Geometric Quantization: The Magnetic Monopole Case and Quantum Hall Effect

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Introduction

Twentieth century physics leave us two great theories: *General Relativity* and *Quantum Mechanics*, both of them with a big amount of mathematical sophistication; *Differential Geometry* is the natural language for the former, *Topology* and *Complex Analysis* are the basis for the second one. Basically all of today's physics, except for general relativity, is formulated in quantum terms. The strong, weak and electromagnetic interactions are described by means of *Quantum Field Theories* whose tendency is the unification of the different descriptions in a sole "theory of everything" that takes into account all interactions as consequences of one universal principle. As a matter of fact, a successful unification theory for weak and electromagnetic interactions, *Electroweak Theory*, was formulated about thirty years ago. The main obstacle in searching for a general "supertheory" is the impossibility of constructing a physically admissible quantum theory of gravity. (Since 1916 its classical formulation has remained practically unchanged and all the research in this direction has been unsuccessful.) Nevertheless, an intense investigation in this area of theoretical physics has been made and it has left many important results that, by the way, have encouraged some areas of mathematics more than physics itself, so much that in the last two decades quantum field theories have become a research subject with increasing interest for mathematicians, and Field medalists are found among theoretical physicists.

Summarizing, the situation at the present time exhibits two different types of settings: a *geometric* theory (for general relativity) and several (*topological*) *quantum* theories (for strong, weak and electromagnetic interactions). In the past few years, the diverse attempts towards unification have undertaken the *quantization* of gravity rather than the *geometrization* of quantum theories. There are some elaborated theories in this last direction with little profit for physics but with a great mathematical interest (for example superstring theories), and the question about the formal meaning of building a quantum description of a physical system, in a *geometrical* setting, has become to be important looking for the solution of that kind of problems.

This thesis considers a geometric setting for quantization, the so-called

Geometric Quantization. Specifically, we apply the standard techniques of *Geometric Quantization* in order to treat two (physically and mathematically) interesting cases: the magnetic monopole and its relation with charge quantization and the *2*-dimensional n electron system. Both of these cases are now recognized as samples of physically sensible *topological effects.* Geometric Quantization was born in the late 60's and 70's, its origin comes from the *Representation Theory* and *Symplectic Geometry.* This scheme does not introduce any new physical idea, but tries to give a formal foundation to the quantum formulation of the description of a physical system from its classical formulation in terms of Hamiltonian (symplectic) dynamics. The more remarkable element of this description is that it shows how quantum phenomena are determined by topological conditions (through "topological numbers ") that characterize the elements in terms of which the quantum description can be carried on, in each particular case.

The first of the cases considered in this work (magnetic monopoles) is historically recognized as the first of the non-trivial topological systems in physics, once again its characteristics are exposed and analyzed, now from the geometric quantization point of view, and the relation with other description is established. The n-electron system in two dimensions, studied in the last chapter, is known between the condensed matter theoreticians as source of a lot of interesting effects (Integer and Fractional Quantum Hall Effect, Anyons, Superconductivity and others) and its study through geometric quantization convince us of the power of the method.

The text is organized as follows. In the first chapter a formal review of the theory is accomplished, almost nothing in this chapter is original except the presentation and some examples and observations, the basic prerequisites for the lecture of this chapter (symplectic geometry, Hamiltonian mechanics, moment mappings, symplectic reduction, etc.) are included in the appendixes at the end of the document, trying to do the text self-contained. Chapters two and three contain the body of the work, the study of the magnetic monopole case and the n electron system, respectively. It is shown how in both of these cases the "quantum numbers" are topological numbers that characterize the corresponding physical description, in the case of magnetic monopoles these numbers correspond with the monopole magnetic charge and in the n electron system case the so-called filling factors for the *Quantum Hall Effect.*

Conventions and Notation

Through this work we assume that all the objects (manifolds, functions, etc.) are infinitely differentiable, even if it is not explicitly mentioned. We use some physical constants that are standard in mathematical physics works, in particular $\hbar = \frac{h}{2\pi}$, where h is the Plank Constant, is extensively used.

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Chapter 1

Geometric Quantization

A fundamental problem in Theoretical Physics is the complete understanding of the quantization process of a physical system, it is the process that, from a classical description of the dynamics of such a system, brings us the corresponding quantum description of its dynamics. Mathematically this process represents the path from a geometric differential description (on a symplectic manifold) to a topological point of view (quantum conditions) and a different scenario: a Hilbert Space.

There exist some standard and "fundamental" schemes for quantization, *canonical* and *functional quantization* for example, but none of them gives in a precise and exact way the construction of the Hilbert space necessary for quantization or the representation of the algebra of classical physical observables in the algebra of self adjoint operators on the Hilbert space from the classical description[1]. The *Geometric Quantization* scheme tries to give necessary and sufficient conditions for the existence of a quantum description of the dynamics of a system classically described using a symplectic manifold, and describes in a precise way the construction of the Hilbert space, operator representation and Lie algebras associated with this description, all this through the methods of the differential geometry and topology. The main scenarios here are the Complex Line Bundles over manifolds and the topological invariants that give its classification.

This chapter develops the three basic steps in the geometric quantization of a classical system: prequantization, introduction of polarizations and half-forms correction, and all the necessary material to carry on the study of the particular cases in the next two chapters. It is assumed a minimum

[1]See L. Ryder ,"Quantum Field Theory", Cambridge University Press (1992).

background in geometrical description of classical mechanical systems at the level of [A1], however a survey of definitions, classical results and the most important examples used through all the work can be found in appendix 1 and 2.

1.1 Prequantization Line Bundle

1.1.1 The Dirac Problem

Given a physical system whose dynamics can be classically described on a symplectic manifold (M, ω), where the set of classical observables $C^\infty(M, \mathbf{R})$ has structure of Lie algebra with the usual sum and scalar product, and the Poisson bracket (denoted by $\{,\}$, see appendix 2), we should build up the ground on which the correspondent quantum description takes place (a Hilbert Space $\mathcal{H}$) and the objects on such ground that determine this description (wave functions, quantum observables, etc.). In a formal way, following Dirac[2], looking for a physically admissible quantum theory, to each classical observable f corresponds a quantum Hermitian operator $\widehat{f}$ on the Hilbert space, the set of these "quantum observables" should have structure of Lie algebra with the Lie bracket ($[\widehat{x}, \widehat{y}] = \widehat{x}\widehat{y} - \widehat{y}\widehat{x}$) and must be verified that

- 1. The application $f \longrightarrow \widehat{f}$ is linear
- 2. If f is constant then $\widehat{f}$ must be the multiplication (by the constant f) operator
- 3. If $\{f, g\} = h$ then $[\widehat{f}, \widehat{g}] = -i\hbar\widehat{h}$

These three conditions, that we will refer to as *Dirac Conditions*, say in other words that we are looking for a *representation* of the classical algebra of observables on M in the algebra of quantum observables on the Hilbert space $\mathcal{H}$. The characteristics of this representation (reducibility, etc.) are determined for the physically admissible results of the quantum description obtained from it.

The quantum description of a physical system is made in terms of *wave functions*, these are complex functions on the phase space of the system ($\psi : T^*Q \to \mathbf{C}$) that have not *direct* physical meaning (do not represent a

[2]P.A.M. Dirac, Proc. Roy. Soc. London A, 109, 642 (1925).

physical magnitude as energy or momentum) but in terms of which all the probabilistic information given by the theory can be found. More precisely these probabilities are given by products as $\psi^*\psi$, where * denotes complex conjugation, and this shows how a "phase" of the type $e^{i\theta}$ in the wave function does not alter the probabilities (and then the physical predictions of the theory), because

$$(e^{i\theta}\psi)^*(e^{i\theta}\psi) = e^{-i\theta}\psi^* e^{i\theta}\psi = \psi^*\psi.$$

In this way it is clear that wave functions in a more general context can be seen as sections of a complex Hermitian line bundle $L \xrightarrow{\pi} M$ on phase space, given that each fiber is isomorphic to the complex numbers, using the Hermitian structure to define, as norms of complex numbers, the quantities whose probabilistic meaning was mentioned.

Definition 1 (L. Van Hove, 1951):

A *prequantization* of the cotangent bundle $(T^*\mathbf{R}^n,\ dp_i \wedge dq_i)$ is an application that to each function $f \in C^\infty(T^*\mathbf{R}^n, \mathbf{R})$ associates an Hermitian operator $\widehat{f}$ on a Hilbert space $\mathcal{H}$ in such way that Dirac conditions are satisfied.

As a matter of fact Van Hove also show that given $(\mathcal{H}, \mathcal{O})$, where $\mathcal{H} = L^2(\mathbf{R}^n, \mathbf{C})$ and $\mathcal{O}$ is the set of Hermitian operators on $\mathcal{H}$, the application

$$\begin{array}{rcl} \wedge : C^\infty(T^*\mathbf{R}^n, \mathbf{R}) & \longrightarrow & \mathcal{O} \\ f & \mapsto & \widehat{f} = i\hbar X_f - \theta(X_f) + f \end{array}$$

where X_f is the Hamiltonian Vector Field defined by f and $\theta = p_i dq_i$ is the symplectic potential[3], is a prequantization for $(T^*\mathbf{R}^n,\ dp_i \wedge dq_i)$. The problem of finding conditions for the existence of a representation of this kind, and to describe precisely such "quantization", in the case of a arbitrary symplectic manifold is one of the *Geometric Quantization* subjects, and begins by choosing the Hilbert space to be used and the representation of the observables.

1.1.2 Existence of the Prequantization Hilbert Space

Let (M, ω) be a symplectic manifold that represents the phase space of a physical system, then $M = T^*Q$ for some manifold Q (the configuration space) and $\omega = d\theta = d(p_i dq_i) = dp_i \wedge dq_i$. Going to the quantum description

[3] See appendix 1

corresponding to this system we need a complex line bundle (in terms of its sections we define wave functions , i.e. the Hilbert space $\mathcal{H}$ of the theory) and an application such that to each function $f \in C^\infty(M, \mathbf{R})$ associate an Hermitian operator $\widehat{f}$ on $\mathcal{H}$ in such way that the Dirac conditions can be satisfied. The conditions for the existence of this bundle (called *prequantization bundle*) are given by the following theorem [W1]:

Theorem 1: Let (M, ω) be a symplectic manifold, then there exists a complex line bundle $L \xrightarrow{\pi} M$ on this manifold and a connection ∇ on L with curvature $\hbar^{-1}\omega$ if and only if the class of $(2\pi\hbar)^{-1}\omega$ in $H^2(M, \mathbf{R})$ is in the image of $H^2(M, \mathbf{Z})$ under the inclusion in $H^2(M, \mathbf{R})$, i.e. if any integral of ω on a oriented 2-surface in M is an integer multiple of $2\pi\hbar$.

Up to isomorphism, the possible choices of the prequantization bundle are parameterized by the cohomology group $H^2(M, U(1))$. In the case of cotangent bundles this condition is automatically satisfied because of its simple connectedness, then the interesting cases are symplectic manifolds that are not, in general, cotangent bundles. Note that this condition of existence of the prequantization bundle (and then of the quantum description itself) is *topological* in character, as all the conditions that appear in the quantum theories of physics, so we can say that all the quantum theories are in essence topological theories whose fundamentals are in the differential-geometric description of the classical models.

1.1.3 Representation of the Algebra of Operators

If the integrality condition is verified on the symplectic manifold correspondent to the classical description, and then exist a Hermitian line bundle $L \xrightarrow{\pi} M$, we define the *Hilbert Space of Prequantization* $H(M, L)$ as the *completion* of the space formed by the square integrable sections[4] $s : M \to L$, noted $\Gamma(L)$, with the inner product

$$(s, s') = \int_M \langle s, s' \rangle \, \epsilon$$

where ϵ is the element of volume of the manifold M ($\epsilon = \frac{1}{2\pi\hbar} dp_1 \wedge \ldots \wedge dp_n \wedge dq_1 \wedge \ldots \wedge dq_n$), and to each observable f we associate an Hermitian

[4] This sections are our *wave functions*.

operator according to the so-called Konstant-Souriau representation

$$\begin{aligned} \wedge : C^\infty(M, \mathbf{R}) &\longrightarrow \mathcal{O} \\ f \quad &\longmapsto \hat{f} = f - i\hbar\nabla_{X_f} \end{aligned}$$

where $\nabla_{X_f}(s)$ denotes the action of the 1-form ∇s on the Hamiltonian vector field generated by f [5]. Note that $\Omega_\nabla = \hbar^{-1}\omega$ implicates that $d\widetilde{\theta} = \hbar^{-1}d\theta$, where θ comes from the symplectic structure and $\widetilde{\theta}$ from the connection on the line bundle, in terms of such this 1-form

$$\begin{aligned} \hat{f} &= f - i\hbar(d - i\widetilde{\theta})(X_f) \\ &= f - i\hbar(d - i\hbar^{-1}\theta)(X_f). \end{aligned}$$

1.1.4 Some Examples

Canonical Operators:

Lets calculate as first example the operators corresponding to position q_i and momentum p_i in phase space, modelled as a cotangent bundle with canonical symplectic form. In this case the correspondent line bundle associated is $P \times \mathbf{C}$ and the representation corresponding to the observables is

$$\hat{f} = f - i\hbar X_f - (p_i dq_i)(X_f)$$

then by definition

$$\overset{\wedge}{p_i} = p_i - i\hbar X_{p_i} - (p_i dq_i)(X_{p_i})$$

so given that $X_{p_i} = \frac{\partial}{\partial q_i}$ then

$$\overset{\wedge}{p_i} = p_i - i\hbar\left(\frac{\partial}{\partial q_i}\right) - p_i = -i\hbar\frac{\partial}{\partial q_i}.$$

In the same way for "position"

$$\overset{\wedge}{q_i} = q_i - i\hbar X_{q_i} - (p_i dq_i)(X_{q_i})$$

and so $X_{q_i} = -\frac{\partial}{\partial p_i}$ gives as the result

$$\overset{\wedge}{q_i} = q_i - i\hbar\left(-\frac{\partial}{\partial p_i}\right) = q_i + i\hbar\frac{\partial}{\partial p_i}.$$

[5] A proof of the hermiticity of this operator can be found in books as [P2] in bibliography.

This result is not in accordance with quantum mechanics (Scrhödinger's version)

$$\overset{\wedge}{p_i} = -i\hbar \frac{\partial}{\partial q_i} \quad \text{and} \quad \overset{\wedge}{q_i} = q_i$$

and in fact this is against the uncertainty principle, which illustrates the necessity of a correction that will be carried on by the introduction of a "polarization" on the prequantization bundle, in the next section.

In the same way that a change in the coordinates $\{q_1, ..., q_n, p_1, ..., p_n\} \to \{z_1, ..., z_n, \overline{z}_1, ..., \overline{z}_n\}$, where $z_i = p_i + iq_i$ and $\overline{z}_i = p_i - iq_i$, on phase space give us new expressions for ω, the Hamiltonian vector fields, etc. ($\omega = \frac{dz_i \wedge d\overline{z}_i}{2i}$, $X_f = \frac{2}{i}\left[\frac{\partial f}{\partial \overline{z}_i}\frac{\partial}{\partial z_i} - \frac{\partial f}{\partial z_i}\frac{\partial}{\partial \overline{z}_i}\right]$, etc.; see appendix 2), there is a correspondent change in the representation of the operators, this is

$$\hat{f} = -i\hbar X_f - \theta(X_f) + f = -i\hbar X_f - \tfrac{2}{i} z_i d\overline{z}_i(X_f) + f.$$

An example that exposes the utility of this change in the coordinates is the following.

n-Dimensional Harmonic Oscillator:

Lets consider an n-dimensional harmonic oscillator as was classically described in appendix 2. An easy calculation, identical to the last one, show us that for the new variables z_i and $\overline{z}_i$ the associated operators are (using $X_{z_i} = -\frac{2}{i}\frac{\partial}{\partial \overline{z}_i}$ and $X_{\overline{z}_i} = \frac{2}{i}\frac{\partial}{\partial z_i}$, through the definitions)

$$\widehat{z_i} = -2\hbar \frac{\partial}{\partial \overline{z}_i} \quad \text{and} \quad \widehat{\overline{z}_i} = \overline{z}_i + 2\hbar \frac{\partial}{\partial z_i}$$

and for the energy operator, the associated Hamiltonian $H = \frac{1}{2} z_i \overline{z}_i$ (by using $X_H = \frac{1}{i}\left[z_i \frac{\partial}{\partial z_i} - \overline{z}_i \frac{\partial}{\partial \overline{z}_i}\right]$), we have then

$$\widehat{H} = \frac{1}{i}\left(z_i \frac{\partial}{\partial z_i} - \overline{z}_i \frac{\partial}{\partial \overline{z}_i}\right).$$

Nevertheless, this former calculation has been done from an observable that contains products of observables (as a matter of fact this Hamiltonian is a quadratic function of coordinates and momenta $H = \frac{1}{2} z_i \overline{z}_i = \frac{1}{2}(p_i^2 + q_i^2)$, and it is clear that $\widehat{H} \neq \frac{1}{2}\widehat{z_i}\widehat{\overline{z}_i}$), so if we want a unique and appropriated representation of this kind of functions we must introduce an adequate extension, this extension will be studied ahead in this chapter.

An Important Example: Minimal Coupling and Geometric Quantization

In appendix 2 has been described as an example the so-called *"minimal coupling"* in the case of the classical dynamics of a charged particle under the influence of an electromagnetic field $F = dA$, in this section we prove the corresponding result for the quantum description. Lets begin by observing that the phase spaces $(TQ^*, \omega_{e,F})$ and (TQ^*, ω), where configuration space is the Minkowskian space-time Q and then the phase space is $TQ^* \simeq \mathbf{R}^8$, equivalently describe this problem by the use of the Hamiltonians H and $\varphi^*_{e,F}(H) = H_{e,A}$, where $H_{e,A}$ denotes the minimal coupling Hamiltonian, which means that exists a simplectomorphism between the symplectic manifolds, as is shown in appendix 2.

Given the form of the Hamiltonian vector fields defined by the modified symplectic 2-form $\omega_{e,F}$

$$X_f^{e,F} = \frac{\partial f}{\partial p_i}\frac{\partial}{\partial q_i} - \left(\frac{\partial f}{\partial q_i} + eF_{ij}\frac{\partial f}{\partial p_j}\right)\frac{\partial}{\partial p_i}$$

where F_{ij} denotes the components of the 2-form F[6], and the correspondent symplectic potential

$$\theta_{e,F} = p_i dq_i + eA_i dq_i$$

it's clear that the connection form in the prequantization line bundle must be modified respect to the standard as

$$\nabla^{e,F} = d - i\theta_{e,F}$$

and then in the Hilbert Space of Prequantization $H_{e,F}(TQ^*, L)$, the Hermitian operator associated to a observable f will be given by the representation of Konstant-Souriau for the new quantization,

$$f \quad \longmapsto \hat{f} = f - i\hbar\nabla^{e,F}_{X_f}$$

so we must also expect changes in the correspondent representation for p_i and q_i. Only it remains to observe that $\overset{\wedge}{p_i} = p_i - i\hbar\nabla^{e,F}_{X_{p_i}}$, where $X^{e,F}_{p_i} = \frac{\partial}{\partial q_i} - eF_{ii}\frac{\partial}{\partial p_i}$, then

$$\nabla^{e,F}_{X_{p_i}} = (d - i\hbar^{-1}\theta_{e,F})\left(\frac{\partial}{\partial q_i} - eF_{ii}\frac{\partial}{\partial p_i}\right)$$

[6]See footnote in appendix 2.

and given the antisymmetry of F ($F_{ii}=0$) we have that

$$\overset{\wedge}{p_i} = p_i - i\hbar\left(\frac{\partial}{\partial q_i} - i\hbar^{-1}p_i - i\hbar^{-1}eA_i\right) = -i\hbar\frac{\partial}{\partial q_i} - eA_i.$$

In the same way $\overset{\wedge}{q_i}= q_i - i\hbar\nabla^{e,F}_{X_{q_i}}$, so with $X^{e,F}_{q_i} = X_{q_i} = -\frac{\partial}{\partial p_i}$ we have that

$$\nabla^{e,F}_{X_{q_i}} = -\frac{\partial}{\partial p_i} - \left(i\hbar^{-1}p_j dq_j - i\hbar^{-1}eA_j dq_j\right)\left(-\frac{\partial}{\partial p_i}\right) = -\frac{\partial}{\partial p_i}$$

and then

$$\overset{\wedge}{q_i} = q_i - i\hbar\left(-\frac{\partial}{\partial p_i}\right) = q_i + i\hbar\frac{\partial}{\partial p_i}$$

which evidences again the necessity of a polarization, but illustrates in $\widehat{p_i}$ the minimal coupling: we have gone from $-i\hbar\frac{\partial}{\partial q_i}$ to $-i\hbar\frac{\partial}{\partial q_i} - eA_i$.

1.2 Polarizations and the Hilbert Space of Quantization

In the last part of the previous section we note how the representation of the canonical operators for coordinates and momenta on the Hilbert space do not correspond with the classical result from quantum mechanics $\widehat{p_i} = -i\hbar\frac{\partial}{\partial q_i}$ and $\widehat{q_i} = q_i$, besides in the position operator an additional term appears that involves momentum coordinates ($\overset{\wedge}{q_i}= q_i + i\hbar\frac{\partial}{\partial p_i}$), undesired fact from the physical point of view. As well as in the operators, we must elude that wave functions (sections in the prequantization line bundle) corresponding to each configuration (state) of the system involving simultaneously the coordinates of one and other physical observables, this means to require that they be "constants along the fibers" of the cotangent bundle (in the case in which the phase space can be modelled as one of them), and then we have to correct this defect in the prequantization process by introducing a new structure called *polarization* in the corresponding phase space. In the case of a phase space different from a cotangent bundle (an arbitrary symplectic manifold), what does mean "fiber" and "constant along fibers"?, the idea of polarization that we must introduce should be as general as the answer to these questions requires. Physically a polarization corresponds to the so-called "representations" of wave functions and observables (momentum, position, Fock, and others) in the quantum description. What is a polarization and how the quantization process takes place with them are the subject of this section.

1.2.1 Distributions, Foliations and Polarizations

Definition 2:

Given an arbitrary manifold M we say that a smooth application D is a *Distribution* if to each point $m \in M$ it associates a linear subspace D_m of T_mM, in such way that:

i) $k = DimD_m$ is constant (independent of m)

ii)$\forall m_o \in M \quad \exists\, U_{m_o} \subset M$, open that contains m_o, and vector fields $X_1, X_2, ..., X_k$ defined on this open and such that $D_m = \langle X_1, X_2, ..., X_k \rangle$ for all $m \in U_{m_o}$[7].

The distribution is called *Integrable* or *Foliation* if for every point m_o in M there exists a submanifold N of M containing it and such that $DimN = k$ and if $m \in N \Rightarrow T_mN = D_m$. This definition is equivalent to say that if $X, Y \in V(M, D) = \{Z \in V(M) : Z(m) \in D_m \ \forall m \in M\}$ then $[X, Y] \in V(M, D)$.

Definition 3:

Let (M, ω) be a symplectic $2n$-dimensional manifold, a *Real Polarization* on M is a foliation D on M such that $\omega(m)(D_m, D_m) = 0$ in all $m \in M$ and no other subspace of T_mM that contains D_m have this property. In this case $DimD_m = n$.

Example 1:

The simplest examples that we can find in this point are the so-called *vertical* (D_v) and *horizontal* (D_h) polarizations of a cotangent bundle T^*Q, defined by the distributions that to each point $\{q_1, q_2, ..., q_n, p_1, p_2, ..., p_n\}$ on this manifold associates the vector spaces of the tangent bundle to T^*Q

$$\left\langle \frac{\partial}{\partial p_1}, \frac{\partial}{\partial p_2}, ..., \frac{\partial}{\partial p_n} \right\rangle$$

and

$$\left\langle \frac{\partial}{\partial q_1}, \frac{\partial}{\partial q_2}, ..., \frac{\partial}{\partial q_n} \right\rangle$$

[7] $\langle a, b, ..., z \rangle$ denotes the subspace generated by $a, b, ..., z$, i.e. $Span\{a, b, ..., z\}$.

respectively. That $\omega(D_v, D_v) = 0$ and $\omega(D_h, D_h) = 0$ follows from the canonical structure of ω.

Unfortunately to work with real polarizations is not enough to solve our problems (as a matter of fact, in many symplectic manifolds modeling admissible phase spaces is not possible to define real polarizations[8]), so we go to define the corresponding structure on a largest manifold, the *Complexification* of the tangent bundle to M.

Definition 4:

Let (M, ω) be a symplectic $2n$-dimensional manifold, a *Complex Polarization* P on M is a complex distribution such that:

(i) $\forall m \in M,\ P_m^{\perp} = \{X \in T_m M^C : \omega(X, Y) = 0\ \forall Y \in P_m\} = P_m$, where $T_m M^C$ denotes the complexification of $T_m M$.

(ii) $D_m \equiv P_m \cap \overline{P}_m \cap T_m M$ have constant dimension k for all m in M.

(iii) P and $P + \overline{P}$ are integrable.

Here $\overline{P}$ denotes the complex conjugation of P. A polarization P is called positive if $-i\omega(X, \overline{X}) \geq 0\ \forall X \in V(M, P)$, and if $P \cap \overline{P} = \{0\}$ P is the *Holomorphic or Kähler Polarization* of M, in this case $T_m M^C = P_m \oplus \overline{P}_m$ $\forall m \in M$.

Example 2:

In the simplest case $(M, \omega) = (T^*\mathbf{R}, dp \wedge dq)$, then on $T(T^*\mathbf{R})^C = T(\{(q_1 + iq_2) + (p_1 + ip_2) : (q_1, p_1), (q_2, p_2) \in T^*\mathbf{R}\})$ we can define the complex polarization given by

$$(p, q) \mapsto P_{(p,q)} = \left\langle \frac{\partial}{\partial p} + i \frac{\partial}{\partial q} \right\rangle,$$

then $DimP = 1$; and on the cotangent bundle $T^*\mathbf{R}^2$, with symplectic form $\omega = dp_i \wedge dq_i$, the distribution that to each point $\{q_1, q_2, p_1, p_2\}$ associate the vector subspace

$$\left\langle \left(\frac{\partial}{\partial p_1} + i \frac{\partial}{\partial q_1} \right) \oplus \frac{\partial}{\partial p_2} \right\rangle$$

define a 2-dimensional polarization.

[8] In $S^2 \subset \mathbf{R}^3$ with canonical volume form cannot be real polarizations because there is no nonsingular vector fields on S^2, and S^2 works as phase space for a lot of physical systems.

In order to solve the problem of defining "fibers" in a symplectic manifold, we can make the following definition:

Definition 5:

A *level* of a foliation D is a maximal connected integral submanifold of M, i.e. a submanifold N of M such that $T_m N = D_m \; \forall m \in M$. The space M/D is defined as the space of all the levels of D, and if there exists a differentiable structure on M/D such that the canonical projection

$$\pi : M \to M/D$$

is a smooth submersion, then the foliation D is called *reducible*.

Example 3:

The real polarizations in example 1 and the complex polarizations in example 2 are reducible. In example 1 we have that $M/D \simeq Q$ in both cases, and in the cases of example 2 that $P \cap \overline{P} = \{0\}$, then $M/P \simeq M$.

1.2.2 Geometric Quantization

From the precedent definitions we are now in position to fix some of the prequantization problems, having more realistic results from the physical point of view. If we have a real or a complex *reducible* polarization P on a symplectic manifold M, we can have a generalized idea of "fibres" on M, the *levels* of such polarization, and with M/P a "generalized configuration space". The question concerning with the criterion of sections "constant along the fibers" can be solved by defining our space as follows,

Definition 6:

If (M, ω) is a prequantizable *symplectic* manifold and L is its prequantum line bundle, given a *reducible* complex polarization P on M, we say that a section $s \in \Gamma(L)$ is *constant on the fibers* of M, or *quantizable*, if and only if $\forall X \in V(M, P)$, $\nabla_X s = 0$, and to the space defined by this sections we will denote as $\Gamma_P(L)$.

Theorem 2:

The smooth complex valued function[9]

$$\begin{aligned} \langle s,t\rangle : M &\longrightarrow \mathbf{C} \\ m &\mapsto \langle s(m),t(m)\rangle_m \end{aligned}$$

where $s,t\in\Gamma_P(L)$, for a given complex reducible polarization P, can be seen as a function on M/P, i.e. $\langle s,t\rangle$ is constant on the levels of P.

*Proof.*Let's see that if $X\in V(M,P)$ then $X\langle s,t\rangle = 0$. Given that ∇ and $\langle,\rangle$ are compatible we know that $\forall\ X\in V(M)$

$$X\langle s,t\rangle = \langle\nabla_X s,t\rangle + \langle s,\nabla_X t\rangle$$

in particular if $X\in V(M,P)$, so if $\nabla_X s=0$ and $\nabla_X t=0$, the result follows$\Box$

In the case of $(M,\omega)=(T^*Q,dp_i\wedge dq_i)$ if we take the vertical distribution D_v then we have that if $s\in\Gamma_{D_v}(L)$ then

$$\nabla_X s = 0$$

$\forall\ X\in V(M,D_v)$, but

$$\nabla_X s = (d-i\hbar^{-1}\theta)(X)(s) = \left(\frac{\partial s}{\partial p_i}dp_i + \frac{\partial s}{\partial q_i}dq_i\right)(X) - i\hbar^{-1}p_i dq_i(X)(s)$$

and $X\in V(M,D_v)$ implies that $X=a_1\frac{\partial}{\partial p_1}+...+a_n\frac{\partial}{\partial p_n}$, so

$$\nabla_X s = \left(\frac{\partial s}{\partial p_i}dp_i + \frac{\partial s}{\partial q_i}dq_i\right)(a_i\frac{\partial}{\partial p_i}) - i\hbar^{-1}p_i dq_i(a_i\frac{\partial}{\partial p_i})(s) = 0$$

i.e.

$$\frac{\partial s}{\partial p_i} = 0$$

for every i, and that corresponds to our idea of "s independent of the momentum p_i", saving in that way our wave functions from violating the uncertainty principle. In fact, to quantize using one or other of the vertical and horizontal polarizations is equivalent to work with the so-called "momentum representation" or "coordinates representation" in the Hilbert space

[9]Do not confuse the inner product of the sections s and t, $\langle s,t\rangle$, with the vector space generated by the vector fields $X_1,...,X_n$, $\langle X_1,...,X_n\rangle$.

from physics literature, respectively. Looking for the same characteristics in the observables we must reduce the set of physically admissible observables for the quantization, then now we are not going to associate an Hermitian operator to each element of $C^{\infty}(M, \mathbf{R})$, besides to each element of an appropriate subalgebra of it.

Definition 7:

The space of *quantizable functions* on M, given a distribution D on M, is the subspace of $C^{\infty}(M, \mathbf{R})$ defined by

$$C^{\infty}(M_D, \mathbf{R}) = \{f \in C^{\infty}(M, \mathbf{R}) : [X_f, X] \in V(M, D)\ \forall X \in V(M, D)\}.$$

This space, as we already say, is a subalgebra of $C^{\infty}(M, \mathbf{R})$ under the Poisson bracket defined by ω. The idea now is to define a *modified Hilbert space* in accordance with the distribution we are working, and an adequate representation of the classical quantizable observables in an operator algebra on Hilbert space. Now, given sections $s, s' \in \Gamma_D(L)$, if the integral $(s, s') = \int\limits_{M/D} \langle s, s' \rangle\, \epsilon_D$ is well defined then we can define the inner product

$$(s, s') = \int\limits_{M/D} \langle s, s' \rangle\, \epsilon_D$$

on the *Hilbert Space of Representation* $\mathcal{H}_D$ defined as the completion of $\{s \in \Gamma_D(L) : \int\limits_{M/D} \langle s, s \rangle\, \epsilon_D < \infty\}$. Finally, we can give the following representation to classical observables on the algebra of Hermitian operators in this modified Hilbert space

$$\begin{aligned} \wedge : C^{\infty}(M_D, \mathbf{R}) &\longrightarrow \tilde{O}_D \\ f \quad &\longmapsto \hat{f}_D \colon \mathcal{H}_D \to \mathcal{H}_D \ = f - i\hbar \nabla_{X_f} \end{aligned}$$

Theorem 3:

With this last definitions follows that: the application $f \longrightarrow \widehat{f}$ is lineal, if f is constant then $\widehat{f}$ is the multiplication by f operator and if $\{f, g\} = h$ then $[\widehat{f}, \widehat{g}] = -i\hbar \widehat{h}$ $(f, g, h \in C^{\infty}(M_D, \mathbf{R}))$. Moreover if the observable f is such that its Hamiltonian vector field X_f is complete then the operator $\widehat{f}_D$ is Hermitian..

*Proof.*See [P2].

This is what we need to build up a quantum theory physically admissible, but as we can see later it is not all the story..

Example 4:

Working in $(M,\omega) = (T^*\mathbf{R}^n, dp_i \wedge dq_i)$, then $L = (M \times \mathbf{C}, \pi_1, M)$ and $\Gamma(L) \simeq C^\infty(M, \mathbf{C})$. The connection form is

$$\nabla_X s = X(s) - i\hbar^{-1}(p_i dq_i)(X)(s)$$

and the Hermitian structure is given by $\langle (x,z_1), (x,z_2)\rangle_x = \overline{z}_1 z_2$. Taken the reducible distribution $D_v = \left\langle \frac{\partial}{\partial p_1}, \frac{\partial}{\partial p_2}, ..., \frac{\partial}{\partial p_n} \right\rangle$, we have that $s \in \Gamma_{D_v}(L) \Leftrightarrow \frac{\partial s}{\partial p_i} = 0$ (s does not depend of p_i, $i = 1, ..., n$), so $\mathcal{H}_{D_v} \simeq L^2(\mathbf{R}^n, \mathbf{C})$. Now, if we take a classical observable $f \in C^\infty(M_{D_v}, \mathbf{R})$, then by definition $\left[X_f, \frac{\partial}{\partial p_i}\right] \in V(\mathbf{R}^2, D_v)$, but $X_f = \frac{\partial f}{\partial p_i}\frac{\partial}{\partial q_i} - \frac{\partial f}{\partial q_i}\frac{\partial}{\partial p_i}$, so $\frac{\partial^2 f}{\partial p_i^2} = 0$ and then

$$f = f_o(q_1, q_2, ..., q_n) + p_1 f_1(q_1, q_2, ..., q_n) + ... + p_n f_n(q_1, q_2, ..., q_n).$$

With this result it is easy to calculate the form of the associate Hermitian operator, because if $f \in C^\infty(M_{D_v}, \mathbf{R})$ then

$$\hat{f}_D\, s = fs - i\hbar\nabla_{X_f} s$$

where $s \in \Gamma_{D_v}(L)$, but then

$$\begin{aligned}
\nabla_{X_f} s &= (d - i\hbar^{-1}\theta)(X_f)(s) \\
&= \left(\tfrac{\partial s}{\partial p_i} dp_i + \tfrac{\partial s}{\partial q_i} dq_i\right)\left(\tfrac{\partial f}{\partial p_i}\tfrac{\partial}{\partial q_i} - \tfrac{\partial f}{\partial q_i}\tfrac{\partial}{\partial p_i}\right) \\
&\quad - i\hbar^{-1} p_i dq_i\left(\tfrac{\partial f}{\partial p_i}\tfrac{\partial}{\partial q_i} - \tfrac{\partial f}{\partial q_i}\tfrac{\partial}{\partial p_i}\right)(s) \\
&= \left\{\left(\tfrac{\partial s}{\partial p_i} dp_i + \tfrac{\partial s}{\partial q_i} dq_i\right)\left(\tfrac{\partial f}{\partial p_i}\tfrac{\partial}{\partial q_i}\right) - i\hbar^{-1} p_i dq_i\left(\tfrac{\partial f}{\partial p_i}\tfrac{\partial}{\partial q_i}\right)(s)\right\} \\
&\quad + \left\{\left(\tfrac{\partial s}{\partial p_i} dp_i + \tfrac{\partial s}{\partial q_i} dq_i\right)\left(\tfrac{\partial f}{\partial q_i}\tfrac{\partial}{\partial p_i}\right) - i\hbar^{-1} p_i dq_i\left(\tfrac{\partial f}{\partial q_i}\tfrac{\partial}{\partial p_i}\right)(s)\right\}
\end{aligned}$$

the last term is the part of $\nabla_{X_f} s$ where the connection acts on the part of X_f generated by $\frac{\partial}{\partial p_1}, \frac{\partial}{\partial p_2}, ..., \frac{\partial}{\partial p_n}$, then this part is zero by definition of s, so

$$\begin{aligned}
\nabla_{X_f} s &= \left(\tfrac{\partial s}{\partial p_i} dp_i + \tfrac{\partial s}{\partial q_i} dq_i\right)\left(\tfrac{\partial f}{\partial p_i}\tfrac{\partial}{\partial q_i}\right) - i\hbar^{-1} p_i dq_i\left(\tfrac{\partial f}{\partial p_i}\tfrac{\partial}{\partial q_i}\right)(s) \\
&= \tfrac{\partial f}{\partial p_i}\tfrac{\partial s}{\partial q_i} - i\hbar^{-1} p_i \tfrac{\partial f}{\partial p_i}(s)
\end{aligned}$$

and

$$\hat{f}_D = f - i\hbar\left(\frac{\partial f}{\partial p_i}\frac{\partial}{\partial q_i} - i\hbar^{-1}p_i\frac{\partial f}{\partial p_i}\right)$$

$$= f - i\hbar\frac{\partial f}{\partial p_i}\frac{\partial}{\partial q_i} - p_i\frac{\partial f}{\partial p_i}$$

in the particular cases of p_i and q_i we recover the usual quantization rules of quantum mechanics:

$$\widehat{p}_{iD} = -i\hbar\frac{\partial}{\partial q_i} \text{ and } \widehat{q}_{iD} = q_i.$$

From the classical commutation relations we deduce then that[10]

$$[\widehat{p}_i, \widehat{q}_j] = -i\hbar\delta_i^j$$

$$[\widehat{p}_i, \widehat{p}_j] = [\widehat{q}_i, \widehat{q}_j] = 0.$$

Example 5: The One Dimensional Harmonic Oscillator

Let $(M,\omega) = (T^*\mathbf{R}, dp\wedge dq)$, $H = \frac{1}{2}(p^2+q^2)$, then $L = (M\times\mathbf{C}, \pi_1, M)$ and $\Gamma(L) \simeq C^\infty(M,\mathbf{C})$. The connection form is

$$\nabla_X s = X(s) - i\hbar^{-1}(pdq)(X)(s)$$

and the Hermitian structure is given by $\langle(x,z_1),(x,z_2)\rangle_x = \overline{z}_1 z_2$. Making the change in the coordinates $z = p+iq$ and $\overline{z} = p - iq$, then $\omega = \frac{d\overline{z}\wedge dz}{2i}$, $H = \frac{1}{2}z\overline{z}$, and $X_H = i\left[z\frac{\partial}{\partial z} - \overline{z}\frac{\partial}{\partial \overline{z}}\right]$. With the *Holomorphic* distribution on $\mathbf{C}$, P_h, given by $P_h = Span\left\{\frac{\partial}{\partial\overline{z}}\right\}$, we have that $\omega(P_h,P_h) = 0$, $DimP_h = 1$, $P_h\cap\overline{P}_h = \{0\} \Rightarrow M/P_h = \mathbf{C}$. $s\in\Gamma_{P_h}(L) \Leftrightarrow \nabla_{X_z}s = 0\;\forall X_z\in V(M,P_h)$, this means that

$$X_z(s) - i\hbar^{-1}(\frac{1}{2i}\overline{z}dz)(X_z)(s) = \frac{\partial s}{\partial\overline{z}}\left(1 - \frac{\hbar^{-1}}{2}\overline{z}dz\right) = 0,$$

i.e. if $\frac{\partial s}{\partial\overline{z}} = 0$ (if s is Holomorphic), so $\mathcal{H}_{P_h} \simeq L^2(R,\mathbf{C})$. Given $f\in C^\infty(M_{P_h},\mathbf{R})$ then

$$X_f = 2i\left(\frac{\partial f}{\partial\overline{z}}\frac{\partial}{\partial z} - \frac{\partial f}{\partial z}\frac{\partial}{\partial\overline{z}}\right)$$

[10] For simplicity in the notation we will omit the D subscript whenever we work with a fixed polarization.

so

$$\nabla_{X_f} s = (d - \tfrac{1}{2}\hbar^{-1}\overline{z}dz)(X_f)(s)$$

$$= \left(\tfrac{\partial s}{\partial z}dz + \tfrac{\partial s}{\partial \overline{z}}d\overline{z}\right)\left[2i\left(\tfrac{\partial f}{\partial \overline{z}}\tfrac{\partial}{\partial z} - \tfrac{\partial f}{\partial z}\tfrac{\partial}{\partial \overline{z}}\right)\right] - i\tfrac{\hbar^{-1}}{2}\overline{z}dz\left[2i\left(\tfrac{\partial f}{\partial \overline{z}}\tfrac{\partial}{\partial z} - \tfrac{\partial f}{\partial z}\tfrac{\partial}{\partial \overline{z}}\right)\right](s)$$

$$= 2i\tfrac{\partial s}{\partial z}\tfrac{\partial f}{\partial \overline{z}} + \hbar^{-1}\overline{z}\tfrac{\partial f}{\partial \overline{z}}(s)$$

where we eliminate the terms annihilated because $s \in \Gamma_{P_h}(L)$ (the part of X_f generated by $\frac{\partial}{\partial \overline{z}}$), we have that

$$\hat{f}_{P_h} = f - i\hbar\nabla_{X_f}$$

$$= f + 2\hbar\tfrac{\partial f}{\partial \overline{z}}\tfrac{\partial}{\partial z} - \overline{z}\tfrac{\partial f}{\partial \overline{z}}$$

deducing from this that

$$\widehat{z}_{P_h} = z$$

and

$$\widehat{\overline{z}}_{P_h} = 2\hbar\frac{\partial}{\partial z}.$$

In physics literature this association of operators to z and $\overline{z}$ is known as *Bergman Representation.*

Now, $\left[i\left(z\frac{\partial}{\partial z} - \overline{z}\frac{\partial}{\partial \overline{z}}\right), \frac{\partial}{\partial \overline{z}}\right] = i\frac{\partial}{\partial \overline{z}} \Rightarrow H \in C^\infty(M_{P_h}, \mathbf{R})$,then we can calculate the Hermitian operator corresponding to H. In this case

$$\widehat{H}_{P_h} = \frac{1}{2}\overline{z}z - i\hbar\left(iz\frac{\partial}{\partial z} - \frac{i\hbar^{-1}}{2}\overline{z}z\right) = \hbar z\frac{\partial}{\partial z}$$

but in this case we are not in agreement with quantum mechanics, because following this result the spectrum of $\widehat{H}$ is $\{n\hbar : n \in \mathbf{N}\}$, but the correct one is $\{\left(n + \frac{1}{2}\right)\hbar : n \in \mathbf{N}\}$. We need some other structure to correct this.

1.3 Half-forms Correction to Geometric Quantization

The correction to geometric quantization that we will see in this section is the so called *Metaplectic* or *Half-Forms* correction, this consists basically of the extension of the structure group of the tangent bundle TM to a $2n$

dimensional symplectic manifold (M, ω), from the symplectic $Sp(2n, \mathbf{R})$ to the metaplectic group $Mp(2n, \mathbf{R})$, which is the connected double covering of $Sp(2n, \mathbf{R})$ [P2]. The problems to be solved with this new structure are basically three:

- Given two different polarizations P and R for (M, ω), how are related $\mathcal{H}_P$ and $\mathcal{H}_R$?
- Is well defined the space $\mathcal{H}_P$ as Hilbert space?
- How to get the right spectrum (from a physical point of view) for some operators, e.g. the Hamiltonian in the harmonic oscillator?

The main idea is to modify the quantum bundle L (and then its connection), and show that the inner product defined on sections of this bundle, as in the previous part of this chapter, is well defined and so the associated Hilbert space. The spectrum correction is consequence of this modification in the connection. The way we will take is not the algebraic one, but the construction of the bundle from the manifold and a complex polarization on it, the "metaplectic way" can be seen in texts as [W1].

1.3.1 Half-Forms

Let (M, ω) be a $2n$ dimensional symplectic manifold and P the Kähler positive polarization on M (then $T_m M^C \simeq P_m \oplus \overline{P}_m$, $\forall m \in M$), lets consider the spaces P^*_m and $\overline{P}^*_m$ of the 1-forms dual to P_m and $\overline{P}_m$. This spaces are sections of the bundles P^* and $\overline{P^*}$, respectively, and lets introduce the determinant bundles[11] $Det_{P^*} = \Lambda^n P^*$ and $Det_{\overline{P}^*} = \Lambda^n \overline{P}^*$.

Theorem 4:

There exist a line bundle Pf_P on M such that $(Pf_P)^2 = Pf_P \otimes Pf_P \simeq Det_{P^*}$ and a bundle Pf^*_P such that $(Pf^*_P)^2 \simeq Det_{\overline{P}^*}$ if and only if the first Chern class of M is even. The sections of the bundles Pf_P and Pf^*_P are called *Half-forms* on M.

Proof. See [W1].

The idea in the process of quantization is to work with the bundle $L \otimes Pf_P$, Pf_P is called the P *canonical bundle on* M, and not with L as before,

[11] For a general discussion on Determinant Line Bundles see [P1] in the bibliography.

extending the connection on L to a connection on $L \otimes Pf_P$ given us a covariant derivative ∇^* acting on sections s^* of this last bundle. In fact, taking the sections of Det_{P^*}, n-forms α on M such that $i_X\alpha = 0 \; \forall X \in V(M,P)$, we define the covariant derivative for $X \in V(M,P)$ by $\nabla^*_X \alpha = i_X d\alpha$, note that in this case $\nabla^*_X\alpha$ and $\mathcal{L}_X\alpha$ coincide. In this corrected quantization the *Quantum Hilbert Space* is the completion (respect to the inner product $\left\langle s\otimes\alpha^{\frac{1}{2}}, t\otimes\beta^{\frac{1}{2}}\right\rangle^* = \left(\left\langle \alpha^{\frac{1}{2}}\otimes\alpha^{\frac{1}{2}}, \beta^{\frac{1}{2}}\otimes\beta^{\frac{1}{2}}\right\rangle\right)^{\frac{1}{2}} \langle s,t\rangle$, where $s\otimes\alpha^{\frac{1}{2}}$, $t\otimes\beta^{\frac{1}{2}} \in C^\infty(M, L\otimes Pf_P)$) of

$$\left\{ s\otimes\alpha^{\frac{1}{2}} : \nabla_X s = \nabla^*_X \alpha^{\frac{1}{2}} = 0, \;\; \forall X\in V(M,P) \right\}$$

so

$$\nabla_X(s\otimes\alpha^{\frac{1}{2}}) = \nabla_X s\otimes\alpha^{\frac{1}{2}} + s\otimes\nabla^*_X\alpha^{\frac{1}{2}} = 0,$$

and the representation to a classical operator $f \in C^\infty(M_P, \mathbf{R})$ is defined by

$$f \quad \mapsto \quad \widehat{(f)}_{\frac{1}{2}} = \left(-i\hbar\nabla_{X_f} + f\right)\otimes 1 + 1\otimes\left(-i\hbar\mathcal{L}_{X_f}\right)$$

so if $s\otimes\alpha^{\frac{1}{2}} \in C^\infty(M, L\otimes Pf_P)$

$$\widehat{(f)}_{\frac{1}{2}}(s\otimes\alpha^{\frac{1}{2}}) = \left(-i\hbar\nabla_{X_f}s + fs\right)\otimes\alpha^{\frac{1}{2}} + s\otimes\left(-i\hbar\nabla^*_{X_f}\alpha^{\frac{1}{2}}\right).$$

Theorem 5:

With this last definitions follows that: the application $f \longrightarrow \widehat{(f)}_{\frac{1}{2}}$ is lineal, if f is constant then $\widehat{(f)}_{\frac{1}{2}}$ is the 'multiplication by f' operator, and if $\{f,g\} = h$ then $[\widehat{(f)}_{\frac{1}{2}}, \widehat{(g)}_{\frac{1}{2}}] = -i\hbar\widehat{(h)}_{\frac{1}{2}}$ $(f,g,h \in C^\infty(M_D, \mathbf{R}))$[12].

*Proof.*See [P2].

1.3.2 Extension of the Algebra of Operators

In appendix 2 we note that the set of classical observables $C^\infty(M,\mathbf{R})$ on a symplectic manifold (M,ω) has structure of Lie algebra under Poisson bracket, defined as $\{f,g\} = \omega(X_f, X_g)$, where $f,g \in C^\infty(M,\mathbf{R})$ and X_f, X_g are the respective Hamiltonian vector fields generated by them. It is clear that the center of this Lie algebra is the set

$$\mathcal{C}_{C^\infty(M,\mathbf{R})} = \{f \in C^\infty(M,\mathbf{R}) : df = 0\}$$

[12]For simplicity in the notation we will omit the subscripts $\frac{1}{2}$, it can be verified that our previous calculation for operators still works in the left side of the tensor product.

i.e the constant functions (if M is connected). But the structure of $C^\infty(M, \mathbf{R})$ is richer than the Lie algebra one, as a matter of fact the identity[13]

$$\{f, g \cdot h\} = \{f, g\} \cdot h + \{f, h\} \cdot g$$

shows that it also has a commutative, associative algebra under ordinary pointwise multiplication *compatible* with the Lie algebra structure, and as we see in the prequantization of the n-dimensional harmonic oscillator (end of section 1.1.4), although the prequantization respects the Lie algebra structure of $C^\infty(M, \mathbf{R})$ (i.e. $[\widehat{f}, \widehat{g}] = -i\hbar\widehat{\{f, g\}}$) it does not keep the commutative and associative one coming from usual multiplication. The question here is how to extend the representation of the Lie algebra of classical observables to a representation of that algebra that includes this algebraic structure, but given that it is not possible to require that

$$\widehat{fg} = \widehat{f}\widehat{g} = \widehat{g}\widehat{f} = \widehat{gf}$$

because if we take the canonical functions p_i and q_i on phase space, for example, it is clear that $[\widehat{p}_i, \widehat{q}_i] = -i\hbar\widehat{\{p_i, q_i\}} = -i\hbar$ and we have that $\widehat{p}_i = -i\hbar\frac{\partial}{\partial q_i}$ and $\overset{\wedge}{q_i} = q_i$, so

$$[\widehat{p}_i, \widehat{q}_i] = \widehat{p}_i\widehat{q}_i - \widehat{q}_i\widehat{p}_i = -i\hbar \neq 0.$$

Following to Axelrod et al.[14], given that the right-hand of this equation is constant and then belongs to $\mathcal{C}_{C^\infty(M,\mathbf{R})}$, defining

$$\widehat{p_i q_j} = \frac{1}{2}\left(\widehat{p}_i\widehat{q}_j + \widehat{q}_j\widehat{p}_i\right)$$

we have that

Proposition:

$$[\widehat{p_i q_j}, \widehat{p_j q_i}] = \frac{-i\hbar}{2}\widehat{\{p_i q_j, p_j q_i\}}$$

and

[13] $\{f, g \cdot h\} = \frac{\partial f}{\partial p_i}\frac{\partial (gh)}{\partial q_i} - \frac{\partial f}{\partial q_i}\frac{\partial (gh)}{\partial p_i} = \frac{\partial f}{\partial p_i}\left(g\frac{\partial h}{\partial q_i} + h\frac{\partial g}{\partial q_i}\right) - \frac{\partial f}{\partial q_i}\left(g\frac{\partial h}{\partial p_i} + h\frac{\partial g}{\partial p_i}\right) = \{f, g\} \cdot h + \{f, h\} \cdot g$

[14] See Axelrod, S., Della Pietra, S., and Witten, E. "Geometric Quantization of Chern-Simons Gauge Theory", J. Differential Geometry, **33** (1991), 787.

$$[\widehat{p_i}, \widehat{p_k q_l}] = \frac{-i\hbar}{2}\widehat{\{p_i, p_k q_l\}}$$

*Proof.*Lets prove the first identity, observe that

$$\widehat{p_i q_j} = \frac{1}{2}\left(-i\hbar\frac{\partial}{\partial q_i}(q_j) + q_j(-i\hbar\frac{\partial}{\partial q_i})\right)$$

$$= \frac{-i\hbar}{2}\left(\delta_{ij} + q_j\frac{\partial}{\partial q_i}\right)$$

so

$$[\widehat{p_i q_j}, \widehat{p_j q_i}] = \left(\frac{-i\hbar}{2}\left(\delta_{ij} + q_j\frac{\partial}{\partial q_i}\right)\right)\left(\frac{-i\hbar}{2}\left(\delta_{ij} + q_i\frac{\partial}{\partial q_j}\right)\right) -$$

$$\left(\frac{-i\hbar}{2}\left(\delta_{ij} + q_i\frac{\partial}{\partial q_j}\right)\right)\left(\frac{-i\hbar}{2}\left(\delta_{ij} + q_j\frac{\partial}{\partial q_i}\right)\right)$$

$$= \left(\frac{-i\hbar}{2}\right)^2\left(q_j\frac{\partial}{\partial q_j} - q_i\frac{\partial}{\partial q_i}\right)$$

but,

$$\{p_i q_j, p_j q_i\} = \frac{\partial(p_i q_j)}{\partial p_k}\frac{\partial(p_j q_i)}{\partial q_k} - \frac{\partial(p_j q_i)}{\partial p_k}\frac{\partial(p_i q_j)}{\partial q_k}$$

$$= q_j p_j - p_i q_i$$

then

$$\widehat{\{p_i q_j, p_j q_i\}} = \widehat{q_j p_j} - \widehat{p_i q_i} = \frac{-i\hbar}{2}\left(\delta_{ij} + q_j\frac{\partial}{\partial q_j}\right) + \frac{i\hbar}{2}\left(\delta_{ij} + q_i\frac{\partial}{\partial q_i}\right)$$

$$= \frac{-i\hbar}{2}\left(q_j\frac{\partial}{\partial q_j} - q_i\frac{\partial}{\partial q_i}\right)$$

so the equality holds. The second identity can be proved exactly in the same way□

We have then that linear and quadratic functions on canonical coordinates on M form, under Poisson bracket, a Lie algebra that is a central extension of $\mathcal{C}_{C^\infty(M,\mathbf{R})}$. For polynomials of a higher order this extension can not be given, because the differences between terms in this case are not central, however we do not need more than this for the geometric quantization of the systems to be studied in this work.

Example 6:

As an example lets see what this implies for the representation of the Hamiltonian function in the case of the n-dimensional harmonic oscillator. As in previous analysis we will work with the complex coordinates $z_i = p_i + iq_i$ and $\overline{z}_i = p_i - iq_i$ on $\mathbf{C}^n$, and then $H = \frac{1}{2} z_i \overline{z}_i$, hence we can calculate the operator

$$\widehat{H}_{P_h} = \frac{1}{2}\widehat{z_i \overline{z}_i} = \frac{1}{4}\left(\widehat{z}_i \widehat{\overline{z}}_i + \widehat{\overline{z}}_i \widehat{z}_i\right)$$

from the last section (example 5) we know that

$$\widehat{z}_{i P_h} = z_i$$

and

$$\widehat{\overline{z}}_{i P_h} = 2\hbar \frac{\partial}{\partial z_i}$$

thus

$$\begin{aligned} \widehat{H}_{P_h} &= \tfrac{1}{4}\left(z_i\left(2\hbar \tfrac{\partial}{\partial z_i}\right) + 2\hbar \tfrac{\partial}{\partial z_i}(z_i)\right) \\ &= \hbar\left(z_i \tfrac{\partial}{\partial z_i} + \tfrac{1}{2}\right), \end{aligned}$$

so its eigenvalues are $\left(n + \frac{1}{2}\right)\hbar$, as we expect.

1.4 Geometric Quantization of Systems with Symmetry

Following the material described in appendix 3, about symmetry and symplectic dynamics, lets summarize some results corresponding to the quantization of a symplectic manifold with a Lie group action on it.

Given a Hamiltonian action $\phi : G \times M \to M$ on a symplectic manifold (M, ω) with the associated moment map $J : M \to \mathcal{G}$, from the homomorphism

$$\begin{aligned} \mathcal{J} : \mathcal{G} &\to C^\infty(M, \mathbf{R}) \\ \xi &\mapsto \mathcal{J}(\xi) \end{aligned}$$

we have a canonical representation of $\mathcal{G}$ on smooth sections of the prequantum line bundle $L \xrightarrow{\pi} M$, in case of its existence. This action is

$$\begin{aligned} \widehat{\mathcal{J}(\xi)} &= \mathcal{J}(\xi) - i\nabla_{X_{\mathcal{J}(\xi)}} \\ &= \mathcal{J}(\xi) - i\nabla_{\xi_M} \end{aligned}$$

where we consider that the action of G on M extends to a global action of G on L.

Consider $\xi \in \mathcal{G}$ a regular value for J, and the reduced phase space $J^{-1}(\xi)/G_\xi \equiv M_\xi$ with the canonical projection

$$\pi_\xi : J^{-1}(\xi) \longrightarrow J^{-1}(\xi)/G_\xi$$

and inclusion

$$i_\xi : J^{-1}(\xi) \longrightarrow M,$$

Theorem:

There exists a unique line bundle $L_G \stackrel{\pi_G}{\to} M_\xi$, with connection form $\nabla^G = \hbar^{-1}\omega_\xi$ such that

$$\pi_\xi^* L_G = i_\xi^* L$$

and

$$\pi_\xi^* \nabla^G = i_\xi^* \nabla.$$

*Proof.*See [GS2].

If there exists a positive Kähler polarization P on M, and the Hamiltonian action of a compact connected Lie group G on M is such that the polarization is G-invariant, i.e. invariant respect the action of G on M, then there exists a positive polarization P_G on M_ξ canonically associated to P [GS2]. Moreover, given that the Hermitian inner product $\langle , \rangle$ on $L \stackrel{\pi}{\to} M$ is G-invariant, there exist a unique Hermitian inner product on $L_G \stackrel{\pi_G}{\to} M_\xi$, notated $\langle , \rangle_G$ such that

$$\pi_\xi^* \langle , \rangle_G = i_\xi^* \langle , \rangle \, .$$

Thus all the necessary quantum data for the quantization of (M_ξ,ω_ξ) are guaranteed by the existence of a G-invariant positive Kähler polarization on M. In this case, if we define the spaces

$$\mathcal{H}_P^G = \{ s \in \mathcal{H}_P : \widehat{\mathcal{J}(\xi)} s = 0 \; \forall \xi \in \mathcal{G} \}$$

and

$$\mathcal{H}_{P_G} = \{ s \in L^2(M_\xi, L_G) : \nabla_X^G s = 0 \; \forall X \in P_G \}$$

we have that

Theorem:

$\mathcal{H}_P^G$ and $\mathcal{H}_{P_G}$ are isomorphic as vector spaces.
*Proof.*See [GS2].

Remark:

The condition of G-invariance for the polarization P is too strong in many cases, but it is possible to weaken it[15]. For our purposes it is enough.

[15] See for example, Popov, A.D. *Constraints, Complex Structures and Quantization.* Dubna Preprint E2-94-174, 1994.

Chapter 2

Magnetic Monopoles and Geometric Quantization

In this chapter we study the quantum description of a particle under the influence of an electromagnetic field and its relation with the Dirac's charge quantization condition. In appendix 2 (from example 2 until the end of the appendix) we have done the classical description of such a system from a symplectic frame of work and in section 1.1.4 we studied the minimal coupling, through the geometrical methods used in prequantization. In the next pages we will see how the Dirac quantization condition for the electron charge can be deduced just from the existence of a quantum description of the system already mentioned, and we will relate this construction with the construction on the configuration space known from the literature[1].

2.1 The Dirac Condition from Geometric Quantization

One of the biggest unresolved questions in theoretical physics is that associated with the quantization of electric charge, i.e. why the observed electric charges in all the electrically charged matter is an integer multiple of a "fundamental charge" e, the electron charge. P. A. M. Dirac[2] found that the existence of a single magnetic charge $\mathtt{m}$ in the universe can be enough to answer that question. As a matter of fact, the quantum dynamics of a particle

[1]See Greub, W. and Petry, H.R. "Minimal Coupling and Complex Line Bundles". J. Math. Phys. **16**, *1347*, 1975.

[2]Dirac, P.A.M. Proc. Roy. Soc. **A133**, *60*, 1931.

with electric charge e under the influence of the magnetic field generated by such particle is well defined if the condition $2e\mathtt{m} = \hbar n$, $n \in \mathbf{Z}$, known as *Dirac Condition*, is satisfied.

Lets consider the symplectic manifold $(T^*Q, \omega_{e,F})$ where $\omega_{e,F} = \omega + eF'$ and ω is the canonical symplectic form on the phase space, corresponding to a particle of charge e under the influence of a field F. Then, the system concerning us can be described with this symplectic manifold and a *"free"* Hamiltonian H (see appendix 2). According with the integrality condition for the existence of a prequantum bundle on such symplectic manifold, our first step in the geometric quantization description, there exists a complex line bundle $L \xrightarrow{\pi} T^*Q$ and a connection ∇ on L with curvature $\hbar^{-1}\omega$ if and only if $\left[(2\pi\hbar)^{-1}\omega_{e,F}\right] \in H^2(T^*Q, \mathbf{Z})$, i.e. if any integral of $\omega_{e,F}$ on an oriented 2-surface in T^*Q is an integer multiple of $2\pi\hbar$. But given that there exists an equivalence between this description and that obtained from the manifold (TQ^*, ω) with the Hamiltonian $\varphi^*_{e,F}(H) = H_{e,A}$ (the so called minimal coupling Hamiltonian), it must be verified simultaneously that $\left[(2\pi\hbar)^{-1}\omega\right] \in H^2(T^*Q, \mathbf{Z})$, and then that $\left[(2\pi\hbar)^{-1}eF'\right] \in H^2(T^*Q, \mathbf{Z})$, thus

$$\int_S eF'\epsilon = 2\pi\hbar n$$

$n \in \mathbf{Z}$, for any closed oriented 2 surface in T^*Q.

Now, the natural projection π from phase space T^*Q on configuration space Q pulls back cohomology classes on Q in cohomology classes on T^*Q, and then the condition for the existence of a quantum description of the system, namely $\left[(2\pi\hbar)^{-1}eF'\right] \in H^2(T^*Q, \mathbf{Z})$, is equivalent to $\left[(2\pi\hbar)^{-1}eF\right] \in H^2(Q, \mathbf{Z})$, where $F' = \pi^*F$ and (see example 2, appendix 2)

$$F = -B_x dy \wedge dz - B_y dz \wedge dx - B_z dx \wedge dy$$

on configuration space. If we consider the magnetic field generated by a magnetic monopole in the origin, then our configuration space is $\dot{\mathbf{R}}^3$, i.e. $\mathbf{R}^3$ without the origin, and the field generated by such particle is given by

$$\overrightarrow{B} = -\mathtt{m}\left(\frac{\overrightarrow{r}}{r^3}\right)$$

with $\overrightarrow{r}$ the position vector in $\mathbf{R}^3$ and $\mathtt{m}$ the magnitude of the monopole magnetic charge. From this we have that

$$F = \frac{\mathtt{m}}{r^3}\left(xdy \wedge dz + ydz \wedge dx + zdx \wedge dy\right)$$

and in this case $\int_S FdVol$ represents the magnetic flux generated by the monopole, an easy calculation shows that

$$\int_S FdVol = 4\pi\mathtt{m}$$

where S is any bounded region in $\mathbf{R}^3$ with regular boundary ∂S such that the origin does not belong to it. Thus, from the integrality condition we deduce that

$$2e\mathtt{m} = \hbar n$$

for $n \in \mathbf{N}$, i.e. the Dirac condition. Clearly, this condition is a *topological restriction* to the existence of a quantum description for the system, given that comes from the topological constraint $\left[(2\pi\hbar)^{-1}eF'\right] \in H^2(T^*Q, \mathbf{Z})$.

2.2 Monopole Description on Configuration Space

In this section we review the construction of line bundles on configuration space that are usually used in the description of the magnetic monopole system. Bundles on configuration space corresponds to the projection $T^*\dot{\mathbf{R}}^3 \xrightarrow{p} \dot{\mathbf{R}}^3$ of the bundles on phase space given by integrality condition in the corresponding symplectic form, as a matter of fact the line bundles $\mathcal{L} \xrightarrow{\pi} T^*\dot{\mathbf{R}}^3$ and $\mathcal{L}' \xrightarrow{\pi'} \dot{\mathbf{R}}^3$ must be such that the diagram

$$\begin{array}{ccc} \mathcal{L} & \xrightarrow{\widetilde{p}} & \mathcal{L}' \\ \downarrow \pi & & \downarrow \pi' \\ T^*\dot{\mathbf{R}}^3 & \xrightarrow{p} & \dot{\mathbf{R}}^3 \end{array}$$

commutes.

It is known that the set of line bundles on $\dot{\mathbf{R}}^3$ is classified by the second de Rham cohomology group $H^2(\dot{\mathbf{R}}^3, \mathbf{Z})$, and this group is isomorphic to $H^2(\mathbf{CP}^1, \mathbf{Z})$ (given that S^2 is a deformation retract of $\dot{\mathbf{R}}^3$ and $S^2 \simeq \mathbf{CP}^1$), i.e. to $\mathbf{Z}$ [KN]. Thus, for any integer n there exist a complex line bundle (corresponding to the physical system defined by a magnetic monopole with "charge" $\frac{n\hbar}{2e}$ in the integrality condition on phase space) from which we can elaborate the corresponding quantum mechanical description of the system. Another (equivalent) classification is the given by Milnor[3], who classifies all the nonequivalent fibre bundles with base space S^n and (arcwise and connected) structure group G in terms of $\Pi_{n-1}(G)$. In fact Milnor shows a one to one correspondence between the equivalent bundles on such manifold with group G and $\Pi_{n-1}(G)$. In the case of electromagnetism these principal bundles have as structure group $U(1) \simeq S^1$ [C1], and $\Pi_{n-1}(S^1) \simeq \mathbf{Z}$, having again the same result.

2.2.1 Dirac Bundles over $\mathbf{CP}^1$

There are many descriptions of monopoles on configuration space[4], all of them equivalent to the Wu-Yang description[5]. Lets present here a equivalent description of all the S^1 bundles on S^2, based on the general *Hopf Fibration* [KN]. For the simplest case (as we will see later, the bundle corresponding with $n = 1$) consider the Hopf map

$$h : S^3 \to S^2$$

where we consider

$$\begin{aligned} S^3 &= \{(x, y, z, t) \in \mathbf{R}^4 : x^2 + y^2 + z^2 + t^2 = 1\} \\ &= \{(z_0, z_1) \in \mathbf{C}^2 : z_0\overline{z}_0 + z_1\overline{z}_1 = 1\} \end{aligned}$$

and we use the polar coordinate transformation given by

$$\begin{aligned} z_0 &= x + iy = \cos\tfrac{\theta}{2} e^{\frac{i}{2}(\chi+\varphi)} \\ z_1 &= z + it = \sin\tfrac{\theta}{2} e^{\frac{i}{2}(\chi-\varphi)} \end{aligned}$$

with $0 \leq \theta < \pi$, $0 \leq \varphi < 2\pi$ and $0 \leq \chi < 4\pi$; and

$$\begin{aligned} S^2 &= \{(x, y, z) \in \mathbf{R}^3 : x^2 + y^2 + z^2 = 1\} \\ &\simeq \mathbf{CP}^1 \simeq \dot{\mathbf{C}}^2 / \sim \end{aligned}$$

[3] See Husemoller, D. *Fibre Bundles*. Springer, 1966.

[4] See Greub, W. and Petry, H.R. "Minimal Coupling and Complex Line Bundles". *J. Math. Phys.* **16**, *1347*, 1975; Quiros, M et al. "On the Topological Meaning of Magnetic Charge Quantization". *J. Math. Phys.* **23**, *873*, 1982; and the standard descriptions of the books quoted in bibliography.

[5] Wu, T. and Yang, C. N. *Nucl. Phys.* **B107**, *365*, 1976 and *Phys. Rev.* **D14**, *437*, 1976. See also [C1] in bibliography.

where $\dot{\mathbf{C}}^2 = \mathbf{C}^2 \backslash \{0,0\}$ with the equivalence relation in $\dot{\mathbf{C}}^2$ given by $(z_0, z_1) \sim \lambda\,(z_0, z_1)\ \ \forall \lambda \in \mathbf{C},\ \lambda \neq 0$. Hopf fibration maps (z_0, z_1) on S^3 to its equivalence class $[z_0, z_1]$ in $\mathbf{CP}^1 \approx S^2$, and this map can be seen as the composition of two maps,

$$\begin{array}{c} h : S^3 \xrightarrow{i_n} \mathbf{C} \simeq \mathbf{R}^2 \xrightarrow{st^{-1}} S^2 \\ (z_0, z_1) \quad \longmapsto \quad [z_0, z_1] \end{array}$$

the first of which defines inhomogeneous coordinates on S^2, the former is a stereographic projection,

$$\begin{array}{c} i_n : S^3 \longrightarrow \mathbf{C} \simeq \mathbf{R}^2 \\ (z_0, z_1) \mapsto \begin{cases} z_1/z_0 \ if \ z_0 \neq 0 \\ z_0/z_1 \ if \ z_1 \neq 0 \end{cases} \end{array}$$

and

$$\begin{array}{c} st : S^2 \longrightarrow \mathbf{C} \simeq \mathbf{R}^2 \\ (\theta, \phi) \mapsto \rho e^{i\phi} \end{array}$$

where $\rho = \frac{\sin\theta}{1-\cos\theta}$.

GRAPHIC

Looking for a connection on this bundle we have at hand a natural connection arising from the line element of S^3

$$ds^2 = 4(dz_0 d\bar{z}_0 + dz_1 d\bar{z}_1)$$

that can be discomposed uniquely into the line element of S^2 ($ds^2 = \sin^2\theta d\phi^2 + d\theta^2$) and the tensorial square of the 1-form

$$\alpha = d\chi + \cos\theta d\phi$$

having

$$ds^2 = \sin^2\theta d\phi^2 + d\theta^2 + \alpha^2.$$

Note that $\frac{1}{2}\alpha$ defines a connection over S^3 [C1], and if we take the two set open covering of S^2 given by $U_+ = S^2 \backslash \{\theta = 0\}$ and $U_- = S^2 \backslash \{\theta = \pi\}$ and the local sections

$$\begin{array}{c} \sigma_+ : U_+ \longrightarrow S^3 \\ (\theta, \phi) \mapsto (z_0, z_1) = \left(e^{i\phi} \cos\frac{\theta}{2}, \sin\frac{\theta}{2}\right) \end{array}$$

$$\begin{array}{c} \sigma_- : U_- \longrightarrow S^3 \\ (\theta, \phi) \mapsto (z_0, z_1) = \left(\cos\frac{\theta}{2}, e^{-i\phi} \sin\frac{\theta}{2}\right) \end{array}$$

then the respective local connection forms are given by

$$A_+ = \sigma_+^* \alpha$$

and

$$A_- = \sigma_-^* \alpha$$

on U_+ and U_-, respectively, where α is the connection 1-form on S^3. So,

$$A_+ = \sigma_+^* \left(d\chi + \cos\theta d\phi\right) = \tfrac{1}{2} \left(1 + \cos\theta\right) d\phi$$

$$A_- = \sigma_-^* \left(d\chi + \cos\theta d\phi\right) = -\tfrac{1}{2} \left(1 - \cos\theta\right) d\phi$$

and from this $A_+ - A_- = d\phi$ on $U_+ \cap U_-$, and the curvature 2-form can be written as

$$\Omega = \frac{1}{2} \sin\theta d\phi \wedge d\theta.$$

Lets show now that this description corresponds to the case n=1 in quantum conditions, and lets build up the general case. Consider the manifolds

$$S^{2n+1} = \{(z_0, z_1, ..., z_n) \in \mathbf{C}^{n+1} : \sum_{k=0}^{n} z_k \overline{z}_k = 1\}$$

and

$$\mathbf{CP}^n \simeq \dot{\mathbf{C}}^{n+1} / \sim$$

where $\dot{\mathbf{C}}^{n+1} = \mathbf{C}^{n+1} \backslash \{0\}$ with the equivalence relation given by $(z_0, z_1, ..., z_n) \sim \lambda\left(z_0, z_1, ..., z_n\right) \ \forall \lambda \in \mathbf{C}, \ \lambda \neq 0$, and the general Hopf fibration

$$S^{2n+1} \xrightarrow{\pi} \mathbf{CP}^n$$

seen as a principal fibre bundle with group S^1 and total space S^{2n+1} [KN], then

$$\pi\left(e^{i\theta}(z_0,...,z_n)\right) = [z_0,...,z_n]\,.$$

On $\mathbf{CP}^n$ a set of open sets commonly used for a local chart description are the sets $U_k = \{[z_0,...,z_n] \in \mathbf{CP}^n : z_k \neq 0\}$, and the local trivializations for this bundle

$$\begin{aligned}\Phi_j : \pi^{-1}(U_j) &\longrightarrow U_j \times S^1 \\ z &\mapsto (\pi(z), z_j/\mid z_j \mid)\end{aligned}$$

where $z = (z_0,...,z_n)$, gives us the transition functions

$$\begin{aligned}\Psi_{kj} : U_j \cap U_k &\longrightarrow S^1 \\ z &\longmapsto z_k \mid z_j \mid / z_j \mid z_k \mid\,.\end{aligned}$$

Now define on U_0 the coordinates $\xi_a = z_a/z_0$ for $a = 1,...,n$, and $z_0 = \rho e^{i\chi}$, then

$$\Psi_{k0}(z) = \xi_k/\mid \xi_k \mid$$

and

$$\rho^2\left(1+\sum_{a=1}^{n}\xi_a\overline{\xi_a}\right) = \rho^2\left(1+\tfrac{z_1\overline{z_1}}{\rho^2}+\tfrac{z_2\overline{z_2}}{\rho^2}+\cdots+\tfrac{z_n\overline{z_n}}{\rho^2}\right)$$

$$= \rho^2\left(1+\tfrac{1-z_0\overline{z_0}}{\rho^2}\right) = \rho^2\left(1+\tfrac{1-\rho^2}{\rho^2}\right) = 1$$

so (summation understood)

$$\rho^2 = \left(1+\xi_a\overline{\xi_a}\right)^{-1}.$$

The connection on this bundle comes from the line element of S^{2n+1}

$$ds^2 = dz_0 d\overline{z}_0 + dz_1 d\overline{z}_1 + \cdots + dz_n d\overline{z}_n$$

that can be written as (summation understood)

$$ds^2 = \left(\rho^2\delta_a^b - \rho^4\overline{\xi}_a\xi_b\right) d\xi_a d\overline{\xi}_b + \alpha^2$$

where

$$\alpha = d\chi + \frac{i}{2}\rho^2\left(\xi_b d\overline{\xi}_a - \overline{\xi}_a d\xi_b\right)$$

and the first term of right hand side defines the so-called *Fubini-Study metric* [KN]. From this we have the curvature 2-form on $\mathbf{CP}^n$ given by

$$\Omega = \left(\rho^2 \delta_a^b - \rho^4 \overline{\xi}_a \xi_b\right) i d\xi_a \wedge d\overline{\xi}_b.$$

Consider now the embedding

$$\begin{array}{rl} \mathbf{CP}^1 & \longrightarrow \mathbf{CP}^n \\ \xi & \mapsto (\xi_0, \xi_1, ..., \xi_n) \end{array}$$

given by

$$\xi_k = \begin{pmatrix} n \\ k \end{pmatrix} \xi^k$$

then, by sustitution in Ω we found

$$\Omega_n = n \left(1 + \xi\overline{\xi}\right)^{-2} i d\xi \wedge d\overline{\xi}$$

so, for $n = 1$,

$$\Omega_1 = \frac{i d\xi \wedge d\overline{\xi}}{\left(1 + \xi\overline{\xi}\right)^2}$$

and defining $\xi = \rho e^{i\phi}$, $\rho = \frac{\sin\theta}{1-\cos\theta}$,

$$\begin{aligned} i d\xi \wedge d\overline{\xi} &= i \left((e^{i\phi} d\rho + i\rho e^{i\phi} d\phi) \wedge (e^{-i\phi} d\rho - i\rho e^{-i\phi} d\phi)\right) \\ &= i^2 \left(\rho d\phi \wedge d\rho - \rho d\rho \wedge d\phi\right) = 2\rho d\rho \wedge d\phi \end{aligned}$$

and

$$\left(1 + \xi\overline{\xi}\right)^2 = \left(1 + \rho^2\right)^2$$

so

$$\Omega_1 = \frac{2\rho d\rho \wedge d\phi}{\left(1 + \rho^2\right)^2}$$

but

$$\begin{aligned} \rho d\rho &= \tfrac{\sin\theta}{1-\cos\theta} d\left(\tfrac{\sin\theta}{1-\cos\theta}\right) = \tfrac{\sin\theta}{1-\cos\theta}\left(\tfrac{\cos\theta - 1}{(1-\cos\theta)^2}\right) d\theta \\ &= \tfrac{-\sin\theta d\theta}{(1-\cos\theta)^2} \end{aligned}$$

and from this

$$\frac{2\rho d\rho}{\left(1+\rho^2\right)^2} = \frac{-2\sin\theta d\theta}{(1-\cos\theta)^2\left(1+\left(\frac{\sin\theta}{1-\cos\theta}\right)^2\right)^2} = -\frac{1}{2}\sin\theta d\theta,$$

then

$$\Omega_1 = -\frac{1}{2}\sin\theta d\theta \wedge d\phi$$

as we expected, the same curvature as in the first Hopf fibration, obviously the case $n = 1$ of $S^{2n+1} \xrightarrow{\pi} \mathbf{CP}^n$.

The general result for the curvature

$$\Omega_n = \frac{n}{2}\sin\theta d\phi \wedge d\theta$$

(that corresponds with the 2-form $F = \frac{\mathtt{m}}{r^3}\left(xdy \wedge dz + ydz \wedge dx + zdx \wedge dy\right)$ in Cartesian coordinates) defined by the local connection 1-forms (on the open sets U_+ and U_-)

$$A_{n+} = \tfrac{n}{2}\left(1+\cos\theta\right)d\phi$$

$$A_{n-} = -\tfrac{n}{2}\left(1-\cos\theta\right)d\phi$$

shows again the Wu-Yang classic result [N1] [C1] , from which we recover the integrality condition for $\mathtt{m}$. Finally, observe that from the curvature we can calculate the first Chern class of each bundle

$$c_1 = -\frac{1}{2\pi}\Omega_n = \frac{n}{4\pi}\sin\theta d\theta \wedge d\phi$$

and thus, its Chern number (also called ”topological index”),

$$C_1 = \int_{S^2} c_1 = \frac{1}{2\pi}\left(\int_{U_+}\Omega_{n+} + \int_{U_-}\Omega_{n-}\right) = -n,$$

just (minus) the integer defining the magnetic monopole charge.

Chapter 3

n-Electron Systems

In this chapter we describe the geometric quantization of a system composed by n electrons (at low temperature) in a two dimensional space, under the action of an external homogeneous magnetic field (perpendicular to the plane of motion of such electrons) and under an uncoupled harmonic potential. The theoretical interest of this system comes from the topological theory of the so-called *Quantum Hall Effect*[1], our goal is to understand filling factors as topological invariants founded through geometric quantization[2].

3.1 n-Electron System: Canonical Description

In this section we study a n-electrons system in a *2* dimensional space with an harmonic potential and (perpendicular) magnetic field, the configuration space Q can be modelled as $\mathbf{R}^{2n}$, giving a phase space $T^*Q \simeq \mathbf{R}^{4n}$ with local canonical coordinates $\{q_{1x}, q_{2x}, ..., q_{nx},\ q_{1y}, q_{2y}, ..., q_{ny},\ p_{1x}, p_{2x}, ..., p_{nx}, p_{1y}, p_{2y}, ..., p_{ny}\}$. Classical Dynamics and Geometric Quantization will be done in these canonical coordinates.

[1] See Hatsugay, Y. *"Topological Aspects of the Quantum Hall Effect"*. J. Phys.: Condens. Matter, **9**, 2507, 1997.

[2] This chapter has benefited from early work of Professor *Simon Scott*, I acknowledge him specially for permission to read his unpublished work about this subject.

3.1.1 Harmonic Potential

Classical Description

On phase space $T^*Q \simeq \mathbf{R}^{4n}$ with local coordinates $\{q_{1x}, q_{2x}, ..., q_{nx}, q_{1y}, q_{2y}, ..., q_{ny}, p_{1x}, p_{2x}, ..., p_{nx}, p_{1y}, p_{2y}, ..., p_{ny}\}$ we take the canonical symplectic form

$$\omega = d\theta = dp_{ix} \wedge dq_{ix} + dp_{iy} \wedge dq_{iy},$$

$(i = 1, 2, ..., n)$. The Hamiltonian function describing this system has a "free" part and the corresponding harmonic potential part

$$H = \frac{1}{2m}(p_{ix}^2 + p_{iy}^2) + \frac{mw_0^2}{2}(q_{ix}^2 + q_{iy}^2)$$

where m denotes the electron mass and in the second term w_0 is the frequency of the oscillators, ahead taken equal to 1 unless it is indicated. In this coordinates the Hamiltonian vector field generated by a function f is then

$$X_f = \frac{\partial f}{\partial p_{ix}}\frac{\partial}{\partial q_{ix}} + \frac{\partial f}{\partial p_{iy}}\frac{\partial}{\partial q_{iy}} - \frac{\partial f}{\partial q_{ix}}\frac{\partial}{\partial p_{ix}} - \frac{\partial f}{\partial q_{iy}}\frac{\partial}{\partial p_{iy}}.$$

Geometric Quantization

Given that the phase space is simply connected there exist a line bundle for the prequantization of this system, lets look for the representation corresponding to the observables $q_{ix}, q_{iy}, p_{ix}, p_{iy}$ and H. Following chapter one, such representation is given by

$$\widehat{f} = f - i\hbar\nabla_{X_f}$$

where

$$\nabla_{X_f} s = \left(ds - i\hbar^{-1}\theta(s)\right)(X_f)$$

it is easy to see that in this case

$$\nabla_{X_f} s = \frac{\partial f}{\partial p_{ix}}\frac{\partial s}{\partial q_{ix}} + \frac{\partial f}{\partial p_{iy}}\frac{\partial s}{\partial q_{iy}} - \frac{\partial f}{\partial q_{ix}}\frac{\partial s}{\partial p_{ix}} - \frac{\partial f}{\partial q_{iy}}\frac{\partial s}{\partial p_{iy}} - i\hbar^{-1}\left(p_{ix}\frac{\partial f}{\partial p_{ix}} + p_{iy}\frac{\partial f}{\partial p_{iy}}\right)(s)$$

and thus

$$\widehat{f} = f - i\hbar\left(\frac{\partial f}{\partial p_{ix}}\frac{\partial}{\partial q_{ix}} + \frac{\partial f}{\partial p_{iy}}\frac{\partial}{\partial q_{iy}} - \frac{\partial f}{\partial q_{ix}}\frac{\partial}{\partial p_{ix}} - \frac{\partial f}{\partial q_{iy}}\frac{\partial}{\partial p_{iy}}\right) - p_{ix}\frac{\partial f}{\partial p_{ix}} - p_{iy}\frac{\partial f}{\partial p_{iy}},$$

from this we have for the observables of our interest that

$$\widehat{p}_{ix} = -i\hbar\tfrac{\partial}{\partial q_{ix}}$$

$$\widehat{p}_{iy} = -i\hbar\tfrac{\partial}{\partial q_{iy}}$$

$$\widehat{q}_{ix} = q_{ix} + i\hbar\tfrac{\partial}{\partial p_{ix}}$$

$$\widehat{q}_{iy} = q_{iy} + i\hbar\tfrac{\partial}{\partial p_{iy}}.$$

As we can observe (see canonical operators in example *1.1.4*) it is necessary the introduction of a polarization, in this case we work with the vertical polarization $P_v = Span\left\{\frac{\partial}{\partial p_{ix}}, \frac{\partial}{\partial p_{iy}}\right\}$, then our physically admissible wave functions s are such that

$$\nabla_X s = 0$$

for every $X \in Span\left\{\frac{\partial}{\partial p_{ix}}, \frac{\partial}{\partial p_{iy}}\right\}$. With this new information it is clear that

$$\widehat{f} = f - i\hbar\left(\frac{\partial f}{\partial p_{ix}}\frac{\partial}{\partial q_{ix}} + \frac{\partial f}{\partial p_{iy}}\frac{\partial}{\partial q_{iy}}\right) - p_{ix}\frac{\partial f}{\partial p_{ix}} - p_{iy}\frac{\partial f}{\partial p_{iy}}$$

having then

$$\widehat{p}_{ix} = -i\hbar\tfrac{\partial}{\partial q_{ix}}$$

$$\widehat{p}_{iy} = -i\hbar\tfrac{\partial}{\partial q_{iy}}$$

$$\widehat{q}_{ix} = q_{ix}$$

$$\widehat{q}_{iy} = q_{iy}.$$

Now we must note that the corresponding operator for H is obtained, following the end of the first chapter, as

$$\widehat{H} = \frac{1}{2m}(\widehat{p}_{ix}^2 + \widehat{p}_{iy}^2) + \frac{m}{2}(\widehat{q}_{ix}^2 + \widehat{q}_{iy}^2)$$

where

$$\widehat{p}_{ix}^2 = -\hbar^2 \tfrac{\partial^2}{\partial q_{ix}^2}$$

$$\widehat{p}_{iy}^2 = -\hbar^2 \tfrac{\partial^2}{\partial q_{iy}^2}$$

$$\widehat{q}_{ix}^2 = q_{ix}^2$$

$$\widehat{q}_{iy}^2 = q_{iy}^2$$

so

$$\widehat{H} = \frac{-\hbar^2}{2m}\left(\frac{\partial^2}{\partial q_{ix}^2} + \frac{\partial^2}{\partial q_{iy}^2}\right) + \frac{m}{2}(q_{ix}^2 + q_{iy}^2).$$

3.1.2 n-Electron System under a Magnetic Field

Classical Description

In this section we will work with a modified symplectic form as is described in the final part of appendix 2. This 2-form is

$$\begin{aligned}\widetilde{\omega} &= \omega + e\pi^*(F)\\ &= dp_{ix} \wedge dq_{ix} + dp_{iy} \wedge dq_{iy} + e\pi^*(Bdx \wedge dy)\\ &= dp_{ix} \wedge dq_{ix} + dp_{iy} \wedge dq_{iy} + \tfrac{eB}{2} dq_{ix} \wedge dq_{iy} - \tfrac{eB}{2} dq_{iy} \wedge dq_{ix},\end{aligned}$$

$(i = 1, 2, ..., n)$, where e is the charge of each electron and we are using the so called *symmetric gauge* in the potential A that defines the field B ($B = dA$, where $A = \frac{B}{2} q_{ix} dq_{iy} - \frac{B}{2} q_{iy} dq_{ix}$). In this way we can work with the "free" Hamiltonian without the introduction of a potential term obtaining an equivalent description (see example 2 in appendix 2), i.e.

$$H = \frac{1}{2m}(p_{ix}^2 + p_{iy}^2)$$

where m denotes the mass of each electron. With the new symplectic form and this gauge we will obtain different expressions for the Hamiltonian vector fields generated by a smooth function f and the symplectic potential $\widetilde{\theta}$, in our coordinate system these Hamiltonian vector fields has the form

$$X_f = \frac{\partial f}{\partial p_{ix}} \frac{\partial}{\partial q_{ix}} + \frac{\partial f}{\partial p_{iy}} \frac{\partial}{\partial q_{iy}} + \left(-\frac{\partial f}{\partial q_{ix}} + \frac{eB}{2} \frac{\partial f}{\partial p_{iy}}\right) \frac{\partial}{\partial p_{ix}} + \left(-\frac{\partial f}{\partial q_{iy}} - \frac{eB}{2} \frac{\partial f}{\partial p_{ix}}\right) \frac{\partial}{\partial p_{iy}}$$

and the symplectic potential is

$$\widetilde{\theta} = p_{ix}dq_{ix} + p_{iy}dq_{iy} + \frac{eB}{2}q_{ix}dq_{iy} - \frac{eB}{2}q_{iy}dq_{ix} = \theta + eA.$$

Geometric Quantization

The existence of the prequantization line bundle for this system is ensured for the same reason that in the harmonic case, so lets look for the representation corresponding to the observables $q_{ix}, q_{iy}, p_{ix}, p_{iy}$ and H in this case, where the change in the symplectic form alters all the canonical expressions. The representation must be given by

$$\widehat{f} = f - i\hbar\nabla_{X_f}$$

where

$$\nabla_{X_f} s = \left(ds - i\hbar^{-1}\widetilde{\theta}(s)\right)(X_f),$$

in this case

$$\nabla_{X_f} s = \frac{\partial f}{\partial p_{ix}}\frac{\partial s}{\partial q_{ix}} + \frac{\partial f}{\partial p_{iy}}\frac{\partial s}{\partial q_{iy}} + \left(-\frac{\partial f}{\partial q_{ix}} + \frac{eB}{2}\frac{\partial f}{\partial p_{iy}}\right)\frac{\partial s}{\partial p_{ix}}$$

$$+\left(-\frac{\partial f}{\partial q_{iy}} - \frac{eB}{2}\frac{\partial f}{\partial p_{ix}}\right)\frac{\partial s}{\partial p_{iy}} - i\hbar^{-1}\left(p_{ix}\frac{\partial f}{\partial p_{ix}} + p_{iy}\frac{\partial f}{\partial p_{iy}}\right)(s)$$

$$+\frac{i\hbar^{-1}eB}{2}\left(q_{iy}\frac{\partial f}{\partial p_{ix}} - q_{ix}\frac{\partial f}{\partial p_{iy}}\right)(s)$$

and thus

$$\widehat{f} = f - i\hbar\left[\frac{\partial f}{\partial p_{ix}}\frac{\partial}{\partial q_{ix}} + \frac{\partial f}{\partial p_{iy}}\frac{\partial}{\partial q_{iy}} + \left(-\frac{\partial f}{\partial q_{ix}} + \frac{eB}{2}\frac{\partial f}{\partial p_{iy}}\right)\frac{\partial}{\partial p_{ix}} + \left(-\frac{\partial f}{\partial q_{iy}} - \frac{eB}{2}\frac{\partial f}{\partial p_{ix}}\right)\frac{\partial}{\partial p_{iy}}\right]$$

$$-p_{ix}\frac{\partial f}{\partial p_{ix}} - p_{iy}\frac{\partial f}{\partial p_{iy}} + \frac{eB}{2}\left(q_{iy}\frac{\partial f}{\partial p_{ix}} - q_{ix}\frac{\partial f}{\partial p_{iy}}\right)$$

having for the observables representation

$$\widehat{p}_{ix} = -i\hbar\left(\frac{\partial}{\partial q_{ix}} - \frac{eB}{2}\frac{\partial}{\partial p_{iy}}\right) + \frac{eB}{2}q_{iy}$$

$$\widehat{p}_{iy} = -i\hbar\left(\frac{\partial}{\partial q_{iy}} + \frac{eB}{2}\frac{\partial}{\partial p_{ix}}\right) - \frac{eB}{2}q_{ix}$$

$$\widehat{q}_{ix} = q_{ix} + i\hbar\frac{\partial}{\partial p_{ix}}$$

$$\widehat{q}_{iy} = q_{iy} + i\hbar\frac{\partial}{\partial p_{iy}}.$$

Again with the vertical polarization $P_v = \left\langle \frac{\partial}{\partial p_{ix}}, \frac{\partial}{\partial p_{iy}} \right\rangle$ we obtain that

$$\widehat{f} = f - i\hbar \left(\frac{\partial f}{\partial p_{ix}} \frac{\partial}{\partial q_{ix}} + \frac{\partial f}{\partial p_{iy}} \frac{\partial}{\partial q_{iy}} \right) + \frac{eB}{2} \left(q_{iy} \frac{\partial f}{\partial p_{ix}} - q_{ix} \frac{\partial f}{\partial p_{iy}} \right)$$

so

$$\widehat{p}_{ix}^A = \widehat{p}_{ix} + \tfrac{eB}{2} q_{iy} = -i\hbar \tfrac{\partial}{\partial q_{ix}} + \tfrac{eB}{2} q_{iy}$$

$$\widehat{p}_{iy}^A = \widehat{p}_{iy} - \tfrac{eB}{2} q_{ix} = -i\hbar \tfrac{\partial}{\partial q_{iy}} - \tfrac{eB}{2} q_{ix}$$

$$\widehat{q}_{ix}^A = \widehat{q}_{ix} = q_{ix}$$

$$\widehat{q}_{iy}^A = \widehat{q}_{iy} = q_{iy}.$$

The operator $\widehat{H}_A$ is obtained as

$$\widehat{H}_A = \frac{1}{2m} \left(\left(\widehat{p}_{ix}^A \right)^2 + \left(\widehat{p}_{iy}^A \right)^2 \right)$$

but in this case

$$\left(\widehat{p}_{ix}^A \right)^2 = \left(-i\hbar \tfrac{\partial}{\partial q_{ix}} + \tfrac{eB}{2} q_{iy} \right) \left(-i\hbar \tfrac{\partial}{\partial q_{ix}} + \tfrac{eB}{2} q_{iy} \right)$$

$$= -\hbar^2 \tfrac{\partial^2}{\partial q_{ix}^2} - \tfrac{i\hbar eB}{2} q_{iy} \tfrac{\partial}{\partial q_{ix}} + \tfrac{e^2 B^2}{2} q_{iy}^2$$

$$\left(\widehat{p}_{iy}^A \right)^2 = -\hbar^2 \tfrac{\partial^2}{\partial q_{iy}^2} + \tfrac{i\hbar eB}{2} q_{ix} \tfrac{\partial}{\partial q_{iy}} + \tfrac{e^2 B^2}{2} q_{ix}^2$$

$$\widehat{q}_{ix}^{A\,2} = q_{ix}^2$$

$$\widehat{q}_{iy}^{A\,2} = q_{iy}^2$$

so

$$\widehat{H}_A = \frac{1}{2m} \left[-\hbar^2 \left(\frac{\partial^2}{\partial q_{ix}^2} + \frac{\partial^2}{\partial q_{iy}^2} \right) - \frac{i\hbar eB}{2} \left(q_{iy} \frac{\partial}{\partial q_{ix}} - q_{ix} \frac{\partial}{\partial q_{iy}} \right) + \frac{e^2 B^2}{2} (q_{ix}^2 + q_{iy}^2) \right]$$

and this operator can be written as

$$\widehat{H}_A = \frac{-\hbar^2}{2m}\left(\frac{\partial^2}{\partial q_{ix}^2} + \frac{\partial^2}{\partial q_{iy}^2}\right) + \frac{eB}{4m}\widehat{L_Z} + \frac{e^2B^2}{4m}(q_{ix}^2 + q_{iy}^2)$$

where $\widehat{L_Z} = \widehat{q}_{iy}\widehat{p}_{ix} - \widehat{q}_{ix}\widehat{p}_{iy}$ is the operator corresponding to the *angular momentum* of the i-th electron around the z axis. A comparison of this Hamiltonian with the corresponding to the harmonic oscillator case shows us that the only differences between them are the constant corresponding to the last term and the angular momentum part.

3.2 Kähler Quantization

3.2.1 Holomorphic Quantum Description

Consider a Kähler manifold (M, J, ω), where J is the almost complex structure defined on M and ω its (compatible) symplectic form. We know then that on M there is a pseudo-Riemannian metric g given by the symplectic and complex structures by $g(X, JY) = \omega(X, Y)$ for any tangent fields X, Y. The complex valued vector fields on M satisfying $J(X) = \pm iX$ are called of type $(1, 0)$ and $(0, 1)$ respectively, and if they are closed under commutation J is called *integrable.*

On M, parametrized with local (complex) coordinates $\{z_1, z_2, ..., z_n\}$ there exist, locally at least, a real function K, called *Kähler Potential*, in terms of which we can write both the metric and symplectic structures as [KN, N1]

$$ds^2 = \omega_{i\bar{j}}(z_i, \bar{z}_j)dz_i d\bar{z}_j$$

$$\omega = \tfrac{i}{2}\omega_{i\bar{j}}(z_i, \bar{z}_j)dz_i \wedge d\bar{z}_j$$

where

$$\omega_{i\bar{j}}(z_i, \bar{z}_j) = \frac{\partial^2 K}{\partial z_i \partial \bar{z}_j}.$$

If we define the operators ∂_i and $\overline{\partial}_i$ by $\partial_i = \frac{\partial}{\partial z_i}dz_i$ and $\overline{\partial}_i = \frac{\partial}{\partial \bar{z}_i}d\bar{z}_i$, follows that $d = \partial_i + \overline{\partial}_i$, we can write the symplectic form as

$$\omega = \frac{i}{2}\partial_i\overline{\partial}_j K$$

and, from the Kähler potential, the symplectic potential can be written

$$\theta = -i\partial_i K$$

or

$$\overline{\theta} = i\overline{\partial}_i K.$$

The Kähler potential is not unique, under gauge transformations of the form

$$K(z_i, \overline{z}_i) \to K(z_i, \overline{z}_i) + \alpha(z_i) + \overline{\alpha}(\overline{z}_i)$$

the metric and symplectic structures are invariant. Introducing the operators

$$\nabla_i = \partial_i + \partial_i K$$

$$\overline{\nabla}_i = \overline{\partial}_i$$

then the commutators

$$[\nabla_i, \overline{\nabla}_j] = [\partial_i + \partial_i K, \overline{\partial}_j] = -i\omega_{ij}$$

$$[\nabla_i, \nabla_j] = [\overline{\nabla}_i, \overline{\nabla}_j] = 0,$$

show us that $(\nabla_i, \overline{\nabla}_i)$ defines a connection on a Holomorphic line bundle $\mathcal{L}$, with first Chern class ω [W2]. This obviously corresponds with the Holomorphic quantization obtained from the Kähler or Holomorphic polarization defined on M by the distribution $\left\langle \frac{\partial}{\partial z_i} \right\rangle$, i.e. Holomorphic sections quantization for the system. In this case

$$\mathcal{H} = \left\{ s \in \Gamma(M) : \overline{\nabla}_{iX} s = 0 \right\}$$

and then polarized sections are Holomorphic functions of z_i, and any two non zero polarized sections s, s' are related by a Holomorphic function ϕ of z_i $(s' = \phi s)$.

In general, under gauge transformations in $K(z_i, \overline{z}_i)$ the Holomorphic sections of our line bundle $\mathcal{L}$ transform as

$$s(z_i) \to e^{\alpha(z_i)} s(z_i)$$

so, on a Kähler manifold M with the Holomorphic polarization a Hermitian structure on $\mathcal{L}$ is given by

$$\langle s, s \rangle = \overline{s} s e^{-\hbar^{-1} K(z_i, \overline{z}_i)}$$

where K is the Kähler potential and $s \in \Gamma_P(M)$. Solutions to $\overline{\nabla}_{iX}\psi = 0$, i.e. wave functions, adopt the form

$$\psi(z_i, \overline{z}_j) = s(z_i)e^{-\frac{1}{4\hbar}Kz_i\overline{z}_j}$$

Thus, the inner product on $\mathcal{H}$ can be written as[3]

$$\langle\psi, \psi'\rangle = \int_M s\overline{s}\; e^{-\frac{1}{2\hbar}Kz_i\overline{z}_j}\epsilon$$

where $\epsilon = \omega^n$ apart from a constant factor.

In the particular cases of our interest, $\mathbf{C}^n$ and $\mathbf{CP}^n$, we have as Kähler potentials the functions

$$K = \frac{1}{2}z_i\overline{z}_i$$

and

$$K = \frac{1}{2}\log(1 + w_1\overline{w}_1 + w_2\overline{w}_2 + \cdots + w_n\overline{w}_n)$$

where $w_k = z_k/z_0$ in the chart U_0 of $\mathbf{CP}^n$ (U_0 is the open defined by the set of points $(z_0, z_1, ..., , z_n)$ in $\mathbf{CP}^n$ where $z_0 \neq 0$, see chapter 2), respectively.

We have made this description of geometric quantization on a Kähler manifold because we will work in this fashion. Our problem is to quantize the n electron system in a *2* dimensional space considering the phase space $T^*Q \simeq \mathbf{R}^{4n}$ (with local coordinates $\{q_{1x}, q_{2x}, ..., q_{nx},\ q_{1y}, q_{2y}, ..., q_{ny}, p_{1x}, p_{2x}, ..., p_{nx},\ p_{1y}, p_{2y}, ..., p_{ny}\}$ and symplectic form $\omega = dp_{ix}\wedge dq_{ix} + dp_{iy}\wedge dq_{iy}$) as the complex $\mathbf{C}^{2n}$ with the complex coordinates $\{z_{q1}, z_{q2}, ..., z_{qn}, z_{p1}, z_{p2}, ..., z_{pn}\}$, where

$$z_{qi} = q_{ix} + iq_{iy}$$

$$z_{pi} = p_{ix} - ip_{iy}.$$

This is the standard coordinate description, so we are looking just to compare our results with the literature ones.

The introduction of a metaplectic structure (half-forms correction) in this description introduces new topological restrictions to the problem, although this case ($\mathbf{C}^{2n}$) is very simple. We have a integrable and reducible polarization on phase space, namely the Holomorphic polarization, so it is

[3]Note that the convergence of this is ensured because of g being positive definite.

direct to verify that the canonical bundle associated is polarized. Take the section of Det_{P^*} given by the n-form

$$dz_{q1} \wedge dz_{q2} \wedge ... \wedge dz_{qn}$$

that polarizes the half-forms bundle Pf_P. Thus the canonical bundle is the bundle of Holomorphic n-forms and the quantization bundle has sections that locally can be written as

$$s(z_{qi}) \otimes \sqrt{dz_{q1} \wedge dz_{q2} \wedge ... \wedge dz_{qn}}$$

on $L \otimes Pf_P$.

Now, if the first Chern class of a Kähler manifold as complex manifold is c, then the corresponding Chern class for its canonical bundle is[4] -c, and then from the topological condition for the existence of the half-forms bundle we have that such structure exist if the 2-form $\frac{\omega}{2\pi\hbar} - \frac{c}{2}$ satisfies the integrality condition

$$\int_R \left(\frac{\omega}{2\pi\hbar} - \frac{c}{2} \right) \epsilon \in \mathbf{Z}$$

where R is any closed oriented 2-surface in $\mathbf{C}^{2n}$.

3.2.2 Holomorphic Description of the n-Electron System and FQHE

From our preceding discussion lets write in complex (Holomorphic) coordinates the corresponding expressions for the quantum Hamiltonian operators and then the wave functions (sections on the corresponding quantum bundle) that describe the system. The aim of this last part is to find this wave functions and to relate topological conditions for the quantization existence with physical effects, in a similar fashion that in the magnetic monopole case, but relating our system with the description of the physically known *Quantum Hall Effect*.

[4] See Morrow, J. and Kodira, K. *Complex Manifolds.* Holt, Rinehart and Winston, 1971.

Harmonic Potential

In coordinates$\{z_{q1}, z_{q2}, ..., z_{qn}, z_{p1}, z_{p2}, ..., z_{pn}\}$, where $q_{ix} = \frac{1}{2}(z_{qi} + \overline{z}_{qi})$ and $q_{iy} = \frac{1}{2i}(z_{qi} - \overline{z}_{qi})$, if we introduce the definitions

$$\partial_{Oi} = \tfrac{i}{2}\left(\widehat{p}_{ix} - i\widehat{p}_{iy}\right)$$

$$\overline{\partial}_{Oi} = \tfrac{i}{2}\left(\widehat{p}_{ix} + i\widehat{p}_{iy}\right)$$

we obtain from the expressions for each operator

$$\partial_{Oi} = \tfrac{\hbar}{2}\left(\tfrac{\partial}{\partial q_{ix}} - i\tfrac{\partial}{\partial q_{iy}}\right)$$

$$\overline{\partial}_{Oi} = \tfrac{\hbar}{2}\left(\tfrac{\partial}{\partial q_{ix}} + i\tfrac{\partial}{\partial q_{iy}}\right)$$

but it is easy to see that the identities

$$\tfrac{\partial}{\partial z_{qi}} = \tfrac{1}{2}\left(\tfrac{\partial}{\partial q_{ix}} - i\tfrac{\partial}{\partial q_{iy}}\right)$$

$$\tfrac{\partial}{\partial \overline{z}_{qi}} = \tfrac{1}{2}\left(\tfrac{\partial}{\partial q_{ix}} + i\tfrac{\partial}{\partial q_{iy}}\right)$$

follow, and then

$$\partial_{Oi} = \hbar\tfrac{\partial}{\partial z_{qi}}$$

$$\overline{\partial}_{Oi} = \hbar\tfrac{\partial}{\partial \overline{z}_{qi}}.$$

Now, given that $\partial_{Oi}\overline{\partial}_{Oi} = -\frac{1}{4}\left(\widehat{p}_{ix}^2 + \widehat{p}_{iy}^2 + i\left[\widehat{p}_{ix}, \widehat{p}_{iy}\right]\right)$, and $[\widehat{p}_{ix}, \widehat{p}_{iy}] = 0$, the Hamiltonian operator $\widehat{H}$ in terms of this variables can be written as

$$\widehat{H} = \tfrac{-2}{m}\partial_{Oi}\overline{\partial}_{Oi} + \tfrac{m}{2}z_{qi}\overline{z}_{qi}.$$

Perpendicular Magnetic Field

In this case for the magnetic potential A we have that (in the symmetric gauge)

$$A = \frac{iB}{4}z_{qi}d\overline{z}_{qi} - \frac{iB}{4}\overline{z}_{qi}dz_{qi}$$

(or $A = \frac{iB}{2}z_{qi}d\overline{z}_{qi} = -\frac{iB}{2}\overline{z}_{qi}dz_{qi}$ in other gauges) and with the definition

$$\partial_{Ai} = \tfrac{i}{2}\left(\widehat{p}_{ix}^A - i\widehat{p}_{iy}^A\right)$$

$$\overline{\partial}_{Ai} = \tfrac{i}{2}\left(\widehat{p}_{ix}^A + i\widehat{p}_{iy}^A\right)$$

where we take the expressions for each operator, we obtain that

$$\partial_{Ai} = \tfrac{\hbar}{2}\left(\tfrac{\partial}{\partial q_{ix}} - i\tfrac{\partial}{\partial q_{iy}}\right) - \tfrac{eB}{4}(q_{ix} - iq_{iy})$$

$$\overline{\partial}_{Ai} = \tfrac{\hbar}{2}\left(\tfrac{\partial}{\partial q_{ix}} + i\tfrac{\partial}{\partial q_{iy}}\right) + \tfrac{eB}{4}(q_{ix} + iq_{iy})$$

i.e.

$$\partial_{Ai} = \partial_{Oi} - \tfrac{eB}{4} z_{qi}$$

$$\overline{\partial}_{Ai} = \overline{\partial}_{Oi} + \tfrac{eB}{4} z_{qi}$$

and from this and $\partial_{Ai}\overline{\partial}_{Ai} = -\frac{1}{4}\left(\widehat{p}_{ix}^{A\,2} + \widehat{p}_{iy}^{A\,2} + i\left[\widehat{p}_{ix}^{A}, \widehat{p}_{iy}^{A}\right]\right)$ follows that the Hamiltonian operator $\widehat{H}_A$ can be written as

$$\widehat{H}_A = \frac{-2}{m}\partial_{Ai}\overline{\partial}_{Ai} - \frac{2\hbar eB}{m},$$

where we use the commutator

$$\left[\widehat{p}_{ix}^{A}, \widehat{p}_{iy}^{A}\right] = \left[\widehat{p}_{ix}, \widehat{p}_{iy}\right] + \tfrac{e^2B^2}{4}\left[\widehat{q}_{ix}, \widehat{q}_{iy}\right] + \tfrac{eB}{2}\left(\left[\widehat{q}_{ix}, \widehat{p}_{ix}\right] + \left[\widehat{q}_{iy}, \widehat{p}_{iy}\right]\right)$$

$$= \tfrac{2i\hbar eB}{2}.$$

Observe that

$$\widehat{H}_A = \frac{-2}{m}\partial_{Oi}\overline{\partial}_{Oi} + \frac{e^2B^2}{8m} z_{qi}\overline{z}_{qi} - \frac{2\hbar eB}{m}$$

and then $\widehat{H}_A$ and $\widehat{H}_O$ have (apart of a constant shift) the same spectrum if

$$\frac{e^2B^2}{8m} = \frac{mw_0^2}{2}$$

i.e. if $\frac{eB}{2m} = w_0$, this give a condition on the intensity of the magnetic field in order to obtain a good approximation in the model of oscillators in change of magnetic field, in a physical sense.

Landau States and Geometric Quantization

Lets relate the system of n electrons in presence of a magnetic field (perpendicular to the plane in which its dynamics take place) with the harmonic potential case, and this with the description of the *Landau States* of a system of n electrons in a magnetic field, which is of interest in the theoretical description of the *Fractional Quantum Hall Effect*. The constraint in the dimension of the system is not arbitrary, besides it corresponds with a physical effect that only appears when this kind of systems is forced to this restriction: the appearance of states that do not behave as *bosons* or *fermions*, but as a mixture between them known as *Anyons*[5].

The Landau states of this kind of system are the states corresponding to the different values of the energy of the system, they are described by the so-called *Laughlin's wave functions* in physics literature, for the *FQHE* the ground state (lowest Landau level) wave functions describe the anyonic states and are specially interesting. Lets use all the calculations we have made in the last few sections to recover the standard quantum mechanical description of lowest Landau states[6] from our geometric quantization scheme. First observe that the Hamiltonians for the (quantum) description of the n electron system under perpendicular magnetic field and harmonic oscillator potential only differs (apart of a constant in the last term) in a multiple of the angular momentum operator (sections *3.1.1* and *3.1.2*) and it is easy note that the commutator between the Hamiltonians and the angular momentum operator is zero (because the z-component angular momentum conservation), so the wave functions and eigenvalues of both Hamiltonians (the energy spectrum of both systems), apart of some constant, are the same, say (example 6, section *1.3*)

$$E_n = \hbar\mathcal{K}\left(n + \frac{1}{2}\right),$$

where $\mathcal{K}$ is a constant ($\mathcal{K} = \frac{eB}{m}$).

Now, it is very clear that each Landau level (each submanifold of the phase space defined by the condition $E = \frac{1}{2m}(p_{ix}^2 + p_{iy}^2) + \frac{m}{2}(q_{ix}^2 + q_{iy}^2) =$ constant) corresponds to a infinitely degenerated state, due to the "rotational symmetry" around the z axis. As a matter of fact if we consider the surface defined by (we have taken away the constants $\frac{1}{2m}$ and $\frac{m}{2}$ corresponding to each term)

$$p_{ix}^2 + p_{iy}^2 + q_{ix}^2 + q_{iy}^2 = 1$$

[5]See Wilczek, F. *Fractional Statistics and Anyon Superconductivity.* World Scientific, 1990.

[6]See Karlhede, A. and Wersterberg, E. *Int. Jour. Mod. Phys. B*, **6**, 1595, 1992.

i.e. $S^{4n-1} \subset \mathbf{R}^{4n}$, which is a coisotropic submanifold of $\mathbf{R}^{4n}$, then its tangent space in a given point is generated by vectors of the form

$$a_i \frac{\partial}{\partial p_{ix}} + b_i \frac{\partial}{\partial p_{iy}} + c_i \frac{\partial}{\partial q_{ix}} + d_i \frac{\partial}{\partial q_{iy}}$$

for which $a_i p_{ix} + b_i p_{iy} + c_i q_{ix} + d_i q_{iy} = 0$. From this, with $a_i = p_{iy}$, $b_i = -p_{ix}$, $c_i = -q_{iy}$ and $d_i = q_{ix}$, we can deduce that this tangent vectors defines a foliation whose leaves are the solution curves to the differential equations

$$\frac{dp_{ix}}{dt} = p_{iy}$$

$$\frac{dp_{iy}}{dt} = -p_{ix}$$

$$\frac{dq_{ix}}{dt} = -q_{iy}$$

$$\frac{dq_{iy}}{dt} = q_{ix}.$$

By identification with $\mathbf{C}^{2n}$, through the change in the coordinates $\{q_{1x}, q_{2x}, ..., q_{nx}, q_{1y}, q_{2y}, ..., q_{ny}, p_{1x}, p_{2x}, ..., p_{nx}, p_{1y}, p_{2y}, ..., p_{ny}\} \to \{z_{q1}, z_{q2}, ..., z_{qn}, z_{p1}, z_{p2}, ..., z_{pn}\}$, this equations reads

$$\frac{dz_{qi}}{dt} = \frac{dq_{ix}}{dt} + i\frac{dq_{iy}}{dt} = iz_{qi}$$

$$\frac{dz_{pi}}{dt} = \frac{dp_{ix}}{dt} - i\frac{dp_{iy}}{dt} = iz_{pi}$$

i.e. the solution curves are given by

$$z_{qi}(t) = e^{it} z_{qi}(0)$$

$$z_{pi}(t) = e^{it} z_{pi}(0)$$

and thus, the Hamiltonian flow generated on $\mathbf{C}^{2n}$ by the function $H = z_{pi}\overline{z}_{pi} + z_{qi}\overline{z}_{qi} = 1$ is given by

$$\begin{aligned} \phi_t : \mathbf{C}^{2n} &\to \mathbf{C}^{2n} \\ \overrightarrow{z} &\mapsto e^{it}\overrightarrow{z} \end{aligned}$$

where $\overrightarrow{z}$ denotes the vector $(z_{q1}, z_{q2}, ..., z_{qn}, z_{p1}, z_{p2}, ..., z_{pn})$ in $\mathbf{C}^{2n}$. This is a Hamiltonian action on phase space, with orbits

$$\mathcal{O}_{\overrightarrow{z}} = \{\phi_t(\overrightarrow{z}) : t \in \mathbf{R}\}$$

$$= \{\exp(it)\overrightarrow{z} : t \in \mathbf{R}\}$$

$$= \{\alpha\overrightarrow{z} : \alpha \in \mathbf{C}\}$$

so the orbit space (reduced phase space) is

$$\mathbf{C}^{2n}/\phi_{action} \simeq \mathbf{CP}^{2n-1}.$$

It is clear now that all we have done is the symplectic reduction procedure described in appendix 3, applied to our symplectic manifold $(\mathbf{C}^{2n}, \omega)$ with the 2-form (in $z_{q_i},\ z_{pi}$ coordinates)

$$\omega = \tfrac{1}{2}\left(dz_{pi} \wedge dz_{qi} + d\overline{z}_{pi} \wedge d\overline{z}_{qi}\right)$$

$$\widetilde{\omega} = \omega + \tfrac{eB}{2i}\left(d\overline{z}_{qi} \wedge dz_{qi}\right)$$

(for the harmonic and magnetic field cases, respectively), with the symplectic action

$$\begin{array}{c} \phi : U(1) \times \mathbf{C}^{2n} \to \mathbf{C}^{2n} \\ (e^{it},\ \overrightarrow{z}) \mapsto e^{it}\overrightarrow{z}. \end{array}$$

This action clearly preserves the Kähler potential $K = \frac{1}{2}(z_{qi}\overline{z}_{qi} + z_{pi}\overline{z}_{pi})$ on $\mathbf{C}^{2n}$ and then the Hermitian and symplectic structures defined in terms of it, so this is a symplectic action. Now, from our results on symplectic reduction in the third appendix, having the canonical identification of the Lie algebra $\mathtt{u(1)}$ with $\mathbf{R}$, and taking the vector field, $\xi = \frac{\partial}{\partial t}$, it is clear that

$$\xi_{\mathbf{C}^{2n}} = q_{ix}\frac{\partial}{\partial p_{ix}} + q_{iy}\frac{\partial}{\partial p_{iy}} - p_{ix}\frac{\partial}{\partial q_{ix}} - p_{iy}\frac{\partial}{\partial q_{iy}}$$

and thus, by definition with the symplectic form $dq_{ix} \wedge dp_{ix} + dq_{iy} \wedge dp_{iy}$,

$$\begin{aligned} d\mathcal{J}(\xi) &= \left(dq_{ix} \wedge dp_{ix} + dq_{iy} \wedge dp_{iy}\right)\left(\xi_{\mathbf{C}^{2n}}\right) \\ &= dq_{ix} \wedge dp_{ix}\left(q_{ix}\tfrac{\partial}{\partial p_{ix}} + q_{iy}\tfrac{\partial}{\partial p_{iy}} - p_{ix}\tfrac{\partial}{\partial q_{ix}} - p_{iy}\tfrac{\partial}{\partial q_{iy}}\right) \\ &\quad + dq_{iy} \wedge dp_{iy}\left(q_{ix}\tfrac{\partial}{\partial p_{ix}} + q_{iy}\tfrac{\partial}{\partial p_{iy}} - p_{ix}\tfrac{\partial}{\partial q_{ix}} - p_{iy}\tfrac{\partial}{\partial q_{iy}}\right) \end{aligned}$$

$$\begin{aligned} dJ(\xi) &= -q_{ix}dq_{ix} - q_{iy}dq_{iy} - p_{ix}dp_{ix} - p_{iy}dp_{iy} \\ &= d\left[-\tfrac{1}{2}\left(z_{qi}\overline{z}_{qi} + z_{pi}\overline{z}_{pi}\right)\right] \end{aligned}$$

and then the momentum map for this action is

$$\begin{array}{c} J : \mathbf{C}^{2n} \longrightarrow \mathbf{R} \\ \overrightarrow{z} \mapsto -\frac{1}{2} \mid \overrightarrow{z} \mid^2 \end{array}$$

where $\overrightarrow{z}$ denotes a vector $(z_{q1}, z_{q2}, ..., z_{qn}, z_{p1}, z_{p2}, ..., z_{pn})$ in $\mathbf{C}^{2n}$. It is clear that $S^{4n-1} = J^{-1}(-\frac{1}{2})$, and in the notation of appendix 3, $M_{-\frac{1}{2}} = S^{4n-1}/U(1) \simeq \mathbf{CP}^{2n-1}$, given that each orbit is the intersection of the sphere with a complex line through the origin[7].

Lets study now quantization on the reduced phase space[8], $\mathbf{CP}^{2n-1}$, with its symplectic structure coming from its Kähler character. We will work now with the coordinates $(\xi_1, \xi_2, ..., \xi_{2n-1})$ defined on open sets $\{U_k\}$ as in chapter two. The Fubini-Studi metric on $\mathbf{CP}^{2n-1}$is [N1][KN]

$$\omega_{FS} = -imE\frac{d\xi_k \wedge d\overline{\xi}_k - \overline{\xi}_k \xi_l d\xi_k \wedge d\overline{\xi}_l}{(1+\xi_k\overline{\xi}_k)^2}$$

so we must to quantize the Kähler manifold $(\mathbf{CP}^{2n-1}, \omega_{FS}, P_h)$, where P_h denotes the Holomorphic polarization (induced by $\mathbf{C}^{2n}$). The first Chern class of $\mathbf{CP}^{2n-1}$is

$$c_1(\mathbf{CP}^{2n-1}) = 2n\,[\omega^\circ]$$

where ω° is a positive generator of $H^2(\mathbf{CP}^{2n-1}, \mathbf{Z})$, so the first Chern class of its canonical bundle is -$2n\,[\omega^\circ]$. From

$$\int\limits_{\mathbf{CP}^{2n-1}} \omega_{FS}^{2n-1} = (2\pi)^{2n-1}$$

we know that $\omega_{FS} \in [\omega^\circ]$, so the existence of a quantum bundle is guaranteed if

$$\Omega = \frac{[\omega_{FS}]}{2\pi\hbar} + \frac{2n\,[\omega^\circ]}{2}$$

is a positive generator of $H^2(\mathbf{CP}^{2n-1}, \mathbf{Z})$, and from this we deduce that then

$$-\frac{mE}{2\pi\hbar} + n \in \mathbf{Z}^+$$

[7]Taking in account the constants in the surface of constant energy defined by $E = \frac{1}{2m}(p_{ix}^2 + p_{iy}^2) + \frac{mw_0^2}{2}(q_{ix}^2 + q_{iy}^2) =$ constant, we must obtain $S^{4n-1} = J^{-1}(-mE)$, and in general the radius of this sphere is given by $2mE = \frac{eBE}{w_0}$.

[8]Without the degeneracy, and then without the symplectic reduction, this quantization corresponds with the *Integral Quantum Hall Effect*, the fractional characteristic of this quantum effect is thus consequence of the symmetry. As a matter of fact Thoules et al. have found the quantization in the Hall conductance in terms of the first Chern number of a vector bundle divided by its rank, where the rank is equal to the degeneracy. See, Thoules, D. et al. *Phys. Rev.* **B31**, *3372*, 1985.

and the possible values for the energies of the electrons in the system are $E_N = \frac{2\pi\hbar}{m}(n+N)$, fact that illustrate the degeneracy in the spectrum.

The reduced Hilbert space in this case could be consider as the completion of the set of Holomorphic sections of the complexificated line (U(1)) bundle (with the Hopf fibration). However the set of Holomorphic line bundles over $\mathbf{CP}^{2n-1}$ is parametrized by $H^1(\mathbf{CP}^{2n-1}, \jmath)$, where $\jmath$ denotes the sheaf of local Holomorphic functions on $\mathbf{CP}^{2n-1}$, and given the map

$$c_1 : H^1(\mathbf{CP}^{2n-1}, \jmath) \longrightarrow H^2(\mathbf{CP}^{2n-1}, \mathbf{Z}) \simeq \mathbf{Z}$$
$$\mathcal{L} \longmapsto c_1(\mathcal{L}) = \tfrac{i[F]}{2\pi}$$

where $[F]$ is the cohomology class of the curvature of any connection on $\mathcal{L}$, we see that there are Holomorphic line bundles over $\mathbf{CP}^{2n-1}$ as many as integer numbers.

Now, if we observe the form of the equations *3.2.1*, for the operators ∇_i and $\overline{\nabla}_i$, and we compare with the result (in a non-symmetric gauge)

$$\partial_{Ai} = \partial_{Oi} + \tfrac{1}{2} eB\overline{z}_{qi}$$

$$\overline{\partial}_{Ai} = \overline{\partial}_{Oi}$$

we can deduce that these operators correspond to the connection operators for a Kähler manifold with Kähler scalar potential

$$K = \frac{eB}{2} z_{qi}\overline{z}_{qi}$$

and this is just (apart from a constant factor) the potential corresponding to $\mathbf{C}^n$ parametrized with local coordinates z_{qi}. Thus, this quantization can be identified with the quantization of $\mathbf{C}^n$, with Kähler form $\omega = \frac{eB}{2} dz_{qi} \wedge d\overline{z}_{qi}$, and then the quantum bundle is a Holomorphic line bundle $\mathcal{L}$ on $\mathbf{C}^n$ with first Chern class $[\omega]$.

Following this idea, our Hilbert space of quantization is the set of Holomorphic sections on $\mathcal{L}$ such that

$$\frac{\partial\psi(z_{qi}, \overline{z}_{qi})}{\partial\overline{z}_{qi}} = 0$$

i.e.

$$\psi(z_{qi}, \overline{z}_{qi}) = \phi(z_{qi}) e^{-\frac{eB}{2\hbar} z_{qi}\overline{z}_{qi}}$$

where the $\phi(z_{qi})$ is a Holomorphic function in coordinates of configuration space (in the usual sense), these sections must be L^2 integrable respect to the Hermitian structure

$$| \psi(z_{qi}, \overline{z}_{qi}) |^2 = \phi\overline{\phi} e^{-\frac{eB}{4\hbar} z_{qi}\overline{z}_{qi}}.$$

For the n-electron system Laughlin[9] proposed the form of the wave functions for different values of the "filling factor" ν defined by the number of electrons in the system divided the magnetic flux (in units of quantum flux Φ_0)

$$\nu = \frac{n}{\Phi/\Phi_0} = \frac{n}{N\Phi}$$

where N is the degeneracy of each Landau Level. This filling factor represent the number of filled levels at $T = 0$, the *IQHE* consider all the (small) integer values of and for *FQHE* it works (respect to the experimental data) with almost all the fractions with odd denominators $m = 2n+1$ and n small. For $\nu = 1/m$ the Laughlin wave functions are[10] (following the Pauli exclusion Principle and angular momentum conservation)

$$\psi_n^{(m)}(z_{qi}, \overline{z}_{qi}) = \prod_{i<j}^{n} (z_{qi} - z_{qj})^m \, e^{-\frac{eB}{2\hbar} z_{qi}\overline{z}_{qi}}.$$

The analytic part of this wave function is a homogeneous polynomial of degree $M = mn(n-1)/2$ (i.e. a eigenstate of angular momentum). The odd denominator rule observed experimentally can be explained by the phase factor $(-1)^m$ that gives the exchange of sign to the wave function under the exchange of two electrons of the system, this is the fermionic statistics involved in electron systems. This results also can be obtained from the geometric quantization of the system, as we can see from the canonical representation of sections of Holomorphic line bundles over $\mathbf{CP}^{2n-1}$.

Now, from $\mathbf{C}^n$ to $\mathbf{CP}^{2n-1}$the symplectic form $\omega = \frac{eB}{2} dz_{qi} \wedge d\overline{z}_{qi}$ can be pushed down to the Fubini-Studi form ω_{FS} through the map $\sigma^E : \mathbf{C}^{2n} \to \mathbf{CP}^{2n-1}$ that takes the surface of constant energy E to its image under the

[9] Lauglin, D. *Phys. Rev. Lett.* **50**, 1395, 1983.

[10] This wave function is the corresponding for the nonsymmetric gauge, in the symmetric gauge it is $\psi_{nm}(z_{qi}, \overline{z}_{qi}) = (z_{qi})^n \, (z_{qj})^m \, e^{-\frac{eB}{2\hbar} z_{qi}\overline{z}_{qi}}$, see Fubini, S. Int. Jour. Mod. Phys. **A5**, 3533, 1990.

reduction quotient, i.e. $\sigma_*^E\left(\frac{eB}{2}dz_{qi}\wedge d\bar{z}_{qi}\right) = \frac{eBE}{w_0\hbar}\omega_{FS}$, and this implies, through the integrality condition,

$$\frac{eBE}{w_0\hbar}\in \mathbf{Z}.$$

For the cases of odd integers (our fermionic system) this means that

$$\frac{eBE}{w_0\hbar}=m$$

where $m=2n+1$, and this is equivalent to

$$\frac{1}{2n+1}=\frac{w_0\hbar}{eBE}$$

for some $n\in\mathbf{Z}$, precisely the filling factor ν[11].

As in the magnetic monopole case lets look for the Holomorphic quantum bundles corresponding to the different quantum numbers, We already know that the first Chern class $c_1(\mathcal{L})=\frac{i[F]}{2\pi}$ classifies this line bundles, and if $c_1(\mathcal{L})=1$ and $c_1(\mathcal{L}')=m$, then $\mathcal{L}'\simeq\mathcal{L}^{\otimes m}$, the mth tensor power of $\mathcal{L}$. From the integrality condition we know that for each integer there is, up to isomorphism, a unique connection corresponding to a Holomorphic line bundle $\mathcal{L}_m$, say $\nabla^m=(\partial_{L_m},\overline{\partial}_{L_m})$ (written in terms of its (1,0) and (0,1) components). In local coordinates

$$\theta_m=\frac{m(2n)}{2\pi i}\frac{\xi_k d\bar{\xi}_k}{1+\xi_k\bar{\xi}_k}$$

is a connection potential on $\mathcal{L}_m$ with curvature $-im\omega_{FS}$, i.e. the expected connection [KN]. Now, given that

$$c_1(\mathcal{L}_m)=\frac{i}{2\pi}\int\limits_{\mathbf{CP}^{2n-1}}-im\omega_{FS}^{2n-1}=m$$

we deduce that $\mathcal{L}_m\simeq\mathcal{L}^{\otimes m}$, where our original $\mathcal{L}$ is $\mathcal{L}_1$, and for each integer m the Hilbert space of quantization is the completion of $\Gamma_m=\Gamma_{P_h}(\mathbf{CP}^{2n-1},\mathcal{L}^{\otimes m})$. Finally, note that is in this sense that the filling factors are topological numbers,

$$\nu=1/c_1(\mathcal{L}_m),$$

whose value is fixed by the integrality condition.

[11]See Hatsugay, Y. *"Topological Aspects of the Quantum Hall Effect"*. J. Phys.: Condens. Matter, **9**, 2507, 1997.

APPENDIXES

Appendix A

Symplectic Manifolds

A.1 Symplectic Forms on Vector Spaces

Definition 1:

Let V be a (real) vector space with finite dimension and V^* its dual space, $\{v_1, v_2, v_3, ..., v_n\}$ a base for V and $\{v^1, v^2, v^3, ..., v^n\}$ the corresponding dual base (i.e. $v^i(v_j) = \delta_{ij}$). Let $L^2(V, \mathbf{R})$ be the set of bilinear functions from $V \times V$ on $\mathbf{R}$ and ω an element of this set, we say that ω is *no-degenerated* if and only if

$$\omega(v_1, v_2) = 0 \qquad \forall v_2 \in V \qquad \Longrightarrow v_1 = 0,$$

this is equivalent to say that the matrix of ω, defined by $\omega_{ij} = \omega(v_i, v_j)$, is no singular, or that the application $\widetilde{\omega} : V \to V^*$ defined by $\widetilde{\omega}(v)(v') = \omega(v, v')$ is an isomorphism.

Definition 2:

A 2-form ω on V is a bilinear application such that for the transposed matrix of ω, defined by $\omega^t(v_1, v_2) = \omega(v_2, v_1)$, follows that $\omega^t = -\omega$. $\Lambda^2(V)$ denotes the set of 2-forms on the vector space.

The structure of a 2-form is determined by its range, if the dimension of the vector space is n and the range of a 2-form on it is r, then $r = 2p$ for some integer p and there exist bases for V and V^* in terms of which

$$\omega = v^1 \wedge v^{p+1} + v^2 \wedge v^{p+2} + \cdots + v^p \wedge v^{2p}$$

and then the matrix of such 2-form is

$$J = \begin{pmatrix} 0 & I_p & 0 \\ -I_p & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

where I_p is the $p \times p$ identity matrix. If ω have maximal range,

$$J = \begin{pmatrix} 0 & I_n \\ -I_n & 0 \end{pmatrix}$$

and then[1]

$$\omega = \sum_{i=1}^{n} v^i \wedge v^{i+n} \equiv v^i \wedge v^{i+n}.$$

If the dimension of V is *even* the ω can be written as in the last equation, and then it is no degenerated. To verify this affirmation it is enough to observe that given a vector space W, defining $V = W \times W^*$, the function $\omega : V \times V \to \mathbf{R}$ defined by $\omega(v_1, v_2) = \omega((w_1, \alpha_1), (w_2, \alpha_2)) = \alpha_2(w_1) - \alpha_1(w_2)$ is a no-degenerated form. In this way if V is an arbitrary vector space and $DimV = 2n$ then $V \cong \mathbf{R}^{2n} \cong \mathbf{R}^n \times \mathbf{R}^n \cong \mathbf{R}^n \times \mathbf{R}^{n*}$, and the result follows from this. Now, given the structure of the 2-form on V mentioned before, it is clear that if ω is a no-degenerated form on V there exist a base $\{v_1, v_2, ..., v_n, w_1, w_2, ..., w_n\}$ for V such that

$$\omega = v^i \wedge w^i$$

and then $\omega(v_i, v_j) = \omega(w_i, w_j) = 0$, $\omega(v_i, w_j) = \omega(w_j, v_i) = \delta_{ij}$. This base is called *canonical* for V.

Definition 3:

A 2-form ω no degenerated on a vector space V is called *Symplectic Form* and to the pair (V, ω) *Symplectic Vector Space*. A function $t : V \to V'$, where (V, ω) and (V', ω') are symplectic vector spaces, is called symplectic or *symplectomorphism* if $t^*\omega' = \omega$ (this is $\omega'(t(v_1), t(v_2)) = \omega(v_1, v_2) \quad \forall v_1, v_2 \in V$).

Every symplectomorphism is an isomorphism (induced by the corresponding 2-form) and the set of symplectomorphisms form a symplectic vector space into itself has group structure, with the composition operation; we denote that group as $Sp(V, \omega)$.

[1] From now we are using the repeated index summation convention.

Theorem 1:

$T \in Sp(V, \omega)$ if and only if $T^t J T = J$, where J is the matrix associated to ω.

A.2 Symplectic Manifolds

Lets go now in our discussion to the case of differentiable smooth (C^∞) manifolds with finite dimension.

Definition 4:

A *Symplectic Manifold* (M, ω) is a pair composed by a smooth manifold M and a differential 2-form ω defined on it, closed and no-degenerated (called *symplectic form*). This means that

$$d\omega = 0$$

and the application

$$\begin{aligned} i : T_m M &\longrightarrow T^*_m M \\ X &\longmapsto i(X) = \omega(X, \cdot) \equiv i_X \omega \end{aligned}$$

is a linear isomorphism for each $m \in M$ between the spaces of tangent and cotangent vectors in m. Every symplectic manifold has even dimension and is orientable, but no every differential manifold with even dimension admits a symplectic structure.

Definition 5:

If a function C^∞, $f : (M, \omega) \to (M', \omega')$, is such that $f^*\omega' = \omega$, this function is called a *symplectomorphism* and is then a local diffeomorphism between the manifolds. As in the case of vector spaces the symplectomorphisms with the composition operation have group structure (denoted $Sp(M, \omega)$).

In the symplectic vector spaces we have a canonical representation for a symplectic 2-form; in this case, with symplectic manifolds, a similar result is the following theorem

Darboux Theorem:

Let (M, ω) be a symplectic manifold and $m \in M$ any point on it, then there exists an open set U_m around m with local coordinates $\{q_1, q_2, ..., q_n, p_1, p_2, ..., p_n\}$ (called canonical coordinates) in such way that on U_m

$$\omega = dp_i \wedge dq_i.$$

Example:

The main example is the *Cotangent Bundle* to a differential manifold. If M is a smooth manifold with local coordinates $\{q_1, ..., q_n\}$ and T_mM, its tangent space at m, then T_mM can be seen in terms of derivations[2] as the space generated by the set $\{\frac{\partial}{\partial q_1}, ..., \frac{\partial}{\partial q_n}\}$ and with this base we have the corresponding to the cotangent space T_m^*M (dual of T_mM) $\{dq_1, ..., dq_n\}$. The *Tangent Bundle* to M is defined as the union (disjoint) $TM = \bigcup_{m \in M} T_mM$ and the cotangent bundle as $T^*M = \bigcup_{m \in M} T_m^*M$, both of them have a natural structure of $2n$-dimensional differential manifold. A "point" in TM can be determined by the $2n$ coordinates $\{q_1, ..., q_n, a_1, ..., a_n\}$, the first n identifies the point m of M where the particular tangent space is taken and the last n give the coefficients identifying the particular point on such tangent space, in this case this "point" is given by the following expression

$$a_1 \frac{\partial}{\partial q_1} + \cdots + a_n \frac{\partial}{\partial q_n}.$$

In a similar way, any element of T^*M is in some of the T_m^*M, thus is determined by n coordinates giving localization to the point m and on it the desired element can be written as

$$p_1 dq_1 + \cdots + p_n dq_n$$

so every local chart on T^*M must have as coordinates $\{q_1, ..., q_n, p_1, ..., p_n\}$.

On the tangent and cotangent bundles described in this way results natural to think in the possibility of defining a symplectic form in each case (given its even dimension), and in fact this forms exists. In the case of

[2]See S.G. Scott "Some Notes on Geometry and Quantization", in Proceedings of 1st. *Encuentro de Geometría Diferencia y Física*, Uniandes, 1995.

the cotangent bundle $T^*M \xrightarrow{\pi} M$, where π denotes the natural projection $\{q_1, ..., q_n, p_1, ..., p_n\} \equiv (q, p) \overset{\pi}{\mapsto} \{q_1, ..., q_n\} \equiv q$, this 2-form

$$\omega = dp_i \wedge dq_i$$

is closed and no-degenerated, and is independent of the chosen coordinates on M. More than closed ($d\omega = 0$) this symplectic form on the cotangent bundle is exact, this means that exists globally a 1-form θ defined on T^*M such that

$$\omega = d\theta,$$

this 1-form is called *canonical* or *symplectic potential* in the literature, and in local coordinates has the form

$$\theta = p_i dq_i.$$

On a general symplectic manifold, given that ω is closed and by the using of the *Poincare lemma*, there exists this symplectic potential only locally (and it is not unique), and in the corresponding canonical coordinates has the structure given by the last equation. In general, the already mentioned Darboux Theorem shows that every symplectic manifold has local structure of cotangent bundle, and then any pair of symplectic manifolds with the same dimension are locally diffeomorphic.

Appendix B

Hamiltonian Dynamics and Symplectic Geometry

The classical description of the dynamics of a given physical system can be build up in many ways, from the Newtonian scheme until the more elaborated and usual, Lagrangian and Hamiltonian, whose incorporation to the structure of the physical theories follows from the facility that they offer in the study of the symmetries and its relation with the conservation laws (through Noethers Theorem in the lagrangian description, for example) and the direct arrive to the corresponding quantum description (in the Hamiltonian scheme). Given a physical system composed by N particles it is necessary, to determine the system configuration in a particular time, to know $3N$ numbers corresponding to the $3N$ spacial coordinates of the system particles, thus we can define the *Configuration Space* for that system as the set of the ordered $3N$ numbers susceptible of being defined for the possible configurations of the system. This coordinates can be position coordinates, but not necessarily they must be[1].

Defined in this way the configuration space Q of a physical system can be seen naturally as a differentiable manifold (differentiability is imposed here in order to define velocities, accelerations and other physical quantities as in Newtonian mechanics) with dimension equal to the number of freedom degree of the system ($3N$ in the case described in last paragraph). If we take the tangent bundle to this manifold we can see it as the space of all

[1] In general any set of numbers that determine completely the system configuration can work as "generalized coordinates" for it, velocities, temperatures and potentials are possible examples.

the possible tangent vectors to trajectories in configuration space, this is the generalized velocities space of the system, where can be defined the *Lagrangian* as a real function of the coordinates and velocities $L(q_i, \dot{q_i})^2$ in terms of which the trajectories followed by the system in configuration space can be determined and thus the equations of motion of the system, known as *Euler-Lagrange* equations[A1]

$$\frac{\partial L}{\partial q_i} - \frac{d}{dt}\frac{\partial L}{\partial \dot{q_i}} = 0.$$

This is the lagrangian description of the dynamics of a physical system, accomplished on the tangent bundle to configuration space, TQ. From the lagrangian of a system we define the "*generalized momentum*" p_i as

$$p_i = \frac{\partial L}{\partial \dot{q_i}}$$

and the *Hamiltonian* function of the system as

$$H(q_i, p_i) = p_i \dot{q_i} - L(q_i, \dot{q_i})$$

in terms of which the motion equations, also called *Hamilton Equations*, are

$$\begin{aligned} \dot{q_i} &= \frac{\partial H}{\partial p_i} \\ \dot{p_i} &= -\frac{\partial H}{\partial q_i}. \end{aligned}$$

The momentum p_i can be seen here as coordinates in the cotangent bundle T^*Q, called *Phase Space* of the system, and we can define an algebraic structure on the set of C^∞ real functions on phase space (called "physical observables") through the Poisson bracket operation[3], that given two physical observables f and g (as energy, momentum, etc.) associate other defined by

$$\{f, g\} = \frac{\partial f}{\partial q_i}\frac{\partial g}{\partial p_i} - \frac{\partial g}{\partial q_i}\frac{\partial f}{\partial p_i}$$

[2]Notation: $\dot{q_i} = \frac{dq_i}{dt}$, where t is a parameter, in physics the time usually.

[3]It is direct to observe that the set of physical observables (smooth real functions on phase space, with the usual sum and multiplication by scalars) have structure of Lie algebra with the Poisson bracket operation, that means that given observables f, g and h, and scalars α and β,

$\{f, g\} = -\{g, f\}$
$\{\alpha f + \beta g, h\} = \alpha\{f, h\} + \beta\{g, h\}$
$\{f, \{g, h\}\} + \{g, \{h, f\}\} + \{h, \{f, g\}\} = 0.$

Another way of writing Hamilton equations, by using the Poisson brackets, follows recognizing that

$$\frac{dA}{dt} = \frac{\partial A}{\partial q_i}\,\dot{q_i} + \frac{\partial A}{\partial p_i}\,\dot{p_i} = \frac{\partial A}{\partial q_i}\frac{\partial H}{\partial p_i} - \frac{\partial H}{\partial q_i}\frac{\partial A}{\partial p_i} = \{A, H\}$$

for any observable A, thus

$$\begin{array}{l} \{q_i, H\} = \frac{\partial H}{\partial p_i} \\ \{p_i, H\} = -\frac{\partial H}{\partial q_i} \end{array}$$

are the same equations. Note that in terms of Poisson bracket we can characterize the constants of motion for the dynamics of the system, because A is a constant of motion if and only if $\{A, H\} = 0$.

The dynamical description of a classical system with higher interest for us is the Hamiltonian description, in first place because it is carried on the cotangent bundle to a manifold (configuration space) and this bundle has a natural symplectic structure, and second because the step from the classical to the corresponding quantum description is made from this formalism, and this can give us some geometrical information about how this process must be accomplished. Lets go on observing how classical dynamical description looks in a geometric fashion, i.e. how the symplectic structure on phase space determines the dynamics. Lets consider a system whose configuration space is the manifold Q and its phase space its cotangent bundle T^*Q, then there exist, defined on phase space, a symplectic form ω that in canonical local coordinates can be written as

$$\omega = dp_i \wedge dq_i.$$

Given a physical observable $f \in C^\infty(T^*Q, \mathbf{R})$ we define the *Hamiltonian Vector Field* associated to f as the field X_f given by the equation

$$i_{X_f}\omega = -df$$

where we use the isomorphism i defined by ω in the precedent appendix. This vector field exists because the symplectic form is nondegenerated (ω is nondegenerated if and only if $i_X\omega = 0 \iff X = 0$).

Theorem 1:

The Lie derivative of the symplectic form along a Hamiltonian vector field is zero.

Proof.

$$\mathcal{L}_{X_f}\omega = d(\omega(X_f)) + d\omega(X_f) = d(-df) + d\omega(X_f) = 0$$

□

This means that along the integral trajectories of the field the symplectic form is "constant".

Given a real function f on phase space T^*Q, its exterior derivative locally is written

$$df = \frac{\partial f}{\partial q_i}dq_i + \frac{\partial f}{\partial p_i}dp_i$$

thus the Hamiltonian vector field defined by this function is $X_f = a_i\frac{\partial}{\partial q_i} + b_i\frac{\partial}{\partial p_i}$, and using the definition

$$\omega(X_f,\cdot) = \omega\left(a_i\tfrac{\partial}{\partial q_i} + b_i\tfrac{\partial}{\partial p_i},\cdot\right) = dp_i \wedge dq_i\left(a_i\tfrac{\partial}{\partial q_i} + b_i\tfrac{\partial}{\partial p_i}\right)$$

$$= dp_i \wedge dq_i\left(a_i\tfrac{\partial}{\partial q_i}\right) + dp_i \wedge dq_i\left(b_i\tfrac{\partial}{\partial p_i}\right)$$

$$= -a_i dp_i + b_i dq_i = -\tfrac{\partial f}{\partial q_i}dq_i - \tfrac{\partial f}{\partial p_i}dp_i$$

so it is verified that

$$X_f = \frac{\partial f}{\partial p_i}\frac{\partial}{\partial q_i} - \frac{\partial f}{\partial q_i}\frac{\partial}{\partial p_i}$$

and then, if $\gamma(t) = (p_i(t), q_i(t))$ is a integral curve of the vector field we have that

$$\begin{aligned} -\tfrac{\partial f}{\partial q_i} &= \dot{p}_i \\ \tfrac{\partial f}{\partial p_i} &= \dot{q}_i \end{aligned}$$

precisely the Hamilton equations in the case in which f is the energy for the system, thus dynamics follows from the 2-form ω.

In the same way the algebraic structure of the set of observables on cotangent bundle T^*Q is determined by the symplectic structure, given that

Poisson bracket can be defined in terms of such form. For any two functions $f, g \in C^\infty(T^*Q, \mathbf{R})$, the *Poisson Bracket* of f with g is defined as

$$\{f, g\} = X_f(g) = -X_g(f) = \omega(X_f, X_g)$$

or, in local coordinates

$$\{f, g\} = X_f(g) = \frac{\partial f}{\partial p_i}\frac{\partial g}{\partial q_i} - \frac{\partial f}{\partial q_i}\frac{\partial g}{\partial p_i}$$

corresponding exactly with the old definition. It is easy to observe that, defining the Lie bracket of vector fields X and Y as $[X, Y] = XY - YX$, follows that

$$\begin{aligned}
[X_f, X_g] &= \left(\tfrac{\partial f}{\partial p_i}\tfrac{\partial}{\partial q_i} - \tfrac{\partial f}{\partial q_i}\tfrac{\partial}{\partial p_i}\right)\left(\tfrac{\partial g}{\partial p_i}\tfrac{\partial}{\partial q_i} - \tfrac{\partial g}{\partial q_i}\tfrac{\partial}{\partial p_i}\right) - \\
&\quad \left(\tfrac{\partial g}{\partial p_i}\tfrac{\partial}{\partial q_i} - \tfrac{\partial g}{\partial q_i}\tfrac{\partial}{\partial p_i}\right)\left(\tfrac{\partial f}{\partial p_i}\tfrac{\partial}{\partial q_i} - \tfrac{\partial f}{\partial q_i}\tfrac{\partial}{\partial p_i}\right) \\
&= \left(\tfrac{\partial^2 f}{\partial p_i^2}\tfrac{\partial g}{\partial q_i} + \tfrac{\partial f}{\partial p_i}\tfrac{\partial^2 g}{\partial p_i \partial q_i} - \tfrac{\partial^2 f}{\partial p_i \partial q_i}\tfrac{\partial g}{\partial p_i} - \tfrac{\partial f}{\partial q_i}\tfrac{\partial^2 g}{\partial p_i^2}\right)\tfrac{\partial}{\partial q_i} \\
&\quad - \left(\tfrac{\partial^2 f}{\partial q_i^2}\tfrac{\partial g}{\partial p_i} - \tfrac{\partial f}{\partial p_i}\tfrac{\partial^2 g}{\partial q_i^2} + \tfrac{\partial^2 f}{\partial p_i \partial q_i}\tfrac{\partial g}{\partial q_i} - \tfrac{\partial f}{\partial p_i}\tfrac{\partial^2 g}{\partial p_i \partial q_i}\right)\tfrac{\partial}{\partial p_i} \\
&= \tfrac{\partial X_f(g)}{\partial p_i}\tfrac{\partial}{\partial q_i} - \tfrac{\partial X_f(g)}{\partial q_i}\tfrac{\partial}{\partial p_i} \\
&= X_{X_f(g)} \\
&= X_{\{f,g\}}\ .
\end{aligned}$$

Very usual in some calculations is the change in the coordinates $\{q_1, ..., q_n, p_1, ..., p_n\} \to \{z_1, ..., z_n, \overline{z}_1, ..., \overline{z}_n\}$, where

$$z_i = p_i + iq_i \qquad \text{and} \qquad \overline{z}_i = p_i - iq_i$$

in terms of which our expressions for the symplectic 2-form, the Hamiltonian vector fields, Poisson brackets and all the expressions, before in coordinates p's and q's, must be written. Observing that

$$q_i = \frac{z_i + \overline{z}_i}{2} \qquad \text{and} \qquad p_i = \frac{z_i - \overline{z}_i}{2i}$$

then

$$\theta = p_i dq_i = \left(\frac{z_i - \overline{z}_i}{2i}\right)\left(\frac{dz_i + d\overline{z}_i}{2}\right) = \frac{z_i d\overline{z}_i}{4i} - \frac{\overline{z}_i dz_i}{4i} = \frac{1}{2i} z_i d\overline{z}_i$$

and

$$\omega = d\theta = dp_i \wedge dq_i = \frac{dz_i \wedge d\overline{z}_i}{2i}.$$

Looking for the expression for the Hamiltonian vector field generated for an observable f and Poisson bracket, in this new coordinates, it is enough to note that

$$\frac{\partial}{\partial z_i} + \frac{\partial}{\partial \overline{z}_i} = \frac{\partial}{\partial q_i} \qquad \text{and} \qquad \frac{\partial}{\partial \overline{z}_i} - \frac{\partial}{\partial z_i} = i\frac{\partial}{\partial p_i}$$

so

$$X_f = \frac{\partial f}{\partial p_i}\frac{\partial}{\partial q_i} - \frac{\partial f}{\partial q_i}\frac{\partial}{\partial p_i} = \frac{2}{i}\left[\frac{\partial f}{\partial \overline{z}_i}\frac{\partial}{\partial z_i} - \frac{\partial f}{\partial z_i}\frac{\partial}{\partial \overline{z}_i}\right]$$

and

$$\{f, g\} = \frac{\partial f}{\partial \overline{z}_i}\frac{\partial g}{\partial z_i} - \frac{\partial f}{\partial z_i}\frac{\partial g}{\partial \overline{z}_i}.$$

Example 1:

An easy example of all this is the one dimensional harmonic oscillator mass unity, case in which the configuration space for the system is $\mathbf{R}$ and phase space $T^*\mathbf{R} \simeq \mathbf{R}^2$, with canonical form $\omega = dp \wedge dq$, thus if we consider the Hamiltonian function (energy) of the system

$$H = \frac{1}{2}(p^2 + q^2)$$

we have as Hamiltonian vector field

$$X_H = \frac{\partial H}{\partial p}\frac{\partial}{\partial q} - \frac{\partial H}{\partial q}\frac{\partial}{\partial p} = p\frac{\partial}{\partial q} - q\frac{\partial}{\partial p}$$

and then the equations of motion are

$$\begin{aligned} \dot{q} &= p \\ \dot{p} &= -q \end{aligned}$$

just the classical definition of momentum ("the mass times the velocity") and Hook force equation for a spring. If we consider the n-dimensional case it is enough to take $\omega = dp_i \wedge dq_i$, with i running from one to n, and the Hamiltonian $H = \frac{1}{2}(p_i^2 + q_i^2)$, obtaining a similar result; and if we use the change of variables described previously we can see for example that the Hamiltonian takes the simplest form $H = \frac{1}{2}z_i\overline{z}_i$, and then its vector field associated is

$$X_H = \frac{1}{i}\left[z_i\frac{\partial}{\partial z_i} - \overline{z}_i\frac{\partial}{\partial \overline{z}_i}\right].$$

Example 2:

Another example, very important for us, is the electromagnetic field. If we take as configuration space the Minkowskian space-time Q for the study of the dynamics of a charged particle under the influence of a field F[4], then phase space $TQ^* \simeq \mathbf{R}^8$ can be parametrized with coordinates $(q_i, p_i) \equiv (q_1, q_2, q_3, q_4, p_1, p_2, p_3, p_4)$, where

$$\pi : TQ^* \longrightarrow Q$$

is the natural projection $\pi(q_i) = x_i$, $p_i = \frac{\partial L}{\partial x_i}$, and L is the lagrangian corresponding to the system. The canonical symplectic form on phase space is

$$\omega = dq_i \wedge dp_i$$

and in terms of it we can find the Hamilton equations (i.e. the dynamics) using the Hamiltonian function of the system, this Hamiltonian must contain a term given count of the interaction between the particle and the field. Alternatively we can describe the dynamics using a Hamiltonian free of interaction terms (called *free Hamiltonian*) $H = p_i \ \dot{q_i} \ -L$, modifying "geometrically" the system, this means modifying the symplectic 2-form, this modification works as follows.

From Maxwell equations

$$*d*F = j \qquad \text{and} \qquad dF = 0$$

where

$$\begin{aligned} F = E_x dt \wedge dx + E_y dt \wedge dy + E_z dt \wedge dz \\ -B_x dy \wedge dz - B_y dz \wedge dx - B_z dx \wedge dy \end{aligned}$$

it is clear that F is closed. Now, taking the pull-back of F by the projection π, say $F' = \pi^*(F)$, we can define the 2-form

$$\omega_{e,F} = \omega + eF'$$

[4] F is a 2-form called *Faraday or Electromagnetic Field Tensor*. In local coordinates on Q

$$F = \frac{1}{2} F^{\mu\nu} dx_\mu \wedge dx_\nu$$

where the coefitients $F^{\mu\nu}$ are given by the matrix

$$F^{\mu\nu} = \begin{pmatrix} 0 & -E^x & -E^y & -E^z \\ E^x & 0 & -B^z & B^y \\ E^y & B^z & 0 & -B^x \\ E^z & -B^y & B^x & 0 \end{pmatrix}$$

Here E and B denotes the electric and magnetic field respectively.

on phase space, where e denotes the particle charge.

Proposition:

The 2-form $\omega_{e,F}$ defined in this way is symplectic, i.e. closed and nondegenerated.

*Proof.*First observe that $\omega_{e,F}$ is closed because

$$d\omega_{e,F} = d\omega + edF' = 0$$

given that ω is closed and so is F, from Maxwell equations, thus its pull-back F' too. To verify the no-degeneracy remember that $\omega_{e,F}$ is nondegenerated if and only if $i_X\omega_{e,F} = 0 \Longleftrightarrow X = 0$, thus the equations $i_{X_f}\omega_{e,F} = -df$ give us an answer respect this. Expanding the vector field X_f in the TQ basis,

$$\begin{aligned} X_f &= a_i \frac{\partial}{\partial q_i} + b_i \frac{\partial}{\partial q_i} \\ &= a_1 \frac{\partial}{\partial q_1} + a_2 \frac{\partial}{\partial q_2} + a_3 \frac{\partial}{\partial q_3} + a_4 \frac{\partial}{\partial q_4} + b_1 \frac{\partial}{\partial p_1} + b_2 \frac{\partial}{\partial p_2} + b_3 \frac{\partial}{\partial p_3} + b_4 \frac{\partial}{\partial p_4} \end{aligned}$$

then

$$\begin{aligned} i_X\omega_{e,F} = \omega_{e,F}(X_f) &= \omega + eF'\left(a\frac{\partial}{\partial q} + b\frac{\partial}{\partial q}\right) \\ &= dq_1 \wedge dp_1 \left(a\frac{\partial}{\partial q} + b\frac{\partial}{\partial q}\right) + dq_2 \wedge dp_2 \left(a\frac{\partial}{\partial q} + b\frac{\partial}{\partial q}\right) + \\ & dq_3 \wedge dp_3 \left(a\frac{\partial}{\partial q} + b\frac{\partial}{\partial q}\right) + dq_4 \wedge dp_4 \left(a\frac{\partial}{\partial q} + b\frac{\partial}{\partial q}\right) + \\ & eE_x dq_1 \wedge dq_2 \left(a\frac{\partial}{\partial q} + b\frac{\partial}{\partial q}\right) + eE_y dq_1 \wedge dq_3 \left(a\frac{\partial}{\partial q} + b\frac{\partial}{\partial q}\right) + \\ & eE_z dq_1 \wedge dq_4 \left(a\frac{\partial}{\partial q} + b\frac{\partial}{\partial q}\right) - eB_x dq_3 \wedge dq_4 \left(a\frac{\partial}{\partial q} + b\frac{\partial}{\partial q}\right) - \\ & eB_y dq_4 \wedge dq_2 \left(a\frac{\partial}{\partial q} + b\frac{\partial}{\partial q}\right) - eB_z dq_2 \wedge dq_3 \left(a\frac{\partial}{\partial q} + b\frac{\partial}{\partial q}\right) \\ &= a_1(dp_1 + eE_x dq_2 + eE_y dq_3 + eE_z dq_4) + b_1 dq_1 + \\ & a_2(dp_2 + eE_x dq_1 - eB_y dq_4 - eB_z dq_3) + b_2 dq_2 + \\ & a_3(dp_3 + eE_y dq_1 - eB_x dq_4 - eB_z dq_2) + b_3 dq_3 + \\ & a_4(dp_4 + eE_z dq_1 - eB_x dq_3 - eB_y dq_2) + b_4 dq_4 \end{aligned}$$

and for f we have that

$$df = \frac{\partial f}{\partial q_i} dq_i + \frac{\partial f}{\partial p_i} dp_i$$

$$= \frac{\partial f}{\partial q_1} dq_1 + \frac{\partial f}{\partial q_2} dq_2 + \frac{\partial f}{\partial q_3} dq_3 + \frac{\partial f}{\partial q_4} dq_4 +$$

$$\frac{\partial f}{\partial p_1} dp_1 + \frac{\partial f}{\partial p_2} dp_2 + \frac{\partial f}{\partial p_3} dp_3 + \frac{\partial f}{\partial p_4} dp_4$$

then, comparing coefficients,

$$\frac{\partial f}{\partial q_1} = -\left(b_1 + eE_x + eE_y + eE_z\right)$$

$$\frac{\partial f}{\partial q_2} = -\left(b_2 + eE_x - eE_y - eE_z\right)$$

$$\frac{\partial f}{\partial q_3} = -\left(b_3 - eE_x + eE_y - eE_z\right)$$

$$\frac{\partial f}{\partial q_4} = -\left(b_4 - eE_x - eE_y + eE_z\right)$$

$$\frac{\partial f}{\partial p_1} = -a_1$$

$$\frac{\partial f}{\partial p_2} = -a_2$$

$$\frac{\partial f}{\partial p_3} = -a_3$$

$$\frac{\partial f}{\partial p_4} = -a_4$$

thus, as we expect, $i_X \omega_{e,F} = 0 \Longleftrightarrow X = 0$ and then $\omega_{e,F}$ is nondegenerated, then the pair $(T^*Q, \omega_{e,F})$ is a symplectic manifold$\square$

The method we have just described (change the symplectic structure on the manifold and leave unaltered the Hamiltonian H of a free particle, and not to modify the Hamiltonian and work with the canonical symplectic form) is equivalent to the so-called *"minimal coupling"* in physics literature, this consists in writing the Hamiltonian of a free particle changing the variable p (momentum) by "$p + eA$", where A is the electromagnetic potential, defined by

$$F = dA$$

(this 1-form exist locally by Poincare Lemma). A verification of the equivalence between both methods can be made looking for a symplectomorphism between the manifolds $(TQ^*, \omega_{e,F})$ and (TQ^*, ω), in such way that $\varphi^*_{e,F}(H) = H_{e,A}$ where $H_{e,A}$ denotes the minimal coupling Hamiltonian.. The symplectomorphism is

$$\begin{aligned} \varphi_{e,F} : (TQ^*, \omega_{e,F}) &\longrightarrow (TQ^*, \omega) \\ (q_i, p_i) &\longmapsto (q_i, p_i + e\pi^*(A)(q_i)) \end{aligned}$$

(then $\varphi^*_{e,F}(\omega) = \omega_{e,F}$), and then the solution trajectories to $H_{e,A}$ relative to ω are the images under $\varphi_{e,F}$ of the solution curves of H relative to $\omega_{e,F}$, thus on Q the curves are the same. There exist a quantum version of this method, very used in physics, also included in this work (see chapter one).

Given the changes in the symplectic form for the description of the electromagnetic field, we must have changes in the local expressions for Hamiltonian vector fields defined by $\omega_{e,F} = \omega + eF'$ according to the equation $df = -X_f \omega_{e,F}$, and the symplectic potential associated to this new symplectic structure. This fields are

$$X_f^{e,F} = \frac{\partial f}{\partial p_i}\frac{\partial}{\partial q_i} - \left(\frac{\partial f}{\partial q_i} + eF_{ij}\frac{\partial f}{\partial p_j}\right)\frac{\partial}{\partial p_i}$$

where the F_{ij} are the components of F, and the symplectic potential 1-forma is

$$\theta_{e,F} = p_i dq_i + eA_i dq_i,$$

as a direct calculation can show.

There are many other applications of the methods of symplectic geometry to physics, additional information about this can be found in [C2] and [GS1].

Appendix C

Symmetries

A classical result in the mathematical physics literature is the so called *Noethers Theorem*, in few words this theorem says that if a physical system is invariant under a symmetry group action (rotations, traslations, etc.) then on such system must be verified a corresponding conservation law (angular momentum, linear momentum, etc.). In this appendix we want to introduce the basic fact we must know from group actions on symplectic manifolds for the geometrical formulation of the dynamics of a system with symmetries, by introducing the concept of *Moment Map* on a symplectic manifold, the main reference for the material reviewed in this appendix is [GS1].

C.1 Actions of Lie Groups on Manifolds

Through this appendix G will note a Lie group, $\mathcal{G}$ its associated Lie algebra ($\mathcal{G} \simeq T_e G$), and the applications

$$\begin{aligned} L_h : G &\to G \\ g &\mapsto hg \end{aligned}$$

and

$$\begin{aligned} R_h : G &\to G \\ g &\mapsto gh \end{aligned}$$

the left and right translation diffeomorphism, respectively.

Definition 1:

Given $\xi \in \mathcal{G}$, let X_ξ denote the left invariant vector field on G defined by

$$X_\xi(g) = T_e L_g(\xi)$$

then there exists a unique integral curve $c_\xi : \mathbf{R} \to G$ for X_ξ starting at e, this means that

$$\begin{cases} \dot{c}_\xi\ (t) = X_\xi(c_\xi(t)) \\ c_\xi(0) = e \end{cases}$$

and $c_\xi(t+s) = c_\xi(t)c_\xi(s)$. Given such curve we define the *Exponential Map* as[1]

$$\begin{aligned} \exp : \mathcal{G} &\to G \\ \xi &\mapsto c_\xi(1). \end{aligned}$$

Examples:

If we take as Lie group a (finite-dimensional) vector space V, its Lie algebra is V itself and in this case the exponential application is just the identity. If $G = GL(n, \mathbf{R})$, then $\mathcal{G} = L(\mathbf{R}^n, \mathbf{R}^n)$ and in this case

$$\exp(A) = \sum_{i=0}^{\infty} \frac{A^i}{i!}.$$

Definition 2:

For an element g of the Lie group G we define the *Adjoint Application* associated to g as

$$Ad_g = T_e(R_{g^{-1}} \circ L_g) : \mathcal{G} \to \mathcal{G}.$$

For $A \in GL(n, \mathbf{R})$, by example, we have that

$$Ad_A(B) = ABA^{-1}$$

for every $B \in L(\mathbf{R}^n, \mathbf{R}^n)$. In general for every group G and Lie algebra $\mathcal{G}$

$$\exp(Ad_g \xi) = g(\exp \xi)g^{-1}$$

where $g \in G$ and $\xi \in \mathcal{G}$.

[1]See by example Warner, J. *Lie Groups, Lie Algebras and Differential Geometry*, Springer Verlag, 1980.

Definition 3:

An *Action* of a Lie Group G on a manifold M is a smooth application

$$\phi : G \times M \to M$$

such that for every $m \in M$ follows that $\phi(e, m) = m$ and $\phi(g, \phi(h, m)) = \phi(gh, m)$, where $g, h \in G$ and $m \in M$. The application

$$\begin{aligned} \phi_g : M &\longrightarrow M \\ m &\mapsto \phi(g, m) \end{aligned}$$

is a diffeomorphism. If for every pair of elements m and m' of M there exist g in G such that $\phi(g, m) = m'$ this action is called *transitive*, if $\phi_g(m) = m$ implies that $g = e$ then is called *free*, if the function $\widetilde{\phi} : G \times M \to M \times M : (g, m) \mapsto (m, \phi(g, m))$ is proper the action is called *proper*. Given an action ϕ we define the *Orbit* of $m \in M$ as the set

$$\mathcal{O}_m = \{\phi_g(m) : g \in G\} \subset M$$

and the *Isotropy Group* of ϕ in m as

$$G_m = \{g \in G : \phi(g, m) = m\}.$$

Thus the action is transitive if and only if there exist just one orbit for the action, and is free if and only if $G_m = \{e\}$ for every m.

Denoting M/G the set of all the orbits of the Lie group G action on M, called *Orbit Space*, and define on it the quotient topology induced by the application

$$\begin{aligned} \pi : M &\to M/G \\ m &\mapsto \mathcal{O}_m \end{aligned}$$

we have then a "reduced" space on which the dynamics of a problem involving symmetries can be carried on.

Example:

Take $M = \mathbf{R}^2 \setminus \{0, 0\}$ and $G = SO(2) = \left\{ \begin{pmatrix} \cos\theta & -sen\theta \\ sen\theta & \cos\theta \end{pmatrix} : \theta \in [0, 2\pi) \right\}$, defining the free action on M as

$$\begin{aligned} \phi\left(\begin{pmatrix} \cos\theta & -sen\theta \\ sen\theta & \cos\theta \end{pmatrix}, \begin{pmatrix} x \\ y \end{pmatrix} \right) &= \begin{pmatrix} \cos\theta & -sen\theta \\ sen\theta & \cos\theta \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} \\ &= \begin{pmatrix} x\cos\theta & -\, ysen\theta \\ xsen\theta & +\, y\cos\theta \end{pmatrix} \end{aligned}$$

it is clear that the corresponding orbits to each point of the plane will be the corresponding concentric circumferences around the origin passing by the respective points, i.e. $M/G \approx \mathbf{R}^+$.

Given an action of a Lie group G on a manifold M it is possible, with the help of the Lie algebra of the group, to define an action of $\mathbf{R}$ on the manifold, this is a "flux" on M.

Definition 4:

If ξ is an element of the Lie algebra $\mathcal{G}$, we define the application

$$\begin{aligned} \phi_\xi : \mathbf{R} \times M &\to M \\ (t, m) &\mapsto \phi(\exp(t\xi), m). \end{aligned}$$

This in fact is an action, and define its infinitesimal generator as the smooth vector field given by

$$\xi_M(m) = \frac{d}{dt}(\phi_{\exp(t\xi)}(m) \mid_{t=0} .$$

A very useful and interesting result give us a relation between the structure of the tangent space to the orbits of an action and the infinitesimal generators of it

$$T_m \mathcal{O}_{m_o} = \{\xi_M(m) : \xi \in \mathcal{G}\}$$

where m and $m_o \in M$.

Two types of actions will be important for us, one on its proper Lie algebra $\mathcal{G}$, defined as

$$\begin{aligned} Ad : G \times \mathcal{G} &\to \mathcal{G} \\ (g, \xi) &\mapsto Ad_g(\xi) \end{aligned}$$

and called *Adjoint Action*, and other on the dual space to the Lie algebra, defined as

$$\begin{aligned} Ad^* : G \times \mathcal{G}^* &\to \mathcal{G}^* \\ (g, \alpha) &\mapsto Ad^*_{g^{-1}}(\alpha) = (T_e(R_{g^{-1}} \circ L_g))^* \alpha \end{aligned}$$

and called *Coadjoint Action*.

Definition 5:

Let M be a manifold and ϕ an action of a Lie group G on M. We say that a function $f : M \to M$ is *equivariant* respect to the action if $f \circ \phi_g = \phi_g \circ f \ \ \forall g \in G$, i.e. $f(\phi_g(m)) = f(\phi(g,m)) = \phi_g(f(m)) = \phi(g, f(m)) \ \forall g \in G$. If f is a equivariant function respect to the action ϕ, then $\forall \xi \in \mathcal{G}$

$$Tf \circ \xi_M = \xi_M \circ f.$$

C.2 The Moment Map

Definition 6:

Given a connected symplectic manifold (M, ω) and a symplectic action ϕ on M (this is an action on M such that ϕ_g is a symplectomorphism for all $g \in G$), if for any $\xi \in \mathcal{G}$ there exist a globally defined map

$$\mathcal{J}(\xi) : M \to \mathbf{R}$$

such that

$$\xi_M = X_{\mathcal{J}(\xi)}$$

(the Hamiltonian vector field generated by $\mathcal{J}(\xi)$), then the map

$$\begin{aligned} J : M &\longrightarrow \mathcal{G}^* \\ m &\longmapsto J(m) \end{aligned}$$

where

$$J(m)(\xi) = \mathcal{J}(\xi)(m)$$

is called *Moment Map* for the action ϕ.

Proposition:

Let $\xi, \eta \in \mathcal{G}$, and suppose there exist a moment map J such that $\mathcal{J}(\xi), \mathcal{J}(\eta) \in C^\infty(M, \mathbf{R})$, then

$$X_{\mathcal{J}([\xi,\eta])} = X_{\{\mathcal{J}(\xi),\mathcal{J}(\eta)\}}$$

where $[,]$ and $\{,\}$ denotes the Lie and Poisson brackets respectively.

The contends of the very known Noethers theorem can be summarized in this context through the following result:

Theorem (Noether):

Let ϕ be a symplectic action on a symplectic manifold (M, ω), with moment map J, and $H : M \to \mathbf{R}$ a G-invariant function (i.e. $H(\phi_g(m)) = H(m)\ \forall m \in M,\ g \in G$), then $\mathcal{J}(\xi)$ is a constant of motion for the dynamics generated by H (i.e $\{H, \mathcal{J}(\xi)\} = 0$ on M) and $J \circ \phi_t = J$, where ϕ_t is the flux of X_H.

Definition 7:

A symplectic action ϕ with moment map J is called *Hamiltonian* if J is Ad^*-equivariant, i.e.

$$J(\phi_g(m)) = Ad^*_{g^{-1}}(J(m))$$

for every $g \in G$ and $m \in M$.

An action is Hamiltonian if and only if $\mathcal{J} : \mathcal{G} \to C^\infty(M, \mathbf{R}) : \xi \mapsto \mathcal{J}(\xi)$ is a Lie algebras homomorphism. In fact can be easily checked that

$$\{\mathcal{J}(\xi), \mathcal{J}(\eta)\} = \mathcal{J}([\xi, \eta]).$$

In the cases of our main interest (cotangent bundles) there are a couple of results that relate the action of the moment map J with the action of the canonical symplectic potential, as a matter of fact if ϕ is a symplectic action of G on (M, ω), where $\omega = d\theta$ and the action preserves θ (i.e. $\phi_g^*\theta = \theta \Rightarrow \mathcal{L}_{\xi_M}\theta = 0$), then the map

$$J : M \to \mathcal{G}$$

such that

$$J(m)(\xi) = (i_{\xi_M}\theta)(m)$$

is an Ad^*-equivariant moment map for this action[2]. If we have an action on the base manifold Q for a cotangent bundle $(TQ^*, \omega = d\theta)$, say

$$\phi : G \times Q \to Q$$

it is possible to lift this action to one on the cotangent bundle

$$\phi^* : G \times TQ^* \to TQ^*$$

[2] See Marsden, J. *Lectures on Geometrical Methods in Mathematical Physics.* S.I.A.M., 1981.

in such way that the maps $\phi_g : Q \to Q$ defines maps $\phi_g^* : TQ^* \to TQ^*$ by

$$\phi_g^*(\alpha)(X) = \alpha(T\phi_g^{-1}(X))$$

where $\alpha \in T_qQ^*$ and $X \in T_{\phi_g(q)}Q$. In this case the moment map is given by $J : TQ^* \to \mathcal{G} : \alpha \mapsto J(\alpha)$, where

$$J(\alpha)(\xi) = \alpha(\xi_Q(q))$$

for $\alpha \in T_qQ^*$, and

$$(i_{\xi_{TQ^*}}\theta)(\alpha) = \alpha(\xi_Q(q)).$$

Example 1: Linear Momentum Conservation

Consider the manifolds $Q = \mathbf{R}^n$ and $TQ^* = \mathbf{R}^{2n}$ and the Lie group $G = \mathbf{R}^n$ acting as[3]

$$\begin{array}{c} \phi : G \times Q \to Q \\ (\overrightarrow{t}, \overrightarrow{q}) \mapsto \overrightarrow{t} + \overrightarrow{q} \end{array}$$

then $\mathcal{G} = \mathbf{R}^n$ and

$$\xi_{\mathbf{R}^n}(\overrightarrow{q}) = \tfrac{d}{dt} \mid_{t=0} \phi_{\exp t\xi}(\overrightarrow{q})$$

$$= \tfrac{d}{dt} \mid_{t=0} (\overrightarrow{q} + \exp t\xi) = \xi$$

so

$$\xi_{\mathbf{R}^n} = \xi \frac{\partial}{\partial q_i}.$$

Now, from our last results on $\mathbf{R}^{2n}$

$$\mathcal{J}(\xi)(m) = J(m)(\xi) = (i_{\xi_{\mathbf{R}^{2n}}}\theta)(m)$$

where $m = p_i dq_i = (\overrightarrow{p}, \overrightarrow{q}) \in \mathbf{R}^{2n}$, so

$$J(\overrightarrow{p}, \overrightarrow{q})(\xi) = (i_{\xi_{\mathbf{R}^{2n}}} p_i dq_i)(\overrightarrow{p}, \overrightarrow{q}) = (\xi_{\mathbf{R}^n}(\overrightarrow{p}, \overrightarrow{q})) = \xi \overrightarrow{p}$$

thus

$$\mathcal{J}(\overrightarrow{p}, \overrightarrow{q}) = \overrightarrow{p}$$

the linear momentum.

[3] Here $\overrightarrow{q}$ denotes the vector $(q_1, q_2, ..., q_n)$.

Example 2: Angular Momentum Conservation

Lets take $Q = \mathbf{R}^3$ and $TQ^* = \mathbf{R}^6$ and the Lie group $G = SO(3)$ with the action

$$\begin{aligned} \phi : G \times Q &\to Q \\ (A, \overrightarrow{q}) &\mapsto A\overrightarrow{q} \end{aligned}$$

then $\mathcal{G} \simeq \mathbf{R}^3$, the isomorphism given by

$$\mathbf{R}^3 \longrightarrow \mathcal{SO}(3)$$

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} \mapsto \begin{pmatrix} 0 & -z & y \\ z & 0 & -x \\ -y & x & 0 \end{pmatrix}$$

as can be checked. Now, in this case

$$\xi_{\mathbf{R}^3}(\overrightarrow{q}) = \tfrac{d}{dt} |_{t=0}\, \phi_{\exp t \overrightarrow{\xi}}(\overrightarrow{q})$$

$$= \tfrac{d}{dt} |_{t=0}\, (\exp t \overrightarrow{\xi} \cdot \overrightarrow{q}) = \tfrac{d}{dt} |_{t=0} \left(\begin{pmatrix} t\xi_1 \\ t\xi_2 \\ t\xi_3 \end{pmatrix} \times \begin{pmatrix} q_1 \\ q_2 \\ q_3 \end{pmatrix} \right)$$

$$= \overrightarrow{\xi} \times \overrightarrow{q}.$$

From this we have on $\mathbf{R}^6$

$$\mathcal{J}(\xi)(m) = J(m)(\xi) = (i_{\xi_{\mathbf{R}^6}}\theta)(m)$$

where $m = p_i dq_i = (\overrightarrow{p}, \overrightarrow{q}) \in \mathbf{R}^6$, so

$$\mathcal{J}(\overrightarrow{p}, \overrightarrow{q}) = \overrightarrow{p} \times \overrightarrow{q}$$

is just the known angular momentum.

C.3 Symplectic Reduction

Given a symmetry on a classical system described through a symplectic manifold (M, ω), and the corresponding group action on this manifold, we want to take only a "effective" part of the system in the corresponding description, reducing the freedom degrees. This can be or not carried on depending on what kind of group defines the action and some properties of the manifold, lets see some results that give sufficient conditions to do this and how this works.

Theorem:

Let G be a Lie group and a transitive action of G on (M, ω), then the image of M under the associated moment map for this action is a coadjoint orbit.

Proof: Because transitivity for any $x, y \in M$ there exist $g \in G$ such that $g(x) = y$, then

$$\begin{aligned} J(M) &= \{J(x) : x \in M\} \\ &= \{J(g(x)) : g \in G\} \\ &= \{Ad^*_{g^{-1}}(J(x)) : g \in G\}. \end{aligned}$$

$\square$

Definition 8:

Suppose a Hamiltonian action $\phi : G \times M \to M$ on a symplectic manifold (M, ω) with the associated (Ad^*-equivariant) moment map $J : M \to \mathcal{G}$, and let $\xi \in \mathcal{G}$ be a regular value for J, then $J^{-1}(\xi)$ is a submanifold of M. Let

$$G_\xi = \{g \in G : Ad^*_{g^{-1}}\xi = \xi\}$$

be the isotropy group of ξ, then G_ξ acts on $J^{-1}(\xi)$ (by Ad^*-equivariance). If $J^{-1}(\xi)/G_\xi \equiv M_\xi$ is a C^∞ manifold for which the canonical projection

$$\pi_\xi : J^{-1}(\xi) \longrightarrow J^{-1}(\xi)/G_\xi$$

is a smooth submersion, the manifold M_ξ is called *Reduced Phase Space* by the action of G.

Theorem (Marsden-Weinstein):

Take M, G, ϕ, J, M_ξ and π_ξ as in the last definition, if all the conditions there follows and

$$i_\xi : J^{-1}(\xi) \longrightarrow M$$

is the inclusion, then there exist a unique symplectic structure ω_ξ on M_ξ such that

$$\pi_\xi^* \omega_\xi = i_\xi^* \omega,$$

thus (M_ξ, ω_ξ) is a symplectic manifold.

Proof: See [GS1] or Marsden, J. quoted in a previous footnote. □

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