b > a$. We want to see how they patch up on $\mathcal{U}_a \cap \mathcal{U}_b$. Let $E_x^{+,(a,b)}$ and $E_x^{-,(a,b)}$ be the vector spaces defined similarly as $E_x^{+,a}$ and $E_x^{-,a}$, but taking into account only eigenvalues $\lambda$ between $a$ and $b$, then

$$E_x^{+,b} = E_x^{+,a} \oplus E_x^{+,(a,b)} \quad \text{and} \quad E_x^{-,b} = E_x^{-,a} \oplus E_x^{-,(a,b)}. \tag{4.21}$$

From this, and the fact that

$$T_x^{(a,b)} := T_x\big|_{E_x^{+,(a,b)}} : E_x^{+,(a,b)} \to E_x^{-,(a,b)} \tag{4.22}$$

is an isomorphism, it follows that

$$L_x^b \cong L_x^a \otimes L_x^{(a,b)}, \tag{4.23}$$

where $L_x^{(a,b)} = (\det E_x^{+,(a,b)})^* \otimes (\det E_x^{-,(a,b)})$. The morphism $T_x^{(a,b)}$ induces an isomorphism

$$\det T_x^{(a,b)} : \det E_x^{+,(a,b)} \to \det E_x^{-,(a,b)}, \tag{4.24}$$

which defines a non-zero *local section* of the line bundle $\mathcal{L}_{(a,b)}$ on $\mathcal{U}_a \cap \mathcal{U}_b$, with fibre $L_x^{(a,b)}$. Thus, $\mathcal{L}_a$ and $\mathcal{L}_b$ patch up via the isomorphism

$$\begin{array}{ccc} L_x^a & \to & L_x^a \otimes L_x^{(a,b)} = L_x^b \\ \psi_a & \mapsto & \psi_a \otimes \det T_x^{(a,b)}. \end{array} \tag{4.25}$$

When $\operatorname{ind} T_x = 0$, given that $\dim E_x^{+,a} = \dim E_x^{-,a}$, the line $\mathcal{L}_a$ has a canonical section

$$\det T^a : \det E_x^{+,a} \to \det E_x^{-,a}, \tag{4.26}$$

and, for $b > a$, the multiplicativity of *finite-dimensional* determinants shows that $\det T^a$ and $\det T^b$ corresponds under the isomorphism (4.25).

Notice that we obtain a canonical global section $\det \mathbf{T}$ of $\mathrm{Det}\mathbf{T}$ picking out the canonical element of the line $L_x$ (which is mapped to 1 by the canonical isomorphism $L_x \cong \mathbb{C}$) when $\operatorname{ind} T_x = 0$ and $T_x$ is invertible, and taking it to be identically zero on components of $X$ for which $\operatorname{ind} T_x \neq 0$ or $T_x$ is not invertible. On components where the dimensions of kernels and cokernels of the family -and hence the index- are constant, (non canonical) sections can also be defined [F88].

Finally, let us point out that the determinant line bundle associated to the elliptic positive-order differential bundle map $\mathbf{T} : \mathcal{E}^+ \to \mathcal{E}^-$ which we denoted by $\mathrm{Det}\mathbf{T}$ (and not $\det \mathcal{E}$, as in the finite-dimensional case), seen as a generalization of the finite-dimensional construction of a determinant line bundle, should be seen here as the determinant line bundle associated to the bundle

$\mathcal{E}^+ \oplus \mathcal{E}^-$ equipped with an additional family of elliptic differential operators $\mathbf{T}$, because it is completely determined by the family of elliptic differential operators. A better notation could be $\mathrm{Det}(\mathcal{E}, \mathbf{T})$, making explicit that the patching has been done with respect to the reference operator $\Delta^+ = T^*T$ (that plays the same role as the weight in the construction of the weighted traces of Chapter 1), and will be the reference operator for the definition of the smooth metric and connection on this bundle, as we shall see in the next section.

**Quillen's Metric on the Determinant Line Bundle**

Following [Q86], let us now use $\zeta$-regularized determinants to construct a *smooth* metric on the determinant line bundle $\mathrm{Det}\mathbf{T}$, associated to the family of elliptic first order differential operators $\{T_x\}_{x\in X}$. There is a natural metric on $\mathrm{Det}\mathbf{T}$, namely the metric $|\cdot|_{L^2}$ induced on it by the hermitian structures on $\mathcal{E}^{\pm}$, but it does not agree on the intersection of two open subsets $\mathcal{U}_a$ and $\mathcal{U}_b$. In fact, consider a section $\sigma$ on the overlap $\mathcal{U}_a \cap \mathcal{U}_b$, with $b > a$. If $\sigma$ on $\mathcal{U}_a$ takes the form

$$\sigma_a = f_a(v_1 \wedge \ldots \wedge v_m) \otimes (w_1 \wedge \ldots \wedge w_n)^{-1}, \tag{4.27}$$

$v_1, \ldots, v_m$ and $w_1, \ldots, w_n$ being basis for $E_x^{+,a}$ and $E_x^{-,a}$, respectively, and $f_a$ a complex function on $\mathcal{U}_a$, the canonical identification of sections given by isomorphism (4.25) yields

$$\sigma_b = \sigma_a \otimes (T^{(a,b)}x_1 \wedge \ldots \wedge T^{(a,b)}x_k) \otimes (x_1 \wedge \ldots \wedge x_k)^{-1}, \tag{4.28}$$

where $x_1, \ldots, x_k$ denotes a basis for $E_x^{+,(a,b)}$. From this it follows that

$$|\sigma_b|^2_{L^2} = |\sigma_a|^2_{L^2} \prod_{a<\lambda_i<b} \lambda_i,$$

where $|\cdot|_{L^2}$ denote the induced metric induced on $L_x^a$ and $L_x^b$ by the hermitian structure on $E$ and $F$, and $\lambda_i$ are the eigenvalues of $\Delta_x^+|_{(a,b)}$. Hence

$$\frac{|\sigma_b|^2_{L^2}}{|\sigma_a|^2_{L^2}} = \det \Delta_x^+|_{(a,b)}, \tag{4.29}$$

so that the metrics $|\cdot|_{L^2}$ do not agree in general over $\mathcal{U}_a$ and $\mathcal{U}_b$.

**Theorem 9** [Q86] *The metric defined on $L_x^a$ by*

$$|\cdot|_Q = (\det_\zeta \Delta_x^+|_{(a,\infty)})^{\frac{1}{2}} |\cdot|_{L^2}, \tag{4.30}$$

*agrees with the corresponding metric on $L_x^b$. All these metrics patch into a smooth metric on* $\mathrm{Det}\mathbf{T}$*, called the* Quillen metric*.*

This result follows from (4.29), using the fact that $\zeta$-determinants coincide with ordinary determinants when restricted to finite rank operators, and hence with the ordinary product of eigenvalues. Notice that, as in the construction of the determinant line bundle itself, the metric is given in terms of (the $\zeta$-determinant of) the Laplacian $\Delta^+$. As a matter of fact, for the canonical section $\det \mathbf{T}$ defined by (4.26), it follows that

$$\| \det \mathbf{T}(x) \|_Q^2 = \det_\zeta \Delta_x^+. \tag{4.31}$$

**The Bismut-Freed Connection**

In this section, following [BF88] (where it was done in the case of a family of Dirac operators, see next section), from the unitary connection $\nabla^{\mathcal{E}}$ on $\mathcal{E}$, we build a connection $\nabla^{^{BF}}$ on Det$\mathbf{T}$. Recall that, for each $x \in X$, the maps

$$T_x^a : E_x^{+,a} \to E_x^{-,a} \qquad \text{and} \qquad T_x^{(a,b)} : E_x^{+,(a,b)} \to E_x^{-,(a,b)},$$

are isomorphisms. Let $\nabla^{a\pm}$ be the projections of the connections $\nabla^{\mathcal{E}^\pm}$ on the bundles $E_x^{\pm,a} \to \mathcal{U}_a$, which are unitary for the restricted metrics, giving rise to connections $\nabla^a$ on $\mathcal{L}^a$, unitary for the $L^2$ metric on $\mathcal{U}_a$. For $a < b$, consider the overlap $\mathcal{U}_a \cap \mathcal{U}_b$. We have two connections on $\mathcal{L}^b$, $\nabla^a$ and $\nabla^b$, such that, on an open set containing $x \in X$,

$$\nabla_x^b = \nabla_x^a + \operatorname{tr}(T_x^{-1} \nabla^{\mathrm{Hom}_a^\pm} T_x|_{a<\lambda<b}), \tag{4.32}$$

where $\nabla^{\mathrm{Hom}_a^\pm}$ is the connection on $\mathrm{Hom}(E_x^{+,a}, E_x^{-,a})$ induced by $\nabla^{a\pm}$. (Compare with (4.5).)

**Proposition 13** [BF88] *Let us set*

$$\bar{\nabla}_x^a = \nabla_x^a + \operatorname{tr}^{\Delta_x^+}(T_x^{-1} \nabla^{\mathrm{Hom}_a^\pm} T_x|_{\lambda>a}), \tag{4.33}$$

*where the second term at the right is the $\Delta_x^+$-weighted trace of $T_x^{-1}(\nabla^{\mathrm{Hom}_a^\pm} T_x)$ restricted to $\lambda > a$, i.e.*

$$\operatorname{tr}^{\Delta_x^+}(T_x^{-1}(\nabla^{\mathrm{Hom}_a^\pm} T_x)|_{\lambda>a}) = \left.\frac{d}{dz}\right|_{z=0} \left\{ z \operatorname{tr}\left( (\Delta_x^+)^{-z} T_x^{-1}(\nabla^{\mathrm{Hom}_a^\pm} T_x)|_{\lambda>a} \right) \right\}. \tag{4.34}$$

*Then $\bar{\nabla}^a$ and $\bar{\nabla}^b$ agree on the overlap $\mathcal{U}_a \cap \mathcal{U}_b$, for $a < b$, and patch together to a connection $\nabla^{^{BF}}$ on $\mathcal{L}$, called the* Bismut-Freed connection.

This fact is a consequence of the independence on $a$ of the term under differentiation in (4.34), and of the fact that $\operatorname{tr}^{\Delta_x^+}$ coincides with the usual trace on finite rank operators, and hence with a finite sum of eigenvalues, i.e. for $a < b$

$$\operatorname{tr}^{\Delta_x^+}(T_x^{-1} \nabla^{\mathrm{Hom}^\pm} T_x|_{\lambda>a}) = \operatorname{tr}^{\Delta_x^+}(T_x^{-1} \nabla^{\mathrm{Hom}^\pm} T_x|_{\lambda>b}) + \operatorname{tr}^{\Delta_x^+}(T_x^{-1} \nabla^{\mathrm{Hom}^\pm} T_x|_{a<\lambda<b}). \tag{4.35}$$

Notice that, as for the construction of the determinant line dundle and its Quillen metric, the Bismut-Freed connection is defined in terms of $\Delta_x^+$, namely through a $\Delta_x^+$-weighted trace.

**Remark.** Let $\det \mathbf{T}$ be the canonical section of $\mathrm{Det}\mathbf{T}$ defined before (see equation (4.31)), then whenever $T_x$ is invertible we have

$$\nabla^{BF} \det \mathbf{T}(x) = (\det T_x)\mathrm{tr}^{\Delta_x^+}\left(T_x^{-1}\nabla^{\mathrm{Hom}^\pm}T_x\right). \tag{4.36}$$

If it had not been choosen using $\Delta_x^+$ as the weight in the definition of the Bismut-Freed connection, it would not be compatible with the Quillen metric.

**Locality of the Curvature of the Bismut-Freed Connection**

Let us now consider the curvature of the Bismut-Freed connection on the determinant line bundle. Recall that the Bismut connection on the bundles $\mathcal{E}^\pm \to X$ was defined by pointwise action. This means that given a section $\psi$ of $\mathcal{E}^\pm$, i.e. a map associating to each $x \in X$ a section $\psi_x$ of the bundle $E_x \to M_x$, and a vector field $\xi$ in $TX$ with horizonatl lift $\xi_M$ in $TM$ (the vector $\xi_x \in T_xX$ lifts to a vector field $\xi_M^x$ along the fibre $M_x$), the Bismut connection associates the section given by

$$\left(\nabla_\xi^{\mathcal{E}^\pm}\psi(x)\right)(m) = \tilde{\nabla}^\pm_{\xi_M^x(m)}\psi_x(m),$$

where $\tilde{\nabla}^\pm$ is the connection on $E_x$ given by (4.9) and $m \in \pi^{-1}(x) = M_x$.

The "point-wise" definition of $\nabla^{\mathcal{E}^\pm}$ implies that its curvature $\Omega_x^{\mathcal{E}^\pm} \in \Lambda^2(S^\pm \otimes W|_{M_x}, S^\pm \otimes W|_{M_x})$ is an endomorphism on the fibres, i.e. that for any section $\psi$ and vector fields $\xi, \eta$

$$\left(\Omega^{\mathcal{E}^\pm}(\xi,\eta)\psi(x)\right)(m) = \tilde{\Omega}^\pm(\xi_M^x(m), \eta_M^x(m))\psi_x(m),$$

where $\tilde{\Omega}^\pm$ denotes the curvature of the connection $\tilde{\nabla}^\pm$ and $m \in \pi^{-1}(x) = M_x$.

**Theorem 10** [PR] *The curvature of the Bismut-Freed connection is local, i.e. it can be written as the integral of a density on the fibre $M/X$.*

*Proof.* Let $\alpha$ be a $\mathcal{C}l(\mathcal{E})$-valued 1-form and $Q$ a weight of order $q$ on $\mathcal{E}$. Then, from Proposition 12 it follows that

$$[\nabla^{\mathcal{E}}, \mathrm{tr}^Q](\alpha) = \frac{1}{q}\mathrm{res}\left(\alpha[\nabla^{\mathcal{E}}, \log Q]\right)$$

so that

$$d\mathrm{tr}^Q(\alpha) = \mathrm{tr}^Q\left([\nabla^{\mathcal{E}}, \alpha]\right) + \frac{1}{q}\mathrm{res}\left(\alpha[\nabla^{\mathcal{E}}, \log Q]\right). \tag{4.37}$$

Applying this equality to the $\mathcal{C}l(\mathcal{E})$-valued 1-form $\alpha = {D^+}^{-1}\nabla^{\mathrm{Hom}^\pm} D^+$ and taking as weight $\Delta^+$ (used in the definition of the Bismut-Freed connection), gives

$$d\,\mathrm{tr}^{\Delta^+}({D^+}^{-1}\nabla^{\mathrm{Hom}^\pm} D^+) \quad = \quad \mathrm{tr}^{\Delta^+}\left([\nabla^{\mathcal{E}^\pm}, {D^+}^{-1}\nabla^{\mathrm{Hom}^\pm} D^+]\right) - \frac{1}{2}\mathrm{res}\,(\Xi)\,,$$

where $\Xi = {D^+}^{-1}\tilde{\nabla}^\pm D^+[\tilde{\nabla}^\pm, \log\Delta^+]$. The first term on the right hand side breaks into two terms as follows

$$\begin{aligned}
\mathrm{tr}^{\Delta^+}\left([\nabla^{\mathcal{E}^\pm}, {D^+}^{-1}\nabla^{\mathrm{Hom}^\pm} D^+]\right) &= \mathrm{tr}^{\Delta^+}({D^+}^{-1}\Omega^{\mathrm{Hom}^\pm} D^+) \\
&- \mathrm{tr}^{\Delta^+}\left({D^+}^{-1}\nabla^{\mathrm{Hom}^\pm} D^+ {D^+}^{-1}\nabla^{\mathrm{Hom}^\pm} D^+\right) \\
&= \mathrm{tr}^{\Delta^-}(\Omega^{\mathrm{Hom}^\mp}) \\
&- \mathrm{tr}^{\Delta^+}\left({D^+}^{-1}\nabla^{\mathrm{Hom}^\pm} D^+ {D^+}^{-1}\nabla^{\mathrm{Hom}^\pm} D^+\right)
\end{aligned}$$

where $\Omega^{\mathrm{Hom}^\mp}$ is the curvature of $\nabla^{\mathrm{Hom}^\mp}$. Since $\Omega^{\mathrm{Hom}^\pm}$ is a multiplication operator, $\mathrm{tr}^{\Delta^-}(\Omega^{\mathrm{Hom}^\mp})$ can be written as an integral of a density on $M/X$, so that to finish the proof it is enough to show that the second term at the right is a Wodzicki residue. Indeed, the evaluation of two tangent vectors on the two form $\mathrm{tr}^{\Delta^+}\left({D^+}^{-1}\nabla^{\mathrm{Hom}^\pm} D^+ {D^+}^{-1}\nabla^{\mathrm{Hom}^\pm} D^+\right)$ gives pointwise the *weighted trace* of a commutator, so this term is a weighted trace anomaly, and follows from our results about tracial anomalies that it is given by a local term. This shows that the curvature of the Bismut-Freed connection on the determinant line bundle Det$\mathbf{T}$ is local.
□

### 4.2.3 The Determinant Line Bundle Associated to a Family of Dirac Operators

In this section we apply the previous construction to a family of Dirac operators on sections of vector bundles over a fibration of spin manifolds. Our main task here is to build a connection on the infinite-rank bundle $\mathcal{E}$.

#### A Family of Dirac Operators

In this section we build a smooth family $\{E_x\}_{x\in X}$ of vector bundles over a particular fibration of spin manifolds $\pi_M : I\!\!M \to X$, considered by Bismut and Freed in [BF88], and we associate to this family a family of first order elliptic differential operators to which we shall apply Quillen's construction of a determinant bundle. The main assumptions we shall make are related with the nature of the *fibre*, a compact even-dimensional manifold throughout denoted $M/X$. As before, let us assume that on the bundle $M/X$ there is a horizontal distribution, i.e. a splitting $T\,\mathbb{M} = T_H\,\mathbb{M} \oplus T(M/X)$ so that the subbundle $T_H\,\mathbb{M} \cong \pi_M^* TX$. Let $T(M/X)$ using the projection $\Pi : T\,\mathbb{M} \to T(M/X)$ be the canonical projection with kernel the chosen horizontal tangent subbundle $T_H\,\mathbb{M}$, from which we lift tangent vectors $\xi(x) \in TX$ to horizontal vector fields $\xi_M^x \in T\,\mathbb{M}$ along the fibres.

From the Riemannian metric $\mathrm{g}_{M/X}$ on the fibre $M/X$ we define a Riemannian metric $\mathrm{g}_M$ on $T\,\mathbb{M}$ by pulling up to $T_H\,\mathbb{M}$ the metric $\mathrm{g}_X$ on $X$, by means of the identification $T_H\,\mathbb{M} \cong \pi_M^* TX$, and letting $\mathrm{g}_M = \mathrm{g}_X \oplus \mathrm{g}_{M/X}$. From the Levi-Civita connection $\nabla^{\mathrm{g}_M}$ associated to this metric, we define a connection $\nabla^{M/X}$ on $T(M/X)$ by $\nabla^{M/X} = \Pi\,\nabla^{\mathrm{g}_M}\,\Pi$. This connection is independent of the choice of $\mathrm{g}_X$ on $TX$ [BF88] [BGV92]. Finally, let us assume that there exists a (fixed) Spin$^c$ structure on the tangent space along the fibres and a Hermitian vector bundle $W$ over $\mathbb{M}$ with compatible connection $\nabla^W$, this means that $\nabla^W$ restricts to a connection on $W_x = W|_{M_x}$, compatible with the Hermitian structure on $W_x$. The metric $\mathrm{g}_{M/X}$ and the spin structure along the fibres determine a Spin$(n)$-bundle of frames of the vertical tangent space over $\mathbb{M}$, we denote it by $S \to T(M/X)$.

Let us now build from this data a family of Hermitian vector bundles and a family of constant order differential operators coupled to $W$. Let $E_x^{\pm} = S^{\pm} \otimes W|_{M_x}$, where $S^{\pm} \to M$ is the spin fibration associated to the construction above, see example 3, and consider the smooth family $\{D_x\}_{x\in X}$ of Dirac operators on $E_x = S \otimes W|_{M_x}$,

$$D_x : \Gamma(S \otimes W|_{M_x}) \to \Gamma(S \otimes W|_{M_x}),$$

i.e. the Dirac operators on $M_x$ coupled to the bundles $W|_{M_x}$ via their corresponding connections. This yields a family of first-order elliptic differential operators, odd with respect to the $\mathbb{Z}_2$-grading $(\pm)$ of $S$,

$$D_x = \begin{pmatrix} 0 & D_x^- \\ D_x^+ & 0 \end{pmatrix}$$

where ${D_x^-}^* = D_x^+$, and a family of generalized Laplacians $\{\Delta_x = D_x^2\}_{x\in X}$.

Let $\mathcal{E}_x^{\pm} = \Gamma(E_x^{\pm})$ be the space of smooth sections of $E_x^{\pm} = S^{\pm} \otimes W|_{M_x} \to M_x$, and consider the infinite-rank vector bundle $\mathcal{E}^{\pm}$ over $X$ whose fibre above $x$ is $\mathcal{E}_x^{\pm}$. From the metrics on $S^{\pm}$ and $W$, together with the volume forms on $M_x$, we define a $L^2$-metric on $\mathcal{E}^{\pm}$ by (4.8). Taking the $L^2$-completions of the spaces of smooth sections, and extending the family $\{D_x^+\}_{x\in X}$ to a family of Fredholm operators, as before, we shall see it as a elliptic first order differential bundle map

$$\mathbb{D}^+ : \mathcal{E}^+ \to \mathcal{E}^-,$$

where $\mathcal{E}^+$ and $\mathcal{E}^-$ are the corresponding Hilbert bundles of sections.

As in (4.9), in order to define a unitary connection on the infinite-dimensional bundles $\mathcal{E}^\pm$ (which will be used in order to build a unitary connection on the determinant line bundle), we modify the connection $\nabla^{M/X}$ on $T(M/X)$, which is not unitary for the $L^2$-inner product defined earlier. Then, taking

$$\tilde{\nabla}^{M/X} = \nabla^{M/X} + \frac{1}{2}\mathrm{div}_{M_x}(m), \tag{4.38}$$

where $\mathrm{div}_{M_x}(m)$ is the divergence of the volume form at $m$ in the base directions, gives a unitary connection $T(M/X)$. Correspondingly, the connections $\tilde{\nabla}^\pm$ on the bundles $S^\pm \otimes W|_{M_x}$ are induced by $\tilde{\nabla}^{M/X}$ and $\nabla^W$, which give rise to the Bismut connection $\nabla_\xi^{\mathcal{E}^\pm}\psi = \tilde{\nabla}^\pm_{\xi_M}\psi$ on the bundles $\mathcal{E}^\pm$. The choice of $\tilde{\nabla}^{M/X}$ made in (4.38) makes this connection unitary for the inner product (4.8).

In the particular case of a family of Dirac operators associated to the fibration considered in this section, the local form of $\Omega^{^{BF}}$ (see Theorem 10) is given by the following theorem, due to Bismut and Freed [BF88]

**Theorem 11** *The curvature of the determinant line bundle is the 2-form component of*

$$\Omega^{^{BF}} = 2\pi i \int_{M/X} \hat{A}(\nabla^{M/X})\,\mathrm{Ch}(\nabla^W), \tag{4.39}$$

*where*

$$\hat{A}(\nabla^{M/X}) = \sqrt{\det\left(\frac{\Omega^{M/X}/4\pi}{\sinh \Omega^{M/X}/4\pi}\right)}$$

*is the $\hat{A}$ genus of the spinor bundle on $M/X$, $\mathrm{Ch}(\nabla^W)$ the Chern form of $W$ and $\nabla^{M/X}$ is the Levi-Civita connection on $M/X$.*

# Part III

# Elliptic Complexes, Gauge Anomalies and Duality

# Chapter 5

# Phase Anomalies in Chern-Simons Models

This part of the work uses the results stated in Section 3.1 to relate phase anomalies in odd dimensions —coming from logarithmic variations of $\zeta$- determinants of Dirac operators— to weighted trace anomalies, thus giving an apriori explanation for the locality we expect from these anomalies. We discuss in detail the Chern-Simons model and we apply the results of Section 3.2 to the study of integrated phase anomalies in this model. Finally we use the Atiyah-Patodi-Singer index theorem to recover the Chern-Simons term as local term corresponding to the associated tracial anomaly.

## 5.1 The Chern-Simons Model and Analytic Torsion

Let $N$ be a three dimensional oriented manifold and $P$ a principal $G$-bundle over $N$ which is assumed trivial. The non-abelian Chern-Simons model is defined by the action functional

$$\mathcal{S}^{CS} = \frac{k}{4\pi}\int_N \mathrm{tr}\left(A\wedge dA + \frac{2}{3}A\wedge A\wedge A\right),$$

where $k$ is a constant (inverse of the Planck constant) and the fields $A$ which are Lie$(G)$-valued one-forms on $N$, elements of the space $\mathcal{A}$ of connections of the principal $G$-bundle $P$ over $N$. The study of the corresponding partition function

$$Z^{CS}\text{``}=\text{''}\int_{\mathcal{A}}[\mathcal{D}A]\exp\left\{\frac{ik}{4\pi}\int_N \mathrm{tr}\left(A\wedge dA + \frac{2}{3}A\wedge A\wedge A\right)\right\}$$

has been carried out by Witten in [W89], who showed by stationary phase method (see Appendix B) that this partition function gives rise, in the "weak coupling limit", to (a sum of) path integrals of the form

$$Z_o = \int_{\mathcal{A}} \exp\left\{\int_N \mathrm{tr}\,(\omega \wedge d\omega)\right\}.$$

Hence the so-called abelian Chern-Simons theory in three dimensions is the semiclassical limit of the non-abelian theory. In the following we shall consider abelian Chern-Simons theories over closed odd-dimensional Riemannian manifolds (see [AS95][S79][W89]).

Let us come back to the context of Section 3.2, i.e. the acyclic de Rham complex (1.53)

$$0 \longrightarrow \Omega^0 \xrightarrow{d_0} \cdots \Omega^{k-1} \xrightarrow{d_{k-1}} \Omega^k \xrightarrow{d_k} \Omega^{k+1} \xrightarrow{d_{k+1}} \cdots \Omega^n \longrightarrow 0,$$

where $\Omega^k = C^\infty(\Lambda^k T^*M \otimes V_\rho)$ and the de Rham operator $D_\nabla = \oplus_{k=0}^n (d_k + d_k^*)$, seen as a Dirac operator, $\nabla$ being the flat connection on $W$. As before we set $\Delta_\nabla = D_\nabla^2$ and $\Delta_k = \Delta_\nabla|_{\Omega^k}$. Recall that acycliclicity implies Hodge decomposition (1.54)

$$\Omega^k = \Omega_k' \oplus \Omega_k'',$$

where $\Omega_k' = \mathrm{Im} d_{k-1} = \ker d_k$ and $\Omega_k'' = \ker d_{k-1}^* = \mathrm{Im} d_{k-1}^*$. Restricting the operator $\Delta_k = d_k^* d_k + d_{k-1} d_{k-1}^*$ to $\Omega_k'$ and $\Omega_k''$, we get invertible operators $\Delta_k' = d_{k-1} d_{k-1}^*|_{\Omega_k'}$ and $\Delta_k'' = d_k^* d_k|_{\Omega_k''}$, and the $\zeta$-function techniques can be extended to define $\det_\zeta(\Delta_k')$ and $\det_\zeta(\Delta_k'')$ (see equations (1.68) to (1.70)). Thus,

$$\det{}_\zeta(\Delta_k'') = \exp\left[\sum_{i=0}^{k} (-1)^{k-i} \log \det{}_\zeta(\Delta_i)\right],$$

with the convention that $d_{-1}^* = d_n = 0$.

The Chern-Simons model in dimension $n = 2k+1$ is modelled, in the "weak coupling limit", in terms of a metric invariant action functional of the type

$$S_k^{CS}(\omega_k) = \langle \omega_k, *d_k \omega_k \rangle = \int_M \omega_k \wedge d_k \omega_k,$$

where $M$ is a $(2k+1)$-dimensional manifold. This action presents a degeneracy on $\Omega_k'$ for, writing $\omega_k = \omega_k' \oplus \omega_k''$ in the above mentioned decomposition, we have $S_k^{CS}(\omega_k) = S_k^{CS}(\omega_k'')$. To deal with this type of degeneracy, we use Schwarz's Ansatz (see equation 2.14) for the corresponding partition function, which yields

$$Z_k^{CS}(*d_k) = \left[\prod_{l=0}^{k-1} \left(\det{}_\zeta(\Delta_l'')\right)^{(-1)^{k-l+1}}\right]^{\frac{1}{2}} \det{}_\zeta\left(*d_k''\right)^{-\frac{1}{2}} \tag{5.1}$$

It is motivated by the formal computation

$$\begin{aligned} Z_k^{CS}(*d_k) &\text{ “ = ” } \int_{\Omega^k} [\mathcal{D}\omega_k] e^{-\langle \omega_k, *d_k \omega_k \rangle} \\ &\text{ “ = ” } \left[ \prod_{l=0}^{k-1} \left( \det{}_\zeta(\Delta_l'') \right)^{(-1)^{k-l+1}} \right]^{\frac{1}{2}} \int_{\Omega_k''} [\mathcal{D}\omega_k''] e^{-\langle \omega_k'', *d_k \omega_k'' \rangle}, \end{aligned}$$

where we have inserted inverted commas around identities involving heuristic objects such as $[\mathcal{D}\omega_k]$, which are to be understood on a heuristic level. However, the right hand side of equation (5.1) is well defined since in $n = 2k+1$ dimensions the operator $*d_k''$ is invertible, self-adjoint and hence has a well-defined determinant. Thus, from the fact that

$$|\det{}_\zeta(*d_l'')| = \det{}_\zeta(| * d_l''|) = \sqrt{\det{}_\zeta(\Delta_l'')}$$

it follows that

$$|Z_k^{CS}(*d_k)| = \left[ \prod_{l=0}^{k-1} \left( \det{}_\zeta \Delta_l'' \right)^{\frac{(-1)^{k-l+1}}{2}} \right] (\det{}_\zeta \Delta_k'')^{\frac{-1}{4}}.$$

Using Hodge duality we have

$$\begin{aligned} |Z_k^{CS}(*d_k)| &= \left[ \prod_{l=0}^{k-1} \left( \det{}_\zeta \Delta_l'' \right)^{\frac{(-1)^{k-l+1}}{4}} \right] (\det{}_\zeta \Delta_k'')^{-\frac{1}{4}} \left[ \prod_{j=0}^{k-1} \left( \det{}_\zeta \Delta_{n-j-1}'' \right)^{\frac{(-1)^{k-j+1}}{4}} \right] \\ &= \left[ \prod_{j=0}^{n-1} \left( \det{}_\zeta \Delta_j'' \right)^{\frac{(-1)^j}{2}} \right]^{\frac{(-1)^{k+1}}{2}} \\ &= T(M)^{\frac{(-1)^{k+1}}{2}}, \end{aligned}$$

where $T(M)$ is the analytic torsion of the manifold $M$. Hence the modulus of the partition function $Z_k^{CS}(*d_k)$, associated to the metric invariant action $S_k^{CS}(\omega_k)$, is metric invariant. From Proposition 3 it follows that

$$\det{}_\zeta(*d_k'') = \sqrt{\det{}_\zeta \Delta_k''} e^{i\frac{\pi}{2}\left\{ \eta_{*d_k''}(0) - \zeta_{|*d_k''|}(0) \right\}}.$$

Using the fact that

$$T(M) = \prod_{l=0}^{k} \left( \det{}_\zeta(\Delta_l'') \right)^{\frac{(-1)^{k-l+1}}{2}},$$

and $\zeta_{|*d_k''|}(0) = 0$ if the dimension of $M$ is $n = 2k+1$, we find

$$Z_k^{CS}(*d_k) = (T(M))^{\frac{(-1)^{k+1}}{2}} e^{-i\frac{\pi}{4}\eta_{*d_k''}(0)}. \tag{5.2}$$

Notice that, for $k = 1$, this yields back the fact that $|Z^{CS}(*d_1)| = \sqrt{T(M)}$, as shown in [W89].

**Proposition 14** *Let $Z_k^{CS}(*d_k)$ and $Z_k^{CS}(d_k^* *)$ denote the partition functions associated to the action functionals $S_k^{CS}(\omega_k) = \langle \omega_k, *d_k\omega_k \rangle$ and $S_k^{CS*}(\omega_k) = \langle \omega_k, d_k^* * \omega_k \rangle$, respectively. Then, for $n = 2k+1$, with $k$ odd,*

$$Z_k^{CS}(*d_k) Z_k^{CS}(d_k^* *)^{-1} = T(M)^{\frac{(-1)^{k+1}}{2}}. \tag{5.3}$$

*Proof.* Both $S_k^{CS}$ and $S_k^{CS*}$ are degenerate action functionals on $\Omega^k$, so that we apply Schwarz's Ansatz's (2.14) in order to define the corresponding partition functions $Z_k^{CS}(*d_k)$ and $Z_k^{CS}(d_k^* *)$. The elliptic resolvent associated to $S_k^{CS}$ is

$$0 \longrightarrow \Omega^0 \xrightarrow{d_0} \cdots \xrightarrow{d_{k-2}} \Omega^{k-1} \xrightarrow{d_{k-1}} \Omega_k' \xrightarrow{*d_k} 0, \tag{5.4}$$

and hence we define the partition function $Z_k^{CS}(*d_k)$ as

$$Z_k^{CS}(*d_k) = \det_\zeta(*d_k)^{\frac{-1}{2}} \prod_{j=0}^{k} (\det_\zeta \Delta''_{k-j})^{\frac{(-1)^{j+1}}{2}}. \tag{5.5}$$

In the same way, the resolvent associated to $S_k^{CS*}$ is

$$0 \longrightarrow \Omega^n \xrightarrow{d_{n-1}^*} \cdots \xrightarrow{d_{k+1}^*} \Omega^{k+1} \xrightarrow{d_k^*} \Omega_k'' \xrightarrow{d_k^* *} 0, \tag{5.6}$$

so

$$Z_k^{CS}(d_k^* *) = \det_\zeta(d_k^* *)^{\frac{-1}{2}} \prod_{j=1}^{k+1} (\det_\zeta \Delta'_{k+j})^{\frac{(-1)^{j+1}}{2}}. \tag{5.7}$$

Since for $k$ odd $d_k * = *d_k^*$ from (5.2) it follows that

$$\begin{aligned} Z_k^{CS}(*d_k) \cdot Z_k^{CS}(d_k^* *)^{-1} &= \left[ \prod_{j=0}^{k} \left(\det_\zeta \Delta''_{k-j}\right)^{\frac{(-1)^{j+k}}{2}} \prod_{j=1}^{k+1} \left(\det_\zeta \Delta'_{k+j}\right)^{\frac{(-1)^{j+k+1}}{2}} \right]^{(-1)^{k+1}} \\ &= T(M)^{\frac{(-1)^{k+1}}{2}}, \end{aligned}$$

where we used Proposition 7 and the equality $\det_\zeta \Delta'_k = \det_\zeta \Delta''_{k-1}$. $\square$

## 5.2 Tracial Anomalies, Phase Anomalies and the Chern-Simons Term

Recall that the (classical) action functional used to model the abelian Chern-Simons theory in odd dimensions is metric invariant, but its associated (quantum) partition function is not, since it contains a phase which depends on the metric on $M$ (see equation 5.2). We will refer to this as a *phase anomaly* in the partition function, since it arises as an anomaly in the quantum level in the sense of section 2.3.

Given a smooth family of connections $\{\nabla_t^\rho, t \in [0,1]\}$ on the exterior bundle $V_\rho$, let $d_t$ be the family of exterior differential operators built from these connections, and let $d_{k,t}$ denote their restriction to $k$-forms. This gives rise to a family $\{Z^{CS}(*d_{k,t})\}_{t\in[0,1]}$ of partition functions of the form (5.2). Another family of partition functions can be built taking a fixed connection on $V_\rho$ and letting the metric $g$ on $M$ (and its associated Levi-Civita connection) vary. It follows from the results of chapter 3 (Proposition 10) that the difference of phases of the $\zeta$-determinants of $d''_{1,t}$ and partition functions at $t=1$ and $t=0$ is given by a Wodzicki residue coming from an integrated tracial anomaly.

In [W89], in order to build a metric invariant partition function, Witten adds to the partition function (5.2) a local counter-term. For this he proceeded in two steps, first fixing the metric and measuring the dependence of the phase on the choice of connection and then, whenever the manifold $M$ has trivial tangent bundle, fixing the connection and measuring the dependence of the phase on the choice of metric. Both these dependences can be measured in terms of tracial anomalies along the lines of Corollary 1, i.e. the variation of the partition function $Z_k(*d''_{k,t})$ induced by a change of metric reads

$$\frac{Z_k(*d''_{k,1})}{Z_k(*d''_{k,0})} = \exp\left\{-i\frac{\pi}{4}(\eta_{*d''_{k,1}}(0) - \eta_{*d''_{k,0}}(0))\right\}$$

where $\{*d''_{k,t}, t \in [0,1]\}$ is the family of operators induced by a family $\{g_t, t \in [0,1]\}$ of Riemannian metrics interpolating $g_0$ and $g_1$, the connection on $V_\rho$ being left fixed. As we will see, for $k=1$, and when the tangent bundle is trivial –in which case we can write the Levi-Civita connection $\nabla^{L.C.} = d + A$– it gives rise, via the Atiyah-Patodi-Singer theorem, to the familiar Chern-Simons term $\int_M \operatorname{tr}\left(A \wedge dA + \frac{2}{3} A \wedge A \wedge A\right)$ arising in topological quantum field theory in dimension 3 (cfr. formula (2.20) in [W99]).

The results of Section 3.1 and equation (3.10) imply that the *phase anomaly* $\phi(*d''_{1,1}) - \phi(*d''_{1,0}) = \log \frac{Z_k(*d''_{k,1})}{Z_k(*d''_{k,0})}$ corresponds to an integrated weighted trace anomaly:

**Theorem 12** *The Chern-Simons phase anomaly between two Riemannian metrics $g_0$ and $g_1$ is an integrated weighted trace anomaly, i.e.*

$$\begin{array}{ccc} \text{phase anomaly} & = & \text{integrated weighted trace anomaly} \\ \downarrow & & \downarrow \\ \log \dfrac{Z_k(*d''_{k,1})}{Z_k(*d''_{k,0})} & = & -i\dfrac{\pi}{4}\displaystyle\int_0^1 \dot{\operatorname{tr}}^{*d''_{k,t}}(\operatorname{sign}(*d''_{k,t}))\, dt. \end{array}$$

*Using APS index theorem* [APSI], *is given by the Chern-Simons term*

$$i\frac{32}{\pi^2}\int_M tr(A\wedge dA+\frac{2}{3}A\wedge A\wedge A).$$

Let us see now how the Atiyah-Patodi-Singer index theorem [APSI, APSII, APSIII] implies that the local term corresponding to this weighted trace anomaly is the classical Chern-Simons term. We restrict ourselves to odd dimensions $n=2k+1$ with $k$ odd so that $n+1$ is a multiple of 4 as required in the Atiyah-Patodi-Singer theorem.

**Proposition 15** *The difference of phases $\phi(*d''_{1,1})-\phi(*d''_{1,0})$ reads*

$$\phi(*d''_{1,1})-\phi(*d''_{1,0})=\frac{\pi}{2}\int_{M\times[0,1]}\mathrm{Ch}(\nabla^W)=i\frac{32}{\pi^2}\int_M tr(A\wedge dA+\frac{2}{3}A\wedge A\wedge A)$$

*Proof.* From Proposition 3 it follows that

$$\phi(*d''_{1,1})-\phi(*d''_{1,0})=\frac{\pi}{2}\left(\eta_{*d''_{1,1}}(0)-\eta_{*d''_{1,0}}(0)\right)$$

so that we are left to compute a difference of $\eta$-invariants which can be expressed using the Atiyah-Patodi-Singer theorem. Let $X=M\times[0,1]$ where $M$ is an $4l-1$ dimensional closed Riemannian manifold and let us equip $X$ with the product metric so that we are in the situation described above. The boundary of $X$ is the odd dimensional manifold $M\times\{0\}\bigcup M\times\{1\}$. With the notations of Theorem 4, where we set $p=k$, since $k$ is odd we have $B_k=*d_k-d_{n-k}*$, where $B_k$ is the restriction of $B$ to the odd $k$ forms. Since $*^2=1$ on $k$ forms in dimension $n=2k+1$, we have $d^*_{n-k}=-*d^*_k$ so that the restriction $B''_k$ to $R(d^*_{k-1})$ coincides with the restriction $*d''_k$.

We therefore need to compute the difference of $\eta$-invariants of $B''_k$. Following Atiyah, Patodi and Singer [APSII], let us first investigate the metric dependence of the eta invariants $\eta_{*d''_k}(0)$ in order to build an invariant independent on the choice of metric. To two metrics $g$ and $g'$ on $M$ correspond two operators $B$ and $B'$, and it follows from the Atiyah-Patodi-Singer index theorem that (see (2.3) in [APSII])

$$\eta_B(0)-\eta_{B'}(0)=n\int_{M\times[0,1]}L(\nabla^{LC})$$

using the fact that $\mathrm{sign}(M\times[0,1])=0$ and that the connection on $W$ is flat. Let us now fix the metric and take two flat connections $\nabla^W_0$ and $\nabla^W_1$ on $W$ restricted to $M$, this leading again to two eta invariants $\eta_{B''_{k,1}}(0)$ and $\eta_{B''_{k,0}}(0)$. From the above it follows that this expression is independent of the choice of metric (see Theorem 2.4 in [APSII]).

We now equip $W$ restricted to $M$ with a one parameter family of connections $\nabla_t^W := (1-t)\nabla_0^W + t\nabla_1^W$ and correspondingly a one parameter family of operators

$$B_t = (-1)^{k+p+1}(\epsilon * d_t - d_t *).$$

We can equip $W$ seen as a bundle over $X = [0,1] \times M$ with the connection $\nabla^W = \frac{d}{dt} + \nabla_t^W$, and build the corresponding Dirac operator

$$D_\nabla^+ = c \circ (\frac{d}{dt} + B_t^{odd}).$$

Because $\eta_{B''_{k,1}}(0) - \eta_{B''_{k,0}}(0)$ does not depend on the choice of metric, we can choose a flat metric. Thus the $L$-form will be trivial. On the other hand $\mathrm{sign}(X) = 0$ for the particular choice of manifold $X = M \times [0,1]$ we took, so that the spectral flow $\Phi(\{B''_{k,t}\})$ vanishes. Applying once again the Atiyah-Patodi-Singer theorem yields

$$\eta_{B''_{k,1}}(0) - \eta_{B''_{k,0}}(0) = \int_{M\times[0,1]} \mathrm{Ch}(\nabla^W).$$

Finally, using Stokes theorem from the existence of a Chern-Simons form $Q_3(A) = -\frac{1}{2}\frac{1}{2\pi^3}\mathrm{tr}(A \wedge dA + \frac{2}{3}A^3)$ such that locally $\mathrm{Ch}(\nabla^W) = dQ_3$, which ends the proof. □

# Chapter 6

# Splitting of the Geometry of Determinants of Families of Complexes and Duality

The aim of this chapter is to extend the splitting in the analytic torsion of the de Rham complex induced by "duality" in AFT's to the curvature of the determinant line bundle associated to a family of complexes of Hermitian vector bundles over a closed Riemannian manifold, parametrized by a smooth manifold. This involves the construction of the determinant line bundle associated to the infinite-rank elliptic complex associated to this family, which follows from the constructions of the precedent chapters.

**Introduction: Duality in Antisymmetric Field Theories**

Consider a closed $n$-dimensional smooth Riemannian manifold $M$ and let $\Omega^k = C^\infty(\Lambda^k T^*M^* \otimes V_\rho)$ be the space of differential $k$-forms on $M$ with values in $V_\rho$, the vector bundle over $M$ defined in Example 2. We assume that the complex

$$0 \longrightarrow \Omega^0 \xrightarrow{d_0} \cdots \Omega^{k-1} \xrightarrow{d_{k-1}} \Omega^k \xrightarrow{d_k} \Omega^{k+1} \xrightarrow{d_{k+1}} \cdots \Omega^n \xrightarrow{d_n} 0,$$

is acyclic. An Antisymmetric Field Theory is a field theory in which the "fields" are modelled by twisted forms on a manifold $M$, i.e. elements of the space of sections $\mathbf{\Omega} = \bigoplus_{k=0}^n \Omega^k$. A $k$-rank (or degree $k$) antisymmetric tensor field is an element of $\Omega^k$, and "duality" establishes an *equivalence* between a particular theory of $(k-1)$ and $(n-k-1)$-rank antisymmetric tensor fields.

A generalization of the electromagnetic action to higher rank twisted forms, gives rise to the classical action for the theory of *antisymmetric tensor fields* that we shall consider. The set of fields is $\Omega^{k-1}$, and the action takes the form

$$\mathcal{S}_{k-1}(\omega_{k-1}) = \langle d_{k-1}\omega_{k-1}, d_{k-1}\omega_{k-1}\rangle = \int_M d_{k-1}\omega_{k-1} \wedge *d_{k-1}\omega_{k-1}, \qquad (6.1)$$

giving rise to the partition function

$$Z(\mathcal{S}_{k-1}) = \int_{\Omega^{k-1}} \exp\{-\langle d_{k-1}\omega_{k-1}, d_{k-1}\omega_{k-1}\rangle\}[\mathcal{D}\omega_{k-1}]. \qquad (6.2)$$

Notice that the classical action $\mathcal{S}_{k-1}$ is a degenerate functional on $\Omega^{k-1}$, in fact $\ker d_{k-1} = \Omega'_{k-1}$ (with the notations of (1.54)). Since we can associate to $Z(\mathcal{S}_{k-1})$ an elliptic resolvent, Schwarz's Ansatz (2.14) is in order. "Duality" conjectures the equivalence of the partition function (6.2) and the one defined on $\Omega^{n-k-1}$ by the action functional

$$\mathcal{S}_{n-k-1}(\omega_{n-k-1}) = \langle d_{n-k-1}\omega_{n-k+1}, d_{n-k-1}\omega_{n-k-1}\rangle, \qquad (6.3)$$

(which is also degenerate). Strictly speaking, two field theories are said to be "dual" if their correlation functions coincide. Here, on the grounds of the "semiclassical approximation" explained in Appendix B, we only require identification of the partition functions. In any case, the identification between two dual antisymmetric field theories involves identifying formal integrals, which we shall interpret as Gaussian integrals since they are defined using quadratic actions. Typically, duality between two theories is exhibed by means of *formal* calculations, using properties of finite-dimensional Gaussian integrals, leading to heuristic identifications. We shall illustrate that kind of manipulations in the next section, before we give an interpretation of this equivalence in the language of Fresnel integrals.

Let us begin by giving a very naive interpretation of this duality at the classical level, which will give us the guise of what we can attempt from the semiclassical analysis through partition functions. Notice that *both* theories, defined by (6.1) and (6.3), can be seen as contained in a (rather trivial) sole theory of antisymmetric tensor fields. Consider the classical action

$$\mathcal{S}_o(\omega_k) = \langle \omega_k, \omega_k\rangle = \|\omega_k\|^2,$$

on $\Omega^k$, which is clearly no degenerate. The Hodge decomposition (1.54) in terms of which $\omega_k \in \Omega^k$ splits into $\omega_k = \omega'_k \oplus \omega''_k$ where $\omega'_k = d_{k-1}\omega_{k-1} \in \Omega'_k$ and $\omega''_k = d^*_k\omega_{k+1} \in \Omega''_k$, for some $\omega_{k-1} \in \Omega^{k-1}$, $\omega_{k+1} \in \Omega^{k+1}$, yields a change of variable so that $\mathcal{S}_o$ reads

$$\mathcal{S}_o(\omega_k) = \mathcal{S}_{k-1}(\omega_{k-1}) \oplus \mathcal{S}^*_{k+1}(\omega_{k+1}), \qquad (6.4)$$

where

$$\mathcal{S}_{k-1}(\omega_{k-1}) = \langle d_{k-1}\omega_{k-1}, d_{k-1}\omega_{k-1}\rangle,$$

and

$$\mathcal{S}^*_{k+1}(\omega_{k+1}) = \langle d_k^*\omega_{k+1}, d_k^*\omega_{k+1}\rangle.$$

Using (1.60) and setting $\eta_{n-k-1} = *\omega_{k+1}$, it follows that

$$\langle d_k^*\omega_{k+1}, d_k^*\omega_{k+1}\rangle = \langle d_{n-k-1}\eta_{n-k-1}, d_{n-k-1}\eta_{n-k-1}\rangle \tag{6.5}$$

so,

$$\mathcal{S}_o(\omega_k) = \mathcal{S}_{k-1}(\omega_{k-1}) \oplus \mathcal{S}_{n-k-1}(\eta_{n-k-1}), \tag{6.6}$$

which puts both classical actions (i.e. both theories) on an equal footing, as complementary parts, in $\mathcal{S}_o$. Notice that the functional $\mathcal{S}_o$ is nothing but the metric on $\Omega^k$, and equation (6.6) gives a splitting of this metric in terms of classical action functionals on $\Omega^{k-1}$ and $\Omega^{k+1} \cong \Omega^{n-k-1}$. This sort of splitting in the geometry will also carried out in the quantum approach.

## 6.1 Duality and Fresnel Integrals

In this section we consider the identification of partition functions on the basis of some heuristic calculations commonly used in the partition function description of duality. We shall give a geometric meaning to some of these formal manipulations in Section 6.2, following the $\zeta$-function approach to partition functions. But for now we follow a measure theoretical approach working with generating functions rather than partition functions and using the language of Fresnel integrals as defined by Albeverio and Høegh-Krohn [AlH76].

### 6.1.1 Heuristics of Duality

The functionals $\mathcal{S}_{k-1}(\omega_{k-1})$ and $\mathcal{S}^*_{k+1}(\omega_{k+1})$ are degenerate but, by restriction on the respective domains, the maps

$$d_k : \Omega''_k \to \Omega'_{k+1} \tag{6.7}$$

and

$$d_k^* : \Omega'_{k+1} \to \Omega''_k, \tag{6.8}$$

are isomorphisms, giving rise to the bijective maps

$$d^*_{k-1}d_{k-1} : \Omega''_{k-1} \ \to \ \Omega''_{k-1},$$

$$d_k d_k^* : \Omega'_{k+1} \to \Omega'_{k+1}.$$

Thus, the functionals

$$\widehat{\mathcal{S}}(\omega''_{k-1}) = \langle d_{k-1}\omega''_{k-1},\, d_{k-1}\omega''_{k-1}\rangle \tag{6.9}$$

and

$$\widehat{\mathcal{S}}^*(\omega'_{k+1}) = \langle d_k^*\omega'_{k+1},\, d_k^*\omega'_{k+1}\rangle, \tag{6.10}$$

are non-degenerate on $\Omega''_{k-1}$ and $\Omega'_{k+1}$, respectively. Let us first recall some formal calculations involved in identifying two dual partition functions. (We use " = " to remark the fact that all the measures involved are ill-defined Lebesgue measures on the $L^2$ spaces of forms, so these calculations are formal.) For example, starting from the partition function corresponding to the action functional $\mathcal{S}_o$ on $\Omega'_k$, using Fourier transform and acyclicity we can write

$$\begin{aligned}\int_{\Omega'_k} & \exp\left\{-\frac{a}{2}\mathcal{S}_o(\eta'_k)\right\}[\mathcal{D}\eta'_k] \\ " = " & \int_{\Omega'_k}[\mathcal{D}\eta'_k]\int_{\Omega^k}[\mathcal{D}\alpha_k]\exp\left\{-\frac{1}{2a}\mathcal{S}_o(\alpha_k)\right\}\exp\left\{i\langle\eta'_k,\alpha_k\rangle\right\} \\ " = " & \int_{\Omega^k}[\mathcal{D}\alpha_k]\exp\left\{-\frac{1}{2a}\mathcal{S}_o(\alpha_k)\right\}\delta\left[\alpha'_k=0\right] \\ " = " & \int_{\Omega''_k}\exp\left\{-\frac{1}{2a}\mathcal{S}_o(\alpha''_k)\right\}[\mathcal{D}\alpha''_k].\end{aligned}$$

Hence doing the change of variables defined by the maps 6.7 and 6.8, $\eta'_k = d_{k-1}\omega''_{k-1}$ and $\alpha''_k = d^*_k\omega'_{k+1}$, we find

$$\begin{aligned}&\int_{\Omega''_{k-1}}\exp\left\{-\frac{a}{2}\langle d_{k-1}\omega''_{k-1}, d_{k-1}\omega''_{k-1}\rangle\right\}\mathcal{J}_{k-1}[\mathcal{D}\omega''_{k-1}] \\ \text{“ =''}\quad&\int_{\Omega'_{k+1}}\exp\left\{-\frac{1}{2a}\langle d^*_k\omega'_{k+1}, d^*_k\omega'_{k+1}\rangle\right\}\mathcal{J}_{k+1}[\mathcal{D}\omega'_{k+1}],\end{aligned} \tag{6.11}$$

where $\mathcal{J}_{k-1}$ and $\mathcal{J}_{k+1}$ denote the associated jacobian determinants $\mathcal{J}_{k-1} := \sqrt{\det(d^*_{k-1}d_{k-1})}$ and $\mathcal{J}_{k+1} := \sqrt{\det(d_k d^*_k)}$.

Let us make a few comments on this computation which, although very formal, gives the gist of the dualization procedure.

1. *Hodge decomposition* in the case of an acyclic complex splits the space of $k$-antisymmetric tensor fields (1.54) and then, through isomorphisms previously defined,
$$\Omega^k \cong \Omega''_{k-1} \oplus \Omega'_{k+1}. \tag{6.12}$$
The $L^2$ scalar product on $\Omega^k$ then gives rise to two (non degenerate) actions $\widehat{\mathcal{S}}$ and $\widehat{\mathcal{S}}^*$, on $\Omega''_{k-1}$ and $\Omega'_{k+1}$ respectively, which are related by a *Fourier transform*. The non-degeneracy in the actions comes from the fact that we restrict ourselves to
$$\Omega''_{k-1} \xrightarrow{d_{k-1}} \Omega^k \xleftarrow{d^*_k} \Omega'_{k+1}. \tag{6.13}$$
Thus, the field $\omega_k \in \Omega_k$ splits into
$$\omega_k = d_{k-1}\omega''_{k-1} \oplus d^*_k\omega'_{k+1}, \tag{6.14}$$
giving rise to two new "fields" (gauge potentials) $\omega''_{k-1}, \omega'_{k+1}$.

2. In the process of taking the Fourier Transform, the coefficient of the quadratic action is inverted ($a \mapsto a^{-1}$), a fact often observed in duality and typical for Fourier transforms of Gaussian functions. A strong coupling can thus be turned into a weak coupling [D98].

3. Finally, if we consider *Hodge star duality* on the complex, through the relation (6.5), we recover the usual "moral" of duality in antisymmetric fields [Q98]: a $(k-1)$-rank antisymmetric tensor field (the "gauge potential" $\omega_{k-1}$) is dual to a $(n-k-1)$-rank antisymmetric tensor field ($\eta_{n-k-1} = *\omega_{k+1}$) or, in "brane" language, a $(k-2)$(electric)-brane is dual to a $(n-k-2)$(magnetic)-brane.

Acyclicity induces a decomposition (6.12), from which we find two possible "potentials" associated to each antisymmetric field in $\Omega^k$, namely $\omega''_{k-1}$ and $\omega'_{k+1}$, the first one for the exterior differential $d_{k-1}$, the second one for $d^*_k$. Writing the partition function of the theory with respect to one or the other give us "dual" formulations of the same theory.

### 6.1.2 Duality through Fresnel Integrals

**The path integrals as Fresnel integrals**

Consider the Hilbert space $\mathcal{H}_k = L^2(\Omega^k)$, the closure (with respect to the $L^2$ inner product) of sections of the bundle $\Omega^k$. The decomposition $\Omega^k = \Omega'_k \oplus \Omega''_k$ induces two Hilbert spaces, namely $\mathcal{H}'_k = L^2(\Omega'_k)$ and $\mathcal{H}''_k = L^2(\Omega''_k)$, $L^2$ closures of sections of $\Omega'_k$ and $\Omega''_k$, respectively. As metric spaces $\mathcal{H}_k$ and $\bar{\mathcal{H}}_k := \mathcal{H}'_k \times \mathcal{H}''_k$ are equivalent, i.e. there exists an isometry

$$\phi : \mathcal{H}_k \to \bar{\mathcal{H}}_k,$$

which is measurable and with measurable inverse as well. Moreover, this isometry is such that if $\mu'_k$ and $\mu''_k$ are measures on $\mathcal{H}'_k$ and $\mathcal{H}''_k$, respectively, then $\nu_k = \phi^{-1}(\mu'_k \otimes \mu''_k)$ is a well-defined measure on $\mathcal{H}_k$. Applying Fubini's theorem for Fresnel integrals (see Theorem 6) it follows that

$$\begin{aligned} \int_{\mathcal{H}_k} g(\omega_k) d\nu(\omega_k) &= \int_{\bar{\mathcal{H}}_k} g(\eta'_k \oplus \eta''_k) d(\mu'_k \otimes \mu''_k)(\eta'_k \oplus \eta''_k) \\ &= \int_{\mathcal{H}'_k} g(\eta'_k) d\mu'_k(\eta'_k) \int_{\mathcal{H}''_k} g(\eta''_k) d\mu''_k(\eta''_k). \end{aligned}$$

From this it is easy to show that $\phi$ gives us canonical embeddings of $\mathcal{F}(\mathcal{H}'_k)$ and $\mathcal{F}(\mathcal{H}''_k)$ into $\mathcal{F}(\mathcal{H})$. In fact, given $f \in \mathcal{F}(\mathcal{H}'_k)$ then there exists $\mu_f \in \mathcal{M}(\mathcal{H}'_k)$ verifying

$$f(\omega'_k) = \int_{\mathcal{H}'_k} e^{i\langle \omega'_k, \eta'_k \rangle} d\mu_f(\eta'_k).$$

Let us take $\mu_k''$ to be $\delta_o''$, the Dirac measure centered in zero on $\mathcal{H}_k''$, and $\mu_k' = \mu_f$, then $\nu_f = \phi^{-1}(\mu_f \otimes \delta_o'') \in \mathcal{M}(\mathcal{H}_k)$ and the last computation reduces to

$$\int_{\mathcal{H}_k} g(\omega_k) d\nu_f(\omega_k) = \int_{\mathcal{H}_k'} g(\eta_k') d\mu_f(\eta_k'), \tag{6.15}$$

where $g \in \mathcal{F}(\mathcal{H}_k)$. Taking $g(\omega_k) = \exp\{-\frac{i}{2}\langle\omega_k, \omega_k\rangle\}$, leads to

$$\int_{\mathcal{H}_k} \exp\{-\frac{i}{2}\langle\omega_k, \omega_k\rangle\} d\nu_f(\omega_k) = \int_{\mathcal{H}_k'} \exp\{-\frac{i}{2}\langle\eta_k', \eta_k'\rangle\} d\mu_f(\eta_k') = \mathcal{F}(f),$$

so $f \in \mathcal{M}(\mathcal{H}_k)$ (a similar argument, taking $\mu_k' = \delta_o'$ and $\mu_k'' = \mu_f$ for $f \in \mathcal{F}(\mathcal{H}_k'')$, shows that $\mathcal{F}(\mathcal{H}_k'')$ can be imbedded into $\mathcal{F}(\mathcal{H}_k)$). Now, if we take in (6.15) $g(\omega_k)$ to be

$$g_a(\omega_k) = \exp\{-\frac{ia}{2}\langle\omega_k, \omega_k\rangle\},$$

where $a$ denotes a real constant, this last Fresnel integral can be written as

$$\mathcal{F}_a(f) = \int_{\mathcal{H}_k'} \exp\{-\frac{ia}{2}\langle\eta_k', \eta_k'\rangle\} d\mu_f(\eta_k') = \int_{\mathcal{H}_k'}^{\sim} \exp\left\{\frac{i}{2a}\langle\eta_k', \eta_k'\rangle\right\} f(\eta_k')\, d\eta_k', \tag{6.16}$$

where the last expression on the right has no meaning as an integral (is the notation of [AlH76]). After the change of variables defined by isomorphism (6.7) this Fresnel integral provides a rigorous realization of the heuristic Feynman integral corresponding to the functional $\widehat{\mathcal{S}}$. The same argument applied to the function $g_{a^{-1}}(\omega_k) = \exp\{-\frac{i}{2a}\langle\omega_k, \omega_k\rangle\}$ (and considering the embedding of $\mathcal{F}(\mathcal{H}_k'')$ into $\mathcal{F}(\mathcal{H}_k)$)), after the change of variables given by (6.8) yields a realization of the "dual" path integral defined by $\widehat{\mathcal{S}}^*$.

### Duality in terms of Fresnel integrals

In this section we shall write down the rigorous expression giving rise to the heuristic interpretation of duality in the literature [Q98]. Let us consider as before a Hilbert space which splits as $\bar{\mathcal{H}}_k := \mathcal{H}_k' \times \mathcal{H}_k''$, then duality in this context will be a consequence of the following result.

**Proposition 16** *Let $f \in \mathcal{M}(\mathcal{H}_k')$ and $g \in \mathcal{M}(\mathcal{H}_k)$, then*

$$\int_{\mathcal{H}_k} g(\omega_k) d\nu_f(\omega_k) = \int_{\mathcal{H}_k'} f(\eta_k') d\mu_g(\eta_k'),$$

*where $\nu_f = \phi^{-1}(\mu_f \otimes \delta_o'') \in \mathcal{M}(\mathcal{H}_k)$ as in the preceding discussion.*

*Proof.* It is a consequence of (6.15) and the obvious equality

$$\int_{\mathcal{H}} g(\omega) d\mu_f(\omega) = \int_{\mathcal{H}} f(\eta) d\mu_g(\eta), \tag{6.17}$$

which follows from the definition. □

Let us now verify that the heuristic argument implying duality can be derived from this fact. We take $\mathcal{H}_k = L^2(\Omega^k)$, where the closure is taken with respect to the $L^2$ hermitian product defined on $\Omega^k$, and we consider as before the induced decomposition $\mathcal{H}_k \cong \mathcal{H}'_k \oplus \mathcal{H}''_k$. Consider $f \in \mathcal{M}(\mathcal{H}_k)$, and $g \in \mathcal{M}(\mathcal{H}'_k)$ such that in some convenient limit

$$\int_{\mathcal{H}'_k} \exp\{i\langle \omega'_k, \eta_k\rangle\} d\mu_g(\omega'_k) \to \delta[\eta'_k = 0].$$

Then, applying Fubini's theorem we have

$$\int_{\mathcal{H}'_k} f(\omega'_k) d\mu_g(\omega'_k) = \int_{\mathcal{H}^k} \left[\int_{\mathcal{H}'_k} \exp\{i\langle \omega'_k, \eta_k\rangle\} d\mu_g(\omega'_k)\right] d\mu_f(\eta_k),$$

which, taking limits in both sides, leads to

$$\int_{\mathcal{H}'_k} f(\omega'_k)[d\omega'_k] \text{``} = \text{``} \int_{\mathcal{H}''_k} d\mu_f(\eta''_k),$$

where $[d\omega'_k]$ denotes a formal Lebesgue measure on $\mathcal{H}'_k$. If for $a$ constant we take $f(\omega_k) = \exp\left\{-\frac{ia}{2}\langle \omega_k, \omega_k\rangle\right\}$, the left side of the previous heuristic identity is $\mathcal{F}_a(g)$, which corresponds to the path integral defined by $\widehat{\mathcal{S}}$, and on the right side we have its dual (recall the finite-dimensional equalities (B.3) to (B.4)), all this after applying the change of variables defined by the maps (6.7) and (6.8) as before.

## 6.2 Analytic Torsion on Riemannian Manifolds and Duality

After this incursion in measure theory, which provides an interpretation for formal path integral computations involved in establishing duality between two antisymmetric field theories, we turn back to the $\zeta$-function approach to partition functions, which provides a geometric interpretation of some of the steps leading to duality relations. Going back to Schwarz's Ansatz (2.14) naturally leads us to consider the analytic torsion of an elliptic complex. From this point of view, a first step in establishing the duality relation is a splitting procedure, briefly mentioned in (6.4), and which we investigate further here.

### Ray-Singer Torsion and Duality

The relation between the analytic torsion of the manifold $M$ and the partition function of an antisymmetric field theory defined on it was pointed out by Schwarz ([S79], see also [ST84]), when studying quantization of antisymmetric tensor field theories defined by the degenerate action (6.1) on $\Omega^{k-1}$. It has also been used in the context of *Topological Quantum Field Theories* [W89][BT91][BBRT91][AS95]. Schwarz shows that the Hodge star duality map and Hodge decomposition on each space $\Omega^k$ imply a factorization of the analytic torsion $T(M)$ in terms of the two partition functions, corresponding to the actions $\mathcal{S}(\omega_{k-1})$ and $\mathcal{S}_{n-k-1}(\omega_{n-k-1})$.

The relation between the two antisymmetric quantum field theories defined by the action functionals $S(\omega_{k-1})$ and $S(\omega_{n-k+1})$, and the Ray-Singer torsion of the manifold $M$, follows from the splitting in the de Rham complex (1.53) induced by Hodge star duality and the two resolvents associated to their corresponding partition functions. Indeed, as follows from (6.5), Hodge star duality implies the equivalence between the action functionals $S(\omega_{n-k+1}) = \langle d_{n-k+1}\omega_{n-k+1}, d_{n-k+1}\omega_{n-k+1}\rangle$ and $S^*(\omega_{k+1}) = \langle d_k^*\omega_{k+1}, d_k^*\omega_{k+1}\rangle$, and the two associated partition functions $Z_k(M)$ and $Z_k^*(M)$ (for the actions $S(\omega_{k-1})$ and $S^*(\omega_{k+1})$, respectively) have resolvents that split the complex at the $k^{th}$ level, namely

$$0 \to \Omega^0 \xrightarrow{d_0} \cdots \xrightarrow{d_{k-2}} \Omega^{k-1} \xrightarrow{d_{k-1}} \Omega^k \xleftarrow{d_k^*} \Omega^{k+1} \xleftarrow{d_{k+1}^*} \cdots \xleftarrow{d_{n-1}^*} \Omega^n \leftarrow 0 \tag{6.18}$$

(compare with (6.13)). Let us stress this more precisely in the following

**Proposition 17** [S79]

$$Z_k(M) \cdot Z_k^*(M)^{-1} = T_{RS}(M)^{(-1)^k}. \tag{6.19}$$

*Proof.* The elliptic resolvent associated to $\mathcal{S}(\omega_{k-1})$ is (see (2.14))

$$0 \longrightarrow \Omega^0 \xrightarrow{d_0} \cdots \xrightarrow{d_{k-3}} \Omega^{k-2} \xrightarrow{d_{k-2}} \Omega'_{k-1} \xrightarrow{\Delta''_{k-1}} 0, \tag{6.20}$$

and hence we define the partition function associated to that action (and resolvent) as

$$Z_k(M) = \prod_{j=0}^{k-1} \left(\det{}_\zeta \Delta''_j\right)^{\frac{(-1)^{k-j}}{2}} \tag{6.21}$$

In the same way, taking the resolvent associated to $\mathcal{S}^*(\omega_{k+1})$,

$$0 \longrightarrow \Omega^n \xrightarrow{d_{n-1}^*} \cdots \xrightarrow{d_{k+2}^*} \Omega^{k+2} \xrightarrow{d_{k+1}^*} \Omega''_{k+1} \xrightarrow{\Delta'_{k+1}} 0, \tag{6.22}$$

we define the associated "dual" partition function by

$$Z_k^*(M) = \prod_{j=0}^{n-k-1} \left(\det{}_\zeta \Delta'_{k+j+1}\right)^{\frac{(-1)^{j+1}}{2}}. \tag{6.23}$$

Thus, from Proposition 7 it follows that

$$
\begin{aligned}
Z_k(M)\cdot Z_k^*(M)^{-1} &= \prod_{j=0}^{k-1}\left(\det\nolimits_\zeta \Delta_j''\right)^{\frac{(-1)^{k-j}}{2}} \cdot \prod_{j=0}^{n-k-1}\left(\det\nolimits_\zeta \Delta_{k+j+1}'\right)^{\frac{(-1)^{j}}{2}} \\
&= \prod_{j=0}^{k-1}\left(\det\nolimits_\zeta \Delta_j''\right)^{\frac{(-1)^{k-j}}{2}} \cdot \prod_{j=k+1}^{n}\left(\det\nolimits_\zeta \Delta_j'\right)^{\frac{(-1)^{k+j-1}}{2}} \\
&= \left[\prod_{j=0}^{k-1}\left(\det\nolimits_\zeta \Delta_j''\right)^{\frac{(-1)^{j}}{2}} \cdot \prod_{j=k+1}^{n}\left(\det\nolimits_\zeta \Delta_{j+}'\right)^{\frac{(-1)^{j-1}}{2}}\right]^{(-1)^k} \\
&= T_{RS}(M)^{(-1)^k},
\end{aligned}
$$

where we used he equality $\det_\zeta \Delta_k' = \det_\zeta \Delta_{k-1}''$. □

Thus, we can say that duality leads to a "factorization" of the Ray-Singer torsion of the space-time manifold in terms of their corresponding partition functions. Hence in even dimensions, since $T_{RS}(M) = 1$, we get the expected identification of the partition function with its dual $Z_k(M) = Z_k^*(M)$. Note that the analytic torsion is a topological invariant of $M$, but there is no reason for $Z_k(M)$ and $Z_k^*(M)$ to have this property.

**Remark.** It follows from (5.2) that, if $n = 2k+1$, the square of the modulus of the Chern-Simons partition function is

$$|Z_k^{CS}(*d_k)|^2 = T_{RS}(M)^{(-1)^{k+1}},$$

and Proposition 17 implies that for any $k$

$$Z_k(M)\cdot Z_k^*(M)^{-1} = T_{RS}(M)^{(-1)^k},$$

so that, if $n = 2k+1$,

$$|Z_k^{CS}(*d_k)|^2 = Z_k(M)^{-1}\cdot Z_k^*(M).$$

(Compare with (5.3)).

## 6.3 Splitting of the Geometry of Determinant Line Bundles in Finite Dimensions

In this section we consider the splitting of the geometry of the determinant line bundle associated to a complex of finite-rank vector bundles over a closed manifold.

### 6.3.1 Milnor's Duality and Splitting of the Torsion

Let $(E_\bullet, T_\bullet)$ be the chain complex of vector spaces

$$0 \to E_0 \xrightarrow{T_0} \cdots \to E_{j-1} \xrightarrow{T_{j-1}} E_j \xrightarrow{T_j} E_{j+1} \to \cdots \xrightarrow{T_{n-1}} E_n \to 0,$$

and consider the "adjoint" chain complex $(E_\bullet, T_\bullet^*)$ to $(E_\bullet, T_\bullet)$, i.e. the chain complex formed by the same collection of vector spaces, but taking as chain maps the formal adjoints to the collection $(T_\bullet)$,

$$0 \to E_n \xrightarrow{T_{n-1}^*} \cdots \to E_{j+1} \xrightarrow{T_j^*} E_j \xrightarrow{T_{j-1}^*} E_{j-1} \to \cdots \xrightarrow{T_0^*} E_0 \to 0. \tag{6.24}$$

It follows from the definition of the torsion (Definition 6) that the element $\tau^*(E_\bullet) = \tau(E_\bullet, T_\bullet^*)$, canonically associated to this "adjoint" complex, belongs to the vector space $\bigotimes_{k=0}^n (\det E_k)^{(-1)^{n-k-1}}$. Then, $\tau(E_\bullet) = (\tau^*(E_\bullet))^{(-1)^{n+1}}$, or

$$\tau(E_\bullet)\,(\tau^*(E_\bullet))^{(-1)^n} = 1, \tag{6.25}$$

which, in particular, implies Milnor's *Duality Theorem* for the Reidemeister torsion [M62]:

$$\tau_R(E_\bullet, \mathbf{e})\,\tau_R(E'_\bullet, \mathbf{e}')^{(-1)^n} = \pm 1. \tag{6.26}$$

We interpret this result saying that the torsion, the analytic torsion and the Reidemeister torsion of the complex $(E_\bullet, T_\bullet)$ can be "factorized" in terms of the torsion, the analytic torsion and the Reidemeister torsion of a (conveniently defined) part of the complex and a part of the torsion, the analytic torsion and the Reidemeister torsion of its "adjoint" complex. This factorization goes as follows. Let us consider the complexes $(E_\bullet^{(j)}, T_\bullet)$, obtained from the complex $(E_\bullet, T_\bullet)$ by cutting at the $j$-th level, i.e.

$$0 \to E_0 \xrightarrow{T_0} \cdots \xrightarrow{T_{j-2}} E_{j-1} \xrightarrow{T_{j-1}} T_{j-1}E_{j-1} \xrightarrow{T_j} 0. \tag{6.27}$$

It is clear that the acyclicity of $(E_\bullet, T_\bullet)$ implies that of $(E_\bullet^{(j)}, T_\bullet)$, so the torsion $\tau_j = \tau(E_\bullet^{(j)})$ is well-defined. Considering the torsions of the truncated complex and its "adjoint" complement, we can recover the torsion of the whole original complex as shown by the following result.

**Proposition 18** *Let $(E_\bullet, T_\bullet)$ be an acyclic chain complex of $n+1$ vector spaces and $(E_\bullet, T_\bullet^*)$ the complex defined by the adjoint maps as before, then*

$$\tau_k(E_\bullet) \otimes \tau_k^*(E_\bullet)^{(-1)^{n+1}} = \tau(E_\bullet), \tag{6.28}$$

*where $\tau_k(E_\bullet)$ and $\tau_k^*(E_\bullet)$ denote, respectively, the torsion of the complexes $(E_\bullet^{(k)}, T_\bullet)$ and $(E_\bullet^{(n-k)}, T_\bullet^*)$.*

*Proof.* From the acyclicity of the complex we have the decomposition $E_k = E'_k \oplus E''_k$, where $E'_k = T_{k-1}E_{k-1}$ and $E''_k = T^*_k E_{k+1}$. We can consider the whole complex $(E_\bullet, T_\bullet)$ as the pasting of the complexes $(E^{(k)}_\bullet, T_\bullet)$ and $(E^{(n-k)}_\bullet, T^*_\bullet)$, i.e.

$$\begin{array}{ccccccccc} 0 & \to E_0 \xrightarrow{T_0} \cdots \xrightarrow{T_{k-2}} E_{k-1} & \xrightarrow{T_{k-1}} & T_{k-1}E_{k-1} & \longrightarrow & 0 & & & \\ & & & \oplus & & & & & \\ & & 0 & \longleftarrow & T^*_k E_{k+1} & \xleftarrow{T^*_k} & E_{k+1} & \xleftarrow{T^*_{k+1}} & \cdots \leftarrow E_n \leftarrow 0. \end{array} \tag{6.29}$$

Observe that

$$\tau_k(E_\bullet) \in \bigotimes_{i=0}^{k-1} (\det E_i)^{(-1)^{i+1}} \otimes (\det E'_k)^{(-1)^{k+1}},$$

and

$$\tau^*_k(E_\bullet) \in \bigotimes_{j=0}^{n-k-1} (\det E_{n-j})^{(-1)^{j+1}} \otimes (\det E''_k)^{(-1)^{n-k-1}}.$$

By (6.25)

$$\bar\tau_{n-k}(E_\bullet) = \tau^*_k(E_\bullet)^{(-1)^{n-k-1}}$$

where $\bar\tau_{n-k}(E_\bullet)$ denotes the torsion of the complex

$$0 \to E''_k \xrightarrow{T_k} E_{k+1} \xrightarrow{T_{k+1}} \cdots \longrightarrow E_{n-1} \xrightarrow{T_{n-1}} E_n \to 0,$$

hence the result follows from the equality

$$\tau_k(E_\bullet) \otimes \bar\tau_{n-k}(E_\bullet)^{(-1)^k} = \tau(E_\bullet).$$

□

**Corollary 2** *The factorization of the torsion goes through to a factorization of the Reidemeister torsion, i.e.*

$$\tau^k_R(E_\bullet, [\mathbf{e}]) \cdot \tau^{k\,*}_R(E_\bullet, [\mathbf{e}])^{(-1)^{n+1}} = \tau_R(E_\bullet, [\mathbf{e}]),$$

*where $\tau_R(E_\bullet, [\mathbf{e}])$, $\tau^k_R(E_\bullet, [\mathbf{e}])$ and $\tau^{k\,*}_R(E_\bullet, [\mathbf{e}])$ denote the Reidemeister torsion of the complexes $(E_\bullet, T_\bullet)$, $(E^{(k)}_\bullet, T_\bullet)$ and $(E^{(n-k)}_\bullet, T^*_\bullet)$, respectively.*

*Proof.* Follows from Proposition 4. □

**Corollary 3** *The factorization of the torsion goes through to a factorization of the analytic torsion*

$$\mathcal{T}_k(E_\bullet) \cdot \mathcal{T}_k{}^*(E_\bullet)^{(-1)^{n+1}} = \mathcal{T}(E_\bullet),$$

*where $\mathcal{T}(E_\bullet)$, $\mathcal{T}_k(E_\bullet)$ and $\mathcal{T}_k{}^*(E_\bullet)$ denote the analytic torsion of the complexes $(E_\bullet, T_\bullet)$, $(E^{(k)}_\bullet, T_\bullet)$ and $(E^{(n-k)}_\bullet, T^*_\bullet)$, respectively.*

*Proof.* Follows from Proposition 5. It can be also seen directly from the definition of analytic torsion, Proposition 6 and the equality $\det \Delta'_k = \det \Delta''_{k-1}$. $\square$

A particular example of this, relevant for our next applications, is the case for which the duality operators are given by a Hodge-type operator.

**Example 9** *Let $V$ be a vector space of dimension $n$ and let us consider a basis $\mathbf{v} = \{v_1, v_2, ..., v_n\}$ of $V$ such that the volume element $\mathrm{vol}_V = v_1 \wedge v_2 \wedge \cdots \wedge v_n \in \det V$ has norm one w.r.t a given inner product on $V$. Let $\Lambda^k V$ be the vector space of alternating $k$-forms on $V$, and let $v$ be an element in $\mathbf{v}$. Define the map*

$$\begin{array}{rcl} f_v : \Lambda^k V & \to & \Lambda^{k+1} V \\ \omega_k & \mapsto & \omega_k \wedge v, \end{array} \tag{6.30}$$

*whose adjoint (w.r.t the inner product in $\Lambda^k V$ given by $\langle \alpha_1 \wedge \alpha_2 \wedge \cdots \wedge \alpha_k, \beta_1 \wedge \beta_2 \wedge \cdots \wedge \beta_k \rangle = \det(\langle \alpha_i, \beta_j \rangle_{i,j})$ ) is the map*

$$f_v^\star = (-1)^{nk} \star f_v \star : \Lambda^{k+1} V \to \Lambda^k V, \tag{6.31}$$

*being $\star$ the Hodge star operator*

$$\star : \Lambda^k V \to \Lambda^{n-k} V, \tag{6.32}$$

*defined by the equation $\langle \star\omega, \eta \rangle \mathrm{vol}_V = \omega \wedge \eta$. It is clear that $f_v^2 = f_v^{\star 2} = 0$, so the sequences*

$$0 \to \Lambda^0 V \xrightarrow{f_v} \cdots \to \Lambda^{j-1} V \xrightarrow{f_v} \Lambda^j V \xrightarrow{f_v} \Lambda^{j+1} V \to \cdots \xrightarrow{f_v} \Lambda^n V \to 0, \tag{6.33}$$

*and*

$$0 \to \Lambda^n V \xrightarrow{f_v^\star} \cdots \to \Lambda^{j+1} V \xrightarrow{f_v^\star} \Lambda^j V \xrightarrow{f_v^\star} \Lambda^{j-1} V \to \cdots \xrightarrow{f_v^\star} \Lambda^0 V \to 0, \tag{6.34}$$

*define acyclic chain complexes $(\Lambda V_\bullet, f_v)$ and $(\Lambda V_\bullet^\star, f_v^\star)$, respectively.* Proposition 17 *and* Corollaries 2 *and* 3 *imply that*

$$\begin{aligned} \tau_k \otimes (\tau_k^\star)^{(-1)^{n+1}} &= \tau, & (6.35) \\ \tau_R^k \cdot \left(\tau_R^{k\star}\right)^{(-1)^{n+1}} &= \tau_R, & (6.36) \\ \mathcal{T}_k \cdot (\mathcal{T}_k{}^\star)^{(-1)^{n+1}} &= \mathcal{T}, & (6.37) \end{aligned}$$

*where $\tau_k$, $\tau_R^k$ and $\mathcal{T}_k$ denote the torsion, Reidemeister torsion and analytic torsion of the complex $(\Lambda V_\bullet, f_v)$ cut at the $k$-th level, and $\tau_k^\star$, $\tau_R^{k\star}$ and $\mathcal{T}_k{}^\star$ denote the torsion of the complex $(\Lambda V_\bullet^\star, f_v^\star)$ cut at the $(n-k)$-th level, respectively.*

**Remark.** Example 9 gives a "finite-dimensional" model of antisymmetric tensor fields (the case in which the manifold $M$ reduces to a point in the de Rham complex (1.53)). Notice that there is a relative sign difference between the alternating product of determinants in (2.14) and (1.57) (see Proposition 7), which comes from the order in which the alternating product is taken. Therefore, the finite-dimensional analytic torsion $\mathcal{T}_k$ of Example 9 coincides only up to a sign (given by the size of the chain complex) with the finite-dimensional "partition function" $Z_k(\{\text{point}\}) = \widehat{Z}_k$ given by (6.21). In fact, the exact relation between $\mathcal{T}_k$ and $\widehat{Z}_k$ is

$$\widehat{Z}_k = (\mathcal{T}_k)^{(-1)^{k+1}},$$

and for the "duals"

$$\widehat{Z}_k^* = (\mathcal{T}_k^*)^{(-1)^{n-k-1}}.$$

Then,

$$\widehat{Z}_k(\widehat{Z}_k^*)^{-1} = \left(\mathcal{T}_k(\mathcal{T}_k^*)^{(-1)^n}\right)^{(-1)^k} = \mathcal{T}^{(-1)^k},$$

which yields back the finite-dimensional analog of Proposition 17.

In the next sections we generalize the previous elementary facts to cover, first the case in which there is not only one complex but a family of complexes (parametrized by a manifold), and then the case when the spaces arising in the complexes are no longer finite-dimensional. We study in particular the geometry induced by the factorization of the torsion –seen as a metric– in those cases.

### 6.3.2 Factorization of the Torsion of a Chain Complex of Vector Bundles

Let us consider a chain complex $(\mathbb{E}_\bullet, \mathbf{T}_\bullet)$ of vector bundles over $X$, i.e. a collection $\{E_k \overset{\pi_k}{\to} X,\ 0 \le k \le n\}$ of finite rank vector bundles over the smooth manifold $X$, and the maps $\{T_k\}_{k=0,\dots,n}$ such that

$$0 \to E_0 \xrightarrow{T_0} \cdots \to E_{k-1} \xrightarrow{T_{k-1}} E_k \xrightarrow{T_k} E_{k+1} \to \cdots \xrightarrow{T_{n-1}} E_n \to 0, \qquad (6.38)$$

is a chain complex fibrewise, where each $T_k$ must be understood as a bundle map. As in Section 1.2.1 we define the *determinant line bundle* associated to $(\mathbb{E}_\bullet, \mathbf{T}_\bullet)$ by $\det \mathbb{E} = \bigotimes_{k=0}^n (\det E_k)^{(-1)^{k+1}}$, where $\det E_k$ is the determinant bundle on $X$ associated to the vector bundle $E_k$. If the fibration of complexes is *acyclic*, which means that for all $x \in X$ the chain complex $(E_{\bullet,x}, T_{\bullet,x})$ given by

$$0 \to E_{0,x} \xrightarrow{T_{0,x}} \cdots \to E_{k-1,x} \xrightarrow{T_{k-1,x}} E_{k,x} \xrightarrow{T_{k,x}} E_{k+1,x} \to \cdots \xrightarrow{T_{n-1,x}} E_{n,x} \to 0,$$

is acyclic, we can associate to each $x \in X$ a canonical element in the fibre $\det \mathbb{E}_x$, namely the torsion $\tau_x(E_\bullet)$ of the acyclic chain complex $(E_{\bullet,x}, T_{\bullet,x})$ defined in (see Definition 6). We call the *torsion* of the fibration of chain complexes the section defined canonically in this way, and we denote it by $\tau(\mathbb{E}_\bullet)$.

Consider the bundles

$$\mathbb{E}^+ = \bigoplus_{k \text{even}} E_k, \qquad \mathbb{E}^- = \bigoplus_{k \text{odd}} E_k,$$

and the family of isomorphisms

$$D_x^+ = \sum_{k=0}^{n} \left(T_{k,x} + T_{k,x}^*\right) : E_x^+ \to E_x^-,$$

which induces a section $\det \mathbb{D}^+$ on the line bundle $(\det \mathbb{E}_x^+)^* \otimes (\det \mathbb{E}_x^-)$. Recall that for each $x \in X$ there is an isomorphism of vector spaces $\det \mathbb{E}^x = \bigotimes_{k=0}^{n} (\det E_k^x)^{(-1)^{k+1}} \cong (\det \mathbb{E}_x^+)^* \otimes (\det \mathbb{E}_x^-)$ defining a vector bundle isomorphism

$$\det \mathbb{E} \cong (\det \mathbb{E}^+)^* \otimes (\det \mathbb{E}^-).$$

However, the sections $\tau(\mathbb{E}_\bullet)$ and $\det(\mathbb{D}^+)$ of such determinant bundles, do not correspond under this isomorphism as shown by (1.46). The analytic torsion –the modulus of the torsion– yields a natural metric on this bundle, and Proposition 17 yields a factorization of this metric in terms of the metrics defined on certain "subbundles" of it. We now describe the geometry of such a determinant bundle, and extend the factorization of the metric to a factorization of the curvature.

Let us assume that each vector bundle $E_k \stackrel{\pi_k}{\to} X$ is equipped with a hermitian structure and a connection, which we denote $h_k$ and $\nabla_k$, respectively. We consider the associated determinant bundle $\det \mathbb{E} = \bigotimes_k (\det E_k)^{(-1)^{k+1}}$, where $\det E_k$ is the determinant line bundle defined by $E_k$, $0 \le k \le n$, with induced hermitian structure $\hat{h}_k$ and connection $\nabla^{\det E_k}$ from those of $E_k$. The connections $\{\nabla^{\det E_k}\}_{0 \le k \le n}$ induce a connection $\nabla^{\det \mathbb{E}}$ on the bundle $\det \mathbb{E}$ defined by

$$\nabla^{\det \mathbb{E}} = \bigoplus_{k=0}^{n} \left(1 \otimes \cdots \otimes \left(\nabla^{\det E_k}\right)^{(-1)^{k+1}} \otimes \cdots 1\right),$$

and the curvature of $\nabla^{\det \mathbb{E}}$ reads

$$\Omega^{\det \mathbb{E}} = \bigoplus_{k=0}^{n} (-1)^{k+1} \Omega^k.$$

Setting

$$\mathbf{\Omega}^- = \bigoplus_{k \text{ odd}} \Omega^k \qquad \text{and} \qquad \mathbf{\Omega}^+ = \bigoplus_{k \text{ even}} \Omega^k,$$

being $\Omega^k$ the curvature on the bundle $\det E_k$, $0 \leq k \leq n$, it follows that $\Omega^{\det \mathbb{E}} = -\,\mathbf{\Omega}^+ \oplus \mathbf{\Omega}^-$.

Considering now the formal adjoints $T_k^* : E_{k+1} \to E_k$, defined by the hermitian structures $h_k$ on each bundle, and the induced complex of bundles $(\mathbb{E}_\bullet, \mathbf{T}_\bullet^*)$, as in (6.24) we have

$$0 \to E_n \xrightarrow{T_{n-1}^*} \cdots \to E_{k+1} \xrightarrow{T_k^*} E_k \xrightarrow{T_{k-1}^*} E_{k-1} \to \cdots \xrightarrow{T_0^*} E_0 \to 0. \tag{6.39}$$

Above each point $x \in X$ we have a factorization of the torsion in terms of the torsion of the "subcomplexes" of $(\mathbb{E}_\bullet, \mathbf{T}_\bullet)$ and $(\mathbb{E}_\bullet, \mathbf{T}_\bullet^*)$ given by

$$(\mathbb{E}_k, \mathbf{T}_\bullet) \qquad 0 \to E_0 \xrightarrow{T_0} \cdots \longrightarrow E_{k-1} \xrightarrow{T_{k-1}} T_{k-1}E_{k-1} \to 0, \tag{6.40}$$

and

$$(\mathbb{E}_k^*, \mathbf{T}_\bullet^*) \qquad 0 \leftarrow T_k^* E_{k+1} \xleftarrow{T_k^*} E_{k+1} \longleftarrow \cdots \xleftarrow{T_{n-1}^*} E_n \leftarrow 0, \tag{6.41}$$

respectively. Let us define the complex line bundles over $X$

$$\det \mathbb{E}_k = \bigotimes_{i=0}^{k-1} (\det E_i)^{(-1)^{i+1}} \otimes (\det E_k')^{(-1)^k}$$

and

$$\det \mathbb{E}_k^* = \bigotimes_{j=0}^{n-k-1} (\det E_{n-j})^{(-1)^{j+1}} \otimes (\det E_k'')^{(-1)^{n-k-1}},$$

where $E_k'$ and $E_k''$ are point-wise defined by $E_k' = T_{k-1}E_{k-1}$ and $E_k'' = T_k^* E_{k+1}$, respectively. Then

$$\det \mathbb{E} \cong \det \mathbb{E}_k \otimes (\det \mathbb{E}_k^*)^{(-1)^{n+1}}. \tag{6.42}$$

Proposition 17 (and its corollaries) on the factorization of the torsion and the metric on these bundles (induced by the torsion) implies the following

**Proposition 19** *Let* $(\mathbb{E}_\bullet, \mathbf{T}_\bullet)$ *be an acyclic chain complex of* $n+1$ *vector bundles over a smooth manifold* $X$*, and let* $\tau(\mathbb{E}_\bullet)$ *be the torsion of the associated determinant line bundle* $\det \mathbb{E}$*, then the splitting* (6.42) *of* $\det \mathbb{E}$ *induce a splitting on the torsion*

$$\tau_k(\mathbb{E}_\bullet) \otimes \tau_k^*(\mathbb{E}_\bullet)^{(-1)^{n+1}} = \tau(\mathbb{E}_\bullet), \tag{6.43}$$

*where* $\tau_k(\mathbb{E}_\bullet)$ *and* $\tau_k^*(\mathbb{E}_\bullet)$ *denote the torsion of the determinant bundles* $\det \mathbb{E}_k$ *and* $\det \mathbb{E}_k^*$*, respectively.*

As a matter of fact, this decomposition of the bundle $\det \mathbb{E}$ induces a decomposition in the metric, connections and curvature. Notice that the modulus of these torsions gives, via Corollary 3, a factorization of the metric (Proposition 5) on $\det \mathbb{E}$ defined by the analytic torsion $\mathcal{T}(\mathbb{E}_\bullet)$ . The corresponding splitting of the connection is given by

$$\nabla^{\det \mathbb{E}} = \nabla^{\det \mathbb{E}_k} \otimes \mathbf{1}_{n-k} \oplus \mathbf{1}_k \otimes \nabla^{\det \mathbb{E}_k^{*}(-1)^{n+1}},$$

where $\mathbf{1}_i$ denote a product $\underbrace{1 \otimes \cdots \otimes 1}_{i \text{ factors}}$, and yields a splitting of the curvature

$$\Omega^{\det \mathbb{E}} = \Omega^{\det \mathbb{E}_k} \oplus (-1)^{n+1} \Omega^{\det \mathbb{E}_k^*},$$

where the connections and curvatures on the determinant bundles are all the induced from the original ones on the bundles $E_k$ (and on $E_k'$ and $E_k''$ by restriction).

Let us stress the importance of *Hodge decomposition*

$$E_k = T_{k-1} E_{k-1} \oplus T_k^* E_{k+1}, \tag{6.44}$$

that lies behind the above constructions. Using this decomposition we can "glue" the two above subcomplexes into a single piece exactly like in (6.18), namely

$$0 \to E_0 \xrightarrow{T_0} \cdots \xrightarrow{T_{k-2}} E_{k-1} \xrightarrow{T_{k-1}} E_k \xleftarrow{T_k^*} E_{k+1} \xleftarrow{T_{k+1}^*} \cdots \xleftarrow{T_{n-1}^*} E_n \leftarrow 0. \tag{6.45}$$

The splitting of the geometry result follows from this.

## 6.4 Splitting of the Geometry of Determinant Line Bundles in Infinite Dimensions

In this section we generalize the results of the previous section to families of elliptic complexes parametrized by a smooth manifold, which give rise to complexes of infinite-rank vector bundles.

Let $\mathbb{M} \stackrel{\pi_M}{\to} X$ be a smooth locally trivial fibration of manifolds, where $X$ is a smooth manifold of finite dimension and the fibre $M_x = \pi_M^{-1}(x)$ a closed Riemannian manifold, for every $x \in X$. Consider an acyclic elliptic complex $(\mathbb{E}_\bullet, T_\bullet)$ of positive-order differential operators acting on sections of Hermitian vector bundles over the manifold $\mathbb{M}$,

$$0 \to E_0 \xrightarrow{T_0} \cdots \to E_{k-1} \xrightarrow{T_{k-1}} E_k \xrightarrow{T_k} E_{k+1} \to \cdots \xrightarrow{T_{n-1}} E_n \to 0. \tag{6.46}$$

For $0 \leq k \leq n$, let $\mathcal{E}_k \to X$ be the infinite-rank vector bundle whose fibre above $x \in X$ is the space of smooth sections $\mathcal{E}_{k,x} = \Gamma(M_x, E_{k,x})$, where $E_{k,x} \to M_x$ denotes the restriction to $M_x$ of the Hermitian vector bundle $E_k \stackrel{\pi_k}{\to} I\!M$ (we are doing here the same assumptions about $E_k$, $I\!M$ and $X$, for $0 \leq k \leq n$, that we do about $E$, $I\!M$ and $X$ in section 4.2.1). Associated to the family $\{E_{k,x}\}_{x \in X}$ there is a family of positive-order differential elliptic operators

$$T_{k,x} : \Gamma(M_x, E_{k,x}) \to \Gamma(M_x, E^x_{k+1}),$$

or, equivalently, a positive-order differential elliptic bundle map $\mathbf{T}_k : \mathcal{E}_k \to \mathcal{E}_{k+1}$ in the sense of Section 4.2. Thus, the acyclic elliptic complex $(I\!E_\bullet, T_\bullet)$ gives rise to an acyclic elliptic complex $(\mathcal{E}_\bullet, \mathbf{T}_\bullet)$ of positive-order differential elliptic bundle maps on infinite-rank vector bundles over $X$, namely

$$0 \to \mathcal{E}_0 \xrightarrow{\mathbf{T}_0} \cdots \to \mathcal{E}_{k-1} \xrightarrow{\mathbf{T}_{k-1}} \mathcal{E}_k \xrightarrow{\mathbf{T}_k} \mathcal{E}_{k+1} \to \cdots \xrightarrow{\mathbf{T}_{n-1}} \mathcal{E}_n \to 0, \tag{6.47}$$

where each map $\mathbf{T}_k$ corresponds to a family $\{T_{k,x}\}_{x \in X}$ of elliptic positive-order differential operators, parametrized by the manifold $X$.

Quillen's construction associates to each positive-order differential elliptic bundle map $\mathbf{T}_k$ (i.e. to the family $\{T_{k,x}\}_{x \in X}$) a determinant line bundle $\mathrm{Det}\mathbf{T}_k \to X$ with smooth Quillen metric and, assuming the existence of a unitary connection on $\mathcal{E}_k$, one can equip $\mathrm{Det}\mathbf{T}_k$ with a Bismut-Freed connection, which is unitary for the Quillen metric. Moreover, as shown in Theorem 10, the curvature of this connection is "local", i.e. it can be written as the integral of a density on the fibre $M/X$.

From the determinant line bundles $\mathrm{Det}\mathbf{T}_k$ thus built, for each $k$, we shall define the determinant line bundle of the elliptic family of acyclic complexes $(\mathcal{E}_\bullet, \mathbf{T}_\bullet)$. As in the finite-dimensional case, there are two possible constructions for the determinant line bundle of the acyclic family. First, the alternating tensor product of the determinant line bundles $\mathrm{Det}\mathbf{T}_k$ yields the line bundle over $X$ given by

$$\mathcal{L}_{\mathbf{T}} = \bigotimes_{k=0}^{n} (\mathrm{Det}\mathbf{T}_k)^{(-1)^{k+1}}.$$

Second, the $\mathbb{Z}_2$-graded Hilbert bundle $\mathcal{E} = \mathcal{E}^+ \oplus \mathcal{E}^-$ over $X$, where

$$\mathcal{E}^+ = \bigoplus_{\mathrm{k\,even}} \mathcal{E}_k \qquad \text{and} \qquad \mathcal{E}^- = \bigoplus_{\mathrm{k\,odd}} \mathcal{E}_k,$$

gives rise to the determinant line bundle $\mathcal{L}_{I\!D} = (\mathrm{Det}\, I\!D^+)^* \otimes \mathrm{Det}\, I\!D^-$, associated to the corresponding family of Dirac operators

$$I\!D^\pm : \mathcal{E}^\pm \to \mathcal{E}^\mp$$

where

$$D_x^+ = \sum_{k=0}^{n} (T_{k,x} + T_{k,x}{}^*) : \mathcal{E}_x^+ \to \mathcal{E}_x^-,$$

$D_x^- = D_x^{+*}$, and for each $x \in X$, and $\mathcal{E}_x^+$ and $\mathcal{E}_x^-$ are the fibres over $x \in X$ of the bundles $\mathcal{E}^+$ and $\mathcal{E}^-$, respectively. Since we work with an acyclic complex, there is a smooth isomorphism between the fibres of $\mathcal{L}_{\mathbf{T}}$ and $\mathcal{L}_{\mathbb{D}}$ which induces a smooth isomorphism of the line bundles $\mathbf{i} : \mathcal{L}_{\mathbb{D}} \to \mathcal{L}_{\mathbf{T}}$ (see e.g. [BGS88]).

Let $\| \cdot \|_{Q,k}$ denote, for $0 \leq k \leq n$, the Quillen metric on the line bundle $\mathrm{Det}\mathbf{T}_k \to X$, and let $\det \mathbf{T}_k$ denote the canonical section of $\mathrm{Det}\mathbf{T}_k$ defined in Section 4.2.2. Then, the natural metric on $\mathcal{L}_{\mathbf{T}}$

$$\| \cdot \|_{\mathcal{L}_{\mathbf{T}}} = \bigotimes_{k=0}^{n} \| \cdot \|_{Q,k}^{(-1)^{k+1}}$$

is the *analytic torsion*, as follows from the definition of the Quillen metric (4.30) and (1.57) (see also Proposition 7). Moreover, if we consider the canonical section of $\mathcal{L}_{\mathbf{T}}$ given by

$$\tau(\mathcal{E}_\bullet, \mathbf{T}_\bullet) = \bigotimes_{k=0}^{n} (\det \mathbf{T}_k)^{(-1)^{k+1}},$$

we have the following infinite-dimensional analog of Proposition 5 in Section 1.2.1

**Proposition 20** *Let $T(\mathcal{E}_{\bullet,x}, \mathbf{T}_{\bullet,x})$ denote the analytic torsion of the elliptic complex $(\mathcal{E}_{\bullet,x}, \mathbf{T}_{\bullet,x})$ defined in* (1.57)*, then*

$$\| \tau(\mathcal{E}_\bullet, \mathbf{T}_\bullet)(x) \|_{\mathcal{L}_{\mathbf{T}}} = T(\mathcal{E}_{\bullet,x}, \mathbf{T}_{\bullet,x}). \tag{6.48}$$

*Proof.* It follows from (4.31) and the definition of $T(\mathcal{E}_{\bullet,x}, \mathbf{T}_{\bullet,x})$.
□

**Remark.** If $\mathbb{M} = M \times X$, i.e. $M_x = M \quad \forall x \in X$, for a closed Riemannian manifold $M$, taking for all $x \in X$, $E_{k,x} = \Lambda^k T^* M \otimes V_\rho$, $T_{k,x} = d_k : \Omega^k \to \Omega^{k+1}$, we recover the de Rham complex $(\Omega^\bullet, d_\bullet)$ of Example 2 as a particular case. The factorization of the Ray-Singer torsion given in Proposition 17 can be then be interpreted as a splitting in the metric of the determinant line associated to $(\Omega^\bullet, d_\bullet)$.

### Splitting of the Determinant Line Bundle

In the previous section we consider the splitting of the geometry of the determinant line bundle associated to an acyclic complex of Finite-rank vector bundles, let us now do the same in the case of the acyclic elliptic complex $(\mathcal{E}_\bullet, T_\bullet)$, ant its associated determinant line bundle $\mathcal{L}_{\mathbf{T}}$.

Recall that, as in Sections 4.2.1 and 4.2.2, from the family of connections $\{\nabla^{E_{k,x}}\}_{x\in X}$, for each $0 \leq k \leq n$, we construct (by point-wise action) a connection $\nabla^{\mathcal{E}_k}$ on the bundle $\mathcal{E}_k \to X$ which induces a connection on $\mathrm{Det}\mathbf{T}_k$, unitary for the Quillen metric $|\cdot|_{Q,k}$; namely the Bismut-Freed connection $\nabla^{BF}_{(k)}$. Theorem 10 shows that, for each $0 \leq k \leq n$, the curvature $\Omega^{BF}_{(k)}$ of the Bismut-Freed connection $\nabla^{BF}_{(k)}$ can be written as the integral of a local density on the fibre $M/X$. This implies that the curvature of the connection $\nabla^{\mathcal{L}_{\mathbf{T}}}$, defined as the induced by alternating tensor product from the unitary connections $\{\nabla^{BF}_{(k)}\}_{0\leq k\leq n}$ (which is clearly unitary for the metric $|\cdot|_{\mathcal{L}_{\mathbf{T}}}$ previously defined) also has a local curvature, denoted by $\Omega^{\mathcal{L}_{\mathbf{T}}}$.

Let $(\mathcal{E}^{(k)}_\bullet, \mathbf{T}_\bullet)$ and $(\mathcal{E}^{(k)*}_\bullet, \mathbf{T}^*_\bullet)$ be the acyclic elliptic complexes given by

$$(\mathcal{E}^{(k)}_\bullet) \qquad 0 \to \mathcal{E}_0 \xrightarrow{\mathbf{T}_0} \cdots \longrightarrow \mathcal{E}_{k-1} \xrightarrow{\mathbf{T}_{k-1}} \mathbf{T}_{k-1}\mathcal{E}_{k-1} \to 0, \tag{6.49}$$

and

$$(\mathcal{E}^{(k)*}_\bullet) \qquad 0 \leftarrow \mathbf{T}^*_k\mathcal{E}_{k+1} \xleftarrow{\mathbf{T}^*_k} \mathcal{E}_{k+1} \longleftarrow \cdots \xleftarrow{\mathbf{T}^*_{n-1}} \mathcal{E}_n \leftarrow 0, \tag{6.50}$$

respectively. The following theorem shows that, in the infinite-dimensional case of the complex $(\mathcal{E}_\bullet, T_\bullet)$, the splitting of the geometry occurs like in the finite-dimensional case, and the locality property of the curvature is conserved.

**Theorem 13** *Let $\mathcal{L}_{\mathbf{T}} \to X$ be the determinant line bundle associated to the family $\{\mathcal{E}_{\bullet,x}, T_{\bullet,x}\}_{x\in X}$ of acyclic elliptic complexes. Then,*

1. *The Quillen metric factorizes according to* (6.19)*, in terms of the metrics of the determinant line bundles associated to the complexes $\mathcal{E}^{(k)}_\bullet$ and $\mathcal{E}^{(k)*}_\bullet$, as*
$$|\cdot|_Q = |\cdot|_{(k)} |\cdot|^{(-1)^{k+1}}_{(k)*},$$
*where $|\cdot|_{(k)}$ and $|\cdot|_{(k)*}$ denote the curvature of the determinant line bundles associated to the complexes $\mathcal{E}^{(k)}_\bullet$ and $\mathcal{E}^{(k)*}_\bullet$, respectively.*

2. *The curvature splits*
$$\Omega^{\mathcal{L}}_{\mathbf{T}} = \Omega_{(k)} \oplus (-1)^{k+1}\Omega^*_{(k)}, \tag{6.51}$$
*where $\Omega_{(k)}$ and $\Omega^*_{(k)}$ denote the curvature of the determinant line bundles associated to the complexes $\mathcal{E}^{(k)}_\bullet$ and $\mathcal{E}^{(k)*}_\bullet$, respectively.*

3. *This splitting respects the locality properties of the curvature given by* Theorem 10.

*Proof.* This is again consequence of acyclicity, which induce a Hodge decomposition like (1.54) of each level of the complex. Then

$$\mathcal{L}_{\mathbf{T}} = \mathrm{Det}\mathcal{E}_\bullet^{(k)} \otimes \left(\mathrm{Det}\mathcal{E}_\bullet^{(k)*}\right)^{(-1)^{k+1}},$$

and this decomposition of the bundle $\mathcal{L}_{\mathbf{T}}$ induces a decomposition in the metric, connections and curvature. Thus, for example, there is a splitting of the connection

$$\nabla^{\mathcal{L}_{\mathrm{T}}} = \nabla^{\mathcal{E}_\bullet^{(k)}} \otimes \mathbf{1}_{n-k} \oplus \mathbf{1}_k \otimes \left(\nabla^{\mathcal{E}_\bullet^{(k)*}}\right)^{(-1)^{n+1}},$$

where $\mathbf{1}_i = \underbrace{1 \otimes \cdots \otimes 1}_{i \text{ factors}}$, and the connections $\nabla^{\mathcal{E}_\bullet{}^{(k)}}$ and $\nabla^{\mathcal{E}_\bullet{}^{(k)}}$, on the determinant bundles $\mathrm{Det}\mathcal{E}_\bullet^{(k)}$ and $\mathrm{Det}\mathcal{E}_\bullet^{(k)*}$ , respectively, are the induced from the original ones on the bundles $\mathcal{E}_k$ (and on $\mathcal{E}_k'$ and $\mathcal{E}_k''$ by restriction), so *1.* and *2.* follow. In order to prove *3.*, i.e. the locality of the curvatures $\Omega_{(k)}$ and $\Omega_{(k)}^*$ of the Bismut-Freed connections on the determinant line bundles associated to the complexes $(\mathcal{E}_\bullet^{(k)}, \mathbf{T}_\bullet)$ and $(\mathcal{E}_\bullet^{(k)*})$, respectively, we have to prove in first place that the splitting does not affect the ellipticity of the complexes. This is indeed the case because $\mathcal{E}_\bullet^{(k)*}$ and $\mathcal{E}_\bullet^{(k)}$ are defined by point-wise restriction on the range of the family of maps $\{T_{k,x}\}_{x\in X}$. On the other hand, the construction of the Bismut-Freed connections on the bundles $\mathrm{Det}\mathcal{E}_\bullet^{(k)}$ and $\mathrm{Det}\mathcal{E}_\bullet^{(k)*}$ is carried out from the families of connections $\{\nabla_{k,x}\}_{x\in X}$ on the *finite-rank* vector bundles $E_k$. Then, Proposition 19 shows that this splitting gives rise to a honest decomposition of the corresponding connections by restriction, so that once again the *pointwise* nature of the definition of the connections on the infinite-rank vector bundles carries out, as well as the corresponding Bismut-Freed connections on $\mathrm{Det}\mathcal{E}_\bullet^{(k)}$ and $\mathrm{Det}\mathcal{E}_\bullet^{(k)*}$ defined from it. The two elliptic complexes $(\mathcal{E}_\bullet^{(k)}, \mathbf{T}_\bullet)$ and $(\mathcal{E}_\bullet^{(k)*})$ are "independent" one of another because of acyclicity.
□

**Concluding Remarks**

1. Let $\pi : \mathbb{M} \to X$ be a holomorphic submersion of complex manifolds, with compact fibre $M/X$, $W \to \mathbb{M}$ a holomorphic vector bundle with connection and $\mathrm{g}^{\mathbb{M}}$ a Kaehler metric on $T\mathbb{M}$. Consider the associated family of Dolbeault complexes $(\Omega^\bullet(M_x, W_x), \bar\partial_x^\bullet)$, parametrized by $X$, where $\Omega^k(M_x, W_x)$ denotes the space of smooth sections of the bundle $\Lambda^k(T^{*(0,1)}M \otimes W|_{M_x})$. In [BGS88] an explicit local expression for

the curvature of the Bismut-Freed connection on the determinant line bundle associated to this family is given, namely

$$\Omega^{\bar{\partial}^\bullet} = 2\pi i \left[ \int_{M/X} \mathrm{Td}(\nabla^{T\,I\!\!M})\mathrm{Ch}(\nabla^W), \right]_{(2)},$$

where $\mathrm{Td}(\nabla^{T\,I\!\!M})$ and $\mathrm{Ch}(\nabla^W)$ denote the Todd form of $T\,I\!\!M$ and the Chern form of the exterior bundle $W$, respectively.

2. The splitting in the geometry of the determinant line bundle stated in Theorem 13 arises essentially because of the acyclicity assumption of the complex. In the non acyclic case the situation is rather different, since no *canonical* splitting is given from the Hodge decomposition. On the other hand, there is no canonical metric on the determinant line bundle associated to a non acyclic family of complexes. As a matter of fact, there are several metrics on the determinant line bundle, defined from the induced metric on the spaces of harmonic sections by the Hermitian structures on the families of bundles, but in all the known cases these metrics depend on the Riemannian and Hermitian structures used to define them (recall that in the acyclic case the analytic torsion is a topological invariant).

3. Holomorphic analogs of topological gauge theories were introduced by A.D.Popov in [Po1]. There, Chern-Simons and BF topological theories are considered on complex, kaehler and Calabi-Yau manifolds. Among the natural holomorphic extensions of the antisymmetric field theories considered in this work are the given by families of complexes of $(p, q)$-forms on complex manifolds, and holomorphic locally Kaehler fibrations of complex manifolds in the sense of [BGS88]. The existence of an *explicit* local expression for the curvature of the Bismut-Freed connection of the determinant bundle in the latter case rises the question in how far the factorization results developed here can be relevant for such theories.

# Appendix A

# Pseudodifferential Operators

In this section we give a brief presentation of the basic tools in our framework, namely classical pseudo-differential operators and particularly elliptic ones, their logarithms and their complex powers, we shall follow [G95] [Sh01].

**Classical elliptic pseudo-differential operators.** Let $U$ be an open subset of $\mathbb{R}^n$. Given $\alpha \in \mathbb{C}$, let us denote by $S^\alpha(U)$ the set of complex valued smooth function

$$\begin{array}{rcl} \sigma : U \times \mathbb{R}^n & \rightarrow & \mathbb{C} \\ (x,\xi) & \mapsto & \sigma(x,\xi) \end{array}$$

satisfiying the following property. Given any compact subset $K$ of $U$ and any muti-indices $\gamma = (\gamma_1, \ldots, \gamma_n)$ and $\beta = (\beta_1, \ldots, \beta_n)$ in $\mathbb{N}^n$, there exists a constant $C^K_{\gamma,\beta}$ such that

$$|D^\gamma_x D^\beta_\xi \sigma(x,\xi)| \leq C^K_{\gamma,\beta}(1 + \|\xi\|)^{\Re\alpha - |\beta|},$$

for all $x$ in $K$, $\xi \in \mathbb{R}^n$, where $\Re\alpha$ is the real part of $\alpha$, $\|\cdot\|$ denotes the norm in $\mathbb{R}^n$ and $|\beta| = \beta_1 + \ldots + \beta_k$. An element of $S^\alpha(U)$ is called a *symbol of order* $\alpha$. Let $S^{(m)}(U)$ denote, for $m \in \mathbb{Z}^+$, the set of symbols of order $\alpha$ with $\Re\alpha \leq m$. A *smoothing symbol* is a symbol in

$$S^{-\infty}(U) \equiv \bigcap_{k \in \mathbb{N}} S^{(-k)}(U)$$

and the relation

$$\sigma \simeq \tilde{\sigma} \Leftrightarrow \sigma - \tilde{\sigma} \in S^{-\infty}(U)$$

defines an equivalence relation on $S(U)$.

The *principal* or leading part of the symbol $\sigma \in S^\alpha(U)$ is defined by

$$\sigma_\alpha(x,\xi) = \lim_{t \rightarrow +\infty} \frac{\sigma(x,t\xi)}{t^\alpha}.$$

A symbol of order $\alpha$ is called a *classical symbol* if there exist $\sigma_{\alpha-j} \in S^{\alpha-j}(U)$, $j \in \mathbb{N}$, such that

$$\sigma(x,\xi) \simeq \sum_{j=0}^{\infty} \sigma_{\alpha-j}(x,\xi)$$

which are positively homogeneous, i.e.

$$\sigma_{\alpha-j}(x,t\xi) = t^{\alpha-j}\sigma_{\alpha-j}(x,\xi) \quad \forall t \in \mathbb{R}^+.$$

Following Kontsevich and Vishik [KV], we say that a classical symbol lies in the *odd-class* if the positively homogeneous components $\sigma_{\alpha-j}$ are moreover homogeneous i.e.

$$\sigma_{\alpha-j}(x,t\xi) = t^{\alpha-j}\sigma_{\alpha-j}(x,\xi) \quad \forall t \in \mathbb{R}.$$

To a symbol $\sigma \in S^{\alpha}(U)$ we associate a *pseudo-differential operator* A of order $\alpha$ on $U$, i.e. a map

$$A : C_o^{\infty}(U) \to C^{\infty}(U)$$

defined by

$$A\,u(x) = \int_{\mathbb{R}^n} \exp\{i\langle \xi, x\rangle\}\sigma(x,\xi)\hat{u}(\xi)d\xi, \tag{A.1}$$

where $\hat{u}$ denote the Fourier transform of the complex valued smooth funcion $u$ with compact support in $U$. Thus, $A\,u$ can also be writen as

$$A\,u(x) = \int_{\mathbb{R}^n \times U} \exp\{i\langle \xi, x-y\rangle\}\sigma(x,\xi)u(y)dyd\xi, \tag{A.2}$$

$\langle\, , \rangle$ denoting the inner product in $\mathbb{R}^n$.

The various classes of symbols introduced previously induce corresponding classes of pseudo-differential operators. A *classical pseudo-differential operator* is a pseudo-differential operator such that its symbol has components given by classical symbols, an *odd-class classical pseudo-differential operator* is a classical pseudo-differential operator such that its symbol has components given by symbols in the odd class and a *smoothing pseudo-differential operator* is a pseudo-differential operator given by a smoothing symbol. Smoothing operators are representable by smooth kernels, i.e. $A$ is smoothing iff there exists $k_A \in C^{\infty}(U \times U)$ such that, for any $u \in C_o^{\infty}(U)$,

$$A\,u(x) = \int_U k_A(x,y)u(y)dy \qquad \forall x \in U. \tag{A.3}$$

The pseudo-differential operator $A$ with principal symbol $\sigma_A$ is said to be *elliptic* if $\sigma_A(x,\xi) \neq 0$ for all $(x,\xi) \in U \times \mathbb{R}^n - \{0\}$. Ellipticity is not altered when adding a smoothing symbol to the symbol of an elliptic operator.

Let us denote by $\Psi^{\alpha}(U)$ the space of all classical pseudo-differential operators of order $\alpha$ on $U$, and by $\Psi^{(m)}(U)$ the space of all classical pseudo-differential operators of order $\alpha$ with $\Re\alpha$ lower or equal to $m$. Thus, a smoothing pseudo-differential operator is an operator in

$$\Psi^{-\infty}(U) \equiv \bigcap_{k \in \mathbb{N}} \Psi^{(k)}(U),$$

and there is an exact sequence

$$0 \to \Psi^{-\infty}(U) \to \Psi^{(m)}(U) \to S^{(m)}(U) \to 0.$$

There is a notion of product of two pseudo-differential operators and the space of classical pseudo-differential operators on $U$, defined as

$$\Psi(U) = \bigcup_{m \in \mathbb{Z}} \Psi^{(m)}(U),$$

is an associative algebra [Sh01]. The product of two elliptic pseudo-differential operators is an elliptic pseudo-differential operator.

An *ordinary differential operator* of order $d \in \mathbb{N}$ is defined by a polynomial symbol (the polynomial being of order $d$) in $\xi$ of the form

$$\sigma(x, \xi) = \sum_{|\alpha| \leq d} a_{\alpha}(x) \xi^{\alpha}, \tag{A.4}$$

where the $a_{\alpha}$ are smooth functions on $U$ and $\xi^{\alpha} = \xi_1^{\alpha_1} \cdot \xi_2^{\alpha_2} \cdots \xi_n^{\alpha_n}$, for $\xi \in \mathbb{R}^n$ with components $(\xi_1, \ldots \xi_n)$. Then the corresponding differential operator is given by

$$Du(x) = \sum_{|\alpha| \leq d} (-i)^{|\alpha|} a_{\alpha}(x) \frac{\partial^{\alpha_1}}{\partial x_1^{\alpha_1}} \cdots \frac{\partial^{\alpha_n}}{\partial x_k^{\alpha_n}} u(x). \tag{A.5}$$

and its principal symbol is $\sigma_D(x, \xi) = \sum_{|\alpha|=d} a_{\alpha}(x) \xi^{\alpha}$. Hence, ordinary partial differential operators of integer order are examples of classical pseudo-differential operators in the odd class.

**Pseudo-differential operators acting on sections of vector bundles.** The definition of pseudo-differential operators can be locally transfered to smooth manifolds as follows. Let $M$ be a closed (i.e. compact and without boundary) oriented smooth Riemannian manifold with dimension $n$, we say that $A : C^{\infty}(M) \to C^{\infty}(M)$ is a pseudo-differential operator of order $\alpha$ on $M$, if for any local chart $(V, \phi)$ of $M$ such that

$$\phi : V \to U$$

is a diffeomorphism of $V$ with an open set $U$ in $\mathbb{R}^n$, the operator $A_V$ defined by the diagram

$$\begin{array}{ccc} C_o^\infty(V) & \stackrel{A}{\to} & C^\infty(V) \\ \phi^* \uparrow & & \uparrow \phi^* \\ C_o^\infty(U) & \stackrel{A_V}{\to} & C^\infty(U) \end{array} \tag{A.6}$$

is a pseudo-differential operator of order $\alpha$ on $U$.

Considering matrices of pseudo-differential operators on $M$ we can define pseudo-differential operators acting on sections of vector bundles over $M$, the context in which we shall work in what follows. Consider a manifold $M$ as before and consider two hermitian vector bundles $E$ and $F$ over $M$, with rank $k$ and $m$, respectively. A *pseudo-differential operator* of order $\alpha$ acting from the space of smooth sections of $E$ to the space of smooth sections of $F$,

$$A : \Gamma(E) \to \Gamma(F)$$

is a linear operator $A$ which can locally be expressed as a $(m \times k)$-matrix of pseudo-differential operators of order $\alpha$ over $M$, i.e. for any local chart $(V, \phi)$ of $M$ and smooth functions $f, g$ with compact support in $V$, there are local trivializations

$$\Phi_E : E|_V \to \phi(V) \times \mathbb{C}^k$$

and

$$\Phi_F : F|_V \to \phi(V) \times \mathbb{C}^m,$$

such that the map

$$\begin{array}{ccc} C_o^\infty(\phi(V), \mathbb{C}^k) & \to & C^\infty(\phi(V), \mathbb{C}^m) \\ u & \mapsto & \pi_F \circ \Phi_F(g\, A\, f)\Phi_E^{-1}\, u, \end{array} \tag{A.7}$$

where $\pi_F$ denotes the projection $\pi_F : F \to M$, is a pseudo-differential operator of order $\alpha$, and hence its symbol takes values in $S^\alpha(\phi(V))$. A change in the local trivializations used to define the pseudo-differential operator does not change its order, it only changes its symbol by a smoothing one. The symbol is obviously defined locally, only the principal symbol of the operator transforms under a change of trivialization as a section of the vector bundle $Sym(\otimes^n T^*M) \otimes \mathrm{Hom}(E, F)$, where $Sym(\otimes^n T^*M)$ denotes the symmetrized $n^{th}$ power of the cotangent bundle to $M$. When $E$ and $F$ have the same rank, we say that $A$ is *elliptic* iff its principal symbol $\sigma_A$, in any local coordinate representation on $M$, is a non-singular matrix for all $x \in U$, $\xi \in T^*M - \{0\}$.

**Example 10** *Consider a closed Riemannian n-manifold $M$, and let $\Omega^k(M)$ denote the space of smooth k-forms on $M$, for $k = 0, 1, \ldots, n$. Let*

$$d_k : \Omega^k(M) \to \Omega^{k+1}(M)$$

*be the de Rham exterior differentiation operator. Then, for all* $0 \leq k \leq n$, $d_k$ *is a differential operator of order* 1*, and its symbol*

$$\sigma_{d_k}(m, \xi) : \Lambda^k T_m^* M \to \Lambda^{k+1} T_m^* M$$

*is simply left exterior multiplication by* $\xi$*. It is elliptic since* $\sigma_{d_k}(m, \xi)$ *is an isomorphism at* $m \in M$ *for each non-zero* $\xi \in T_m^* M$*.*

Let us denote by $\Psi^\alpha(E, F)$ the space of all classical pseudo-differential operators of order $\alpha$ taking smooth sections of the vector bundle $E$ to sections of $F$, and by $\Psi^{(m)}(E, F)$ the space of all classical pseudo-differential operators of order lower or equal to $m$. The *symbol set* $S^{(m)}(E, F)$ is defined by the exact sequence

$$0 \to \Psi^{-\infty}(E, F) \to \Psi^{(m)}(E, F) \to S^{(m)}(E, F) \to 0,$$

where $\Psi^{-\infty}(E, F) = \bigcap_{k \geq 0} \Psi^{(-k)}(E, F)$. When $E = F$, we shall denote these spaces by $\Psi^\alpha(E)$ and $\Psi^{(m)}(E)$, respectively, and when $E$ is the trivial bundle $M \times \mathbb{C}$ by $\Psi^\alpha(M)$ and $\Psi^{(m)}(M)$, respectively. When $M$ is compact, there is a notion of product of two pseudo-differential operators and

$$\Psi(E) = \bigcup_{m \in \mathbb{Z}} \Psi^{(m)}(E)$$

defines an associative algebra [Sh01]. As before, the product of two elliptic pseudo-differential operators is an elliptic pseudo-differential operator.

*Notation.* Let $E$ be a vector bundle above a smooth $n$-dimensional Riemannian manifold $M$, and let $\mathcal{C}l(E)$ denote the algebra of classical pseudo-differential operators acting on smooth sections of $E$. We shall denote by $Ell(E)$, $Ell^*(E)$, $Ell^*_{ord>0}(E)$, $Ell^{s.a}_{ord>0}(E)$ and $Ell^+_{ord>0}(E)$ the class of elliptic, invertible elliptic, invertible elliptic with strictly positive order, self-adjoint elliptic and positive self-adjoint elliptic operators with strictly positive order acting on sections of $E$, respectively.

**Admissible elliptic pseudo-differential operators.** If $M$ is compact, and we shall asume that in what follows, the spectrum of $A \in Ell^*_{ord>0}(E)$, denoted spec$(A)$, consists of isolated eigenvalues with finite multiplicity [Sh01]. There is therefore a disc $D_R$ of positive radius around the origin which does not contain any point of spec$(A)$. We shall say that $A$ has a *spectral cut* $L_\theta$ if there is a ray $L_\theta = \{\lambda \in \mathbb{C}, \arg(\lambda) = \theta\}$ in the complex plane which does not intersect spec$(A)$. Such an operator will be called *admissible* and we shall denote by $\mathcal{A}d(E)$ the set of admissible operators acting on sections of $E$. Any element of $Ell^*_{ord>0}(E)$ such that the matrix given by its principal symbol has no eigenvalues in some non empty conical neighborhood $\Lambda$ of a ray in the spectral plane is admissible, since in that case at most a finite number of eigenvalues of the operator are contained in $\Lambda$ [Sh01]. We have following inclusions

$$Ell^{*+}_{ord>0}(E) \subset Ell^{*s.a}_{ord>0}(E) \subset \mathcal{A}d(E),$$

where the superscript $*$ means that we are restricting to the subset of invertible operators in each one of the considered classes.

**Complex powers and logarithms of elliptic operators.** Let $A \in \mathcal{A}d(E)$ with spectral cut $L_\theta$. For $\Re z < 0$, the *complex power* $A^z_\theta$ of $A$ is the bounded operator on any space $H^s(E)$ of sections of $E$ of Sobolev class $H^s$, defined by the contour integral

$$A^z_\theta = \frac{i}{2\pi}\int_{\Gamma_\theta} \lambda^z (A - \lambda I)^{-1} d\lambda$$

where $\Gamma_\theta = \Gamma_{1,\theta} \cup \Gamma_{2,\theta} \cup \Gamma_{3,\theta}$ is the path on the complex plane given by $\Gamma_{1,\theta} = \{\lambda = re^{i\theta}, r \geq R\}$, $\Gamma_{2,\theta} = \{\lambda = Re^{i\phi}, \theta \geq \phi \geq -\theta\}$, $\Gamma_{3,\theta} = \{\lambda = re^{i(\theta-2\pi)}, r \geq R\}$, $R$ being the radius of a disc around the origin which does not intersect $\mathrm{spec}(A)$. Here $\lambda^z = \exp\{z \log \lambda\}$ with $\log \lambda = \log|\lambda| + i\theta$ on $\Gamma_{1,\theta}$, and $\log \lambda = \log|\lambda| + i(\theta - 2\pi)$ on $\Gamma_{3,\theta}$.

The definition of $A^z_\theta$ is independent of the choice of $R$ but depends on the choice of the spectral cut $L_\theta$ and yields, for any $z \in \mathbb{C}$, an elliptic operator of order $z \cdot \mathrm{ord}(A)$. When $z = -k$, with $k \in \mathbb{N}$, $A^z$ coincides with the usual operator $A^{-k}$ of order $-k \cdot \mathrm{ord}(A)$. The operator $A^z_\theta$ is independent of the choice of $\theta$ only if $A$ is essentially self-adjoint, in which case it coincides with the corresponding complex power defined using spectral representations. However, in the following we shall focus on operators in $Ell^{+}_{ord>0}(E)$ and use the principal branch of the logarithm, taking $\theta = \pi$, and dropping the mention of $\theta$.

For arbitrary $k \in \mathbb{Z}$, the map $z \to A^z_\theta$ defines a holomorphic function from $\{z \in \mathbb{C}, \Re z < k\}$ to the space of bounded linear maps from $H^s(E)$ to $H^{s-k\mathrm{ord}(A)}(E)$ for any real $s$. We set the *logarithm* of $A \in \mathcal{A}d(E)$ to be

$$\log_\theta A = \frac{\partial}{\partial z} A^z_\theta \bigg|_{z=0},$$

which defines a *non classical* pseudo-differential operator of zero order, and hence a bounded operator from $H^s(E)$ to $H^{s-\epsilon}(E)$ for any $\epsilon > 0$ and any $s \in \mathbb{R}$. In local coordinates $(x, \xi)$ on $T^*M$, the symbol of the operator $\log_\theta A$ is the sum of $\mathrm{ord}(A) \cdot \log|\xi|\, \mathrm{I}_d$ with the symbol of a classical pseudo-differential operator of order 0. Hence, although the logarithm of an injective admissible elliptic classical pseudo-differential operator with spectral cut $L_\theta$ is not itself a classical pseudo-differential operator, for two operators $A, B \in$

$\mathcal{A}d(E)$, admitting spectral cuts $L_{\theta_1}$ and $L_{\theta_2}$,

$$\frac{\log_{\theta_1} A}{\mathrm{ord}(A)} - \frac{\log_{\theta_2} B}{\mathrm{ord}(B)} \in \mathcal{C}l(E).$$

In the same way, for $A, Q \in \mathcal{A}d(E)$, the bracket $[\log Q, A]$ is a classical pseudo-differential operator.

# Appendix B

# The Partition Function in Quantum Field Theory

### The Action in Classical and Quantum Field Theory

A field theory, from a physical point of view, is defined by a (finite-dimensional) *space-time manifold* $M$ (with a Riemannian or Minkowskian metric structure), a *target space* $F$ (usually a manifold playing the role of fibre in a given vector fibration $E$ over the space-time manifold), the set of *fields* $\Phi = \Gamma(E)$ (vector valued functions or more generally sections of the above mentioned fibration), and relating $M, E, F$ and $\Phi$, and defining the dynamics of the theory, a functional on the set of fields, $S : \Phi \to \mathbb{R}$ (or $\mathbb{C}$), called the *Action.* Aditional information cames from the possible symmetries of the theory, that can be incorporated by means of group actions (on the space-time, the target space or the total space of the fibration) or by symmetries in the functional form of the action (see Freed and Deligne Lectures in [Dea99]).

The action is taken in general to be a functional of the form

$$S = \int_M L(\phi, \partial\phi)\, d\mu_M,$$

where $d\mu_M$ denotes the volume element defined by the metric on $M$ and $L$ is the *lagrangian density* of the theory, function of the fields (and its derivatives). Classical dynamics of the fields is determined by the solution to a variational problem, or "least action principle" [BS80] [IZ88]. This means that the *physical fields* $\phi_p$ are those described by the solutions to the extremal problem $\left.\frac{\delta S}{\delta \phi}\right|_{\phi_p} = 0$ or, equivalently, the Euler-Lagrange equations of the field.

A quantum field is a generalized function $\hat{\phi}$ taking values in a space of operators acting on a Hilbert space, satisfying some particular axioms (which we shall no discuss here, see Kazhdan's Lectures in [Dea99]). Thus, from an empty state $|0\rangle$ in the Hilbert space, all the dynamical information of the quantum field theory can be obtained through the quantities $\langle 0| : \hat{\phi}(x_1)\hat{\phi}(x_2)\cdots \hat{\phi}(x_n) : |0\rangle$, called correlation functions, which represent *probability amplitudes*, the $:\,:$ denoting a decrasing temporal order in $x_1, x_2, \ldots, x_n \in M$. From a path integral point of view, quantum dynamics can be also entirely defined by the action, this time through the corresponding *path integrals*. The basic idea in functional or path integral quantization is to consider that the probability amplitudes can be expressed in terms of integrals over the space of *all* the possible dynamical trajectories of the system, i.e.

$$\langle 0| : \hat{\phi}(x_1)\hat{\phi}(x_2)\cdots \hat{\phi}(x_n) : |0\rangle = \frac{1}{Z_o}\int_\Phi \phi(x_1)\phi(x_2)\cdots\phi(x_n)\exp\{-S(\phi)\}\,[\mathcal{D}\phi],$$

where $[\mathcal{D}\phi]$ denotes a *formal* measure on the space of all the fields $\phi$ and $S$ the classical action of the theory under consideration. We sall then consider *generating functionals*, i.e. formal objects of the form

$$Z(S,F) = \frac{1}{Z_o}\int_\Phi F(\phi)\exp\{-S(\phi)\}\,[\mathcal{D}\phi], \tag{B.1}$$

where $F$ a functional on the space of fields. Here $Z_o$ denotes the *partition function* of the theory, given by equation (B.1) when $F(\phi) = 1$, a normalization factor in order to have $\langle 0|0\rangle = 1$. $\Phi$b eing tipically an infinite-dimentional manifold , the formal Lebesgue-type measure on $\Phi$, $[\mathcal{D}\phi]$, is ill-defined in general.

### Perturbative Expansion and Stationary Phase Approximation

An action functional generally contains two kinds of terms: kinematical and interaction terms. Hence, it can be wirtten as $S = S_o + S_i$, where $S_o$ denotes a quadratic (kinematical) functional in the fields and $S_i$, assumed to contain all the information about the interactions of the fields. Thus, the formal integral in equation (B.1), through a formal series expansion of the exponential containing the interaction term, can be written as

$$\int_\Phi F(\phi)e^{-iS}\,[\mathcal{D}\phi] = \int_\Phi F(\phi)e^{-iS_o}\left(\sum_{k=0}^{\infty}\frac{1}{k!}(-iS_i)^k\right)[\mathcal{D}\phi], \tag{B.2}$$

giving the whole path integral as a sum of (infinite-dimensional) Gaussian integrals. Even if $Z_o$, $\int_\Phi$ and $[\mathcal{D}\phi]$ make no sense as mathematical objects, physicists make manipulations of the whole object imitating the well-known techniques and results about Gaussian integrals in finite dimensions (change of variable formulae, Fourier transforms, ...), obtaining numerical results which are in extraordinary accordence with experimental measures. This indicates that, even if the heuristic object has no mathematical meaning, a well defined mathematical object must be behind it.

Perturbation theory, through Feynman rules, shows how to associate to each term in the formal sum (B.2) of heuristic path integrals a well-defined integral, which sometimes diverges but can be made finite by the use of *renormalization*. A great amount of physical information can be so obtained perturbatively. However, not less relevant (physical and mathematical) information can be obtained from the theoretical study of the formal object (B.1), through the constructive point of view or the so-called non-perturbative methods. Constructive field theory tries to make sense of path integrals as properly defined integrals, i.e. through the study of measure theory on functional spaces, considering each single model and giving a sense to the corresponding measure, partition function and generating functionals. The idea, following the Wiener measure approach in stochastic analysis, is to consider the formal measure thogether with the exponential of the quadratic part of the action as defining a *Gaussian measure* $d\mu_\phi$ on the corresponding functional space, thus taking of the fact that Gaussian measures (contrary to Lebesgue measures) on infinite-dimensional spaces do exists. It has been developed by many physicists and mathematicians –E. Nelson, A. Jaffe, S. Albeverio and many others– and has lead to many interesting results in theoretical physics as in stochastic analysis (see e.g. [AlH76][Al97]).
Non-perturbative methods have lead to not only physically relevant results, but also –and most at all– mathematical results, in particular about the geometry and the topology of the underlying components of the field theoretical description (the manifold playing the role of space-time, the fibration,...). In the last twenty years the study of these methods, initiated by the pioniering works of A.Schwarz and E. Witten, and developed by many others after, gave rise to a whole branch of mathematical physics called *Topological Quantum Field Theories* (for a review, containing abundant reference to the original bibliography see [BBRT91]). Regularized determinants of differential or pseudodifferential operators acting on infinite-dimensional vector spaces are a fundamental component of this point of view, where they model the partition funtions.

In the following section we shall consider the main facts about finite-dimensional Gaussian integrals which promped some of the formal manipulations of heuristic path integrals.

**Gaussian Measures and Fourier Transforms on Finite-dimensional Vector Spaces**

Let $V$ be a $n$-dimensional oriented vector space, with inner product $\langle\,,\rangle$ and a basis $\{v_1, v_2, \dots, v_n\}$ such that

$$v_1 \wedge v_2 \wedge \cdots \wedge v_n = vol_{(V)},$$

where $vol_{(V)}$ (also denoted $dv$) denotes the oriented volume element on $V$, or "Lebesgue measure", defined by the metric induced on $V$ by its inner product. The Gaussian approach we follow here is based in the finite-dimensional Gaussian integral

$$\int_V e^{-\frac{1}{2}Q(v)}\, dv = (\det T_Q)^{-\frac{1}{2}}, \tag{B.3}$$

where $Q(v) = \langle T_Q v, v\rangle$ is a symmetric and positive quadratic form defined on $V$.

To each inner product on $V$ there is an associated *Gaussian Measure* given by

$$\mu(X) = \frac{1}{(2\pi)^{\frac{n}{2}}} \int_X e^{-\frac{1}{2}\langle v,v\rangle} dv.$$

Let $\mathcal{S}(V)$ denote the Schwartz space of rapidly decreasing smooth functions on $V$. Then the *Fourier Transform* of a function $f \in \mathcal{S}(V)$ is the function $\widehat{f} \in \mathcal{S}(V^*)$ given by

$$\widehat{f}(x) = \frac{1}{(2\pi)^{\frac{n}{2}}} \int_V f(v) e^{-i\langle x,v\rangle} dv.$$

The Fourier inversion formula

$$f(v) = \frac{1}{(2\pi)^{\frac{n}{2}}} \int_{V^*} \widehat{f}(x) e^{i\langle x,v\rangle} dx$$

implies that $F^2(f)(v) = f(-v)$, where

$$\begin{array}{rcl} F : \mathcal{S}(V) & \rightarrow & \mathcal{S}(V^*) \\ f & \mapsto & \widehat{f}. \end{array}$$

For example, the Fourier transform of the function $f(v) = e^{\frac{i}{2}\langle Av,Av\rangle}$ (where $A$ is a nonsingular and symmetric matrix) is

$$\widehat{f}(x) = \frac{1}{(2\pi)^{\frac{n}{2}} \det A}\; e^{-\frac{i}{2}\langle Ax,Ax\rangle}.$$

The *Fourier Transform* of a (positive) measure $\mu$ in $V$ is the function on $V^*$ (identified with $V$ through the inner product $\langle\, ,\rangle$) defined by

$$\widehat{\mu}(x) = \int_V e^{-\frac{1}{2}\langle x,v\rangle} d\mu(v). \tag{B.4}$$

**The Stationary Phase Method for Finite Dimensional Path Integrals**

The stationary phase method studies the asymptotic behavior of integrals of the form

$$I(\hbar) = \int_V e^{\frac{i}{\hbar}S(x)} a(x) dx,$$

when $\hbar \to 0$, where $S : V \to \mathbb{R}$ is a $C^\infty$ function and $a \in C_o^\infty(V, \mathbb{C})$, and $V$ is a $n$-dimensional vector space. The constant $\hbar$ is the analog to the Planck constant in path integrals, and the limit $\hbar \to 0$ physically means taking the classical limit of the theory. The following Proposition shows that the main (asymptotic) contribution to $I(\hbar)$ come from the non-degenerate critical points of $S$.

**Proposition 21** [Hor] *Let $\{x_1, x_2, \ldots, x_m\} \in \mathrm{Supp}(a)$ be the only non-degenerate critical points of $S(x) = \langle T^S x, x\rangle$, then for each $N \in \mathbb{Z}^+$*

$$I(\hbar) = (2\pi\hbar)^{\frac{n}{2}} \sum_{i=1}^{m} \left\{ \frac{e^{\frac{i\pi}{4}\mathrm{sign}(S)}}{|\det T^S|^{\frac{1}{2}}} \sum_{k=0}^{N} \frac{1}{k!} \left( D^k a \right)(x_i) \hbar^k \right\} + \mathrm{O}(\hbar^{N+1+\frac{n}{2}}), \quad \text{(B.5)}$$

*where $D$ is the differential operator given by*

$$D = \frac{i}{2} \sum_{i,j} T_{ij}^{S^{-1}} \frac{\partial}{\partial x_i} \frac{\partial}{\partial x_j}.$$

It follows from this that

$$\int_V e^{\frac{i}{\hbar}S(x)} a(x) dx \sim (2\pi\hbar)^{\frac{n}{2}} \sum_{i=1}^{m} \frac{e^{\frac{i\pi}{4}\mathrm{sign}(S)}}{|\det T^S|^{\frac{1}{2}}} a(x_i), \quad \text{(B.6)}$$

which, in the case of a positive definite matrix $T^S$ give us

$$\int_V e^{\frac{i}{\hbar}S(x)} a(x) dx \sim (2\pi\hbar)^{\frac{n}{2}} \sum_{i=1}^{m} (\det T^S)^{-\frac{1}{2}} a(x_i).$$

Notice that in classical mechanics the critical values of the action are the classical trajectories (or fields). Here, if there is only one non-degenerate critical value at $x_o$

$$\int_V e^{\frac{i}{\hbar}S(x)} a(x) dx \sim (2\pi\hbar)^{\frac{n}{2}} (\det T^S)^{-\frac{1}{2}} a(x_o),$$

so, in the limit $\hbar \to 0$, the quntum dynamics can be seen as a perturbation of the classical dynamics. Moreover, recall that in our finite-dimensional model, the partition function of the path integral $I(\hbar)$ is $(\det T^S)^{-\frac{1}{2}}$. Applying this to the heuristics of the path integrals, the stationary phase method shows that from the *partition function* of the theory the "semiclassical" information of the field theory can be obtained through perturbative methods [IZ88]. Thus, the computation of many path integrals in field theory are done from the partition function of the theory, i.e. generalizating the well-known properties of purely Gaussian integrals in finite dimensions (change of variables, Fourier transforms,...), where they can be given a mathematical meaning.

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