GEOMETRIC AND MATAPLECTIC QUANTIZATION

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Abstract. Geometric quantization gives a representation of the algebra of classical observables of a dynamical system, described through a symplectic manifold, in the Lie algebra of operators acting on a Hilbert space. We review and compare the geometrical and topological key ingredients of two approaches to this type of quantization, the original Kostant-Souriau-Kirillov prequantization procedure and the more recent metaplectic quantization introduced by K. Habermann, which uses some properties of the symplectic Dirac operator. The comparison shows that, besides the difference between the two constructions, they fit into the same framework and give rise to standard features of the quantization problem.

1. Introduction

Quantizing geometrically a classical physical system means to represent its classical Poisson algebra of observables in a Lie algebra of hermitian operators acting on a Hilbert space built from the geometrical (and topological) features of the system. Throughout this paper we will consider two approaches to this problem, the original prequantization program of B. Kostant, J-M. Souriau, A. Kirillov and others, see e.g. [13][10][8][14]), and the more recent metaplectic quantization introduced by K. Habermann in [7], where the construction of symplectic Dirac operators is used to give an (infinite-rank) alternative to the above mentioned geometric quantization procedure of Konstant and Souriau. Indeed, in [7] an attempt is done to replace the usual prequantization plus polarization plus half-density procedure by one involving no longer a line bundle with connection, but the symplectic spinor bundle defined in [9]. In this paper we recall these procedures and we compare their main features showing that, even if different, the later contains capital features of the former, and that together they give a complete geometric quantization program.

The paper is organized as follows. In the next section we give a definition of geometric quantization, and we recall how the geometry and topology of a symplectic manifold describing a classical physical system determine such geometric quantization. Section 3 is devoted to review the geometrical constructions giving rise to the approaches by Kostant-Souriau and Habermann, and a comparison of them. A natural question arises about the behavior of this geometric quantization procedure with respect to geometrical and topological aspects of the symplectic Dirac operators which inspired the construction of [7], this question will be addressed in a forthcoming paper.

2. Geometric Quantization

A classical dynamical system is described by a symplectic manifold (which we will assume to be compact), and it is in the symplectic form where all the dynamical information is contained (physical trajectories are the integral curves of vector fields along which the symplectic form is constant). The *geometric quantization* program tries to build, from the geometric objects defining the classical dynamics of the system, a representation of the classical Poisson algebra of observables into a *quantum* algebra of operators acting on a Hilbert space of sections of a vector fibration over the symplectic manifold.

2.1. **Symplectic Geometry and Classical Dynamics.** Let (M,ω) be a compact *symplectic manifold*, i.e. ω is a closed *symplectic* 2-form on M such that the application

$$\begin{array}{rcl} i_\omega : T_m M & \rightarrow & T_m^* M \\ X & \mapsto & i_\omega(X) = \omega(X,\cdot) \end{array}$$

is a linear isomorphism, for each $m \in M$, between the spaces of tangent and cotangent vectors in m. By Darboux theorem, around any point of M there exists an open set with local coordinates $\{q_1,...,q_n,p_1,...,p_n\}$ such that on it

$$\omega = \sum_{i=1}^{n} dp_i \wedge dq_i,$$

where $\dim M = 2n$. Very often in mathematical physics dynamical systems are described in configuration spaces modelled by manifolds Q and phase spaces modelled by their cotangent bundles T^*Q. Given a physical observable, i.e. a function $f \in C^\infty(T^*Q,\mathbb{R})$, we define the *Hamiltonian Vector Field* associated to f, denoted X_f, by the equality

$$i_\omega(X_f) = -df,$$

where we use the isomorphism i_ω. In local Darboux coordinates

$$X_f = \frac{\partial f}{\partial p_i}\frac{\partial}{\partial q_i} - \frac{\partial f}{\partial q_i}\frac{\partial}{\partial p_i}.$$

For any two functions $f,g \in C^\infty(T^*Q,\mathbb{R})$, the *Poisson Bracket* of f with g is defined by

$$\{f,g\} = X_f(g) = -X_g(f) = \omega(X_f,X_g) \tag{1}$$

which gives, in local coordinates,

$$\{f,g\} = X_f(g) = \frac{\partial f}{\partial p_i}\frac{\partial g}{\partial q_i} - \frac{\partial f}{\partial q_i}\frac{\partial g}{\partial p_i}.$$

It follows that this operation gives the structure of Lie algebra to $C^\infty(M = T^*Q,\mathbb{R})$ and, moreover [1],

$$[X_f,X_g] = X_{\{f,g\}}. \tag{2}$$

2.2. **Quantization.** Let (M, ω) be a symplectic manifold and consider the Lie Poisson algebra of classical observables $C^\infty(M, \mathbb{R})$. A *quantization* of this algebra means is a representation of it in the Lie algebra of quantum observables (operators) acting on a Hilbert space $\mathcal{H}$ (with the Lie bracket), verifying the *Dirac conditions*:

1. The application $f \mapsto \widehat{f}$ is linear
2. If f is constant then $\widehat{f}$ must be the multiplication (by the constant f) operator
3. If $\{f, g\} = h$ then

$$[\widehat{f}, \widehat{g}] = -i\hbar\widehat{h}. \tag{3}$$

We say that we quantize *geometrically* the classical system described by (M, ω) if we find a map

$$\begin{array}{rcl} C^\infty(M) \times \Gamma(E) & \to & \Gamma(E) \\ (f, \psi) & \mapsto & \hat{f}\psi \end{array}$$

satisfying the last three conditions, where $\Gamma(E)$ denotes the space of sections of a Hermitian vector bundle $E \to M$, modelling wave functions.

There are two families of vector bundles considered in the geometric quantization literature: line bundles —i.e. rank one complex vector bundles (e.g. in the classical Kostant-Souriau geometric prequantization [8] in which E is the complex line bundle $L \to M$ on the phase space defined when the symplectic form is integral, also in the metaplectic quantization of Robinson-Rawnsley [12] in which the complex line bundle is defined by a character of the metaplectic representation, etc.), and infinite-rank Hilbert bundles —i.e. bundles having as fibre an infinite-dimensional Hilbert space (e.g. in the metaplectic quantization setting of Habermann [7], in which the bundle E is the infinite-rank bundle of symplectic spinors). In both cases the representation is found through the geometry of such bundles induced by the symplectic structure on the manifold, i.e. through connections defined by the symplectic form ω.

Given a covariant derivative

$$\nabla^E : \Gamma(E \otimes TM) \to \Gamma(E)$$

on E, there is a natural geometric association $f \mapsto \hat{f}$, originally due to Souriau and Konstant, giving rise to a good quantization; namely,

$$\begin{array}{rcl} C^\infty(M) & \to & \mathcal{O}(\Gamma(E)) \\ f & \mapsto & \hat{f} = f - i\hbar\nabla^E_{X_f} \ , \end{array}$$

where X_f denotes the hamiltonian vector field associated to f. Notice that the covariant derivative should be non-trivial (i.e. non-flat) since it must carry part of the information about the symplectic form ω (i.e. about the classical dynamics of the system). Indeed, since $X_{\{f,g\}} = [X_f, X_g]$, an easy calculation shows that

$$\begin{array}{rcl} [\hat{f}, \hat{g}] & = & -2i\hbar\{f, g\} - \hbar^2[\nabla^E_{X_f}, \nabla^E_{X_g}] \\ & = & -i\hbar\widehat{\{f, g\}} + i\hbar\left(i\hbar R(X_f, X_g) - \{f, g\}\right). \end{array} \tag{4}$$

Thus, in order the third condition above (equality (3)) to be verified, natural constraints appear on the curvature R of ∇^E. In the next section we will review how this constraints appear –from the geometry of the symplectic manifold– in two particular settings, the so-called Kostant-Souriau prequantization and Habermann's metaplectic quantization.

3. Two Approaches to Geometric Quantization

Let (M, ω) be a symplectic manifold representing the phase space of a physical system (e.g. $M = T^*Q$ for some manifold Q –the configuration space, see [3]). In this section we review the geometrical constructions giving rise to Kostant-Souriau and Habermann's geometric quantization, emphasizing remarkable features of some ingredients of the later, and comparing their scope.

3.1. **Kostant-Souriau Prequantization.** The Kostant-Souriau prequantization, which is the first step in their geometric quantization program, is based on the following observation due to A. Weil.

Theorem 3.1. *Let (M, ω) be a symplectic manifold, then there exists a complex line bundle $L \xrightarrow{\pi} M$ on this manifold and a connection ∇ on L with curvature $\hbar^{-1}\omega$ if and only if the class of $(2\pi\hbar)^{-1}\omega$ in $H^2(M, \mathbb{R})$ is in the image of $H^2(M, \mathbb{Z})$ under the inclusion in $H^2(M, \mathbb{R})$, i.e. if any integral of ω on a oriented 2-surface in M is an integer multiple of $2\pi\hbar$.*

Up to isomorphism, the possible choices of the prequantization bundle are parameterized by the cohomology group $H^2(M, U(1))$, see e.g. [8][14].

If the integrality condition is verified on the symplectic manifold correspondent to the classical description, and then exist a Hermitian line bundle $L \xrightarrow{\pi} M$, the *Hilbert Space of prequantization* $H(M, L)$ is the completion of the space formed by the square integrable sections $s : M \to L$, noted $\Gamma(L)$, with the inner product

$$(s, s') = \int_M \langle s, s' \rangle \, \epsilon$$

where $\epsilon = \frac{1}{2\pi\hbar} dp_1 \wedge \cdots \wedge dp_n \wedge dq_1 \wedge \ldots \wedge dq_n$ is the element of volume of the manifold M. In this setting, to each observable f we associate an Hermitian operator according to the Konstant-Souriau representation

$$\hat{f} = f - i\hbar\nabla_{X_f},$$

where X_f denotes the Hamiltonian vector field generated by f. From equality (4), the equality (3) follows only if the following identity for the curvature tensor of ∇ is verified

$$R(X_f, X_g) = [\nabla_{X_f}, \nabla_{X_g}] - \nabla_{X_{\{f,g\}}} = -\frac{i}{\hbar}\{f, g\}.$$

Nevertheless, this holds since when the integrality condition is satisfied we can chose the curvature of ∇ to be proportional to the symplectic form, $\Omega_\nabla = \hbar^{-1}\omega$.

Example 3.1. *If a classical system is modelled as a cotangent bundle with canonical symplectic form (so that the corresponding line bundle is $M \times \mathbb{C}$), for the operators corresponding to position q_i and momentum p_i the representation is given by*

$$\hat{f} = f - i\hbar X_f - (p_i dq_i)(X_f).$$

Then, since $X_{p_i} = \frac{\partial}{\partial q_i}$ and $X_{q_i} = -\frac{\partial}{\partial p_i}$,

$$\overset{\wedge}{p_i} = -i\hbar \frac{\partial}{\partial q_i} \qquad \text{and} \qquad \overset{\wedge}{q_i} = q_i + i\hbar \frac{\partial}{\partial p_i},$$

which fails to be in accordance with quantum mechanics (Scrhödinger's version):

$$\overset{\wedge}{p_i} = -i\hbar \frac{\partial}{\partial q_i} \quad \text{and} \quad \overset{\wedge}{q_i} = q_i$$

and illustrates the necessity of a correction by means of a polarization, see e.g. [14].

3.2. **Habermann's Metaplectic Quantization.** The Segal-Shale-Weil representation of the metaplectic group —the two-fold covering of the symplectic group— gives rise to a infinite rank vector bundle, this is the symplectic spinor construction due to Konstant [9]. In [7], a geometric quantization is defined on symplectic spinors using, within the Kostant-Souriau recipe, a perturbation of the spinor derivative induced by a symplectic connection on M; the perturbation is given by a symplectic Clifford multiplication used by the Habermann in his definition of symplectic Dirac operators (see [5]).

The SSW representation and Symplectic Spinors. Let (V, ω) be a $2m$-dimensional symplectic vector space, and let $\mathsf{h}(V)$ denote its associated Heisenberg Lie algebra. Then $\mathsf{h}(V) \cong \mathbb{R}^{2m} \times \mathbb{R}$, and there is a natural representation of the group[1] $\mathrm{Mp}^c(V)$ in the Hilbert space $L^2(\mathbb{R}^m)$ given as follows. By the Stone-von Neumann theorem there is a unique (up to equivalence) irreducible unitary representation

$$\varrho : \mathsf{h}(V) \;\; \rightarrow \;\; U(L^2(\mathbb{R}^m)),$$

where $U(\mathbb{R}^m)$ denotes the group of unitary operators acting on the Hilbert space $\mathcal{H} = L^2(\mathbb{R}^m)$, satisfying

$$\varrho(0, t) = e^{it}\mathrm{Id}_{L^2(\mathbb{R}^m)}.$$

Since the symplectic group acts on Heisenberg algebra by

$$\begin{array}{rcl} Sp(V) \times \mathsf{h}(V) & \rightarrow & \mathsf{h}(V) \\ (g, (v, t)) & \mapsto & (gv, t), \end{array}$$

it induces new irreducible unitary representations ϱ_g of $\mathsf{h}(V)$, for all $g \in Sp(V)$, such that $\varrho_g(0, t) = e^{it}\mathrm{Id}_{L^2(\mathbb{R}^m)}$. Then, there exists a unique (up to a phase) unitary bijective operator $U_g : L^2(\mathbb{R}^m) \rightarrow L^2(\mathbb{R}^m)$ such that $\varrho = U_g \circ \varrho \circ U_g^{-1}$, and the map $g \mapsto U_g$ defines a projective unitary representation of the symplectic group in $L^2(\mathbb{R}^m)$, which lifts uniquely to a unitary representation

$$\rho : \mathrm{Mp}^c(V) \;\; \rightarrow \;\; U(\mathcal{H}),$$

[1]Following [7] we shall work with the metaplectic-c group rather than with $\mathrm{Mp}(V)$ itself.

this is the called *Segal-Shale-Weil representation* of $\mathrm{Mp}^c(V)$. This representation stabilizes the Schwartz space of rapidly decreasing smooth functions on $\mathbb{R}^m$, denoted $\mathbf{S}(\mathbb{R}^m)$, which is used to model the symplectic spinor space in [9]. However, since it is required in the quantization process to end up with an algebra of operators acting on a Hilbert space, later on we will use the Hilbert space $\mathcal{H} = L^2(\mathbb{R}^m)$ to model our symplectic spinor space.

Let us now consider a $2m$-dimensional compact connected symplectic manifold (M, ω). There is a canonical principal $Sp(2m)$-bundle over M, namely the bundle of symplectic frames over it. A metaplectic structure on M is a principal $\mathrm{Mp}(2m)$-bundle over M together with an an equivariant morphism (with respect to the double covering map) from it into the symplectic frame bundle. There is a topological obstruction to the existence of metaplectic structures on a symplectic manifold (similar to those relative to the existence of spin structures on Riemannian manifolds, see e.g. [11]), but there is no topological constraint to the existence of Mp^c-structures on M. However, working with the former involves to fix a connection form on the associated line bundle to the Mp^c-structures, see [12].
The bundle of symplectic spinors is the one associated to the $\mathrm{Mp}^c(2m)$-bundle P over M through the SSW representation:

$$\mathcal{S} = P \times_\rho L^2(\mathbb{R}^m).$$

Symplectic Clifford Multiplication and Symplectic Dirac Operators. The symplectic spinor bundle is the analogue of the Riemannian spinor bundle in the construction of the Dirac operator. In the symplectic case, given a symplectic vector space (V, ω), the roll of the complex Clifford algebra is played by the Weil algebra $\mathsf{W}_{\mathbb{C}}(V)$, which is infinite-dimensional and is represented in a infinite-dimensional space. Consider the space $L^2(\mathbb{R}^m)$ of square integrable smooth functions on $\mathbb{R}^m$, there is an irreducible representation of the Heisenberg algebra $\mathsf{h}(V)$ on $L^2(\mathbb{R}^m)$ —which is not very different in spirit to the usual (Riemannian) Clifford case— and gives rise to the so-called *symplectic Clifford multiplication*. This representation

$$\begin{array}{rcl} \mathsf{c} : \mathsf{h}(V) & \to & End_{\mathbb{C}}(L^2(\mathbb{R}^m)) \\ a & \mapsto & \mathsf{c}(a) : L^2(\mathbb{R}^m) \to L^2(\mathbb{R}^m), \end{array}$$

comes from the following map from V into $End_{\mathbb{C}}(L^2(\mathbb{R}^m))$. Let $\{q_1, \ldots, q_m, \, p_1, \ldots, p_m\}$ be a symplectic basis of V, i.e. a basis such that $\omega(p_i, q_j) = \delta_i^j$ and $\omega\mid_{V^\pm} = 0$, where

$$V = V^+ \oplus V^-$$

and $V^\pm$ are the m-dimensional Lagrangian subspaces of V generated by $\{q_1, \ldots, q_m\}$ and $\{p_1, \ldots, p_m\}$, respectively. Putting, for $\psi \in L^2(\mathbb{R}^m)$,

$$\begin{array}{rcll} \mathsf{c}(p_i)\psi & = & \frac{\partial \psi}{\partial x_i} & \\ \mathsf{c}(q_j)\psi & = & x_j \psi & \qquad \text{(multiplication operator by } x_j\text{)}, \end{array}$$

where $\{x_1, \ldots, x_m\}$ denote the coordinates in $\mathbb{R}^m$, and extending c to all of $\mathsf{h}(V)$ by $\mathsf{c}(x_o)\psi = i\psi$, where x_o denotes the generator of the central part of $\mathsf{h}(V)$, and $\mathsf{c}(vw) = \mathsf{c}(v) \circ \mathsf{c}(w)$, it is easy to verify that c preserves the relations in $\mathsf{W}_{\mathbb{C}}(V)$, i.e.

$$\mathsf{c}(v) \circ \mathsf{c}(w) - \mathsf{c}(w) \circ \mathsf{c}(v) = -i\omega(v, w), \tag{5}$$

so it give us a linear map into $End_{\mathbb{C}}(L^2(\mathbb{R}^m))$, which extends to a one-to-one algebra homomorphism

$$\mathsf{c} : \mathsf{W}_{\mathbb{C}}(V) \to End_{\mathbb{C}}(L^2(\mathbb{R}^m)),$$

from which we obtain an irreducible representation of $\mathsf{h}(V)$ on $L^2(\mathbb{R}^m)$. By Mp^c-equivariance, this Clifford multiplication in the fibers defines a symplectic Clifford multiplication of symplectic spinors by tangent vectors (see [6])

$$\begin{array}{rcl} \mathsf{c} : \Gamma(TM) \otimes \Gamma(\mathcal{S}) & \to & \Gamma(\mathcal{S}) \\ (X, \varphi) & \mapsto & \mathsf{c}(X)\varphi. \end{array}$$

Remark 3.1. *Notice that symplectic Clifford multiplication is degenerate since, for $\varphi \in \mathcal{S}$ and for any $X \in \Gamma(TM)$*

$$c(X)\varphi = 0,$$

gives rise to a differential equation with possible non trivial solutions.

In the symplectic case there is no canonical choice for a covariant derivative (i.e. no uniqueness property as for Levi-Civita type connections in Riemannian geometry) so, in order to go on into the construction of symplectic Dirac operators, we begin by fixing a *symplectic connection*, i.e. a covariant derivative $\nabla : \Gamma(TM) \to \Gamma(T^*M \otimes TM)$ such that

$$\nabla\omega = 0 \qquad \Leftrightarrow \qquad X(\omega(Y, Z)) = \omega(\nabla_X Y, Z) + \omega(Y, \nabla_X Z), \tag{6}$$

for any vector fields X, Y and Z. From ∇, we induce a covariant derivative on spinor fields which we denote ∇^S.

From a symplectic connection on M and the symplectic Clifford multiplication introduced before we can define symplectic Dirac operators in the usual way, i.e. composing the c with the induced covariant derivative on the spinor bundle. There are two possible ways of coupling these two maps, depending on the isomorphism used to identify TM with $T * M$. Here we use the identification given through the symplectic form (i.e. the map i_ω), the identification using a Riemannian structure gives rise to symplectic Dirac operators studied in [4]. A direct computation in local coordinates shows that, for any symplectic spinor φ,

$$D\varphi = \sum_{i=1}^{n} \mathsf{c}(e_i)\nabla^S_{f_i}\varphi - \mathsf{c}(f_i)\nabla^S_{e_i}\varphi, \tag{7}$$

where $\{e_i, f_i\}$ denotes a (local) symplectic basis on the tangent space of M induced by Darboux coordinates.

Proposition 3.1. *For any classical observable $f \in C^\infty(M, \mathbb{R})$,*

$$[D, f] = c(X_f), \tag{8}$$

where X_f denotes the associated Hamiltonian vector field.

Proof. Let $f \in C^\infty(M,\mathbb{R})$ and $\varphi \in \Gamma(\mathcal{S})$, then

$$\begin{aligned} D(f\varphi) &= \sum_{i=1}^{n} \mathsf{c}(e_i)(df(f_i)\varphi + f\nabla^S_{f_i}\varphi) - \mathsf{c}(f_i)(df(e_i)\varphi + f\nabla^S_{e_i}\varphi) \\ &= \mathsf{c}(X_f)\varphi + f\nabla^S_{X_f}\varphi. \end{aligned}$$

□

Remark 3.2. *This symplectic Dirac operator is* not *elliptic, as follows from the degeneracy of the symplectic Clifford multiplication remarked above.*

Metaplectic Quantization. Let us come back to the discussion of the possible quantization recipes on the symplectic spinor bundle $\mathcal{S}$ associated to a symplectic manifold (M,ω). As before, the starting point must be the choice of a covariant derivative on $\mathcal{S}$.

In the symplectic spinor bundle we can try the same Kostant-Sternberg trick for geometric quantization, namely to each observable f we associate

$$\begin{aligned} C^\infty(M,\mathbb{R}) &\rightarrow \mathcal{O}(\Gamma(\mathcal{S})) \\ f &\mapsto \hat{f} \ = f - i\hbar\nabla^S_{X_f} \ , \end{aligned}$$

where X_f denotes the Hamiltonian vector field generated by f and ∇^S denotes the symplectic spinor derivative on $\Gamma(\mathcal{S})$ mentioned above. However, (3) will not be satisfied, since we have no mens to cancel the second term in equation (4).

Remark 3.3. *Notice that in the metaplectic setting the symplectic Clifford multiplication can be chosen to be proportional to the symplectic form (by choosing the constant to which the generator of the central term in the Heisenberg algebra, x_o, is sent by Clifford multiplication). For example, from (5) we can have*

$$\mathsf{c}([X_f, X_g]) = -\frac{i}{\hbar}\omega(X_f, X_g), \tag{9}$$

and we can actually try

$$f \mapsto \hat{f} \ = f - i\hbar\,[D, f] \ , \tag{10}$$

which gives $\widehat{\{f,g\}} = 2\{f,g\}$, and then the singular relation

$$[\hat{f}, \hat{g}] = -\frac{i\hbar}{2}\widehat{\{f,g\}}.$$

However, it is clear that the map (10) does not define a good quantization.

Habermann's metaplectic quantization introduces a perturbation on the spinor covariant derivative of the form (10) in order to get a weak version of (3).

Definition 3.1. *Let us define the* Clifford covariant derivative *on symplectic spinors by*

$$\nabla^H_X\varphi = \nabla^S_X\varphi + \mathsf{c}(X)\varphi, \tag{11}$$

where $X \in \Gamma(TM)$, $\varphi \in \Gamma(\mathcal{S})$.

Proposition 3.2. *Let ∇^S and ∇^H denote the above defined covariant derivatives on the bundle of symplectic spinors. Then, the corresponding curvatures $R(X,Y)$ and $R^H(X,Y)$ are related by*

$$R^H(X,Y) = R(X,Y) + \mathsf{c}(T(X,Y)) + \mathsf{c}([X,Y]),$$

where $\mathsf{c}(T(X,Y))$ denotes the symplectic Clifford multiplication by the torsion $T(X,Y) = \nabla^S_X Y - \nabla^S_Y X - [X,Y]$ of ∇^S.

Proof. It follows from the identity

$$[\nabla^H_X, \nabla^H_Y] = [\nabla^S_X, \nabla^S_Y] + (\mathsf{c}(\nabla^S_X Y) - \mathsf{c}(\nabla^S_Y X)) + \mathsf{c}([X,Y]).$$

□

Notice that, given any classical observable f, it follows from (8) that the Clifford covariant derivative of a symplectic spinor, evaluated in the direction of the Hamiltonian vector field associated to f, is

$$\nabla^H_{X_f} = \nabla^S_{X_f} + [D, f]. \tag{12}$$

From (4)

$$[\hat{f}, \hat{g}] = -2i\hbar\{f,g\} - \hbar^2 R^H(X_f, X_g) - \hbar^2 \nabla^H_{[X_f, X_g]},$$

so it follows from (9) that the Kostant-Souriau recipe $\hat{f} = f - i\hbar\nabla^H_{X_f}$ in this case gives the relation [7]

$$[\hat{f}, \hat{g}] = -i\hbar\widehat{\{f,g\}} - \hbar^2 \left[R(X_f, X_g) + \mathsf{c}(T(X_f, X_g))\right], \tag{13}$$

where R and T denote the curvature and torsion associated to ∇^S, respectively. This is en example of the weaker version of condition (3) used, for example, in deformation quantization [2].

Remark 3.4. *Notice that using the Clifford covariant derivative on symplectic spinors ∇^H instead of ∇^S, in the construction of the symplectic Dirac operator (keeping the use of the symplectic form in the identification between tangent and cotangent spaces), indices a new symplectic Dirac operator D^H such that,*

$$D^H\varphi = (D + \frac{\dim M}{2})\varphi, \tag{14}$$

as follows from (5) when applied to a symplectic basis.

3.3. **Discussion and Final Comments.** As mentioned earlier, the prequantization presented in Section 2 gives only the first part within the geometric quantization approach due to Kirillov, Kostant, Souriau and others. Indeed, a polarization and a half-density correction are in order to finish the task, the former reduces to a half the number of "degrees of freedom" of the wave functions (sections of the prequantization bundle) and the later gives the required Hilbert space structure on the space of wave functions. This approach introduces topological constraints to the existence of quantizations (related with the integrality condition on symplectic forms). On the other hand, the approach suggested by Habermann seems to apply to any symplectic manifold, having at once these two features: First, since the SSW representation of the metaplectic group of a $2m$-dimensional symplectic vector space is done into (L^2 of) a m-dimensional vector space, then there are only half of the dimensions from the beginning. Second, the Hilbert space structure for symplectic spinors come automatically from the construction. However, this approach seems to be geometrically 'rigid', i.e. several objects must be fixed, a connection on the line bundle associated to the $Mp^{\mathbb{C}}$-structure and a symplectic connection (in order to define the symplectic Dirac operator).

There are other important differences between both constructions. For example, while the usual set up by Souriau and Kostant gives rise to wave functions taking values in a finite-rank vector bundle, in the second approach these wave functions take values in an infinite-dimensional vector bundle. Nevertheless, let us remark that the symplectic bundle construction of Kostant is used to obtain the half-density bundle in the usual construction: it comes as a subbundle of the symplectic spinor bundle (isomorphic to the subbundle determined by the first eigenstates of the symplectic Dirac operator). Thus, even though the geometric frame work of Haberman's construction is not a novelty, the use of the symplectic Dirac operator in the perturbation of the covariant derivative used in the quantum representation of observables seems to be geometrically interesting by itself. Notice that, for example, this perturbation in the connection gives rise to some kind of spectral shift in the symplectic Dirac operator (equation (14)) very similar to the one obtained, e.g. in the harmonic oscillator Hamiltonian, after the usual metaplectic correction (see [14]).

Finally, recall that the first approach gives an exact quantization condition (3), and the second only a weak version of it, namely equation (13).

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