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From Tracial Anomalies to Anomalies in Quantum Field Theory

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Abstract: ζ-regularized traces, resp. super-traces, are defined on a classical pseudo-differential operator A by:

$$\mathrm{tr}^Q(A) := \mathrm{f.p.}\, \mathrm{tr}(AQ^{-z})_{|z=0}, \quad \text{resp.} \quad \mathrm{str}^Q(A) := \mathrm{f.p.}\, \mathrm{str}(AQ^{-z})_{|z=0},$$

where f.p. refers to the finite part and Q is an (invertible and admissible) elliptic reference operator with positive order. They are commonly used in quantum field theory in spite of the fact that, unlike ordinary traces on matrices, they are neither cyclic nor do they commute with exterior differentiation, thus giving rise to *tracial anomalies*. The purpose of this article is to show, on two examples, how tracial anomalies can lead to anomalous phenomena in quantum field theory.

Introduction

In the path integral approach to quantum field theory, ζ-regularizations are used to make sense of partition functions as ζ-determinants. Similarly, ζ-regularization procedures are used to investigate the geometry of determinant bundles associated to families of elliptic operators [Q1, BF]. Underlying these ζ-regularizations is the idea of extracting a finite part from an a priori divergent expression, such as infinite dimensional integrals and infinite dimensional traces.

Path integration in quantum field theory often gives rise to anomalies, which we shall refer to as *quantum field anomalies*. Quantum field anomalies typically arise from the fact that some symmetry on the classical level reflected in the invariance of the classical action under some symmetry group, is not conserved on the quantum level, namely in the path integral built up from this classical action. Such anomalous phenomena can often be read off the geometry of determinant bundles (see e.g. [Fr, BF, EM, E]) associated to families of operators involved in the classical action or arising from the action of the symmetry group on the classical action. Here are a few milestones of the long story of the development of the concept of anomaly; see [Ad, BJ, Bar, GJ] for a perturbative approach, see [Fu] for a path integral approach, see [Ba, Ber, N and TJZW] for a review.

On the other hand, regularized traces of the type tr^Q (where $\mathrm{tr}^Q(\cdot) := \text{f.p.}\ \mathrm{tr}(\cdot\ Q^{-z})_{|z=0}$, Q being the weight) give rise to another type of anomaly, which we refer to here as *tracial anomalies*, such as

- the coboundary $\partial \mathrm{tr}^Q$ of the regularized trace tr^Q [M, MN, CDMP],
- the dependence on the weight measured by $\mathrm{tr}^{Q_1} - \mathrm{tr}^{Q_2}$, where Q_1 and Q_2 are two weights with the same order [CDMP, O],
- the fact that it does not commute with the exterior differentiation namely $[d, \mathrm{tr}^Q] := d \circ \mathrm{tr}^Q - \mathrm{tr}^Q \circ d \neq 0$, where Q is a family of weights parametrized by some manifold (when this manifold is one dimensional, we use instead the notation $\dot{\mathrm{tr}}^Q$) [CDMP, P1, PR].

Our first aim in this article, is to show how the use of regularized traces and determinants in the path integral approach to quantum field theory can lead to tracial anomalies, and how the latter relate to quantum field anomalies. Since tracial anomalies can be expressed in terms of Wodzicki residues [Wo], they have some local feature which is in turn reflected on the locality of anomalies in quantum field theory.

Our second aim, which is strongly linked with the first one, is to show how local terms arising in some index theorems can be seen as tracial anomalies; this indirectly leads back to some well-known relations between anomalies in quantum field theory and local terms in index theorems (see e.g. [AG, AGDPM]).

A first hint towards a relation between tracial anomalies and index type theorems is the fact –already observed in [MN]– that the index of an elliptic operator A can be interpreted as the coboundary $\partial \mathrm{tr}^Q(A, A^{-1})$ of the regularized trace tr^Q, where Q is an arbitrary weight and A^{-1} a parametrix of A. This relation between tracial anomalies and index type theorems extends to families of operators, relating this time the local term in the index theorem to variations $\dot{\mathrm{tr}}^Q$ and $[d, \mathrm{tr}^Q]$ of regularized traces. In fact, quantum field anomalies can also lie in a combination of these two types of tracial anomalies $[d, \mathrm{tr}^Q]$ and $\partial \mathrm{tr}^Q$ (which can e.g. couple to form $[\nabla, \mathrm{tr}^Q]$, where ∇ is some connection). As an illustration we shall see how

1. the local term in the Atiyah-Patodi-Singer theorem [APS II] which, for a particular family of Dirac operators, measures a *phase anomaly* of a partition function on one hand,
2. and on the other hand, the local term in the index theorem for families from which the curvature on a determinant bundle associated to a family of Dirac operators can be derived [BF], describing a *(local geometric) chiral anomaly*

can both be interpreted as tracial anomalies.

In the latter case we focus on non gravitational anomalies, restricting ourselves to the case of a determinant bundle associated to a family of twisted chiral Dirac operators acting on a fixed manifold. For gravitational anomalies, one needs to consider a fibration of manifolds instead of a fixed manifold. The curvature arises there as a combination of tracial anomalies and local terms involving the underlying geometry of the fibration of manifolds from which the determinant bundle is built. Tracial anomalies mix with the geometry of the underlying fibration of manifolds to build geometric characteristics of the determinant bundle, such as the curvature [PR], and the relation between the two types of anomalies, tracial and quantum field anomalies, is less straightforward.

Combining the relations we establish between quantum field anomalies and tracial anomalies on one hand, local terms in index theorems and tracial anomalies on the other hand, leads to the following relations corresponding to points 1 and 2 above:

1.

$$\boxed{\begin{array}{l}\textit{phase}\text{ anomaly of}\\ \text{a partition function}\end{array}} \leftrightarrow \boxed{\begin{array}{c}\textit{tracial}\text{ anomaly}\\ \int_0^1 \dot{\mathrm{tr}}^Q\end{array}} \leftrightarrow \boxed{\begin{array}{l}\text{local term in the}\\ \textit{APS index theorem}\end{array}}$$

2. and

$$\begin{array}{ccc}
\boxed{\begin{array}{l}\text{obstruction to the \textit{Wess-}}\\ \text{\textit{Zumino consistency} re-}\\ \text{lations for a covariant}\\ \text{gauge anomaly}\end{array}} & \leftrightarrow & \boxed{\begin{array}{l}\text{(pull-back on the gauge}\\ \text{Lie agebra of)}\\ \text{the \textit{curvature} on a deter-}\\ \text{minant bundle}\end{array}}\\
\updownarrow & & \updownarrow\\
\boxed{\begin{array}{c}\textit{tracial}\text{ anomalies}\\ d\mathrm{tr}^Q \text{ and } \partial\mathrm{tr}^Q\end{array}} & \leftrightarrow & \boxed{\begin{array}{l}\text{(pull-back on the gauge}\\ \text{Lie agebra of)}\\ \text{the local term of degree 2}\\ \text{in the \textit{index theorem} for}\\ \text{families}\end{array}}
\end{array}$$

In particular, these relations tell us, before even computing the various anomalies using index theorems, that these should be local, since they correspond to tracial anomalies which are local as Wodzicki residues. This approach to anomalies seen as Wodzicki residues is closely related in spirit to works by J. Mickelsson and his coworkers (see e.g. [LM, M, MR] and very recently [AM]).

Having set up these relations between tracial anomalies and anomalies in quantum field theories gives another insight on the latter type of anomaly. A natural question is to try to circumvent tracial anomalies, one way being to pick the most divergent term instead of the finite part [PR2]. Another approach inspired from the analogy with anomalies in quantum field theory, would be to introduce counterterms to compensate the tracial anomalies, just as one introduces counterterms in classical actions to compensate anomalies arising at the quantum level. A first step in this direction was made in [PR] but at this stage we are still unable set up a consistent goemetric framework which would incorporate counterterms and take care of tracial anomalies arising from taking finite parts.

The article is organized as follows. We first recall from previous works [CDMP, MN, P1] (Sect. 1) how tracial anomalies occur from taking finite parts of otherwise divergent traces. We recall in this section the relation mentioned above between the index of an operator and the coboundary of a regularized trace. We then briefly describe (Sect. 2) related anomalies such as multiplicative anomalies (first described in [KV, O] and further investigated in [Du]) of ζ-determinants and discuss what we call a *pfaffian anomaly*, namely an obstruction preventing the square of the pfaffian of an operator from coinciding with its determinant. In Sect. 3 we describe variations of η-invariants as a tracial anomaly $\int_0^1 \dot{\mathrm{tr}}^Q$, thus giving an interpretation of the local term arising in the Atiyah-Patodi-Singer theorem for families [APS I, APS II, APS III] as an integrated tracial anomaly. In Sect. 4, we discuss the geometry of determinant bundles associated to families of elliptic operators in relation to tracial anomalies in the spirit of [PR], but focussing here on the case of a determinant bundle built from a trivial fibration of manifolds, relevant for gauge theories. In Sect. 5, we illustrate the results of Sect. 4 by the example of families of signature operators in dimension 3, which give rise to a phase anomaly interpreted here as an integrated tracial anomaly. It leads, via the APS theorem, to the well-known Chern-Simon term in topological quantum field theory (TQFT). In Sect. 6, we investigate a covariant chiral gauge anomaly which can be read off the geometry of the determinant bundle associated to a family of chiral Dirac operators parametrized by connections. It differs from the consistent chiral gauge anomaly discussed in [AS] by a tracial anomaly of type $\mathrm{tr}^{Q_1} - \mathrm{tr}^{Q_2}$ which is a local expression. The pull-back on the gauge Lie algebra of the curvature of this determinant bundle can be interpreted as an obstruction to the Wess-Zumino consistency relations. Here again this obstruction arises as a tracial anomaly, but this time of the type $[d, \mathrm{tr}^Q]$. It is a local expression given by the index theorem for families.

Finally in Appendix A, we discuss the relevance of the multiplicative anomaly in the computation of the jacobian determinants corresponding to a change of variable in a gaussian path integral which underlies the computation of anomalies in quantum field theory. We refer the reader to [AM] for the interpretation of some gauge anomalies in odd dimensions in terms of the multiplicative anomaly for what we call weighted determinants, and [CZ, ECZ, EFVZ, Do] for further discussions concerning the relevance of the multiplicative anomaly for ζ-determinants in quantum field theory.

In Appendix B, following [At, Wi], for the sake of completeness, we briefly recall how the Chern-Simon term [CS] in TQFT in three dimensions can be derived from the APS theorem [APS II].

Notations. In what follows M is a smooth closed n-dimensional manifold and E a $\mathbb{Z}_2$-graded vector bundle above M (this includes ordinary bundles E which can be seen as graded bundles $E \oplus \{0\}$). $Cl(M, E)$ denotes the algebra of classical pseudo-differential operators (P.D.O.s) acting on smooth sections of E and $Ell(M, E)$, resp. $Ell^*(M, E)$, resp. $Ell^*_{ord>0}(M, E)$, resp. $Ell^{*adm}_{ord>0}(M, E)$ the set of elliptic, resp. invertible elliptic, resp. invertible elliptic with positive order, resp. invertible admissible elliptic classical pseudo-differential operators which have positive order. A *weight* is an element of $Ell^{*adm}_{ord>0}(M, E)$ often denoted by Q and with order q (in the self-adjoint case, one can drop the invertibility condition as we explain further along).

1. Weighted Trace Anomalies

Given a weight Q and A in $Cl(M, E)$, the map $z \mapsto \mathrm{tr}(AQ^{-z})$ is meromorphic at $z = 0$ with a pole of order 1 and following [CDMP] we call a *Q-weighted trace of A*, resp. *Q-weighted super-trace of A* the expression:

$$\mathrm{tr}^Q(A) := \mathrm{f.p.} \left(\mathrm{tr}(AQ^{-z})\right)_{|z=0}, \quad \text{resp.} \quad \mathrm{str}^Q(A) := \mathrm{f.p.} \left(\mathrm{str}(AQ^{-z})\right)_{|z=0}, \tag{1}$$

where f.p. means we take the finite part of the expansion at $z = 0$ of the meromorphic function $\mathrm{tr}(AQ^{-z})$, resp. $\mathrm{str}(AQ^{-z})$ and where $\mathrm{str}(\cdot) := \mathrm{tr}(\Gamma \cdot)$, Γ denoting the grading operator which can be seen as a multiplication operator acting fibrewise on the fibres of E.

Remark. The definition of a complex power Q^{-z} involves a choice of spectral cut for the admissible operator Q. In order to simplify notations we drop the explicit mention of the spectral cut in the definition of the weighted trace. In the case when Q is a positive operator, any ray in $\mathbb{C}$ different from the positive real half line serves as a ray in the spectrum of the leading symbol and an easy computation yields $\mathrm{tr}^{Q^k} = \mathrm{tr}^Q$ for any positive integer k.

We also define the *Wodzicki residue* of A:

$$\mathrm{res}(A) := \mathrm{ordQ} \cdot \mathrm{Res}_{z=0}\left(\mathrm{tr}(AQ^{-z})\right),$$

resp. the *super Wodzicki residue* of A:

$$\mathrm{sres}(A) := \mathrm{ord}Q \cdot \mathrm{Res}_{z=0}\left(\mathrm{str}(AQ^{-z})\right) = \mathrm{res}(\Gamma A),$$

where the order of the operator Q is denoted by $\mathrm{ord}Q$. Unlike weighted traces, the Wodzicki residue does not depend on the choice of Q and defines a trace on the algebra of classical P.D.Os. Another important feature of the Wodzicki residue is that it can be described as an integral of local expressions involving the symbol of the operator [Wo]:

$$\mathrm{res}(A) = \frac{1}{(2\pi)^n}\int_M \int_{|\xi|=1} \mathrm{tr}_x\left(\sigma_{-n}(x,\xi)\right) d\xi\, d\mu(x), \tag{2}$$

where n is the dimension of M, μ the volume measure on M, tr_x the trace on the fibre above x and σ_{-n} the homogeneous component of order $-n$ of symbol of the classical pseudo-differential operator A.

When Q has positive leading symbol, we can recover the ζ-regularized trace (1) using a heat-kernel expansion. Indeed, via a Mellin transformation [BGV], one can show that (see e.g. [P1]):

$$\mathrm{f.p.}\left(\mathrm{tr}(AQ^{-z})\right)_{|z=0} = \mathrm{f.p.}\left(\mathrm{tr}(Ae^{-\epsilon Q})\right)_{|\epsilon=0} - \frac{\gamma}{\mathrm{ord}Q}\cdot \mathrm{res}(A),$$

$$\text{resp.}\quad \mathrm{f.p.}\left(\mathrm{str}(AQ^{-z})\right)_{|z=0} = \mathrm{f.p.}\left(\mathrm{str}(Ae^{-\epsilon Q})\right)_{|\epsilon=0} - \frac{\gamma}{\mathrm{ord}Q}\cdot \mathrm{sres}(A),$$

where γ is the Euler constant. Thus, if $\mathrm{res}(A) = 0$, resp. $\mathrm{sres}(A) = \mathrm{res}(\Gamma A) = 0$ in the $\mathbb{Z}_2$-graded case, we find:

$$\mathrm{tr}^Q(A) = \mathrm{f.p.}\left(\mathrm{tr}(Ae^{-\epsilon Q})\right)_{|\epsilon=0},$$

$$\text{resp.}\quad \mathrm{str}^Q(A) = \mathrm{f.p.}\left(\mathrm{str}(Ae^{-\epsilon Q})\right)_{|\epsilon=0}.$$

The notion of weighted trace can be extended to the case when Q is a non-injective self-adjoint elliptic operator with positive order. Being elliptic, such an operator has a finite dimensional kernel and the orthogonal projection P_Q onto this kernel is a P.D.O. of finite rank. Hence, since Q is an elliptic operator so is the operator $Q + P_Q$, for the ellipticity is a condition on the leading symbol which remains unchanged when adding P_Q. Moreover, Q being self-adjoint the range of Q is given by $R(Q) = (\ker Q^*)^\perp = (\ker Q)^\perp$ so that $Q' := Q + P_Q$ is onto. Q' being injective and onto is invertible and being self-adjoint, and therefore admissible, it lies in $Ell^{*adm}_{ord>0}(M, E)$ (it has the same order as Q) and we can define $\mathrm{tr}^{Q'}(A)$, resp. $\mathrm{str}^{Q'}(A)$. A straightforward computation shows that:

$$\mathrm{tr}^{Q'}(A) = \mathrm{f.p.}\left(\mathrm{tr}(Ae^{-\epsilon Q})\right)_{|\epsilon=0}, \quad \text{resp.}\quad \mathrm{str}^{Q'}(A) = \mathrm{f.p.}\left(\mathrm{str}(Ae^{-\epsilon Q})\right)_{|\epsilon=0}. \tag{3}$$

We pay a price for having left out divergences when taking the finite part of otherwise diverging expressions, namely the occurrence of *weighted trace anomalies*. They will play an important role in what follows and we shall show later on how they relate to chiral (gauge) anomalies.

In order to describe weighted trace anomalies, it is useful to recall properties of logarithms of admissible elliptic operators. The logarithm of a classical P.D.O. $A \in Ell^{*adm}_{ord>0}(M, E)$ is defined by $\log A = \frac{d}{dz}_{|z=0} A^z$, and depends on the spectral cut one chooses to define the complex power A^z. Although the logarithm of a classical P.D.O. is not classical, the bracket $[\log Q, A]$ and the difference $\frac{\log Q_1}{q_1} - \frac{\log Q_2}{q_2}$ of two such logarithms are classical P.D.O.s.

A first weighted trace anomaly: The coboundary. It is by now a well known fact that, despite their name, weighted traces are not traces; given $A, B \in Cl(M, E)$ we have [M, MN, CDMP]:

$$\partial \mathrm{tr}^Q(A, B) = \mathrm{tr}^Q([A, B]) = -\frac{1}{\mathrm{ord} Q} \mathrm{res}\,(A[\log Q, B])\,, \tag{4}$$

where $\partial \mathrm{tr}^Q$ denotes the coboundary of the linear functional tr^Q on the Lie algebra $Cl(M, E)$ in the Hochschild cohomology. This coboundary corresponds to the Radul cocycle in the physics literature [R, M].

Remark. An elliptic pseudo-differential operator $A \in Cl(M, E)$ of positive order has a parametrix $A^{-1} \in Cl(M, E)$ such that

$$AA^{-1} = \mathrm{Id} - P_{A^*} \qquad \text{and} \qquad A^{-1}A = \mathrm{Id} - P_A,$$

where A^* is the formal adjoint of A and, as before, P_B denotes the orthogonal projection onto the kernel of B. Applying the weighted trace tr^Q, where Q is a weight, to the difference of pseudo-differential operators $AA^{-1} - A^{-1}A$ yields,

$$\begin{aligned} \partial \mathrm{tr}^Q(A, A^{-1}) &= \mathrm{tr}^Q(AA^{-1} - A^{-1}A) \\ &= \mathrm{tr}^Q(P_A - P_{A^*}) = \mathrm{tr}(P_A - P_{A^*}) \\ &= \mathrm{ind}(A), \end{aligned} \tag{5}$$

where in the second line we have used the fact that the weighted trace coincides with the usual trace on finite-rank operators. As already observed in [MN], this relates the index to a tracial anomaly of type (4). This is a first hint to further relations we shall establish between local terms in index type theorems and other types of tracial anomalies.

The coboundary anomaly (4) extends to weighted super-traces:

Lemma 1. *Let $A, B \in Cl(M, E)$ be two P.D.O.s and let Q be an even admissible elliptic invertible operator, all acting on sections of some super-vector bundle $E := E^+ \oplus E^-$. Then*

$$\partial \mathrm{str}^Q(A, B) = \mathrm{str}^Q(\{A, B\}) = -\frac{1}{\mathrm{ord} Q} \mathrm{sres}\,(A\{\log Q, B\})\,, \tag{6}$$

where $\{A, B\} := AB + (-1)^{|A|\cdot|B|} BA$ with $|A| = 0$, resp. $|A| = 1$ if A is even, resp. A is odd.

Proof. Writing $A := \begin{bmatrix} A_{++} & A_{+-} \\ A_{-+} & A_{--} \end{bmatrix}$, $B := \begin{bmatrix} B_{++} & B_{+-} \\ B_{-+} & B_{--} \end{bmatrix}$, one easily sees it is sufficient to check the formula for the odd operators $\begin{bmatrix} 0 & A_{+-} \\ A_{-+} & 0 \end{bmatrix}$ and $\begin{bmatrix} 0 & B_{+-} \\ B_{-+} & 0 \end{bmatrix}$, since the result for the even part follows from (4).

Let us therefore consider two odd operators $A = \begin{bmatrix} 0 & A^- \\ A^+ & 0 \end{bmatrix}$ and $B = \begin{bmatrix} 0 & B^- \\ B^+ & 0 \end{bmatrix}$ acting on sections of some super-vector bundle $E := E^+ \oplus E^-$. We have:

$$\begin{aligned}
\mathrm{str}^Q(\{A, B\}) &= \mathrm{tr}^Q(\Gamma\{A, B\}) \\
&= \mathrm{tr}^Q(\Gamma AB + \Gamma BA) \\
&= \mathrm{tr}^Q(-A^+B^- + B^-A^+ - B^+A^- + A^-B^+) \\
&= \mathrm{tr}^Q([B^-, A^+]) + \mathrm{tr}^Q([A^-, B^+]) \\
&= \frac{1}{\mathrm{ord}\, Q}\mathrm{res}(A^+[\log Q, B^-]) - \frac{1}{\mathrm{ord}\, Q}\mathrm{res}(A^-[\log Q, B^+]) \\
&\quad \text{where we have used (4)} \\
&= -\frac{1}{\mathrm{ord}\, Q}\mathrm{res}(\Gamma A^+\{\log Q, B^-\}) - \frac{1}{\mathrm{ord}\, Q}\mathrm{res}(\Gamma A^-\{\log Q, B^+\}) \\
&\quad \text{where we have used the fact that Q (and hence log Q) is even} \\
&= -\frac{1}{\mathrm{ord}\, Q}\mathrm{sres}(A\{\log Q, B\}).
\end{aligned}$$

□

A second weighted trace anomaly: The dependence on the weight. Weighted traces depend on the choice of the weight in the following way. For $Q_1, Q_2 \in Ell^{*adm}_{ord>0}(M, E)$ with orders q_1, q_2 we have [CDMP]:

$$\mathrm{tr}^{Q_1}(A) - \mathrm{tr}^{Q_2}(A) = -\mathrm{res}\left(A\left(\frac{\log Q_1}{q_1} - \frac{\log Q_2}{q_2}\right)\right). \tag{7}$$

In a similar way, for weighted supertraces we have:

$$\mathrm{str}^{Q_1}(A) - \mathrm{str}^{Q_2}(A) = -\mathrm{sres}\left(A\left(\frac{\log Q_1}{q_1} - \frac{\log Q_2}{q_2}\right)\right). \tag{8}$$

This extends to variations of traces of one parameter families of operators $\{Q_x, x \in X\}$ in $Ell^{*adm}_{ord>0}(M, E)$ with constant order q, and common spectral cut, X being some smooth manifold. For a given $A \in Cl(M, E)$ we have [CDMP, PR, P1]:

$$[d, \mathrm{tr}^Q](A) := d\,\mathrm{tr}^Q(A) = -\frac{1}{q}\mathrm{res}(A\, d\log Q), \tag{9}$$

and similarly for weighted supertraces:

$$[d, \mathrm{str}^Q](A) := d\,\mathrm{str}^Q(A) = -\frac{1}{q}\mathrm{sres}(A\, d\log Q). \tag{10}$$

Using the Fréchet Lie group structure on the set $Cl_0^*(M, E)$ of zero order invertible P.D.O.s to define $e^{tB}, t \in \mathbb{R}$ for a zero order P.D.O. B and applying (9) to $Q_t := e^{-tB}Qe^{tB}$ yields:

$$\dot{\mathrm{tr}}^{Q_t}(A) := \left[\frac{d}{dt}\right]\mathrm{tr}^{Q_t}(A) = \frac{1}{q}\mathrm{res}(A[B, \log Q]) = \partial\mathrm{tr}^Q(A, B),$$

so that the anomaly (4) can be seen as a manifestation of the anomaly (9). A similar computation would lead us from (10) to (6). Note that since the difference of two logarithms of admissible operators of same order is classical, so is the differential of the logarithm of a family of such operators.

Combining (4) and (9) yields, for a smooth family of operators $\{A_x\}_{x\in X} \subset Cl(M, E)$ parametrized by X and a connection $\nabla = d + [\theta, \cdot]$ on the trivial bundle $X \times Cl(M, E)$:

$$\begin{aligned}
[\nabla, \mathrm{tr}^Q](A) &= [d, \mathrm{tr}^Q](A) - \mathrm{tr}^Q([\theta, A]) \\
&= \frac{-1}{q}\left\{\mathrm{res}(Ad\ \log Q) + \mathrm{res}(A[\theta, \log Q])\right\} \\
&= \frac{-1}{q}\mathrm{res}(A[\nabla, \log Q]) \qquad (11)
\end{aligned}$$

and similarly for supertraces.

An important observation in view of what follows is that all these tracial anomalies (4), (7), (9), (11) (resp. (6), (8), (10)) being Wodzicki residues (resp. superresidues) of some operator, can be expressed in terms of integrals on the underlying manifold M of local expressions involving the symbols of that operator.

Terminology. Inspired by the terminology used for anomalies in quantum field theory, we shall refer to $A \mapsto [d, \mathrm{tr}^Q](A)$, $A \mapsto [d, \mathrm{str}^Q](A)$ and $A \mapsto \dot{\mathrm{tr}}^Q(A)$ as *infinitesimal trace anomalies* and to $A \mapsto \int_0^1 \dot{\mathrm{tr}}^Q(A)$ as *integrated trace anomalies*. Strictly speaking, as we shall see in the sequel, anomalies in quantum field theory arise not so much as maps $[d, \mathrm{tr}^Q]$ but rather as their value $[d, \mathrm{tr}^Q](A)$ for specific operators A; the sign of a Dirac operator in odd dimensions is one example of pseudo-differential operator A we shall come across in the expression of the phase anomaly described in Sect. 5.

Extending weighted traces to logarithms. In finite dimensions, determinants are exponentiated traces of logarithms; we extend weighted traces to logarithms of pseudo-differential operators in order to define determinants in infinite dimensions.

Given $A, Q \in Ell^{*adm}_{ord>0}(M, E)$ we set (see [KV, O, Du, L]):

$$\mathrm{tr}^Q(\log A) := \mathrm{f.p.}\left(\mathrm{tr}((\log A)Q^{-z})\right)_{|z=0}. \qquad (12)$$

As before, Q is referred to as the weight and $\mathrm{tr}^Q(\log A)$ as the Q-weighted trace of $\log A$. Underlying this definition, is a choice of a determination of the logarithm which we shall not make explicit in the notation unless it is strictly necessary.

Theorem [O] (see also [Du]). *For $Q_1, Q_2, A \in Ell^{*adm}_{ord>0}(M, E)$ with orders q_1, q_2 and a respectively,*

$$\begin{aligned}
\mathrm{tr}^{Q_1}(\log A) - \mathrm{tr}^{Q_2}(\log A) = &-\frac{1}{2}\mathrm{res}\left[\left(\log A - \frac{a}{q_1}\log Q_1\right)\left(\frac{\log Q_1}{q_1} - \frac{\log Q_2}{q_2}\right)\right] \\
&-\frac{1}{2}\mathrm{res}\left[\left(\log A - \frac{a}{q_2}\log Q_2\right)\left(\frac{\log Q_1}{q_1} - \frac{\log Q_2}{q_2}\right)\right]. \qquad (13)
\end{aligned}$$

2. From Multiplicative Anomalies for ζ-Determinants to Pfaffian Anomalies

We recall here some basic properties of ζ-determinants of admissible operators. For an admissible elliptic operator $A \in Ell^{adm}_{ord>0}(M, E)$ of positive order with non-zero eigenvalues, the function $\zeta_A(z) := \sum_{\lambda \in Spec(A)} \lambda^{-z}$ is holomorphic at $z = 0$ and we can define the ζ-determinant of A:

$$\det_\zeta(A) := \exp\left(-\zeta_A'(0)\right) = \exp \mathrm{tr}^A(\log A). \tag{14}$$

Remark. In fact physicists often consider *relative determinants*, i.e. expressions of the type

$$\frac{\det^Q(A)}{\det_\zeta(Q)} = \exp \mathrm{tr}^Q\left(\log A - \log Q\right),$$

combining a weighted determinant $\det^Q(A) := \exp \mathrm{tr}^Q(\log A)$ (a notion introduced in [Du]) with the ζ-determinant of a fixed reference operator (the weight Q here). Weighted and ζ-determinants are related by a Wodzicki residue

$$\det_\zeta(A) = \det^Q(A) \exp\left(-\frac{a}{2}\mathrm{res}\left(\frac{\log Q}{q} - \frac{\log A}{a}\right)^2\right).$$

The ζ-determinant is invariant under inner automorphisms of $Ell^*_{ord>0}(M, E)$. Indeed, let A be an operator in $Ell^{*adm}_{ord>0}(M, E)$ and let $C \in CL(M, E)$ be invertible, then CAC^{-1} lies in $Ell^*_{ord>0}(M, E)$ and is also admissible since an inner automorphism on P.D.Os induces an inner automorphism on leading symbols $\sigma_L(CAC^{-1}) = \sigma_L(C)\sigma_L(A)\sigma_L(C)^{-1}$ and hence leaves both the spectra of the operator and of its leading symbol unchanged. Given $Q \in Ell^*_{ord>0}(M, E)$ admissible, we have $\log CAC^{-1} = \log A$ and $\mathrm{tr}^{CQC^{-1}}(C \log AC^{-1}) = \mathrm{tr}^Q(\log A)$, a fact which can easily be deduced from the definition of weighted traces (see [CDMP]). It follows that:

$$\det_\zeta(CAC^{-1}) = \det_\zeta(A). \tag{15}$$

Multiplicative anomaly [KV]. Another type of anomaly which is closely related to weighted trace anomalies is the multiplicative anomaly of ζ-determinants. The Fredholm determinant is multiplicative but the ζ-determinant is not, this leading to an anomaly $F_\zeta(A, B) := \frac{\det_\zeta(AB)}{\det_\zeta(A)\det_\zeta(B)}$ which reads [KV, Du]:

$$\begin{aligned} \log F_\zeta(A, B) = {} & \frac{1}{2a}\mathrm{res}\left(\left(\log A - \frac{a}{a+b}\log(AB)\right)^2\right) \\ & + \frac{1}{2b}\mathrm{res}\left(\left(\log B - \frac{b}{a+b}\log(AB)\right)^2\right) \\ & + \mathrm{tr}^{AB}\left(\log(AB) - \log A - \log B\right) \end{aligned} \tag{16}$$

for any two operators $A, B \in Ell^{*adm}_{ord>0}(M, E)$ of order a and b, respectively. Specializing to $B = A^*$, the adjoint of A for the L^2 structure induced by a Riemannian metric on M and a Hermitian one on E, in general we have $F_\zeta(A, A^*) \neq 0$ and hence:

$$\det_\zeta(A^* A) \neq |\det_\zeta(A)|^2. \tag{17}$$

Weighted determinants are not multiplicative either and their *multiplicative anomaly* can be expressed using a Campbell-Hausdorff formula for P.D.O.s, see [O, Du], see also [AM] where such expressions are used to derive gauge anomalies in quantum field theory.

ζ-determinants for self-adjoint operators. ζ-determinants take a specific form for self-adjoint operators, which involves the η-invariant.

Let $A \in Ell^*_{ord>0}(M, E)$ be a self-adjoint elliptic (classical) pseudo-differential operator. The η-invariant first introduced by Atiyah, Patodi and Singer [APS I, APS II, APS III] is defined by:

$$\eta_A(0) := \mathrm{tr}^{|A|}(\mathrm{sgn}(A)),$$

where the classical P.D.O. $\mathrm{sgn}(A) := A|A|^{-1}$ can be seen as the sign of A. Since res (sgn A) $= 0$ [APS I], the renormalized limit f.p. $\left(\mathrm{tr}(\mathrm{sgn}A|A|^{-z})\right)_{|z=0}$ is in fact an ordinary limit so that $\eta_A(0) = \lim_{z\to 0}\left(\mathrm{tr}(\mathrm{sgn}A|A|^{-z})\right)$.

The ζ-determinant of a self-adjoint operator can be expressed in terms of the η-invariant as follows:

Proposition 1. *Let $A \in Ell^*_{ord>0}(M, E)$ be any self-adjoint elliptic pseudo-differential operator. Then*

$$\mathrm{tr}^A(\log A) = \mathrm{tr}^{|A|}(\log A) \tag{18}$$

and

$$\det_\zeta(A) = \exp \mathrm{tr}^{|A|}(\log A) = \det_\zeta |A| \cdot e^{\frac{i\pi}{2}(\eta_A(0) - \zeta_{|A|}(0))}. \tag{19}$$

We call $\phi(A) := \frac{\pi}{2}\left(\eta_A(0) - \zeta_{|A|}(0)\right)$ the phase of $\det_\zeta(A)$.

Proof. Although (19) is a well known result, we derive it here as a consequence of (7) using the language of weighted traces. Formula (18) relies on the fact (recalled above) that $\mathrm{res}(\mathrm{sgn}(A)) = 0$. Using the polar decomposition $A = |A|U = U|A|$, where $U := \mathrm{sgn}(A)$ one can write $\log A = \log|A| + \log U$ since $[|A|, U] = 0$. Applying the results of (13), we get (with a the order of A):

$$\begin{aligned}
\mathrm{tr}^A(\log A) - \mathrm{tr}^{|A|}(\log A) &= -\frac{a}{2}\mathrm{res}\left((\log U)^2\right) \\
&= a\frac{\pi^2}{8}\mathrm{res}((U-I)^2) \\
&= a\frac{\pi^2}{8}\mathrm{res}(U^2 - 2U + I) \\
&= a\frac{\pi^2}{4}\mathrm{res}(I - U) \\
&= -a\frac{\pi^2}{4}\mathrm{res}(U) = 0.
\end{aligned}$$

In the second line we used the fact that $U = \exp\left(\frac{i\pi}{2}(U - I)\right)$, as can easily be seen applying either side of the equality to eigenvectors of A. In the fourth line we used the fact that $U^2 = I$ since A is self-adjoint, and in the last line we used the fact that $\mathrm{res}(U) = 0$ as proved by Atiyah, Patodi and Singer [APS I]. From this it follows that

$$\det_\zeta(A) = \exp\left(\mathrm{tr}^{|A|}(\log A)\right) = \det_\zeta|A| e^{i\phi(A)} \tag{20}$$

with $\phi(A) = -i\mathrm{tr}^{|A|}\log_{(\frac{\pi}{2})} U = \frac{\pi}{2}\left(\eta_A(0) - \zeta_{|A|}(0)\right)$. The expression in terms of the η-invariant follows inserting $\eta_A(0) = \mathrm{tr}^{|A|}(U)$. $\square$

Remark. This proposition yields back the definition of ζ-determinants for self-adjoint operators introduced by [AS, Si] and often used in the physics literature.

In the particular case when A is (formally) self-adjoint, the anomaly expressed in (17) vanishes:

$$\det_\zeta(A^*A) = \det_\zeta(A^2) = \det_\zeta(|A|^2) = |\det_\zeta(A)|^2.$$

The last equality follows from (19) since $\eta_A(0)$ and $\zeta_{|A|}(0)$ are real.

A Pfaffian anomaly.

Definition 1. *The Pfaffian of* $A := \begin{bmatrix} 0 & -D \\ D & 0 \end{bmatrix}$ *– where* $D \in Ell^{*adm}_{ord>0}(M, E)$ *is a self-adjoint operator– is defined by:*

$$\mathrm{Pf}_\zeta(A) := \det_\zeta(D).$$

The following result points to a *Pfaffian anomaly* in this infinite dimensional setting since it shows that the determinant is not in general the square of the Pfaffian.

Theorem 1. *The square of the Pfaffian of* $A = \begin{bmatrix} 0 & -D \\ D & 0 \end{bmatrix}$ *with D self-adjoint does not in general coïncide with the determinant of A for we have:*

$$\mathrm{Pf}_\zeta(A)^2 = \det_\zeta(A) F_\zeta(D, D)^{-1} = \det_\zeta(A) e^{i\pi\left(\eta_D(0) - \zeta_{|D|}(0)\right)},$$

where $F_\zeta(A, B)$ *is the multiplicative anomaly described in* (16).

Remark. Note the fact that $e^{i\pi\left(\eta_D(0) - \zeta_{|D|}(0)\right)}$ is exactly the square of the phase of the ζ-determinant of the self-adjoint operator D described in Proposition 1.

Proof. First notice that $\log A - \log|A| = -\frac{i\pi}{2}\epsilon(iA)$, $\epsilon(iA) := \frac{iA}{|A|}$ being the sign of iA where we have cut the plane along some axis L_θ with $\frac{\pi}{2} < \theta < \frac{3\pi}{2}$. Using this relation we can compare $\det_\zeta(A)$ and $\det_\zeta(|A|)$:

$$\begin{aligned}
\log\det_\zeta(A) - \log\det_\zeta(|A|) &= \mathrm{tr}^A(\log A) - \mathrm{tr}^{|A|}(\log|A|) \\
&= \mathrm{tr}^A(\log A) - \mathrm{tr}^{|A|}(\log A) + \mathrm{tr}^{|A|}\left(\log A - \log|A|\right) \\
&= \frac{\pi^2}{8a}\mathrm{res}\left((\epsilon(iA))^2\right) - \frac{i\pi}{2}\mathrm{tr}^{|A|}\left(\epsilon(iA)\right) \\
&= \frac{\pi^2}{8a}\mathrm{res}\,(I) - \frac{i\pi}{2}\eta_{iA}(0) \\
&= -\frac{i\pi}{2}\eta_{iA}(0).
\end{aligned}$$

Let us compute $\eta_{iA}(0)$. If $\{\lambda_n, n \in \mathbb{N}\}$ denotes the spectrum of D, then the spectrum of A is given by $\{i\lambda_n, n \in \mathbb{N}\} \cup \{-i\lambda_n, n \in \mathbb{N}\}$ as can be shown considering the action of A on the orthonormal basis of eigenvectors $z_n := u_n + iv_n$, $\bar{z}_n := u_n - iv_n$, where $u_n := e_n \oplus 0$, $v_n := 0 \oplus e_n$ and $e_n, n \in \mathbb{N}$ is a basis of eigenvectors of D associated to the eigenvalues λ_n. Thus $\mathrm{tr}(A|A|^{-z-1}) = i\sum_n \lambda_n|\lambda_n|^{-z-1} - i\sum_n \lambda_n|\lambda_n|^{-z-1} = i\mathrm{tr}(D|D|^{-z-1}) + i\mathrm{tr}(-D|D|^{-z-1}) = 0$, where we have used the fact that $|A| = |D| \oplus |D|$, and hence $\eta_{iA}(0) = i\mathrm{tr}(A|A|^{-z-1})_{|z=0} = 0$. Finally we find

$$\det_\zeta(A) = \det_\zeta(|A|) = (\det_\zeta|D|)^2.$$

We are now ready to compare $\det_\zeta(|A|)$ with $\mathrm{Pf}_\zeta(A)^2$. Since the latter is $\det_\zeta(D)^2$, it differs from the former by the quotient

$$\frac{\mathrm{Pf}_\zeta(A)^2}{\det_\zeta(A)} = \frac{\left(\det_\zeta D\right)^2}{\det_\zeta(|D|)^2} = \frac{\left(\det_\zeta D\right)^2}{\det_\zeta(D^2)} = F_\zeta(D, D)^{-1},$$

where we have used the fact that $D^2 = |D|^2$ and $\det_\zeta(D^2) = \det_\zeta(|D|^2) = \left(\det_\zeta|D|\right)^2$, a relation which can easily be derived from the triviality of the multiplicative anomaly $F_\zeta(|D|, |D|)$. $\square$

3. Variations of η-Invariants as Integrated Trace Anomalies

Given two invertible self-adjoint elliptic operators A_1 and A_0, the spectral flow of a continuous family of self-adjoint elliptic operators $\{A_t, t \in [0, 1]\}$ interpolating them measures the net number of times the spectrum $\bigcup_{t\in[0,1]} \mathrm{Spec}(A_t)$ of the family $\{A_t, t \in [0, 1]\}$ crosses the zero axis [APS III]. Making this definition precise requires some care since there might well be an infinite number of crossings of the zero axis. There are different ways of defining the spectral flow, see e.g. [BLP, Me]. In the following, continuity and differentiability of families of classical pseudo-differential operators with fixed order uses the usual Fréchet topology on the set of classical pseudo-differential operators with fixed order given by a countable set of semi-norms on the symbol and on the remainder (see [Gr, KV]). Let us observe that [Me]

Lemma 2. *Given a continuous family* $\{A_t, t \in [0, 1]\}$ *of self-adjoint elliptic operators, there is a partition* $t_0 = 0 < t_1 < \cdots < t_N = 1$ *of the interval* $[0, 1]$ *and there are real numbers* $\lambda_i, i = 1, \cdots, N$, $\lambda_0 = \lambda_{N+1} = 0$ *such that the spectrum of* A_t *avoids* λ_i *for any* t *in the interval* $[t_i, t_{i+1}]$.

Proof. It follows from the discreteness of the spectrum $\mathrm{Spec}(A_t)$ of A_t that, given any $t_0 \in]0, 1[$, there is some $\lambda_0 \in \mathbb{R}$ which avoids the spectrum of A_{t_0}. For $\lambda \in \mathbb{R}$, let $U_\lambda := \{t \in]0, 1[, \lambda \notin \mathrm{Spec}(A_t)\}$. From the continuity of the family $\{A_t\}$, it follows that U_λ is an open subset of $]0, 1[$. Since U_{λ_0} contains t_0, it also contains the closure of some open interval I_{λ_0} centered at t_0. It is clear from the construction that $[0, 1] \subset \bigcup_{\lambda\in\mathbb{R}} \bar{I}_\lambda$. Since $[0, 1]$ is compact, one can extract from this covering a finite covering $I_{\lambda_i} := [t_{i-1}, t_i], i = 1, \cdots, N$, where $\lambda_0 = \lambda_{N+1} = 0$ (recall that A_0 and A_1 are invertible), $t_0 = 0 < t_1 < \cdots < t_N = 1$, such that λ_i does not belong to $\{\mathrm{Spec}(A_t), t \in [t_{i-1}, t_i]\}$. $\square$

Let $t_i, i = 0, \cdots, N, \lambda_j, j = 0, \cdots, N+1$ be as in the above lemma. The spectral flow of the family $\{A_t\}$ is defined by [Me] (formula (8.134)):

$$\mathrm{SF}(\{A_t\}) := \sum_{i=0}^{N} \sum_{\lambda\in\mathrm{Spec}(A_{t_i})\cap[\lambda_i,\lambda_{i+1}]} \mathrm{sgn}(\lambda_{i+1} - \lambda_i) m(\lambda, t_i), \tag{21}$$

where $m(\lambda, t)$ denotes the multiplicity of λ in the spectrum of A_t and $\mathrm{sgn}(\alpha)$ is $-1, 0$ or 1 as α is negative, 0 or positive. One can check that this definition is independent of the chosen partition. It also follows from the definition that if A_t is invertible for any $t \in [0, 1]$, then $\mathrm{SF}(A_t) = 0$ as expected. As a further consequence of the definition, given $\alpha \in \mathbb{R}$, then

$$\mathrm{SF}(\{A_t - \alpha\}) := \mathrm{SF}(\{A_t\}) + \mathrm{sgn}(\alpha)\left[\mathrm{tr}(P_{1,\alpha}) - \mathrm{tr}(P_{0,\alpha})\right]. \tag{22}$$

(Compare with formula (8.135) in [Me].) Here $P_{t,\alpha}$ denotes the orthogonal projection onto the finite dimensional space generated by eigenvectors of A_t with eigenvalues in $[0, \alpha]$ or $[\alpha, 0]$, according to whether α is positive or negative.

In order to relate the difference of the η-invariants $\eta_{A_1}(0) - \eta_{A_0}(0)$ to the spectral flow, we need the following

Lemma 3. *Let $\{A_x, x \in X\}$ be a smooth family of self-adjoint elliptic operators with constant positive order a parametrized by some manifold X. On an open subset $U \subset X$ chosen such that the operators A_x are invertible for all $x \in U$, the map $x \mapsto \mathrm{tr}^{|A_x|}(\mathrm{sgn}(A_x))$, where $\mathrm{sgn}(A_x) := A_x|A_x|^{-1}$, is differentiable and we have:*

$$\begin{aligned} d\left(\mathrm{tr}^{|A|}(\mathrm{sgn}(A))\right) &= [d\,, \mathrm{tr}^{|A|}](\mathrm{sgn}(A)) \\ &= -\frac{1}{a}\mathrm{res}(A^{-1}d|A|) = -\frac{1}{a}\mathrm{res}(|A|^{-1}dA), \end{aligned} \tag{23}$$

where we have set $[d, \mathrm{tr}^Q] := d \circ \mathrm{tr}^Q - \mathrm{tr}^Q \circ d$.

Proof. On one hand it follows from (9) that

$$\begin{aligned} \left[d\,, \mathrm{tr}^{|A|}\right](\mathrm{sgn}(A)) &= -\frac{1}{a}\mathrm{res}(\mathrm{sgn}(A) d \log |A|) \\ &= -\frac{1}{a}\mathrm{res}(\mathrm{sgn}(A)|A|^{-1}d|A|) \\ &= -\frac{1}{a}\mathrm{res}(A^{-1}d|A|), \end{aligned}$$

where we have used the fact that $[|A|, \mathrm{sgn}A] = 0$. On the other hand, by [APS III], Proposition (2.10), we have:

$$\begin{aligned} d\left(\mathrm{tr}^{|A|}(\mathrm{sgn}(A))\right) &= d\left(\mathrm{f.p.}\left(\mathrm{tr}(\mathrm{sgn}A|A|^{-z}\right)\right)_{|z=0} \\ &= -\mathrm{f.p.}\left(z\left(\mathrm{tr}(dA|A|^{-(z+1)}\right)\right)_{|z=0} \\ &= -\frac{1}{a}\mathrm{res}\left(dA|A|^{-1}\right). \end{aligned}$$

But by Proposition (2.11) in [APS III], the map $\mathrm{res}(\mathrm{sgn}(A))$ is constant for a continuous variation of A and hence

$$\begin{aligned} \mathrm{res}(A^{-1}d|A|) - \mathrm{res}(dA|A|^{-1}) &= d\,\mathrm{res}(A^{-1}|A|) \\ &= d\,\mathrm{res}(\mathrm{sgn}A) \\ &= 0\,, \end{aligned}$$

so that finally

$$d\left(\operatorname{tr}^{|A|}(\operatorname{sgn}(A))\right) = [d\,,\operatorname{tr}^{|A|}](\operatorname{sgn}(A)) = -\frac{1}{a}\operatorname{res}(A^{-1}d|A|) = -\frac{1}{a}\operatorname{res}(|A|^{-1}dA)$$

as claimed in the lemma. □

The following theorem relates the variation of η invariants to an integrated trace anomaly.

Theorem 2. *Let* $\{A_t, t \in [0,1]\}$ *be a smooth family of self-adjoint invertible elliptic operators with constant order in* $Cl(M,E)$*. Then*

$$\begin{aligned}\eta_{A_1}(0) - \eta_{A_0}(0) &= \int_0^1 \dot{\operatorname{tr}}^{|A_t|}(\operatorname{sgn}(A_t))dt \\ &= -\frac{1}{a}\int_0^1 \operatorname{res}\left(\dot{A}_t|A_t|^{-1}\right)dt\end{aligned}$$

which relates the difference of the η*-invariants* $\eta_{A_1}(0) - \eta_{A_0}(0)$ *to an integrated trace anomaly* $\int_0^1 \dot{\operatorname{tr}}^{|A_t|}(\operatorname{sgn}(A_t))dt$*, where we have set* $\dot{\operatorname{tr}}^{|A_t|} := \frac{d}{dt}\operatorname{tr}^{|A_t|}$*.*

Proof. Applying the first identity in (23) to a family parameterized by [0, 1] yields

$$\frac{d}{dt}\eta_{A_t}(0) = \frac{d}{dt}\left(\operatorname{tr}^{|A_t|}(\operatorname{sgn}(A_t))\right) = \dot{\operatorname{tr}}^{|A_t|}(\operatorname{sgn}(A_t))$$

and hence

$$\eta_{A_1}(0) - \eta_{A_0}(0) = \int_0^1 \frac{d}{dt}\eta_{A_t}(0)dt = \int_0^1 \dot{\operatorname{tr}}^{|A_t|}(\operatorname{sgn}(A_t))\,dt.$$

□

The following corollary of Theorem 2 is a reformulation of the Atiyah-Patodi-Singer theorem in terms of weighted trace anomalies. We derive it from Theorem 2, closely following the proof of Proposition 8.43 in [Me].

Corollary 1. *Let* A_1 *and* A_0 *be two invertible elliptic self-adjoint operators with common order a and let* $\{A_t, t \in [0,1]\}$ *be a smooth family of self-adjoint (possibly non-invertible) elliptic operators with fixed order a interpolating* A_0 *and* A_1*. Then*

$$\begin{aligned}\eta_{A_1}(0) - \eta_{A_0}(0) &= 2\mathrm{SF}(\{A_t\}) + \int_0^1 \dot{\operatorname{tr}}^{|A_t|}(\operatorname{sgn}(A_t))dt \\ &= 2\mathrm{SF}(\{A_t\}) - \frac{1}{a}\int_0^1 \operatorname{res}\left(\dot{A}_t|A_t|^{-1}\right)dt, \qquad (24)\end{aligned}$$

which relates the difference of the η*-invariants* $\eta_{A_1}(0) - \eta_{A_0}(0)$ *to the spectral flow, via an integrated trace anomaly* $\int_0^1 \dot{\operatorname{tr}}^{|A_t|}(\operatorname{sgn}(A_t))dt$ *involving* $\dot{\operatorname{tr}}^{|A_t|}(\operatorname{sgn} A_t)$*.*

Remark. The residue on the r.h.s. of (24) corresponds to the local term $\int_0^1 \dot{\eta}^c_{A_t}(0)dt$ –where $\eta^c_{A_t}$ is the "continuous" part of the η-invariant– which arises in the Atiyah-

Patodi-Singer theorem for a family of self-adjoint Dirac operators $\{A_t\}$. In other words we have the following schematic correspondence:

$$\boxed{\text{local term in the Atiyah-Patodi-Singer theorem for families}} \leftrightarrow \boxed{\text{an integrated } \textit{tracial} \text{ anomaly}}$$

Proof of the Corollary. We show that one can reduce the proof of the corollary to the case of a family of invertible operators, and then apply Theorem 2 which yields the desired formula in that case. In order to reduce the proof to the case of a family of invertible operators, let us first observe that formula (24) is invariant under a shift $A_t \mapsto A_t - \alpha$, $\alpha \in \mathbb{R}$. Let us first consider the case $\alpha \geq 0$. Since A_t has only a finite number of eigenvalues (counted with multiplicity) contained in $[0, \alpha]$, under a shift $A_t \mapsto A_t - \alpha$ its η invariant will change by minus this number of eigenvalues and we have

$$\eta_{A_t - \alpha}(0) = \eta_{A_t}(0) - \mathrm{tr}(P_{t,\alpha}),$$

where as before $P_{t,\alpha}$ denotes the orthogonal projection onto the finite dimensional space generated by eigenvectors of A_t with eigenvalues in $[0, \alpha]$ or $[\alpha, 0]$ according to the sign of α. In a similar way, for $\alpha \leq 0$ we have

$$\eta_{A_t - \alpha}(0) = \eta_{A_t}(0) + \mathrm{tr}(P_{t,\alpha}),$$

and hence for any $\alpha \in \mathbb{R}$,

$$\eta_{A_t - \alpha}(0) = \eta_{A_t}(0) + \mathrm{sgn}(\alpha)\mathrm{tr}(P_{t,\alpha}). \tag{25}$$

As a consequence, we find:

$$\eta_{A_1 - \alpha}(0) - \eta_{A_0 - \alpha}(0) = \eta_{A_1}(0) - \eta_{A_0}(0) + \mathrm{sgn}(\alpha)\left[\mathrm{tr}(P_{1,\alpha}) - \mathrm{tr}(P_{0,\alpha})\right]. \tag{26}$$

Let us now investigate how $\mathrm{res}(\dot{A}_t |A_t|^{-1})$ changes under such a shift. From Lemma 3 it follows that for $\alpha \in \mathbb{R}$ then:

$$\begin{aligned} \dot{\mathrm{tr}}^{|A_t - \alpha|}(\mathrm{sgn}(A_t - \alpha)) - \dot{\mathrm{tr}}^{|A_t|}(\mathrm{sgn}(A_t)) &= \frac{d}{dt}\left(\eta_{A_t - \alpha}(0) - \eta_{A_t}(0)\right) \\ &= -\mathrm{sgn}(\alpha)\frac{d}{dt}\mathrm{tr}(P_{t,\alpha}), \end{aligned}$$

and hence that:

$$\begin{aligned} &\int_0^1 \left[\dot{\mathrm{tr}}^{|A_t - \alpha|}(\mathrm{sgn}(A_t - \alpha)) - \dot{\mathrm{tr}}^{|A_t|}(\mathrm{sgn}(A_t))\right] dt \\ &\quad = -\mathrm{sgn}(\alpha)\left[\mathrm{tr}(P_{1,\alpha}) - \mathrm{tr}(P_{0,\alpha})\right]. \end{aligned} \tag{27}$$

Combining formulae (25), (27) and (22), giving the variation of the various ingredients of formula (24) under a shift by α, shows that a shift of the family of operators by α does not modify Eq. (24).

Using the partition of [0, 1] introduced in Lemma 3, Eq. (24) can be seen as a combination of the following equations:

$$\eta_{A_{t_i}}(0) - \eta_{A_{t_{i-1}}}(0) = 2\mathrm{SF}\left(\{A_t\}_{t \in [t_{i-1}, t_i]}\right) + \int_{t_{i-1}}^{t_i} \dot{\mathrm{tr}}^{|A_t|}(\mathrm{sgn}(A_t))dt, \quad i = 1, \cdots, N.$$

By the above preliminary remarks, it suffices to show this for any shift $A_{t_i} - \alpha$ of A_{t_i}. Since by Lemma 2 we know the existence of $\lambda_i \in \mathbb{R}, i = 1, \cdots, N$ such that $\mathrm{Spec}(A_t - \lambda_i)$ does not meet the zero axis on $[t_{i-1}, t_i]$, the proof of the theorem indeed reduces to the case when all the operators in the family $\{A_t\}$ are invertible, considered in Theorem 2. □

As we shall see later on, the Atiyah-Patodi-Singer index theorem gives an explicit description of the local term arising from the Wodzicki residue in (24) for classes of Dirac operators. As a consequence of the above discussion we have:

Corollary 2. *Let M be an odd dimensional manifold and let $\{A_t, t \in [0,1]\}$ be a smooth family of self-adjoint elliptic pseudo-differential operators of positive constant order a with vanishing spectral flow interpolating two invertible differential (more generally odd-class pseudo-differential) operators A_0, A_1. Then the difference of phases $\phi(A_1) - \phi(A_0)$ of the ζ-determinants of A_1 and A_0 can be expressed in terms of an integrated weighted trace anomaly involving $\dot{\mathrm{tr}}^{A_t}(\mathrm{sgn}(A_t))$:*

$$\begin{aligned}\phi(A_1) - \phi(A_0) &= \frac{\pi}{2}\left(\eta_{A_1}(0) - \eta_{A_0}(0)\right)\\ &= \frac{\pi}{2}\int_0^1 \dot{\mathrm{tr}}^{|A_t|}(\mathrm{sgn}(A_t))dt\\ &= -\frac{\pi}{2a}\int_0^1 \mathrm{res}(|A_t|^{-1}\dot{A}_t)dt, \end{aligned}\tag{28}$$

where we have kept the notations of Theorem 2.

Proof. The phase $\phi(A_i), i = 0, 1$ given by (see (20)) $\phi(A_i) = \frac{\pi}{2}\left(\eta_{A_i}(0) - \zeta_{|A_i|}(0)\right)$ reduce here to $\frac{\pi}{2}\eta_{A_i}(0)$ since $\zeta_{|A_i|}(0) = 0$ vanishes in odd dimensions [Si]. □

4. Determinant Bundles and Trace Anomalies

We first need to recall the construction of determinant bundles for families of elliptic operators on closed manifolds. We shall not recall it in full detail, referring the reader to [Q1, BF, BGV] for a precise description of the local trivializations involved in the construction of the determinant bundle. In order to avoid technicalities, here we only state the results at points for which the operator is invertible, which simplifies the presentation of the formulae. Let $\mathbb{M} \to X$ be a smooth (locally trivial) fibration of manifolds based on a smooth manifold X modelled on some closed Riemannian manifold M. Let $\mathbb{E}^+ \to \mathbb{M}$, resp. $\mathbb{E}^- \to \mathbb{M}$, be a Hermitian finite rank vector bundle on $\mathbb{M}$ and let $\mathcal{E}^+ \to X$, resp. $\mathcal{E}^- \to X$ be the induced infinite rank superbundle on X with fibre above x given by $\mathcal{E}_x^+ := C^\infty(M_x, E_x^+)$, resp. $\mathcal{E}_x^- := C^\infty(M_x, E_x^-)$, M_x, resp. E_x^+, resp. E_x^- being the fibre of $\mathbb{M}$, resp. of $\mathbb{E}^+$, resp. $\mathbb{E}^-$ above x.

A metric on $\mathcal{E}^+$, $\mathcal{E}^-$. The Hermitian metric on $\mathbb{E}^+$, resp. $\mathbb{E}^-$ induces a metric on $\mathcal{E}^+$, resp. $\mathcal{E}^-$:

$$\langle\sigma^+, \rho^+\rangle_x^+ = \int_{\mathbb{M}/X} \langle\sigma^+(x), \rho^+(x)\rangle_m^{E_x^+} d\mu_x(m), \tag{29}$$

resp.

$$\langle\sigma^-, \rho^-\rangle_x^- = \int_{\mathbb{M}/X} \langle\sigma^-(x), \rho^-(x)\rangle_m^{E_x^-} d\mu_x(m), \tag{30}$$

where $\mu_x(m)$ is the volume element on the fibre M_x, $\sigma^+, \rho^+ \in C^\infty(X, \mathcal{E}^+)$, resp. $\sigma^-, \rho^- \in C^\infty(X, \mathcal{E}^-)$ and $\langle\cdot,\cdot\rangle_m^+$, resp. $\langle\cdot,\cdot\rangle_m^-$ are Hermitian products on the fibres E_m^+ and E_m^-.

A connection on $\mathcal{E}^+$, $\mathcal{E}^-$. Given a horizontal distribution on $\mathbb{E}^+$, resp. $\mathbb{E}^-$, one can build a connection $\tilde{\nabla}^{\mathcal{E}^+}$, resp. $\tilde{\nabla}^{\mathcal{E}^-}$ on $\mathcal{E}^+, \mathcal{E}^-$ from a connection $\nabla^{\mathbb{E}^+}$, resp. $\nabla^{\mathbb{E}^-}$ on $\mathbb{E}^+$, resp. $\mathbb{E}^-$:

$$\left(\tilde{\nabla}_U^{\mathcal{E}^+}\sigma\right)(m) := \nabla^{E^+}_{\tilde{U}(m)}\sigma(x), \tag{31}$$

resp.

$$\left(\tilde{\nabla}_U^{\mathcal{E}^-}\sigma\right)(m) := \nabla^{E^-}_{\tilde{U}(m)}\sigma(x), \tag{32}$$

where $U \in T_x X$ and $\tilde{U}(m)$ is the horizontal lift of U at point $m \in M_x$.

This connection needs to be slightly modified to become compatible with the above metric on $\mathcal{E}^+$ and $\mathcal{E}^-$ [BGV, BF]:

$$\nabla^{\mathcal{E}^+} := \tilde{\nabla}^{\mathcal{E}^+} + \frac{1}{2}\mathrm{div}_{M_x}$$

resp.

$$\nabla^{\mathcal{E}^-} := \tilde{\nabla}^{\mathcal{E}^-} + \frac{1}{2}\mathrm{div}_{M_x},$$

where div_{M_x} is the divergence of the volume form in the direction of the base manifold X.

The Quillen determinant bundle. Let $\{A_x^+ : \mathcal{E}_x^+ \to \mathcal{E}_x^-, x \in X\}$ be a smooth family of elliptic admissible operators with constant positive order a. They yield a smooth family of Fredholm operators $\{A_x^{+,s} : H^s(M_x, E_x^+) \to H^{s-a}(M_x, E_x^-), x \in X\}$ with $s \in \mathbb{R}$. Following Quillen [Q1], to this family of Fredholm operators, we can associate a determinant bundle $\mathcal{L}_{A^+}$.

There is a metric on $\mathcal{L}_{A^+}$ called the Quillen metric [Q1] defined at a point x where A_x^+ is invertible by:

$$\|Det A^+\|_x^Q := \det_\zeta |A_x^+|, \tag{33}$$

where $Det A^+$ is a section of $\mathcal{L}_{A^+}$ and where we have set as before $|A_x^+| := \sqrt{\left(A_x^+\right)^* A_x^+}$.

A connection on the determinant bundle. Following [BF] let us now equip the determinant bundle with a connection. It arises as a natural extension of the well-known formula for the logarithmic variation of the determinant of a family of invertible elliptic operators, which we recall here and prove using the language of weighted traces.

Lemma 4. *Let $E \to M$ be a fixed Hermitian vector bundle over a fixed closed Riemannian manifold. Let $A_x \in Ell^{adm}_{ord>0}(M, E)$ be a smooth family parametrized by some smooth manifold X with a common spectral cut and constant order a. Then, at a point $x \in X$ at which A_x is invertible we have for $h \in T_x X$:*

$$d \log \det_\zeta(A)(h) = \mathrm{tr}^{A_x}(A_x^{-1} dA(h)). \tag{34}$$

Proof. Let $\{\gamma_x(t), t \in [0, t_0]\}$ be a curve on X driven by h and starting at x at time $t = 0$,

$$\begin{aligned} d\log\det_\zeta(A)(h) &= d\mathrm{tr}^A(\log A)(h) \\ &= \mathrm{tr}^{A_x}(d\log A)(h) + [d, \mathrm{tr}^A](h)(\log A_x) \\ &= \mathrm{tr}^{A_x}(A_x^{-1} dA(h)) + \lim_{t\to 0} t^{-1}\left(\mathrm{tr}^{A_{\gamma_x(t)}}(\log A_x) - \mathrm{tr}^{A_x}(\log A_x)\right) \\ &= \mathrm{tr}^{A_x}(A_x^{-1} dA(h)) - \frac{1}{2}\lim_{t\to 0} t^{-1}\mathrm{res}\left((\log A_{\gamma_x(t)} - \log A_x)^2\right) \\ &= \mathrm{tr}^{A_x}(A_x^{-1} dA(h)), \end{aligned}$$

where we have used formula (13). □

- When $\mathbb{E}^+ = \mathbb{E}^- = \mathbb{E}$, setting $\mathcal{E} := \mathcal{E}^+ = \mathcal{E}^-$ and letting $\{A_x := A_x^+, x \in X\}$ be a family of formally self-adjoint operators, the above computation gives a hint for the choice of a connection on $\mathcal{L}_A$. We define it at a point $x \in X$ where A_x is invertible by:
$$(Det A_x)^{-1}\,\nabla_U^{Det}\, Det A := \mathrm{tr}^{A_x}\left(A_x^{-1}[\nabla_U^{\mathcal{E}}, A]\right), \qquad \forall U \in T_x X. \tag{35}$$

This connection is compatible with the Quillen metric as the following lemma shows:

Lemma 5. *Let $\{A_x, x \in X\}$ be a smooth family of formally self-adjoint elliptic operators and $\mathcal{L}_A$ the associated determinant bundle on X. The connection* (35) *is compatible with the Quillen metric. Namely:*

$$\mathcal{R}e\left(\mathrm{tr}^{A_x}\left(A_x^{-1}\nabla_U^{Hom(\mathcal{E})}A\right)\right) = d_U \log \|Det A\|_Q, \qquad \forall U \in T_x X$$

at a point x where A_x is invertible. Moreover the imaginary part coincides with an infinitesimal tracial anomaly of type (11). *For $U \in T_x X$,*

$$\begin{aligned} \mathcal{I}m\left(\mathrm{tr}^{A_x}\left(A_x^{-1}\nabla_U^{Hom(\mathcal{E})}A\right)\right) &= \frac{\pi}{2}[\nabla_U, \mathrm{tr}^{|A|}](\mathrm{sgn}A - I) \\ &= -\frac{\pi}{2}\mathrm{res}\left((\mathrm{sgn}A - I)|A|^{-1}[\nabla_U, |A|]\right). \end{aligned}$$

Proof. Writing $\nabla^{\mathcal{E}} = d + \theta^{\mathcal{E}}$ locally, it follows from (34) that:

$$\begin{aligned} d\log\det_\zeta(A) &= \mathrm{tr}^A(A^{-1}dA) \\ &= \mathrm{tr}^A(A^{-1}dA) + \mathrm{tr}^A(A^{-1}[\theta^{\mathcal{E}}, A]) \\ &= \mathrm{tr}^A(A^{-1}[\nabla^{\mathcal{E}}, A]) = \mathrm{tr}^A(A^{-1}\nabla^{Hom\mathcal{E}}(A)). \end{aligned}$$

Thus, differentiating (19) yields:

$$\mathrm{tr}^A\left(A^{-1}\nabla^{Hom(\mathcal{E})}A\right) = d\log\det_\zeta|A| + \frac{i\pi}{2}d\,\left(\eta_A(0) - \zeta_{|A|}(0)\right).$$

Since $\eta_A(0) - \zeta_{|A|}(0)$ is real, the first part follows using (33) with $A^+ = A$.
As for the second part of the lemma, we have:

$$\begin{aligned} \mathcal{I}m\left(\mathrm{tr}^A\left(A^{-1}\nabla^{Hom(\mathcal{E})}A\right)\right) &= \frac{\pi}{2}d\mathrm{tr}^{|A|}(\mathrm{sgn}A - I) \\ &= \frac{\pi}{2}\left[d, \mathrm{tr}^{|A|}\right](\mathrm{sgn}A - I) \qquad \text{by formula (20)} \\ &= \frac{\pi}{2}\left[\nabla^{\mathcal{E}}, \mathrm{tr}^{|A|}\right](\mathrm{sgn}A - I), \end{aligned}$$

where we have used the fact that

$$\begin{aligned}\mathrm{tr}^{|A|}\left([\theta^{\mathcal{E}}, \mathrm{sgn}A - I]\right) &= -\frac{1}{a}\mathrm{res}([\log|A|, \theta^{\mathcal{E}}](\mathrm{sgn}A - I)]) \\ &= -\frac{1}{a}\mathrm{res}([\mathrm{sgn}A - I, \log|A|]\theta^{\mathcal{E}}) = 0.\end{aligned}$$

Here as before, $\theta^{\mathcal{E}}$ is the local one form arising in a local description of the connection $\nabla^{\mathcal{E}}$. □

- When $\mathbb{E}^+ \neq \mathbb{E}^-$, letting $\mathbb{E} := \mathbb{E}^+ \oplus \mathbb{E}^-$ be the finite rank supervector bundle built from the direct sum, and $\mathcal{E} := \mathcal{E}^+ \oplus \mathcal{E}^-$ the corresponding infinite rank supervector bundle, following Bismut and Freed [BF], we equip the bundle $\mathcal{L}_{A^+}$ with a connection whose expression is a generalization of the r.h.s. of (34) up to the fact that the weight A_x is replaced by $|A_x|$. At a point x at which A_x^+ is invertible, the Bismut-Freed connection reads, for any $U \in T_xX$,

$$\left(Det A_x^+\right)^{-1} \nabla_U^{Det} Det A^+ := \mathrm{tr}^{|A_x^+|}\left(\left(A_x^+\right)^{-1} \nabla_U^{Hom(\mathcal{E}^+,\mathcal{E}^-)} A^+\right). \tag{36}$$

Lemma 6 ([BF]). *Let $\{A_x^+, x \in X\}$ be a smooth family of elliptic operators and $\mathcal{L}_{A^+}$ the associated determinant bundle on X. The Bismut-Freed connection is compatible with the Quillen metric, namely*

$$\mathcal{R}e\left(\mathrm{tr}^{|A_x^+|}\left(\left(A_x^+\right)^{-1} \nabla_U^{Hom(\mathcal{E}^+,\mathcal{E}^-)} A^+\right)\right) = d_U \log \|Det A^+\|_Q,$$

for any $U \in T_xX$ at a point x where A_x^+ is invertible.

Proof.

$$\begin{aligned}
&2\mathcal{R}e\left(\mathrm{tr}^{|A^+|}\left(\left(A^+\right)^{-1} \nabla^{Hom(\mathcal{E}^+,\mathcal{E}^-)} A^+\right)\right) \\
&= \mathrm{tr}^{|A^+|}\left(\left(A^+\right)^{-1} \nabla^{Hom(\mathcal{E}^+,\mathcal{E}^-)} A^+\right) + \left(\mathrm{tr}^{|A^+|}\left(\left(A^+\right)^{-1} \nabla^{Hom(\mathcal{E}^+,\mathcal{E}^-)} A^+\right)^*\right) \\
&= \mathrm{tr}^{|A^+|}\left(\left(A^+\right)^{-1} \nabla^{Hom(\mathcal{E}^+,\mathcal{E}^-)} A^+\right) + \mathrm{tr}^{|A^+|}\left(\nabla^{Hom(\mathcal{E}^+,\mathcal{E}^-)} A^- \left(A^-\right)^{-1}\right) \\
&= \mathrm{tr}^{|A^+|}\left(\left(A^+\right)^{-1} \nabla^{Hom(\mathcal{E}^+,\mathcal{E}^-)} A^+\right) + \left(\mathrm{tr}^{|A^-|}\left(\left(A^-\right)^{-1} \nabla^{Hom(\mathcal{E}^+,\mathcal{E}^-)} A^-\right)\right) \\
&= \mathrm{tr}^{|A^+|}\left(\left(A^- A^+\right)^{-1} \nabla^{Hom(\mathcal{E}^+)} A^- A^+\right) \\
&= 2\mathrm{tr}^{|A^+|}\left(|A^+|^{-1} \nabla^{Hom(\mathcal{E}^+)} |A^+|\right) \\
&= 2\mathrm{tr}^{|A^+|}\left(|A^+|^{-1} d|A^+|\right) + 2\mathrm{tr}^{|A^+|}\left(|A^+|^{-1}[\theta^{\mathcal{E}^+}, |A^+|]\right) \\
&= 2\mathrm{tr}^{|A^+|}\left(|A^+|^{-1} d|A^+|\right) \\
&= 2d \log \det_\zeta |A^+|,
\end{aligned}$$

where we have set $A^- := \left(A^+\right)^*$ and written $\nabla^{\mathcal{E}^+} = d + \theta^{\mathcal{E}^+}$ locally. □

Note that one could also have equipped the bundle $\mathcal{L}_A$ with the Bismut-Freed connection in the self-adjoint case, which would amount to taking the weight $|A|$ instead of the weight A chosen in formula (35).

Lemma 7. *In the self-adjoint case, the Bismut-Freed connection 1-form*

$$(Det\,A)^{-1}\,\tilde{\nabla}^{Det}\,Det\,A := \mathrm{tr}^{|A|}\left(A^{-1}[\nabla^{\mathcal{E}}, A]\right) = d\log\det_{\zeta}(|A|)$$

is a purely real exact form given by the exterior differential of the Quillen metric.

Proof. The result follows from the fact that $\overline{\mathrm{tr}^{|A|}(B)} = \mathrm{tr}^{|A|}(B^*)$ as the following computation shows:

$$\begin{aligned}
2\mathcal{I}m\left((Det\,A)^{-1}\,\tilde{\nabla}^{Det}\,Det\,A\right) &= \mathrm{tr}^{|A|}\left(A^{-1}\nabla^{Hom(\mathcal{E})}A\right) - \overline{\mathrm{tr}^{|A|}\left(A^{-1}\nabla^{Hom(\mathcal{E})}A\right)}\\
&= \mathrm{tr}^{|A|}\left(A^{-1}\nabla^{Hom(\mathcal{E})}A\right) - \mathrm{tr}^{|A|}\left(\nabla^{Hom(\mathcal{E})}A^*\left(A^{-1}\right)^*\right)\\
&= \mathrm{tr}^{|A|}\left(A^{-1}\nabla^{Hom(\mathcal{E})}A\right) - \mathrm{tr}^{|A|}\left(A^{-1}\nabla^{Hom(\mathcal{E})}A\right)\\
&= 0.
\end{aligned}$$

□

The curvature on the determinant bundle. The following theorem relates the curvature on the determinant bundle to trace anomalies.

Theorem 3. *1. When $\mathbb{E}^+ = \mathbb{E}^- = \mathbb{E}$, setting $\mathcal{E} := \mathcal{E}^+ = \mathcal{E}^-$ and letting $\{A_x := A_x^+, x \in X\}$ be a smooth family of formally self-adjoint operators, the connection 1-form differs from an exact form by a trace anomaly of type* (7)*:*

$$\begin{aligned}
(Det\,A)^{-1}\,\nabla^{Det}\,Det\,A &= d\log\det_{\zeta}(|A|) + \left(\mathrm{tr}^{A} - \mathrm{tr}^{|A|}\right)\left(A^{-1}[\nabla^{\mathcal{E}}, A]\right)\\
&= d\log\det_{\zeta}(|A|) - \frac{1}{a}\mathrm{res}\left(A^{-1}[\nabla^{\mathcal{E}}, A](\log A - \log|A|)\right).
\end{aligned} \tag{37}$$

In particular, the curvature is a differential of a trace anomaly residue.

2. When $\mathbb{E}^+ \neq \mathbb{E}^-$, letting $\mathbb{E} := \mathbb{E}^+ \oplus \mathbb{E}^-$ and $\mathcal{E} := \mathcal{E}^+ \oplus \mathcal{E}^-$, we set $A := \begin{bmatrix} 0 & A^+ \\ A^- & 0 \end{bmatrix}$ with A^- the formal adjoint of A^+. Under the further assumption that the bundle $\mathcal{E}$ is trivial, letting $\nabla^{\mathcal{E}} := d$ be the exterior differential, the curvature on the determinant bundle reduces to a tracial anomaly. Let $x \in X$ be a point for which A_x is invertible, then for any $U, V \in T_xX$,

$$\begin{aligned}
\left(Det\,A_x^+\right)^{-1}\,\Omega^{Det}\,Det\,A^+(U,V) &= -\frac{1}{2}\partial\mathrm{str}^{Q_x}\left((A_x)^{-1}\,d_U A, (A_x)^{-1}\,d_V A\right)\\
&\quad + \frac{1}{2}[d, \mathrm{str}^{Q}]\left((A_x)^{-1}\,dA\right)(U,V).
\end{aligned} \tag{38}$$

This corresponds to a Wodzicki (super)residue by (6) *and* (10).

Remark. Equation (38) is a particular case of a more general formula obtained in [PR], where no assumption was made on the triviality of the fibration of manifolds $\mathbb{M} \to X$:

$$\begin{aligned}
\left((Det\,A_x)^{-1}\,\Omega^{Det}\,Det\,A\right)(U,V) &= -\mathrm{str}^{Q_x}(\Omega^{\mathcal{E}})(U,V)\\
&\quad - \frac{1}{2}\partial\mathrm{str}^{Q_x}\left(A_x^{-1}[\nabla_U^{\mathcal{E}}, A], A_x^{-1}[\nabla_V^{\mathcal{E}}, A]\right)\\
&\quad + \frac{1}{2}[\nabla^{\mathcal{E}}, \mathrm{str}^{Q}]\left(A_x^{-1}[\nabla^{\mathcal{E}}, A^+]\right)(U,V))
\end{aligned}$$

which yields back (38) when taking $\nabla^{\mathcal{E}} := d$. The bracket $[\nabla^{\mathcal{E}}, \mathrm{str}^Q]$ reflects the graded version of the tracial anomaly (11). The particular case under consideration here of a trivial fibration of manifolds $\mathbb{M} \to X$ is sufficient when studying gauge anomalies while the more general setting of [PR] would be necessary to investigate gravitational anomalies.

Proof.

1.

$$\begin{aligned}
&(Det A)^{-1}\,\nabla^{Det} Det A - (Det A)^{-1}\,\tilde{\nabla}^{Det} Det A \\
&= \left[\mathrm{tr}^A - \mathrm{tr}^{|A|}\right]\left(A^{-1}[\nabla^{\mathcal{E}}, A]\right) \\
&= -\frac{1}{a}\mathrm{res}\left(A^{-1}[\nabla^{\mathcal{E}}, A]\,(\log A - \log|A|)\right).
\end{aligned}$$

This combined with Lemma 7 yields (37). Differentiating on either side yields the expression of the curvature as the differential of a trace anomaly residue.

2. A straightforward computation in the spirit of that of Lemma 6 yields:

$$\left(Det A^+\right)^{-1}\nabla^{Det} Det A^+ = d\log\det{}_\zeta|A^+| + \frac{1}{2}\mathrm{str}^{Q_x}\left(A^{-1}[\nabla^{\mathcal{E}}, A]\right),$$

the weighted supertrace corresponding to the purely imaginary part of the connection, the exact form to the real part as shown in Lemma 6. Here $Q := A^2$. Specializing to $\nabla^{\mathcal{E}} = d$ in the case of a trivial bundle $\mathcal{E}$ and differentiating this expression yields:

$$\begin{aligned}
\left(Det A^+\right)^{-1}\Omega^{Det} Det A^+ &= \frac{1}{2}d\left(\mathrm{str}^Q\left(A^{-1}dA\right)\right) \\
&= \frac{1}{2}[d, \mathrm{str}^Q]\left(A^{-1}dA\right) - \frac{1}{2}\mathrm{str}^Q\left(A^{-1}dAA^{-1}dA\right).
\end{aligned}$$

Formula (38) then follows applying this formula to the vectors U and V. □

5. The Chern Simons Term as an Integrated Trace Anomaly

In this section and the next one, we specialize to the case of a *trivial* fibration $\mathbb{M} \to X$, with *constant* fibre given by a closed spin manifold M. Let $\mathbf{W} \to \mathbb{M}$ be a vector bundle with constant fibre above $(x, m) \in \mathbb{M}$ given by $\mathbf{W}_{x,m} := M \times W$, where W is an exterior vector bundle on M and let $\mathbb{E} \to \mathbb{M}$ be a Hermitian Clifford vector bundle with *constant* fibre given by a Hermitian Clifford vector bundle $\mathbb{E}_{x,m} := M \times E$, where $E = S \times W$, S being the spin bundle on M. Thus

$$\mathcal{E} \simeq X \times C^\infty(M, E) = X \times C^\infty(M, S \times W).$$

Note that in the context of gauge theory, $W = ad P$, where P is typically an $SU(N)$ (non-abelian case) or an $U(1)$ (abelian case) principal bundle on M.

We specialize here to the odd dimensional case, leaving the even dimensional case for the next section.

To a smooth family of Hermitian connections $\{\nabla_x^W, x \in X\}$ on W, we associate a smooth family of Clifford connections $\{\nabla^{L.C.} \otimes 1 + 1 \otimes \nabla_x^W, x \in X\}$, where $\nabla^{L.C.}$ is the Levi-Civita connection on M given by a Riemannian metric. These Clifford connections, combined with the Clifford multiplication c, yields a family of Dirac operators acting on smooth sections $C^\infty(M, E)$ of the Clifford module E (see e.g. [BGV, LaMi, Fr]):

$$\{D_x := c \circ (\nabla^{L.C.} \otimes 1 + 1 \otimes \nabla_x^W), x \in X\}. \tag{39}$$

Since the underlying manifold is odd-dimensional they are formally self-adjoint.

The signature operator on a 3-dimensional manifold. We apply the result of Theorem 2 and its corollary to the signature operator on an *odd dimensional* manifold M. Let ρ be a representation of the fundamental group of M on an inner product space V and let W be the vector bundle over M defined by ρ. The bundle $E := \oplus_k \Lambda^k T^*M \otimes W$ is a Clifford module for the following Clifford multiplication:

$$\begin{aligned} C^\infty(T^*M) \times C^\infty(E) &\to C^\infty(E) \\ (a, \alpha) &\mapsto c(a)\alpha = \epsilon(a) \wedge \alpha - i(a)\alpha, \end{aligned}$$

where $\epsilon(a)$ denotes exterior product, $i(a)$ interior product. It can also be equipped with a Hermitian structure coming from that on W and the natural inner product on forms induced by the Riemannian structure on M. The Clifford bundle is naturally graded by the parity on forms:

$$E := E^+ \oplus E^- = \left(\oplus_i \Lambda^{2i} T^*M \otimes W\right) \oplus \left(\oplus_i \Lambda^{2i+1} T^*M \otimes W\right).$$

Let $\Omega^k := C^\infty(\Lambda^k T^*M \otimes W)$ be the space of smooth W-valued k-forms on M. The bundle W comes with a flat (self-adjoint) connection ∇^ρ that couples with the Levi-Civita connection $\nabla^{L.C.}$ to give a (self-adjoint) connection $\nabla = \nabla^{L.C.} \otimes 1 \oplus 1 \otimes \nabla^\rho$ on E from which we can construct a Dirac operator D_∇. On the other hand, the exterior differentiation d coupled with the flat connection ∇^ρ yields an operator $d_\rho := d \otimes 1 + 1 \otimes \nabla^\rho : C^\infty(E) \to C^\infty(E)$ such that $d_\rho^2 = 0$. We henceforth assume the corresponding twisted de Rham complex $0 \to \Omega^0 \to \Omega^1 \to \cdots \to \Omega^n$ is acyclic. Identifying d_ρ with $\epsilon \circ \nabla^{L.C.}$, d_ρ^* identifies to $-i \circ \nabla^{L.C.}$, from which it easily follows that $d_\rho + d_\rho^* = (\epsilon - i) \circ \nabla^{L.C.} = c\dot{\nabla}^{L.C.}$ and hence

$$D_\nabla := c(\nabla^{L.C.} \otimes 1 + 1 \otimes \nabla^\rho) = d_\rho + d_\rho^*.$$

In the following we drop the explicit mention of the representation ρ in the notation writing d instead of d_ρ and denoting by d_k its restriction to k forms.

Note that in dimension $n = 2k+1$, the operator $*d_k$, where $*$ denotes the Hodge star operator, is a formally self-adjoint elliptic operator of order 1. We need to further restrict it in order to get an invertible operator. The complex $0 \to \Omega^0 \to \Omega^1 \to \cdots \to \Omega^n \to 0$ being acyclic, we can write $\Omega^k = \Omega_k' \oplus \Omega_k''$, where $\Omega_k' = \mathrm{Im} d_{k-1} = \ker d_k$ and $\Omega_k'' = \ker d_{k-1}^* = \mathrm{Im}\, d_{k-1}^*$. Restricting the operator $*d_k$ to Ω_k'':

$$*d_k'' := *d_k{}_{|_{\Omega_k''}}$$

yields in dimension $n = 2k + 1$, an invertible formally self-adjoint elliptic operator of order 1. In the following proposition, we first let the connection ∇^W vary, then the metric g on M vary, which give rise to two families of self-adjoint operators to which we shall apply Corollary 2 of Sect. 3.

Proposition 2. *Let M be a 3-dimensional closed Riemannian manifold. Using the above construction, with $n = 3$ ($k = 1$), one can build a smooth family of self-adjoint operators $\{D_t := *d''_{1,t}, t \in [0,1]\}$ from:*

- *a smooth family of connections $\{\nabla_t^W := \nabla_t^\rho, t \in [0,1]\}$ on W and a fixed metric on M*
- *or a smooth family of Riemannian metrics $\{g_t, t \in [0,1]\}$ (inducing a family of Levi-Civita connections) and a fixed connection ∇^W on W.*

In both cases, the phases $\phi(D_0), \phi(D_1)$ of the ζ-determinants of D_t at the end points $t = 0$ and $t = 1$, given by (20), differ by a Wodzicki residue coming from an integrated trace anomaly:

$$\begin{aligned}\phi(D_1) - \phi(D_0) &= \frac{\pi}{2}(\eta_{D_1}(0) - \eta_{D_0}(0)) \\ &= \frac{\pi}{2}\int_0^1 \dot{\mathrm{tr}}(D_t|D_t|^{-1})dt \\ &= -\frac{\pi}{2}\int_0^1 \mathrm{res}(\dot{D}_t|D_t|^{-1})dt. \end{aligned} \tag{40}$$

Remark. The local expression on the right-hand side corresponds to the local term given by the Atiyah-Patodi-Singer theorem [APS II] in terms of underlying characteristic classes as we shall see in Appendix B.

Proof. Since the signature of $M \times [0,1]$ vanishes, so does the spectral flow of the family $\{D_t, t \in [0,1]\}$, so that the assumptions of Corollary 2 are satisfied. Applying Corollary 2 yields the result. □

The Chern Simons model. Let us give an interpretation of formula (40) in the context of gauge theory as a *phase anomaly* of some partition function.

Following Witten [Wi] (see also [AdSe]), to build the Chern-Simons model in dimension $n = 2k + 1$, one starts from a classical action functional of the type $S_k(\omega_k) = \langle \omega_k, *d_k\omega_k\rangle$, which presents a degeneracy. Here $\langle\alpha,\beta\rangle = \int \alpha \wedge *\beta$ for any p-forms α and β, where $*$ is the Hodge star operator. Indeed, writing $\omega_k = \omega_k' \oplus \omega_k''$ in the above mentioned decomposition, we have $S_k(\omega_k) = S_k(\omega_k'')$. To deal with this type of degeneracy, A. Schwarz [Sc] suggested –in analogy with the Faddeev-Popov procedure– to define the partition function associated to the classical action functional S_k by the following:

Ansatz.

$$\begin{aligned} Z_k \text{ ``}:=\text{''} &\int_{\Omega^j} \mathcal{D}\omega_k e^{-\langle\omega_k, *d_k\omega_k\rangle} \\ \text{``}:=\text{''} &\left(\prod_{l=0}^{k-1}\left(\det{}_\zeta(\Delta_l'')\right)^{(-1)^{k-l+1}}\right)^{\frac{1}{2}} \int_{\Omega_k''} \mathcal{D}\omega_k'' e^{-\langle\omega_k'', *d_k\omega_k''\rangle} \\ = &\left(\prod_{l=0}^{k-1}\left(\det{}_\zeta(\Delta_l'')\right)^{(-1)^{k-l+1}}\right)^{\frac{1}{2}} \det{}_\zeta\left(*d_k''\right)^{-\frac{1}{2}}, \end{aligned}$$

where we have inserted inverted commas around identities involving heuristic objects such as $\mathcal{D}\omega_k$, which are to be understood on a heuristic level. However, the last formula is well defined since in $n = 2k+1$ dimensions the operator $*d_k$ is self-adjoint and hence has a well-defined determinant. Using Hodge duality and the fact that $|\det_\zeta(*d_l'')| = \sqrt{\det_\zeta(\Delta_l'')}$ it follows that:

$$|Z_k| = \sqrt{T(M)^{(-1)^{k+1}}},$$

where $T(M)$ is the analytic torsion of M [RS]:

$$T(M) := \prod_{j=0}^{k} \det_\zeta(\Delta_j'')^{\frac{(-1)^{j-l+1}}{2}}. \tag{41}$$

Let us comment on the notations used in this formula, in particular on the meaning of the ζ-determinants involved in the formula. Restricting the operator $\Delta_k := \Delta_{\nabla_{|_{\Omega^k}}} = d_k^* d_k + d_{k-1} d_{k-1}^*$ to Ω_k'', we get an invertible operator $\Delta_k'' := d_k^* d_{k_{|_{\Omega_k''}}}$. As the restriction to Ω_k'' of a self adjoint elliptic operator, the operator Δ_k'' has purely discrete real spectrum $\{\lambda_n'', n \in \mathbb{N}\}$ and the usual ζ-function techniques can be extended to define $\det_\zeta(\Delta_k'') := \exp\left(-\zeta'_{\Delta_k''}(0)\right)$, where $\zeta_{\Delta_k''}(s) := \sum_n \left(\lambda_n''\right)^{-s}$ see [RS].

Writing $\det_\zeta(*d_k'') = \sqrt{\det_\zeta \Delta_k''} e^{i\frac{\pi}{2}\left(\eta_{*d_k''}(0) - \zeta_{*d_k''}(0)\right)}$ as in formula (19) we find:

$$Z_k = \sqrt{T(M)^{(-1)^{k+1}}} e^{-i\frac{\pi}{4}\eta_{*d_k''}(0)}, \tag{42}$$

where we have used the fact that $\zeta_{|*d_l''|}(0) = 0$ in odd dimensions. This yields back the fact that $|Z_k| = \sqrt{T(M)^{(-1)^{k+1}}}$.

A variation of the underlying metric on M induces a variation of the partition function. The analytic torsion being a topological invariant, its modulus remains constant and it follows from Proposition 2 that the phase of the partition function changes by some local Wodzicki residue term. In [Wi] (see also [At]), Witten suggested to modify this partition function adding such local counterterms in order to build a regularized partition function independent of the metric on M. For this he proceeded in two steps, first fixing the metric and measuring the dependence of the phase on the choice of connection and then, whenever the manifold M has trivial tangent bundle, fixing the connection and measuring the dependence of the phase on the choice of metric. Both these dependences can be measured in terms of tracial anomalies along the lines of Proposition 2. Since the classical action for the abelian Chern-Simons model $\langle \omega_k'', *d_k\omega_k''\rangle = \int \omega_k'' \wedge d_k\omega_k''$ is independent of the choice of the metric, the dependence of the phase of the partition function on the metric arises as an anomaly on the quantum level, which we shall refer to as a *phase anomaly* of the partition function. By Proposition 2, the variation of the partition function $Z_k(g_0) \to Z_k(g_1)$ induced by a change of metric $g_0 \to g_1$ reads:

$$\frac{Z_k(g_1)}{Z_k(g_0)} = \exp\left(-i\frac{\pi}{4}(\eta_{*d_{k,1}}(0) - \eta_{*d_{k,0}}(0))\right),$$

where as in Proposition 2, $\{g_t, t \in [0,1]\}$ is a family of Riemannian metrics interpolating g_0 and g_1, the connection ∇^W on W being left fixed. For $k = 1$, and when the tangent bundle is trivial – in which case we can write the Levi-Civita connection $\nabla^{L.C.} = d + \omega -$

it gives rise, via the Atiyah-Patodi-Singer theorem (see Appendix B), to the familiar non-abelian Chern-Simons term $\int_M \operatorname{tr}\left(\omega \wedge d\omega + \frac{2}{3}\omega \wedge \omega \wedge \omega\right)$ arising in topological quantum field theory in dimension 3 (cf. formula (2.20) in [Wi]).

Proposition 2 thus establishes a correspondence between:

phase anomaly for the Chern-Simons partition function	$\leftrightarrow$	*tracial* anomaly $\int_0^1 \operatorname{tr}^{A_t}(\dot{\operatorname{sgn}} A_t)dt$	$\leftrightarrow$	local term in the Atiyah-Patodi-Singer *index theorem* for families

6. Chiral (Gauge) Anomalies

The *consistent* chiral gauge anomaly derived by Atiyah and Singer [AS] can be described in terms of the geometry of the determinant bundle associated to a family of chiral Dirac operators. Here we discuss a *covariant* chiral gauge anomaly in terms of the geometry on that line bundle, and show how it differs from the consistent gauge anomaly by a local term given by some tracial anomaly which is responsible for the lack of "consistency" of the covariant anomaly.

A determinant bundle on the space of connections. We consider here an *even dimensional* closed Riemannian manifold M in which case the spinor bundle S splits $S = S^+ \oplus S^-$ and the Clifford module $E = S \otimes W$ splits accordingly into $E = E^+ \oplus E^-$.

Let $X := \mathcal{C}(W)$ denote the affine space of connections on the exterior bundle W based on M. $\mathcal{C}(W)$ is an affine Fréchet space with vector space $\Omega^1(M, Hom(W))$, the space of $Hom(W)$-valued one forms on M. Concretely, this means that fixing a reference connection $\nabla_0^W \in \mathcal{C}(W)$ (e.g. the ordinary exterior differentiation if W is trivial), any other connection reads $\nabla^W = \nabla_0^W + A$, where A is a $Hom(W)$-valued one form on M. We henceforth use this reference connection to identify ∇_A^W with the 1-form A.

To the smooth family of connections $\{\nabla_A^W, A \in \mathcal{C}(W)\}$ on W, we associate a smooth family of Clifford connections $\{\nabla^{L.C.} \otimes 1 + 1 \otimes \nabla_A^W, A \in \mathcal{C}(W)\}$, which combined with the Clifford multiplication c yields a smooth family of *chiral* Dirac operators acting from $C^\infty(M, E^+)$ to $C^\infty(M, E^-)$:

$$\{D_A^+ := c(\nabla^{L.C.} \otimes 1 + 1 \otimes \nabla_A^W), A \in \mathcal{C}(W)\}. \tag{43}$$

Associated to the family $\{D_A^+, A \in \mathcal{C}(W)\}$, there is a determinant bundle $\mathcal{L}_{D^+}$ on $X = \mathcal{C}(W)$.

We set as before

$$D_A^- := \left(D_A^+\right)^*, \quad \Delta_A^+ := D_A^- D_A^+, \quad \Delta_A^- := D_A^+ D_A^-, \quad \Delta_A := \Delta_A^+ \oplus \Delta_A^-.$$

The gauge group action. The gauge group $\mathcal{G} := C^\infty(M, Aut(W))$ is a Fréchet Lie group with Lie algebra $Lie\,(\mathcal{G}) := C^\infty(M, Hom(W))$. If $W = ad P$, where $P \to M$ is a trivial principal G bundle, G the structure group, then $Lie\,(\mathcal{G}) := C^\infty(M, Lie(G))$, where $Lie(G)$ is the Lie algebra of G.

The gauge group acts on $\mathcal{C}(W)$ by:

$$\begin{aligned} \Theta : \mathcal{G} \times \mathcal{C}(W) &\to \mathcal{C}(W) \\ (g, \nabla^W) &\mapsto g^* \nabla^W \end{aligned}$$

and induces a map:

$$\begin{aligned} \theta_A : \mathcal{G} &\to \mathcal{C}(W) \\ g &\mapsto g^* \nabla_A^W. \end{aligned}$$

This map is not injective unless the connection A is irreducible.

Identifying the tangent space $T_e\mathcal{G}$ at the unit element e of $\mathcal{G}$ with the Lie algebra $Lie\,(\mathcal{G})$, the tangent map reads:

$$\begin{aligned} d_e\theta_A : Lie\,(\mathcal{G}) &\to T_A\mathcal{C}(W) \\ u &\mapsto \frac{d}{dt}_{|t=0}(g_t^* \nabla_A^W) = [\nabla_A^W, u], \end{aligned} \tag{44}$$

where $g_t := \exp tu$, exp being the exponential map on the gauge group $C^\infty(M, Aut(W))$ (which one might want to complete into a Hilbert Lie group at this stage but we shall skip these technicalities here).

The BRS (Becchi-Rouet-Stora) operator is defined by:

$$\begin{aligned} \delta : \Omega^1(\mathcal{G}, \Omega^1(M, Hom(W))) &\to \Omega^2(\mathcal{G}, \Omega^1(M, Hom(W))), \\ \alpha \otimes A &\mapsto d\alpha \otimes A - \alpha \otimes d\theta_A, \end{aligned}$$

where $A \in \Omega^1(M, Hom(W))$. It is clear from its definition that $\delta^2 = 0$ so that one can define the corresponding cohomology, called BRS cohomology. It moreover follows from the above definition that:

$$\delta A = -d\theta_A(\omega), \quad \delta\omega = -\frac{1}{2}[\omega, \omega] = -\omega \wedge \omega = -\omega^2,$$

where ω is the Maurer-Cartan form on $\mathcal{G}$, namely the left invariant $LieG$ valued one form on $\mathcal{G}$ defined by $\omega_e(v) = v$ for $v \in Lie\,(\mathcal{G})$. It is called the Faddeev-Popov ghost and written $\omega = g^{-1}dg$ in the BRS context.

The pull-back of the Bismut-Freed connection by the gauge group action. Since the vector bundles $\mathcal{E}^+$ and $\mathcal{E}^-$ on $\mathcal{C}(W)$ are trivial, we can take $\nabla^{\mathcal{E}^+} = \nabla^{\mathcal{E}^-} = d$ and equip the corresponding determinant bundle with the Bismut-Freed connection ∇^{Det} defined in (36) with d instead of $\nabla^{Hom(\mathcal{E}^+,\mathcal{E}^-)}$.

Given a connection A, the Bismut-Freed connection on the line bundle $\mathcal{L}_{D^+} \to \mathcal{C}(W)$ can be pulled back by the map θ_A to a one form on the gauge group $\mathcal{G}$:

$$\left(\theta_A^* \nabla^{Det}\right)_u = \nabla^{Det}_{\bar{U}_A}, \tag{45}$$

where $\bar{U}_A := d\theta_A u$ is the canonical vector field on $\mathcal{C}(W)$ generated by $u \in Lie\,(\mathcal{G})$. The following proposition expresses the pull-back of the Bismut-Freed connection in the direction of $u \in Lie\,(\mathcal{G})$:

Proposition 3. *Given an irreducible connection A, the pull-back $\theta_A^* \nabla^{Det}$ of the Bismut-Freed connection on the gauge group in the direction $u \in Lie\,(\mathcal{G})$ is a local expression* *which can be interpreted as a chiral gauge anomaly. Given a section* $Det D^+$ *of* $\mathcal{L}_{D^+}$ *which is invertible at a point* A*:*

$$\left(Det D_A^+\right)^{-1}\left(\theta_A^* \nabla^{Det}\right)_u Det D^+ = \mathrm{str}^{\Delta_A}(u) = (2\pi i)^{-\frac{n}{2}} \int_M \left[\hat{A}(\nabla^{L.C.})\, \mathrm{tr}_m(e^{-\Omega_A^W} u(m))\right]_{\mathrm{vol}}, \tag{46}$$

where n *is the dimension of* M*,* Ω_A^W *is the curvature of* ∇_A^W*,* tr_m *the trace on the fibre* W_m *above* m *and* $\hat{A}(\nabla^{L.C.})$ *the* $\hat{A}$*-genus on* M*.*

Remark. This anomaly $\omega(u) := \theta_A^* \nabla^{Det}(u) = \mathrm{tr}^{\Delta_A^+}\left(\left(D_A^+\right)^{-1} dD^+(\bar{U}_A)\right)$ differs from the anomaly discussed in [AS]. There the authors consider instead (see Theorem 3)

$$\bar{\omega}(u) := d \log \det_\zeta(\bar{\Delta}_A^+)(\bar{U}_A) = \mathrm{tr}^{\bar{\Delta}_A^+}\left(\left(D_A^+\right)^{-1} dD^+(\bar{U}_A)\right),$$

where we have set $\Delta_A^+ := \left(D_A^+\right)^* D_A^+$ and $\bar{\Delta}_A^+ := \left(D_{A_0}^+\right)^* D_A^+$ with $A_0 \in \mathcal{C}(W)$ and A "close enough" to A_0 so that $\bar{\Delta}_A^+$ is admissible and hence has a well-defined ζ-determinant. The two anomalies clearly differ by a tracial anomaly of the type (7), where $Q_1 = \Delta_A^+$ and $Q_2 = \bar{\Delta}_A^+$ and hence by a local expression. The above proposition gives a local expression for ω from which it therefore follows that $\bar{\omega}$ also has a local expression, a particular feature here since one does not generally expect the differential of a ζ-determinant to be local. Since $\bar{\omega}$ is closed in the cohomology on $\mathcal{G}$ ($\delta\bar{\omega}) = 0$), the consistent anomaly $\bar{\omega}$ indeed satisfies the Wess-Zumino consistency relations. The differential in the cohomology on $\mathcal{G}$ of the local term given by the (differential of the) tracial anomaly $\delta(\omega - \bar{\omega}) = \delta\omega$ measures the obstruction preventing the covariant anomaly ω from being consistent.

Proof. It follows from definition (36) that:

$$\begin{aligned}
\left(Det D_A^+\right)^{-1}\left(\theta_A^* \nabla^{Det}\right)_u Det D^+ &= \mathrm{tr}^{|D_A^+|}\left(\left(D_A^+\right)^{-1}(dD_A^+)(\bar{U}_A)\right)\\
&= \mathrm{tr}^{|D_A^+|}\left(\left(D_A^+\right)^{-1} c(d\nabla^W(\bar{U}_A))\right)\\
&= \mathrm{tr}^{|D_A^+|}\left(\left(D_A^+\right)^{-1}\left(c[\nabla_A^W, u]\right)\right)\\
&= \mathrm{tr}^{|D_A^+|}\left(\left(D_A^+\right)^{-1}[D_A^+, u]\right)\\
&= \mathrm{tr}^{|D_A^+|}\left(\left(D_A^+\right)^{-1} D_A^+ u\right) - \mathrm{tr}^{|D_A^+|}\left(\left(D_A^+\right)^{-1} u D_A^+\right)\\
&= \mathrm{tr}^{|D_A^+|}(u) - \mathrm{tr}^{\Delta_A^+}\left(\left(D_A^+\right)^{-1} u D_A^+\right)\\
&= \mathrm{tr}^{|D_A^+|}(u) - \mathrm{tr}^{\Delta_A^-}\left(D_A^+\left(D_A^+\right)^{-1} u\right)\\
&= \mathrm{tr}^{\Delta_A^+}(u) - \mathrm{tr}^{\Delta_A^-}(u)\\
&= \mathrm{str}^{\Delta_A}(u),
\end{aligned}$$

where we have used the fact that $D_A^+ \Delta_A^+ = \Delta_A^+ D_A^-$ as can easily be checked from the definition of Δ_A^+. This proves the first equality in (46). The local version of the Atiyah-

Singer theorem then yields a local expression for the term $\mathrm{str}^{\Delta_A}(u)$. Indeed it follows from results by Patodi and Gilkey that (see e.g. Theorem 4.1 in [BGV])

$$k_\epsilon(m,m) \sim (4\pi t)^{-\frac{n}{2}} \sum_{i=0}^{\infty} t^i k_i(m),$$

where $k_\epsilon(m,n)$, $m,n \in M$, is the kernel of the heat-operator $e^{-\epsilon\Delta_A}$ and $k_i \in C^\infty(M, C_{2i}(T^*M) \otimes Hom(W))$. Thus, applying $u(m)$ fibrewise, taking the trace on the fibre above m and then integrating along m we get:

$$\int_M \mathrm{str}_m(u(m)k_\epsilon(m,m)) \sim (4\pi t)^{-\frac{n}{2}} \sum_{i=0}^{\infty} t^i \int_M \mathrm{str}_m(u(m)k_i(m)), \tag{47}$$

where str_m means we have taken the supertrace along the ($\mathbb{Z}_2$-graded) fibre E_m of E above $m \in M$. On the other hand, the pointwise supertrace $\mathrm{str}_m(a \otimes b)$ of $a \otimes b \in C(T_m^*M) \otimes Hom(W_m)$ is equal to a Berezin integral (see e.g. Prop 3.21 in [BGV]):

$$\mathrm{str}_{E_m}(a \otimes b) = (-2i)^{\frac{n}{2}} \sigma_n(a(m)) \mathrm{str}_{W_m} b(m),$$

where σ is the symbol map taking Clifford elements to forms. Combining this with (47) eventually yields the local expression

$$\mathrm{str}^{\Delta_A}(u) = (2\pi i)^{-\frac{n}{2}} \int_M \left[\hat{A}(\nabla^{L.C.})\mathrm{tr}_m\left(e^{-\Omega_A^W} u(m)\right)\right]_{\mathrm{vol}},$$

after making the usual identifications with the underlying geometric data. □

When the manifold M is a $n = 2d$ dimensional unit sphere S^{2d}, the $\hat{A}$ genus is trivial and the covariant gauge anomaly reads:

$$\omega_A(u) = (2\pi i)^{-d} \int_{S^{2d}} \left[\mathrm{tr}_m\left(e^{-\Omega_A^W} u(m)\right)\right]_{\mathrm{vol}} = \frac{(-1)^d}{(2i\pi)^d d!} \int_{S^{2d}} \mathrm{tr}_m \left[\left(\Omega_A^W\right)^d u(m)\right].$$

When $n = 2$ and $M = S^2$, writing $\Omega_A = dA + A \wedge A$ we get:

$$\omega_A(u) = \frac{i}{2\pi} \int_{S^2} \mathrm{tr}_m\left((dA + A \wedge A)u(m)\right),$$

when $n = 4$ and $M = S^4$ we get:

$$\omega_A(u) = -\frac{1}{8\pi^2} \int_{S^4} \mathrm{tr}_m\left((dA + A \wedge A)^2 u(m)\right),$$

and when $n = 6$ and $M = S^6$ we get:

$$\omega_A(u) = \frac{1}{48 i\pi^3} \int_{S^6} \mathrm{tr}_m\left((dA + A \wedge A)^3 u(m)\right).$$

The pull-back $\theta_A^* \nabla^{Det}$ on the gauge group measures a covariant chiral gauge anomaly; unlike in the case of the consistent gauge anomaly investigated in [AS], there is an apriori obstruction preventing it from being consistent, namely the pull-back of the curvature $\theta_A^* \Omega^{Det}$ of the Bismut-Freed connection which measures the obstruction to the Wess-Zumino consistency relations for this gauge anomaly.

Proposition 4. *The obstruction to the* Wess-Zumino consistency relations *for the covariant gauge anomaly given by its differential on the gauge group $\delta\omega$ coincides with the pull-back $\theta_A^*\Omega^{Det}$ of the curvature of the Bismut-Freed connection. It corresponds to a Wodzicki residue arising from trace anomalies and is therefore local; it can be expressed as an integral on M of some local form:*

$$\left(Det D_A^+\right)^{-1} \theta_A^*\Omega^{Det} Det D_A^+(u,v) = (2\pi i)^{-\frac{n}{2}} \left[\int_M \hat{A}(\nabla^{L.C.})\mathrm{tr}\left(e^{-\Omega^{\mathbf{W}}}\right)\right]_{[2]} (\bar{U}_A, \bar{V}_A),$$

where $u, v \in Lie\,(\mathcal{G})$, $\bar{U}_A := d\theta_A u$, $\bar{V}_A = d\theta_A v$ and $\Omega^{\mathbf{W}}$ the curvature of the connection $\nabla^{\mathbf{W}}$ on the bundle $\mathbf{W}$.

When the manifold M is a $n = 2d$ dimensional unit sphere S^{2d}, the $\hat{A}$ genus is trivial and the obstruction to the Wess-Zumino consistency relations reads

$$\begin{aligned}
&\left(Det D_A^+\right)^{-1} \theta_A^*\Omega^{Det} Det D_A^+(u,v) \\
&\quad = \frac{(-1)^{d+1}}{(d+1)!(2\pi i)^d}\left[\int_{S^{2d}} \mathrm{tr}\left(\Omega^{\mathbf{W}}\right)^{d+1}\right]_{[2]} (\bar{U}_A, \bar{V}_A) \\
&\quad = \frac{(-1)^{d+1}}{(d+1)!(2\pi i)^d}\left[\int_{S^{2n}} \mathrm{tr}\left(d\nabla^W + \Omega^W\right)^{d+1}\right]_{[2]} (\bar{U}_A, \bar{V}_A),
\end{aligned}$$

where we have used the fact that given $V \in C^\infty(TX)$ and $Z \in C^\infty(TM)$ we have $\nabla^{\mathbf{W}}(V,Z) = dV + \nabla^W Z$, so that the curvature is given by $\Omega^{\mathbf{W}} = d\nabla^W + \Omega^W$. Setting $n = 2$, $d = 1$ on $M = S^2$ we get:

$$\left(Det D_A^+\right)^{-1} \theta_A^*\Omega^{Det} Det D_A^+(u,v)(4\pi i)^{-1}\int_{S^2} \mathrm{tr}\left(d\nabla^W \wedge d\nabla^W\right)(\bar{U}_A, \bar{V}_A).$$

Setting $n = 4$, $d = 2$, on $M = S^4$ we get:

$$\left(Det D_A^+\right)^{-1} \theta_A^*\Omega^{Det} Det D_A^+(u,v) = (8\pi^2)^{-1}\int_{S^4} \mathrm{tr}\left(d\nabla^W \wedge d\nabla^W \wedge \Omega^W\right)(\bar{U}_A, \bar{V}_A).$$

Setting $n = 6$, $d = 3$, on $M = S^6$ we get:

$$\begin{aligned}
&\left(Det D_A^+\right)^{-1} \theta_A^*\Omega^{Det} Det D_A^+(u,v) \\
&\quad = (32 i\pi^3)^{-1}\int_{S^6} \mathrm{tr}\left(d\nabla^W \wedge d\nabla^W \wedge \Omega^W \wedge \Omega^W\right)(\bar{U}_A, \bar{V}_A).
\end{aligned}$$

Proof. The curvature of the Bismut-Freed connection described in formula (38) reads:

$$\begin{aligned}
\left(Det D_A^+\right)^{-1} \Omega^{Det} Det D_A^+(U,V) = &-\frac{1}{2}\partial \mathrm{str}^{\Delta_A}\left(D_A^{-1}dD_A(U), D_A^{-1}dD_A(V)\right) \\
&+\frac{1}{2}[d, \mathrm{str}^{\Delta_A}]\left(D_A^{-1}dD_A\right)(U,V)
\end{aligned}$$

which we saw was a combination of trace anomalies; applying this to $\bar{U}_A := d\theta_A u$, $\bar{V}_A = d\theta_A v$, $u, v \in Lie\,(\mathcal{G})$ yields the fact that its pull-back can also be interpreted as a combination of trace anomalies and can therefore be expressed in terms of Wodzicki residues using the results of Sect. 1. The computation of the curvature Ω^{Det} carried out in [AS] for Dirac operators parametrized by connections and later in [BF] in the case of Dirac operators parametrized by metrics yields (taking $\nabla^{\mathcal{E}} = d$ with the notations of Sect. 4):

$$\begin{aligned}\left(Det D_A^+\right)^{-1} \Omega^{Det} Det D_A^+(U, V) &= \lim_{\epsilon \to 0} \operatorname{str}\left(e^{-(\sqrt{\epsilon} D_A + \epsilon[d, D_A])^2}\right)_{[2]} (U, V)\\ &= \left[(2\pi i)^{-\frac{n}{2}} \int_M \hat{A}(\nabla^{L.C.}) \operatorname{tr}\left(e^{-\Omega^{\mathrm{W}}}\right)\right]_{[2]} (U, V)\end{aligned}$$

thus leading to the second part of the proposition. Here $\sqrt{\epsilon} D_A + \epsilon[d, D_A]$ denotes the part of degree 1 of the family parametrized by ϵ of superconnections associated to the family D_A [Q2, BF, BGV]. $\square$

A similar result would hold for gravitational chiral anomalies described in [BF] as the curvature on a determinant bundle associated to a family of Dirac operators parametrized by metrics. The essential difference is that the geometric setting there involves a non-trivial fibration of Riemannian (spin) manifolds. Hence the vector bundles $\mathcal{E}^+$ and $\mathcal{E}^-$ are not trivial and are equipped with non-trivial connections $\nabla^{\mathcal{E}^+}, \nabla^{\mathcal{E}^-}$. As a result the curvature on the determinant bundle is a combination of a local term given by some trace anomalies and a local term arising from the underlying geometry of the fibration of manifolds $\mathbb{M}$; the tracial anomaly mixes with the underlying geometry to build a chiral anomaly.

Concluding Remark. Proposition 4 shows once again how closely related (chiral) quantum anomalies and tracial anomalies are, thus leading to the following schematic correspondence:

obstruction to the *Wess-Zumino consistency* relations for a covariant gauge anomaly	$\leftrightarrow$	(pull-back on the gauge Lie agebra of) the *curvature* on a determinant bundle
$\updownarrow$		$\updownarrow$
tracial anomalies $d\mathrm{tr}^Q$ and $\partial\mathrm{tr}^Q$	$\leftrightarrow$	(pull-back on the gauge Lie agebra of) the local term of degree 2 in the *index theorem* for families

Appendix

A. The Multiplicative Anomaly for ζ-Determinants and Anomalies in Physics

In finite dimensions, determinants naturally arise from Gaussian integration:

$$\frac{1}{(2\pi)^{\frac{n}{2}}} \int_{\mathbb{R}^n} e^{-\frac{1}{2}<Qx,x>} dx = (\det Q)^{-\frac{1}{2}},$$

where Q is a positive definite symmetric matrix, $\langle\cdot,\cdot\rangle$ the euclidean inner product on $\mathbb{R}^n$. Mimicking the finite dimensional setting, one computes Gaussian integrals in infinite dimensions substituting for the ordinary determinant, the ζ-determinant:

$$\int_{\text{configurations}\,\varphi} e^{-\frac{1}{2}<Q\varphi,\varphi>}\mathcal{D}^Q[\varphi] = (\det{}_\zeta Q)^{-\frac{1}{2}}, \tag{A.1}$$

where Q is an invertible admissible elliptic operator with positive order. The integrals on the infinite dimensional configuration space of the physical system are therefore to be understood as the r.h.s. well-defined ζ-determinant. The "volume measures" $\mathcal{D}^Q[\varphi]$– which are there to remind us that we are mimicking the finite dimensional integration procedure– can a priori depend on Q, a dependence one needs to take into account in the following.

Just as the operator Q "weights" a priori divergent traces in a way that enables us to extract a finite part, it serves here to "extract a finite part" of a priori ill-defined formal path integrals.

Let us see how this Q-dependence can affect the computations. Starting from the finite dimensional setting, let us make the change of variable $\tilde{x} = Cx$ in a gaussian integral and denote by J the corresponding jacobian determinant:

$$\begin{aligned}(\det Q)^{-\frac{1}{2}} &= \int_{\mathbb{R}^n} e^{-\frac{1}{2}<Q\tilde{x},\tilde{x}>}d\tilde{x}\\ &= \int_{\mathbb{R}^n} e^{-\frac{1}{2}<QCx,Cx>}J dx\\ &= J\cdot\det(C^*QC)^{-\frac{1}{2}}.\end{aligned}$$

Furthermore

$$J := \frac{(\det(C^*QC))^{\frac{1}{2}}}{(\det Q)^{\frac{1}{2}}} = \sqrt{\det(C^*C)} = |\det C|.$$

Similarly, replacing ordinary determinants by ζ-determinants, one could expect the modulus of the jacobian determinant of a $\tilde{\varphi} = C\varphi$ in (A.1) to correspond to a quotient of ζ-determinants. But at this point the multiplicative anomaly comes into the way.

Let C be an invertible elliptic operator (with possibly zero order), C^* its formal adjoint (with respect to an L^2 structure on the space of sections it is acting on), assuming that Q is positive (or "sufficiently close" to a positive operator [KV, Du]), then C^*QC is a positive elliptic operator (or "sufficiently close" to a positive operator) with positive order in such a way that we can define its ζ-determinant. Applying a computation similar to the finite dimensional one would yield:

$$J_Q := \frac{\det_\zeta(C^*QC)^{\frac{1}{2}}}{(\det_\zeta Q)^{\frac{1}{2}}}.$$

But this does not generally coincide with

$$\tilde{J} := \sqrt{\det_\zeta(C^*C)}.$$

In any case the latter determinant is only defined if C has non-vanishing positive order, which is not always the case in applications where C could typically be a multiplication operator. The fact that $J \neq J_Q$ is a consequence of the multiplicative anomaly for ζ-determinants recalled in (16) as the following computation shows:

$$\begin{aligned} J_Q^2 &= \frac{\det_\zeta(C^* Q C)}{\det_\zeta Q} \\ &= \frac{\det_\zeta(Q C^* C)}{\det_\zeta Q} = F_\zeta(Q, C^*C)\det_\zeta(C^*C) = F_\zeta(Q, C^*C) \cdot \tilde{J}^2. \end{aligned} \tag{A.2}$$

The second identity follows from interpolating C^*QC and QC^*C by the family $Q_t := Q^t C^* Q^{1-t} C, t \in [0, 1]$ of constant order elliptic operators which have a constant determinant since: $\frac{d}{dt} \log \det_\zeta Q_t = 0$. The third identity follows from (16).

B. Computation of the Chern-Simons Term in TQFT in Dimension 3 Using the Atiyah-Patodi-Singer Theorem

Theorem [APS II]. *Let X be an oriented Riemannian manifold of dimension $4l$ with boundary M such that X is isometric to a product $M \times I$, $I \subset \mathbb{R}$ near the boundary. Let ∇^W be a connection on the exterior bundle W based on X and $\nabla^{L.C.}$ the Levi-Civita connection on X. Let $D_\nabla := d_\nabla + d_\nabla^*$, where $d_\nabla = d \otimes 1 + 1 \otimes \nabla^W$ and $d_\nabla^* = d^* \otimes 1 + 1 \otimes \nabla^W$ as in Sect. 5, and let D_∇^+ denote the restriction of D_∇ to the even forms on X. Near the boundary,*

$$D_\nabla^+ = c \circ (\frac{d}{dt} + B^{odd}),$$

where B^{odd} is the restriction to odd forms on the boundary of the operator defined on $2p$ or $2p+1$ forms by:

$$B_\nabla = (-1)^{k+p+1}(\epsilon * d_\nabla - d_\nabla *),$$

ϵ denoting the grading operator on forms. We let the operator D_∇^+ act on sections f of the vector bundle satisfying the Atiyah-Patodi-Singer (APS) boundary condition $Pf(\cdot, 0) = 0$, where P is the spectral projection of B^{odd} corresponding to non-negative eigenvalues. Then

$$\mathrm{ind} D_\nabla^+ = \int_X L(\nabla^{L.C})\mathrm{tr}_x\left(e^{-\Omega^W}\right) + \eta_B(0),$$

where L is the Hirzebruch L polynomial, Ω^W the curvature on W, and where η_B denotes the η invariant of B_{odd}.

Let us apply this result to $X = M \times [0, 1]$, where M is an $4l - 1$ dimensional closed Riemannian manifold and let us equip X with the product metric. The boundary of X is the odd dimensional manifold $M \times \{0\} \bigcup M \times \{1\}$. With the notations of the above theorem where we set $p = k$, since k is odd, we have $B_k = *d_k - d_{n-k}*$, where B_k is the restriction of B to the odd k forms. Since $*^2 = 1$ on k forms in dimension $n = 2k+1$, we have $d_{n-k}^* = - * d_k^*$ so that the restriction B_k'' to $R(d_{k-1}^*)$ coincides with the restriction $*d_k''$.

In order to compute the r.h.s of (40) we need to compute the difference of η-invariants of B_k''. Following Atiyah, Patodi and Singer, let us first investigate the metric dependence of the eta invariants $\eta_{*d_k''}(0)$ in order to build an invariant independent on the choice of metric.

To two metrics g and g' on M correspond two operators B and B' and it follows from the Atiyah-Patodi-Singer index theorem that (see (2.3) in [APS II]):

$$\eta_B(0) - \eta_{B'}(0) = n \int_{M\times[0,1]} L(\nabla^{L.C}) \tag{B.3}$$

using the fact that $\mathrm{sign}(M \times [0, 1]) = 0$ and that the connection on W is flat.

Let us now fix the metric and take two flat connections ∇_0^W and ∇_1^W on W restricted to M, this leading again to two η invariants $\eta_{B''_{k,1}}(0)$ and $\eta_{B''_{k,0}}(0)$. From the above it follows that this expression is independent of the choice of metric (see Theorem 2.4 in [APS II]).

We now equip W restricted to M with a one parameter family of connections $\nabla_t^W := (1-t)\nabla_0^W + t\nabla_1^W$ and correspondingly a one parameter family of operators:

$$B_t = (-1)^{k+p+1}(\epsilon * d_t - d_t *).$$

We can equip W seen as a bundle over $X = [0, 1] \times M$ with the connection $\nabla^W := \frac{d}{dt} + \nabla_t^W$ and build the corresponding Dirac operator:

$$D_\nabla^+ = c \circ (\frac{d}{dt} + B_t^{odd}).$$

Because $B''_{k,1}(0) - B''_{k,0}(0)$ does not depend on the choice of metric, we can choose a flat metric. Thus the L form will be trivial. On the other hand $\mathrm{sgn}(X) = 0$ for the particular choice of manifold $X = M \times [0, 1]$ we took so that the spectral flow vanishes. Applying once again the Atiyah-Patodi-Singer theorem yields:

$$\eta_{B''_{k,1}}(0) - \eta_{B''_{k,0}}(0) = \int_{M\times[0,1]} \mathrm{tr}_x \left(e^{-\Omega^W}\right). \tag{B.4}$$

Combining (B.3) and (B.4) where the Levi-Civita connection reads $d + \omega$ and the connection on W reads $\nabla^W = d + A$ (provided both the tangent bundle and the bundle E are trivial) yields the expression of the Chern-Simons term computed by Witten (see formula (2.23) in [Wi]).

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