GEOMETRIC QUANTIZATION OF GENERALIZED COMPLEX MANIFOLDS

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Abstract. In this paper we define the geometric quantization of a generalized complex manifold and we show how this quantization procedure contains as a particular case the geometric quantization program for symplectic and Poisson manifolds.

1. Introduction

Let M be a $2n$-dimensional compact manifold and let $\mathbb{J}$ be a generalized complex structure on it, i.e. an integrable almost complex structure on the generalized tangent bundle $\mathbb{T}M = TM \oplus T^*M$, which is orthogonal with respect to the natural symmetric pairing on $\mathbb{T}M$ or, equivalently, a Dirac structure $L \subset \mathbb{T}_{\mathbb{C}}M$ such that $L \oplus \bar{L} = \mathbb{T}_{\mathbb{C}}M$, $L \cap \bar{L} = \{0\}$, where integrability is meant with respect to the Courant bracket [5][6][8]. Generalized complex manifolds include, as particular cases, complex, Poisson and symplectic manifolds, and they have attached to them important algebraic structures. In particular, the maximal isotropic L, with the Courant bracket, defines a Lie algebroid on M whose anchor map is the projection onto the tangent part. The associated Lie algebroid cohomology is the usual Dolbeault cohomology if the generalized complex structure comes from an actual complex structure on M and, in the symplectic case, it is nothing but the de Rham cohomology of differential forms on the manifold. Also, there is a natural Poisson algebra structure defined on (a convinient subalgebra of) the algebra of smooth functions on M which, when considered the generalized structures defined by the usual ones, coincides with their corresponding Poisson algebras [5]. In this paper we consider the geometric quantization of such algebras, seen as a continuation of the geometric prequantization program of Weinstein and Zambon for Dirac structures [18] and a generalization of the quantization for compact symplectic manifolds introduced by Vergne et al [16][17].

After some preliminaries on the algebraic characterizations of Dirac structures and the Poisson algebra associated to generalized complex manifolds (following closely the presentation of [5] and [8]) we will recall in section 3 the main ingredients in our definition of geometric quantization and, in 3.2, we give our definition and study its implications in some basic examples. Throughout all the paper we work with Courant brackets *without* twisting, this is because the prequantization condition we assume on the generalized complex manifolds cannot be stated the way we use it when there is a twisting by a 2-form in the bracket defining the Lie algebroid. The main features of this quantization setting work, in consequence, on exact Courant algebroids without twisting.

2. Generalized Complex Geometry

2.1. **Local model.** Let V be a finite-dimensional real vector space and consider the *doubled* vector space $\mathbb{V} = V \oplus V^*$. There is on $\mathbb{V}$ a the natural symmetric pairing

$$\langle X \oplus \xi, Y \oplus \eta \rangle_+ = \frac{1}{2}\left(\xi(Y) + \eta(X)\right), \tag{1}$$

that we will use to identify $\mathbb{V} \cong \mathbb{V}^*$, and a natural antisymmetric pairing given by

$$\langle X \oplus \xi, Y \oplus \eta \rangle_- = \frac{1}{2} \left(\xi(Y) - \eta(X) \right). \tag{2}$$

Definition 2.1. *A* Generalized Complex Structure *on V is an endomorphism*

$$\mathbb{J} : \mathbb{V} \to \mathbb{V}$$

satisfying:

(i) $\mathbb{J}^2 = -1$,
(ii) $\mathbb{J}^* = -\mathbb{J}$.

It is easy to see from this definition that having a generalized complex structure on V is equivalent to having a complex structure $\mathbb{J}$ on $\mathbb{V}$ which is orthogonal with respect to the inner product (1), i.e. $\mathbb{J}^*\mathbb{J} = 1$.

There are other characterizations of generalized complex structurs on vector spaces, based on the so-called linear Dirac structures [5][6]. Indeed, let us consider now the complexification of $\mathbb{V}$, namely $\mathbb{V}^{\mathbb{C}} = \mathbb{V} \otimes \mathbb{C}$, and the extensions of $\mathbb{J}$ and $\langle , \rangle_+$ to $\mathbb{V}^{\mathbb{C}}$, which we will continue to denote by $\mathbb{J}$ and $\langle , \rangle$, respectively. Let

$$L = \left\{ \alpha \in \mathbb{V}^{\mathbb{C}} \mid \mathbb{J}(\alpha) = i\alpha \right\},$$

then, by orthogonality $\langle \alpha, \beta \rangle = \langle \mathbb{J}\alpha, \mathbb{J}\beta \rangle = -\langle \alpha, \beta \rangle$, so that $\langle \alpha, \beta \rangle = 0$. Thus, L is isotropic and $\dim L = \dim V = m$, so it is maximal isotropic and $L \cap \bar{L} = \{0\}$. Conversely, let $L < \mathbb{V}^{\mathbb{C}}$ be a maximal isotropic complex subspace such that $L \cap \bar{L} = \{0\}$, then $\bar{L} \cong L^*$, $L \oplus \bar{L} \cong V \oplus V^*$ and we can associate to it a generalized complex structure $\mathbb{J}_L$ on V by multiplication by i on L and multiplication by $-i$ on $\bar{L}$. Since lineal Dirac structures are precisely maximal isotropic subspaces with respect to (1), in general,

Proposition 2.1. *A generalized complex structure on V is completely determined by a linear Dirac structure L of $\mathbb{V}^{\mathbb{C}}$ such that $L \cap \bar{L} = \{0\}$.*

Natural examples of linear Dirac structures are the graphs of presymplectic and Poisson structures on vector spaces, they were inroduced by Weinstein in order to study the geometry of Dirac theory of constraints [6]. Notice that, if W is a vector subspace of V, taking its annihilator W° in V^* we have a maximal isotropic $L_W = W \oplus W^\circ$. In general, if $i : W \hookrightarrow V$ denotes the inclusion and $\varepsilon \in \Lambda W^*$, the space

$$L_W^\varepsilon = \{ X \oplus \xi \in W \oplus V^* \mid i^*\xi = i_X \varepsilon \}$$

is an extension of the form

$$0 \to W^\circ \to L_W^\varepsilon \to W \to 0,$$

then $L_W^\varepsilon \cong W \oplus W^\circ$ is maximal isotropic. Moreover, every maximal isotropic in $\mathbb{V}$ is of the form L_W^ε for some W and ε [8]. The codimension of W in V, $k = n - \dim W$, is called the *type* of the generalized complex structure associated to L_W^ε. It follows from these characterizations of linear Dirac structures that a vector space has a generalized complex structure if and only if it has even dimension $n = \dim V = 2m$.

Example 2.1. *Let (V, J) be a vector space together with a complex structure. Then J induces a natural generalized complex structure $\mathbb{J}_J$ on V given by*

$$\mathbb{J}_J = \begin{pmatrix} -J & 0 \\ 0 & J^* \end{pmatrix}, \tag{3}$$

where $J^* : V^* \to V^*$ denotes the dual map. In this case the maximal isotropic associated to $\mathbb{J}_J$ is the given by

$$L_J = L^0_{V_{0,1}} = V_{0,1} \oplus V^*_{1,0},$$

where $V_{1,0} = \{X \in V^{\mathbb{C}} \mid J(X) = iX\} = \overline{V}_{0,1}$. Thus, the associated generalized complex structure associated to a complex structure is type m.

Example 2.2. *Let (V,ω) be a symplectic vector space. Then ω induces a natural generalized complex structure $\mathbb{J}_\omega$ on V given by*

$$\mathbb{J}_\omega = \begin{pmatrix} 0 & -\omega^{-1} \\ \omega & 0 \end{pmatrix}, \tag{4}$$

where ω^{-1} denotes the inverse isomorphism to the natural map $\omega : V \to V^* : X \mapsto \iota_X\omega$. Then, the maximal isotropic space associated to it is

$$L_\omega = L^\omega_{V^{\mathbb{C}}} = \{X - i\omega(X) \mid X \in V^{\mathbb{C}}\}.$$

Thus, the associated generalized complex structure associated to a symplectic structure is type 0.

Spinors on generalized complex vector spaces. Let V be a finite-dimensional real vector space and consider the doubled vector space $\mathbb{V} = V \oplus V^*$ and its Clifford algebra $Cl(\mathbb{V})$ defined by the quotient of the tensor algebra by elements of the form $\alpha^2 = \langle \alpha, \alpha \rangle_+$, where $\alpha = X \oplus \xi \in \mathbb{V}$ and the inner product is the one given in (1). There is a natural spinor space, $S = \Lambda^\bullet V^*$, and a natural *spinor representation* for $Cl(\mathbb{V})$ given as follows

$$(X \oplus \xi) \cdot \varphi = \iota_X \varphi + \xi \wedge \varphi, \tag{5}$$

where $\varphi \in \Lambda^\bullet V^*$. Since the natural volume element in $Cl(\mathbb{V})$ satisfies $\omega^2 = 1$, it induces a grading in the spinor space $S = S^+ \oplus S^- \cong \Lambda^{ev} V^* \oplus \Lambda^{odd} V^*$. Now, given a nonzero element φ in S, the space

$$L_\varphi = \{\alpha \in \mathbb{V} \mid \alpha \cdot \varphi = 0\}$$

is isotropic in $\mathbb{V}$. If we consider $V^{\mathbb{C}}$, we call the spinor $\varphi \in \Lambda^\bullet V^* \otimes \mathbb{C}$ a *pure spinor* when L_φ is maximal isotropic and, in such a case, it is possible to describe each maximal isotropic L in $\mathbb{V}$ through a unique line $K_L \subset \Lambda^k V^* \otimes \mathbb{C}$ of pure spinors, where k denotes the type of the associated generalized complex structure [4]. Namely, if L^ε_W is a maximal isotropic and W° the annihilator of W, taking a basis $\{\theta_1, \dots, \theta_k\}$ for W° and $B \in \Lambda^2 V^* \otimes \mathbb{C}$ such that $i^*B = -\varepsilon$, then the characteristic pure spinor line $K_{L^\varepsilon_W}$ is the generated by $\varphi = e^B \theta_1 \wedge \cdots \wedge \theta_k$ [8].

2.2. **Generalized Complex Structures on Manifolds.** Let M be a smooth manifold and consider its *generalized tangent bundle* $\mathbb{T}M = TM \oplus T^*M$ with inner product induced by (1). We will say that $\mathbb{J}$ is a *generalized almost complex structure* on M if it is an almost complex structure on $\mathbb{T}M$ which is orthogonal with respect to the inner product introduced above. Equivalently, considering the complexification $\mathbb{T}_{\mathbb{C}}M = \mathbb{T}M \otimes \mathbb{C}$ of $\mathbb{T}M$, a generalized almost complex structure is defined by an almost Dirac structure $L < \mathbb{T}_{\mathbb{C}}M$ such that $L \cap \bar{L} = \{0\}$ so that, via the inner product, $L^* \cong \bar{L}$ and $\mathbb{T}_{\mathbb{C}}M \cong L \oplus \bar{L}$.

Definition 2.2. [5] *Consider an even-dimensional manifold M. The* Courant bracket *on $\mathbb{T}M$ is the defined by*

$$[X \oplus \xi, Y \oplus \eta]_C = [X,Y] + \mathcal{L}_X \eta - \mathcal{L}_Y \xi - \frac{1}{2} d(\iota_X \eta - \iota_Y \xi), \tag{6}$$

where $X \oplus \xi, Y \oplus \eta \in \Gamma(\mathbb{T}M)$. We say that a generalized almost complex structure $\mathbb{J}$ on M is a generalized complex structure *if its maximally isotropic subbundle L is Courant involutive, i.e. its space of sections $\Gamma(L)$ is closed under the Courant bracket.*

Recall that a *Lie algebroid* with *anchor* map $a : \mathcal{A} \to TM$ is a vector bundle $\mathcal{A} \to M$ on M with Lie bracket $[,]_{\mathcal{A}}$ on $\Gamma(\mathcal{A})$ such that $a : \Gamma(\mathcal{A}) \to \Gamma(TM) \cong Vect(M)$ defines a Lie algebra homomorphism, i.e.

$$a([A,B]_{\mathcal{A}}) = [a(A), a(B)], \qquad A, B \in \Gamma(\mathcal{A}) \tag{7}$$

and the Leibniz rule

$$[A, fB]_{\mathcal{A}} = f[A,B]_{\mathcal{A}} + (a(A)f)B, \tag{8}$$

is satisfied, where $A, B \in \Gamma(\mathcal{A}), f \in C^\infty(M)$ (for general facts on Lie algebroids see [2]). Given a Lie algebroid $\mathcal{A} \to M$ with Lie bracket $[,]_\mathcal{A}$ and anchor a on $\Gamma(\mathcal{A})$, it is possible to generalize the usual definition of exterior derivative to get a Lie algebroid derivative

$$d_\mathcal{A} : \Omega^k_\mathcal{A}(M) \to \Omega^{k+1}_\mathcal{A}(M),$$

where $\Omega^k_\mathcal{A}(M) = \Gamma(\Lambda^k \mathcal{A}^*)$, by

$$\begin{aligned} d_\mathcal{A}\sigma(A_0, \ldots, A_k) &= \sum_{i=0}^{k} (-1)^i a(A_i)\, \sigma(A_0, \ldots, \hat{A}_i, \ldots, A_k) \\ &\quad + \sum_{i<j} (-1)^{i+j} \sigma([A_i, A_j]_\mathcal{A}, A_0, \ldots, \hat{A}_i, \ldots, \hat{A}_j, \ldots, A_k), \end{aligned} \tag{9}$$

where $\sigma \in \Omega^k_\mathcal{A}(M), A_i \in \Gamma(\mathcal{A})$.

Given an even-dimensional manifold M and a generalized complex structure $\mathbb{J}$ with associated maximally isotropic subbundle $L < \mathbb{T}_\mathbb{C}M$, so that $\mathbb{T}_\mathbb{C}M \cong L \oplus \bar{L}$, restricting the Courant bracket on $\mathbb{T}_\mathbb{C}M$ to L we have a Lie algebroid structure [5][8]. Let $(L, [\,,\,]_C, \pi_{TM})$ be the Lie algebroid on M associated with a generalized complex structure $\mathbb{J}$. Putting for $f \in C^\infty(M, \mathbb{C})$

$$d_L f = df + i\mathbb{J}(df), \tag{10}$$

which corresponds to the L^*-part of df in the decomposition $\mathbb{T}_\mathbb{C}M \cong L \oplus L^*$, we extend d_L to complex functions on M, obtaining a complex

$$0 \to C^\infty(M, \mathbb{C}) \cong \Omega^0_L(M) \xrightarrow{d_L} \Gamma(L^*) \cong \Omega^1_L(M) \xrightarrow{d_L} \cdots \xrightarrow{d_L} \Omega^n_L(M) \to 0,$$

where n denotes the rank of the bundle L.

Proposition 2.2. *The Lie algebroid exterior derivative is a first order elliptic differential operator, it satisfies $d_L^2 = 0$, so that $(\Omega^\bullet(L), d_L)$ is an elliptic differential complex.*

Proof. Notice that, by definition, if $A, B \in \Gamma(L)$ then $\mathbb{J}(A) = iA$ and $\mathbb{J}(B) = iB$, so

$$d_L^2 f(A, B) = d_L(df + i\mathbb{J}(df))(A, B),$$

thus, as follows from (9),

$$d_L^2 f(A, B) = a(A)(df + i\mathbb{J}(df))(B) - a(B)(df + i\mathbb{J}(df))(A) - (df + i\mathbb{J}(df))([A, B]_C),$$

but, decomposing $A = X - i\mathbb{J}(X), B = Y - i\mathbb{J}(Y)$ gives

$$\begin{aligned} d_L^2 f(A, B) &= X(df + i\mathbb{J}(df))(Y - i\mathbb{J}(Y)) - Y(df + i\mathbb{J}(df))(X - i\mathbb{J}(X)) \\ &- (df + i\mathbb{J}(df))([X, Y] - i\mathbb{J}([X, Y])), \end{aligned}$$

but $(df + i\mathbb{J}(df))(Y - i\mathbb{J}(Y)) = df(Y)$ and so on, so that $d_L^2 f(A) = d^2 f(X, Y) = 0$. The same holds for any rank higher than 1 using the definition of d_L.

In order to check ellipticity, let us compute the symbol of d_L. By definition [1], if P is a differential operator of order k its principal symbol σ_P is given by

$$\sigma_P(x, \xi) = \lim_{t \to \infty} \frac{1}{t^k} \left(e^{-itf} P e^{itf}\right)(x),$$

where $f \in C^\infty(M)$ is such that $df(x) = \xi$. Then,

$$\sigma_{d_L}(x, \xi) = \lim_{t \to \infty} \frac{1}{t} \left(e^{-itf} d_L e^{itf}\right)(x) = \lim_{t \to \infty} \frac{1}{t} \left(e^{-itf}[ite^{itf}(df)_{L^*} \wedge + e^{itf} d_L]\right)(x),$$

so that

$$\sigma_{d_L}(x, \xi) = i(d_L f)(x) \wedge = i(df)_{L^*}(x) \wedge,$$

which can also be written

$$\sigma_{d_L}(x, \xi) = i\pi^*(\xi) \wedge,$$

where $\pi^* : T^*_{\mathbb{C}}M \to L^*$ denotes the dual to the projection $\pi : L \to T_{\mathbb{C}}M$, i.e. $\pi^*(\xi) = \xi + i\mathbb{J}(\xi)$. It is clear that $\pi^*(\xi) \neq 0$ whenever $\xi \in T^*M$ is not zero, because $\mathbb{T}_{\mathbb{C}}M = L \oplus \bar{L}$ and then $\xi = x + \bar{x}$ for some $x \neq 0$ in L. Therefore, d_L is elliptic □

We denote by $H^\bullet_L(M)$ the cohomology associated to the elliptic complex $(\Omega^\bullet_L(M), d_L)$, often called L-cohomology of M. It follows that

Proposition 2.3. [8] *The generalized cohomology associated to the generalized complex structure $\mathbb{J}_\omega$, which is Courant involutive since $d\omega = 0$ for a symplectic manifold (M, ω), is the complex de Rham cohomology of M, i.e.*

$$H^*_{L_\omega}(M) = H^*_{dR}(M, \mathbb{C}).$$

The generalized cohomology associated to the generalized complex structure $\mathbb{J}_J$ of a complex manifold (M, J) is the Dolbeault cohomology of M, i.e.

$$H^*_{L_J}(M) = H^*_{\bar{\partial}}(M).$$

2.3. **The Poisson Algebra.** The Poisson algebra associated to a Dirac structure L on a manifold M has been defined by Courant and Weinstein (see [5][6]), generalizing the usual definition for symplectic and Poisson manifolds.

Definition 2.3. *A function $f \in C^\infty(M)$ is called* L-admissible *if there exists a vector field X_f on M such that $X_f \oplus df$ is a section of L. When such a vector field exists, it is called a* Hamiltonian vector field *associated to f, and the* Poisson bracket *between functions $f, g \in C^\infty_L(M)$ (the space of L-admissible functions over M) is defined by*

$$\{f, g\} = X_f(g). \tag{11}$$

Proposition 2.4. [5] *With the above defined Poisson bracket, the algebra $C^\infty_L(M)$ is a Poisson algebra, the Poisson algebra associated to the Dirac structure L.*

Remark 2.1. *In the case we work with a Dirac structure L_ω coming from a symplectic structure ω on M, see (4), we have that, since ω is non degenerate, $C^\infty_L(M) = C^\infty(M)$ and the Poisson bracket coincides (up to a constant) with the usual Poisson bracket in symplectic geometry. However, in the case of generalized structures coming from a complex structure J on M, the Dirac structure is given by $L_J = T_{0,1}M \oplus T^*_{1,0}M$, where $T_{1,0}M = \{X \in TM \mid J(X) = iX\}$. Since, point by point, the annihilator of $T_{0,1}M$ is $T^*_{1,0}M$, the Poisson bracket in this case is identically zero, so the Poisson algebra is trivial.*

Remark 2.2. *In the symplectic case $\mathbb{J}_\omega(df) = -\omega^{-1}(df) = -X_f$, where X_f denotes the* usual symplectic Hamiltonian vector field *associated to f, so that $d_L f = -iX_f + df$. In* [8] *the* generalized Hamiltonian vector field *associated to $f \in C^\infty(M)$ is defined as the section of $\mathbb{T}M$ given by*

$$X_f + \xi_f = \mathbb{J}(df). \tag{12}$$

The Poisson bracket of two smooth functions f and g is then defined as the smooth map given by $\{f, g\} = X_f(g)$. In this framework we recover exactly the symplectic case as before, but in the complex case there is a difference: the Poisson bracket is zero because there are no hamiltonian vector fields after this definition in the complex case.

3. Geometric Quantization of Generalized Complex Structures

Given a symplectic manifold (M, ω) with an associated Poisson algebra $(C^\infty(M), \{,\})$, modeling the classical phase space for a dynamical system. A *geometric quantization* of such a Poisson algebra means a map

$$\begin{array}{rcl} C^\infty(M) & \to & \mathsf{End}(\Gamma(E)) \\ f & \mapsto & \hat{f}, \end{array}$$

where $\Gamma(E)$ denotes the space of sections of a Hermitian vector bundle $E \to M$, modelling wave functions, which satisfies the Dirac quantization conditions [19]:

1. The application $f \mapsto \widehat{f}$ is linear
2. If f is constant then $\widehat{f}$ must be the multiplication (by the constant f) operator
3. If $\{f, g\} = h$ then

$$[\widehat{f}, \widehat{g}] = -i\widehat{h}. \tag{13}$$

In the Kostant-Souriau geometric approach to quantization [9][11][14], the first step towards a geometric quantization of a symplectic manifold (M, ω) is to build a prequantization bundle, i.e. a complex line bundle $\mathcal{L} \xrightarrow{\pi} M$ endowed with a connection ∇ with curvature $2\pi i\omega$. Such a bundle exists if and only if the class of $\frac{1}{2\pi}\omega$ in $H^2(M, \mathbb{R})$ is in the image of $H^2(M, \mathbb{Z})$ under the inclusion in $H^2(M, \mathbb{R})$ (see e.g. [11][19]). When this integrality condition is verified, the Hilbert space of prequantization $\mathcal{H}(M, \mathcal{L})$ is the completion of the space formed by the square integrable sections $s : M \to \mathcal{L}$, with the inner product

$$(s, s') = \int_M \langle s, s' \rangle \, \epsilon,$$

where $\epsilon = \frac{1}{2\pi} dp_1 \wedge \cdots \wedge dp_n \wedge dq_1 \wedge \ldots \wedge dq_n$ is the element of volume of the symplectic manifold M. In this setting, to each observable $f \in C^\infty(M)$ we associate an Hermitian operator according to the *Konstant-Souriau representation*

$$\hat{f} \; = f - 2\pi i \nabla_{X_f},$$

where X_f denotes the Hamiltonian vector field generated by f. This prequantization should be promoted to a quantization of the symplectic manifold by means of a polarization (see [9][19] for details).

This setting for quantization has been generalized by Vergne, Guillemin and others (see [16][17] and references therein), stressing the importance of the almost complex structure J on M and the induced Spin^c-Dirac operator on the Clifford bundle of forms. Indeed, given an almost complex structure J on a prequantizable manifold (M, ω), compatible with the symplectic form and inducing a decomposition of the (complexification of the) tangent bundle as $T_{\mathbb{C}}M = T_{1,0}M \oplus T_{0,1}M$, the canonical Clifford bundle associated to the Spin^c-structure on M is [12]

$$S_J(M) = \Lambda^\bullet T_{\mathbb{C}}M \otimes K_J^{\frac{1}{2}} \cong \Omega^{0,\bullet}(M) \otimes (\Lambda^n T^*_{\mathbb{C}}M)^{\frac{1}{2}}\,, \tag{14}$$

where $K_J = \Lambda^n T^*_{\mathbb{C}}M$ is the canonical bundle associated to (M, J). Thus, provided a Hermitian connection on the canonical bundle K_J, there exists an associated Spin^c-Dirac operator

$$\partial : \Omega^{0,+}_{\mathcal{L}}(M) \to \Omega^{0,-}_{\mathcal{L}}(M)$$

over the bundles of $\mathcal{L}$-valued differential forms on M, where $\mathcal{L} \to M$ denotes the prequantization line bundle. If $\mathsf{Ind}(\partial)$ denotes the index bundle of ∂ over M, the *quantization* of the (Poisson algebra on the) manifold M is defined as the bundle

$$Q(M) = \mathsf{Ind}(\partial) = \mathsf{Ker}(\partial) - \mathsf{CoKer}(\partial) \tag{15}$$

over M. This definition of quantization is the suitable one in connection with symplectic reduction and representation theory, i.e. given an equivariant Hamiltonian action on M with a moment map, in this context it has been shown that quantization commutes with reduction [13].

3.1. **The Weinstein-Zambon prequantization representation.** Let $(M, \mathbb{J})$ be a generalized complex manifold, let $L < \mathbb{T}_{\mathbb{C}}M$ be the Lie algebroid associated to $\mathbb{J}$ on M, $(C_L^\infty(M), \{,\})$ the Poisson algebra of L-admissible functions and $H_L^\bullet(M)$ the L-cohomology associated to the elliptic complex $(\Omega^\bullet(L), d_L)$. Consider the natural antisymmetric pairing $\langle\, ,\, \rangle_-$ on $\mathbb{T}M$ given by (2), and let us denote by Ω_L its associated 2-form on L, i.e. the given by its restriction to L. Recall that the pullback of the anchor map $\pi_{TM} : \Gamma(L) \to \Gamma(TM)$ induces a map

$$\pi_{TM}^* : \Omega_{dR}^\bullet(M) \to \Omega^\bullet(L)$$

descending to a map between de Rham and Lie algebroid cohomology $\pi_{TM}^* : H_{dR}^\bullet(M) \to H_L^\bullet(M)$ [2] and $d_L\Omega_L = 0$. Following [18], we say the generalized complex manifold $(M, \mathbb{J})$ to be *prequantizable* if

$$[\Omega_L] \in \pi_{TM}^*(i_*(H_{dR}^2(M, \mathbb{Z}))), \tag{16}$$

where $i : \mathbb{Z} \to \mathbb{R}$ denotes the inclussion, i.e. if there exist a real integral 2-form Ω on M such that

$$\Omega_L = \pi_{TM}^*\Omega + d_L\beta \tag{17}$$

where $\beta \in \Gamma(L^*)$.

Remark 3.1. *Notice that, if the generalized complex structure is the one defined by a symplectic structure, the* 2*-form* Ω *is the symplectic form and* $\beta = 0$*, so that the prequantization condition becomes the usual geometric Kostant-Souriau prequantization condition for* M*. In the same manner, when the generalized complex structure is the graph of a Poisson bivector over* M*, this condition coincides with Vaisman's geometric prequantization condition for Poisson manifolds* [15]*. However, notice that the prequantization condition cannot be verified in the case in which the generalized complex structure comes from a genuine complex structure on* M*, so that* there is no prequantization in the complex case*. This fact is compatible with the fact pointed out previously that the Poisson algebra itself is trivial in the complex case.*

As it was the case in the symplectic setting, the significance of the prequantization condition above is that, whenever it is satisfied, it is possible associate with the prequantizable manifold a line bundle $\mathcal{L} \to M$ with a Hermitian connection ∇ such that its curvature form is Ω and, as a consequence, we obtain a representation of the Poisson algebra (11) on the space of operators acting on sections of such a bundle. In the case of Dirac structures, using covariant derivatives associated to Lie algebroids (see [7]), a similar representation is obtained in [18]. Namely, given a vector bundle $E \to M$ and a Lie algebroid $\mathcal{A}$ over M, a $\mathcal{A}$-*covariant derivative* on sections of E is a map

$$\nabla^{\mathcal{A}} : \Gamma(\mathcal{A}) \otimes \Gamma(E) \to \Gamma(E)$$

giving rise to a linear differential operator, satisfying the usual rules, and such that

$$\nabla_\alpha^{\mathcal{A}}(f\varphi) = f\nabla_\alpha^{\mathcal{A}}\varphi + (a(\alpha)(f))\,\varphi, \tag{18}$$

where a denotes the anchor map and $\alpha \in \Gamma(\mathcal{A})$, $\varphi \in \Gamma(E)$ and $f \in C^\infty(M)$. In particular, Lemma 6.2 in [18] follows the same in the case of generalized complex structures to have

Proposition 3.1. *Let* $(M, \mathbb{J})$ *be a generalized complex manifold and let* $L < \mathbb{T}_{\mathbb{C}}M$ *be the Lie algebroid associated to* $\mathbb{J}$ *on* M*. If* Ω *is a closed integral* 2*-form on* M *and* ∇ *a connection on a Hermitian line bundle* $\mathcal{L}$ *with curvature* $2\pi i\Omega$ *and the integrality condition*

(17) *is satisfied, then* $\pi_{TM}^*\Omega = \Omega_L + d_L\alpha$ *for* $\alpha \in \Gamma(L^*)$, *and the* L*-connection* ∇^L *defined by*

$$\nabla^L_{\cdot} = \nabla_{\pi_{TM}(\cdot)} - 2\pi i \langle \cdot, \alpha \rangle, \tag{19}$$

where $\langle , \rangle$ *denotes the duality pairing in* L, *has curvature* $2\pi i \Omega_L$.

With the result above at hand, from section 6 in [18] it follows that, given a prequantizable generalized complex manifold $(M, \mathbb{J})$ with associated Dirac structure L, the Poisson algebra of L-admissible functions on the manifold can be represented on L-flat sections of $\mathcal{L}$ with respect to the L-connection ∇^L. Namely, if we define

$$\Gamma_L(\mathcal{L}) = \{ s \in \Gamma_{loc}(\mathcal{L}) \mid \nabla^L_{Y \oplus 0} = 0 \quad \forall Y \in L \cap TM \} \tag{20}$$

where ∇^L is the L-connection given by (19), then

$$\begin{array}{rcl} C^\infty_L(M) & \to & \mathsf{End}(\Gamma_L(\mathcal{L})) \\ f & \mapsto & \hat{f} = f - 2\pi i \nabla^L_{X_f \oplus df}, \end{array}$$

where X_f denotes any Hamiltonian vector field associated to f, is a representation of the Poisson algebra of L-admissible functions. It is clear that in the symplectic case we retrieve the Kostant-Souriau representation mentioned earlier, altogether with its generalization to the Poisson case given in [15].

3.2. **Quantization of generalized complex structures.** Let us now consider a prequantizable generalized complex manifold $(M, \mathbb{J})$, with prequantum bundle and Hermitian connection $(\mathcal{L}, \nabla)$. In particular, $\mathbb{J} : \mathbb{T}M \to \mathbb{T}M$ is an (integrable, with respect to the Courant bracket) almost complex structure on the generalized tangent bundle, and it is characterized by a Dirac structure $L < \mathbb{T}_{\mathbb{C}}M$. Since $L \cap \bar{L} = \{0\}$, via the inner product, $L^* \cong \bar{L}$ and $\mathbb{T}_{\mathbb{C}}M \cong (L \oplus \bar{L})$. Recall that L is a Lie algebroid on M and therefore there is a differential complex associated to it, namely $(\Omega^\bullet_L(M), d_L)$, the Lie algebroid differential complex defined in (9), whose L-cohomology is denoted $H^\bullet_L(M)$.

Recall that the Dirac structure L can be also specified through a spinor line subbundle $K_L \subset \Lambda^\bullet T^*_{\mathbb{C}}M$, which is annihilated by the action of L by Clifford multiplication. This line K_L is the *canonical line bundle* associated to the generalized complex structure specified by L. There is a relation between the complex of differential forms on M —in which sections of K_L are included— and the L-complex defined by the Lie algebroid structure of L that we describe as follows [8]. Consider the spaces

$$\Omega^j_{K_L}(M) = \Gamma(\Lambda^j L^* \cdot K_L) \cong \Gamma(\Lambda^j L^* \otimes K_L), \tag{21}$$

of K_L-valued L-forms on M. Then,

$$\Omega^0_{K_L}(M) = \Gamma(K_L), \quad \Omega^1_{K_L}(M) = \Gamma(L^* \cdot K_L) \cong \Gamma(\mathbb{T}M) \otimes \Gamma(K_L), \tag{22}$$

since L-sections annihilate K_L and $\Omega^{2m}_{K_L}(M) = \Gamma(\Lambda^{2m} L^* \cdot K_L) \cong \Gamma(\det L^*) \otimes \Gamma(K_L) = K_{L^*}$, the pure spinor line associated to the complementary Dirac structure L^*. As a matter of fact [8], there is a chain complex

$$0 \to \Omega^0_{K_L}(M) \xrightarrow{\bar{\partial}_L} \Omega^1_{K_L}(M) \xrightarrow{\bar{\partial}_L} \cdots \xrightarrow{\bar{\partial}_L} \Omega^{2m}_{K_L}(M) \to 0, \tag{23}$$

defined as follows: Since $K_L \subset \Omega^\bullet(M)$ we can take usual exterior derivatives on elements of the canonical line associated to the generalized complex structure and, moreover, Courant integrability of L is equivalent to ask

$$d : \Gamma(K_L) \to \Gamma(L^* \otimes K_L),$$

to satisfy $d^2 = 0$. Thus, coupling d to the Lie algebroid exterior derivative d_L to extend it to $\Omega^j_{K_L}(M) = \Gamma(\Lambda^j L^* \otimes K_L)$ as

$$\bar{\partial}_L(\lambda \otimes \varphi) = d_L\lambda \otimes \varphi + (-1)^{\deg(\lambda)}\lambda \otimes d\varphi, \tag{24}$$

we have a Lie algebroid module structure on K_L over the differential algebra $(\Omega^\bullet_L(M), d_L)$, i.e. it gives rise to a differential operator

$$\bar{\partial}_L : \Omega^j_{K_L}(M) \to \Omega^{j+1}_{K_L}(M) \tag{25}$$

called *generalized Dolbeault operator*, which is actually a Lie algebroid connection for the module K_L [7][8]. Since in the complex case it reproduces (with a different grading) the usual Dolbeault cohomology for complex manifolds, the cohomology associated to the differential complex $(\Omega^\bullet_{K_L}(M), \bar{\partial}_L)$, which is elliptic from the definition (24), is called *generalized Dolbeault cohomology*.

Remark 3.2. *Since $\Omega^0_{K_L}(M) = K_L$, $\Omega^j_{K_L}(M) = \Gamma(\Lambda^j L^* \otimes K_L)$, we have a $\mathbb{Z}$-grading on $\Omega^\bullet(M) \otimes \mathbb{C}$, namely*

$$\Omega^\bullet(M) = \Omega^0_{K_L}(M) \oplus \Omega^1_{K_L}(M) \oplus \cdots \oplus \Omega^{2m}_{K_L}(M).$$

It follows that there is an isomorphism

$$\Omega^\bullet(M) \cong \Omega^\bullet_L(M) \otimes K_L. \tag{26}$$

Consider now the Mukai pairing of differential forms

$$(,) : \Omega^\bullet(M) \otimes \Omega^\bullet(M) \to \det(T^*M)$$

given by $(\varphi, \rho) = [\varphi^\top \wedge \rho]_{\dim M}$. This pairing induces a nondegenerate pairing

$$(,) : \Omega^j_{K_L}(M) \otimes \Omega^{2m-j}_{K_L}(M) \to \det(T^*M)$$

which, for $j = 2m$, gives rise to an isomorphism $\det T^*M \cong \Omega^{2m}_{K_L}(M) \otimes \Omega^0_{K_L}(M)$. It follows from (26) and the definition of $\Omega^0_{K_L}(M)$ that

$$\Omega^\bullet(M) \otimes \det(TM)^{\frac{1}{2}} \cong \Omega^\bullet_L(M) \otimes \det(L)^{\frac{1}{2}}, \tag{27}$$

which indicates that $S_L(M) = \Omega^\bullet(M) \otimes \det(TM)^{\frac{1}{2}}$ is a natural choice for the spin bundle associated to the Clifford algebra of $\mathbb{T}M$ through the action (5) (compares with (14) in the complex case). Moreover, the Mukai pairing can be seen as a nondegenerate bilinear form

$$(,) : S_L(M) \otimes S_L(M) \to C^\infty(M)$$

on such a bundle.

Finally, we arrive to our definition of quantization for generalized complex manifolds. Let $\Omega^j_{K_L}(M, \mathcal{L}) = \Gamma(\Lambda^j L^* \otimes K_L \otimes \mathcal{L})$ the space of "L-forms" on M with values in the line bundle $K_L \otimes \mathcal{L}$, and consider the spaces

$$\Omega^+_{K_L}(M, \mathcal{L}) = \bigoplus_j \Omega^{2j}_{K_L}(M, \mathcal{L}), \qquad \Omega^-_{K_L}(M, \mathcal{L}) = \bigoplus_j \Omega^{2j+1}_{K_L}(M, \mathcal{L}).$$

Coupling the generalized Dolbeault operator (25) with a connection ∇ on $\mathcal{L}$ gives rise to an elliptic operator

$$\bar{\partial}_{\mathcal{L}} : \Omega^+_{K_L}(M, \mathcal{L}) \to \Omega^-_{K_L}(M, \mathcal{L}). \tag{28}$$

Definition 3.1. *Let $(M, \mathbb{J})$ be a prequantizable generalized complex manifold, with prequantum bundle and Hermitian connection $(\mathcal{L}, \nabla)$. Let L be the Dirac structure characterizing $\mathbb{J}$ and $(\Omega^\bullet_{K_L}(M, \mathcal{L}), \bar{\partial}_{\mathcal{L}})$ the associated elliptic complex described before. Then, the* quantization *of the Poisson algebra given by (11) is the bundle*

$$Q(M, \mathcal{L}) = \mathrm{Ind}(\bar{\partial}_{\mathcal{L}}) = \mathrm{Ker}(\bar{\partial}_{\mathcal{L}}) - \mathrm{CoKer}(\bar{\partial}_{\mathcal{L}}). \tag{29}$$

As a consequence of this definition we have that, for a manifold with a generalized complex structure induced from a symplectic form, the quantization coincides with the one given by Vergne in (15). However, notice that if the generalized complex structure is purely complex, there is no quantization associated to it.

Remark 3.3. *Notice that we have worked the case in which there is no twisting in the Courant bracket given by* (6). *Indeed, if the Courant bracket has a twisting given by a closed* 2*-form on* M*, the* 2*-form* Ω_L *used to define the prequantization condition* (17) *is no longer* d_L*-closed.*

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