ABSTRACT OF THE TALK "DEFORMATION OF G-STRUCTURES"

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References

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1. G-structures

Let G be a subgroup in $GL(n)$. A *G-structure on a manifold M* is a principal G-subbundle $L_G(M) \to M$ of the linear frame bundle $L(M)$. Each G-structure is defined by a section $s : M \to P_G = L(M)/G$.

A G-structure on a manifold M is mathcalled *integrable* if, for each point $p \in M$, there exists a chart on $U \ni p$ whose natural frame is a section of $L_G(M)$.

Example 1. Let

$$G = \begin{pmatrix} A & B \\ 0 & C \end{pmatrix}, \tag{1}$$

where A is $k \times k$-matrix, B is $k \times (n-k)$-matrix, and C is $(n-k) \times (n-k)$-matrix.

Then a G-structure $L_G(M) \to M$ is a k-dimensional distribution Δ on M. The corresponding section $s : M \to P_G = L(M)/G$ is a section of the Grassmann bundle $G_k(M)$ of k-dimensional subspaces in TM.

If this G-structure is integrable, then the distribution Δ is also integrable, hence determines a foliation.

2. Deformations of integrable G-structures

2.1. **Deformation of G-structures.** A *deformation of a G-structure s* is a one-parametric family of sections $s_t : M \to L(M)/G$ such that $s_0 = s$. A deformation is mathcalled *inessential* if $s_t = f_t^*(s)$ for a one-parametric family f_t of diffeomorphisms of M.

An *infinitesimal deformation* of a section $s : M \to P_G$ is a vertimathcal vector field $V = \frac{d}{dt}|_{t=0} s_t$ along $s(M)$, or a section of the bundle $s^*V(P_G)$. That is

$$\mathcal{D}(s) = \Gamma(M; s^*V(P_G)) \tag{2}$$

An *inessential infinitesimal deformation* of a section $s : M \to P_G$ is a vertimathcal vector field $V = \frac{d}{dt}|_{t=0} f_t^*(s_0)$ along $s(M)$. One can prove that the space of inessential infinitesimal deformations is

$$\mathcal{D}_0 = \{L_X s \mid X \in \mathfrak{X}(M)\}. \tag{3}$$

Hence the space of essential deformations of s is

$$\mathcal{D}_{ess} = \frac{\mathcal{D}(s)}{\mathcal{D}_0(s)} = \frac{\Gamma(M; s^*V(P_G))}{\{L_X s \mid X \in \mathfrak{X}(M)\}}. \tag{4}$$

Remark 1. This can be done for sections of natural bundle.

2.2. **Deformations of integrable structures.** A *deformation of an integrable G-structure s* is a one-parametric family of sections $s_t : M \to L(M)/G$ such that each s_t is integrable and $s_0 = s$.

An *infinitesimal deformation* of a section $s : M \to P_G$ is a vertimathcal vector field $V = \frac{d}{dt}|_{t=0} s_t$ along $s(M)$, or a section of the bundle $s^* V(P_G)$ but now the sections s_t are *integrable* G-structures.

First approach. Any integrable G-structure s on M determines a *pseudogroup structure* on M.

Γ is a pseudogroup of transformations of $f : \mathbb{R}^n \to \mathbb{R}$ such that Df lies in G. Then existence of G-structure is equivalent to existence of a Γ-atlas (an atlas on M with transition functions in Γ).

We can repeat the construction of deformations of complex structure.

We introduce the sheaf $\mathfrak{X}_a$ of infinitesimal automorphisms of s. Then the vector fields

$$V_{\beta\alpha} = \frac{d}{dt}|_{t=0} \varphi_\beta^{-1} \varphi_{\beta\alpha}(t) \varphi_\alpha, \tag{5}$$

give us a cocycle in $\check{C}^1(M; \mathfrak{X}_a$.

If the $\varphi_{\beta\alpha}(t)$ correspond to an unessential deformation, then $V_{\beta\alpha} = W_\beta - W_\alpha$, where $W_\alpha \in \mathfrak{X}_a(U_\alpha)$, so the cocycle is a coboundary. Thus

$$\mathcal{D}_{ess} = \check{H}^1(M; \mathfrak{X}_a). \tag{6}$$

Example 2. If s is foliation, then the transition functions have the form:

$$\bar{x}^i = \varphi^i(x^j), \quad \bar{x}^\alpha = \varphi^\alpha(x^j, x^\beta) \tag{7}$$

and the sheaf of infinitesimal isometries of a foliation $\mathcal{F}$ consists of so-called basic vector fields of $\mathcal{F}$:

$$V^i(x^j)\partial_i + V^\alpha(x^j, x^\beta)\partial_\alpha \tag{8}$$

Second approach A section $s : M \to P_G$ can be represented by a covering U_α such that on each U_α there is given a local frame e_α and $e_\alpha = e_\beta g_{\beta\alpha}$, where $g_{\beta\alpha} \in G$. If s is an integrable G-structure, then the frames e_α are natural frames of charts on M.

The vector bundle $s^*(VP_G)$ is isomorphic to $F_G = T_1^1(M)/E_G$, where E_G are linear operators whose matrices with respect to e_α lie in $\mathfrak{g}$.

It will be convenient to see it in the following way. Let ∂_i be the frame determining the section s, and $e_i(t) = A_i^j(t)\partial_j$ be the frame which determines the deformation. Then $\frac{d}{dt}|_{t=0} e_i = \frac{d}{dt}|_{t=0} A_i^j \partial_j$, so we have linear operators $V_i^j = \frac{d}{dt}|_{t=0} A_i^j$. However, $\bar{e}_i(t) = g_i^m(t) A_m^j(t) \partial_j$ determine exactly the same deformation, and if we take the derivative we get that $\bar{V}_i^j = V_i^j + W_i^j$, where W_i^j is in E_G.

Now, if we take deformation, consisting of integrable structures, then $[e_i(t), e_j(t)] = 0$, from where we get that $\partial_{[i} V_{j]}^k = 0$ (of course, this equality is not invariant). However, if we take into account that we are working with classes and use the torsion free connection ∇ such that $\nabla s = 0$, or equivalently, the connection form takes values in $\mathfrak{g}$, then we arrive at the result that the deformation space is

$$\mathcal{D} = \ker(D : \frac{\Omega^1 M \otimes TM}{\Omega^0 M \otimes E_G} \to \frac{\Omega^2 M \otimes TM}{Alt(\Omega^1 M \otimes E_G)} \tag{9}$$

Example 3. For a foliation $\mathcal{F}$ we get that $\mathcal{D} = \{V_\alpha^i : T\mathcal{F} \to TM/T\mathcal{F} | \partial_{[\beta} V_{\alpha]}^k = 0\}$.

Now if $s_t = f_t^*(s)$, where f_t is a flow of a vector field X, is unessential deformation, then the vertical vector field $\frac{d}{dt}|_{t=0}$ can be written in terms of T_1^1/E_G as $[\partial_i X^j]$, so

$$\mathcal{D}_{ess} = \ker(D : \frac{\Omega^1 M \otimes TM}{\Omega^0 M \otimes E_G} \to \frac{\Omega^2 M \otimes TM}{Alt(\Omega^1 M \otimes E_G)})/im(D : TM \to \frac{\Omega^1 M \otimes TM}{\Omega^0 M \otimes E_G}) \tag{10}$$

Example 4. For a foliation $\mathcal{F}$ we get that $\mathcal{D}_{ess} = \{V_\alpha^i : T\mathcal{F} \to TM/T\mathcal{F} | \partial_{[\beta} V_{\alpha]}^k = 0\}/\{\partial_\alpha X^i\}$.

Relation between these approaches.

3. P-complex for the Lie derivative

For an involutive differential operator Spencer constructed a differential complex (Spencer, Pommaret).

Theorem 1. *The P-complex of the Lie derivative is isomorphic to the complex $(C^q(P), d)$, where*

$$C^q(P) = \frac{\Omega^q(M) \otimes TM}{Alt(\Omega^{q-1}(M) \otimes E_{\mathfrak{g}})},$$

and the differential $d\colon C^q(P) \to C^{q+1}(P)$ is induced by the differential operator $D = Alt \circ \nabla$, where Alt is the alternation and ∇ is the covariant derivative of a torsion-free connection adapted to Q (i. e. $(D\omega)^j_{i_1\dots i_{q+1}} = \nabla_{[i_1}\omega^j_{i_2\dots i_{q+1}]}$ with respect to local coordinates adapted to Q).

Because, the kernel of the Lie derivative is exactly the sheaf $\mathfrak{X}_a$ of infinitesimal automorphisms of the G-structure s, we get a resolution for the $\mathfrak{X}_a$.

If this resolution is fine (the Poincaré Lemma holds true), then $\check{H}^1(M; \mathfrak{X}_a) \cong H^1(C^*, d)$.

I do not know the general proof of the fact that this resolution is fine for arbitrary integrable G-structure, and possibly this is not true. However, for all the "classical" structures this resolution gives the corresponding "classical cohomology theory", and is fine.

Example 5. Let $\mathcal{F}$ be a foliation structure on a smooth manifold M, and let Δ be the corresponding integrable distribution. Then $E_{\mathfrak{g}} = \{A \in T^1_1(M) \mid A(\Delta) \subset \Delta\}$. Therefore $C^p = \Omega^p(\Delta) \otimes (TM/\Delta)$. If (x^i, x^α) are adapted local coordinates, i.e., if Δ is given by the equations $dx^i = 0$, then d can be written locally as $(d\omega)_{\alpha_1\dots\alpha_{q+1}}{}^i = \partial_{[\alpha_1}\omega_{\alpha_2\dots\alpha_{q+1}]}{}^j$. Thus we arrive at Vaisman's foliated cohomology.