ζ-FUNCTION REGULARIZATION AND INDEX THEORY IN NONCOMMUTATIVE GEOMETRY

ALEXANDER CARDONA *
Mathematics Department
Faculty of Science and Technology
Keio University
3-14-1, Hiyoshi, Kohoku-ku
Yokohama, 223-8522, Japan.

In this paper we review the ζ-function regularization approach to noncommutative index theory. In particular, we show how, through the use of a suitable generalization of ζ-function regularized quantities (as the weighted traces used in[4,5,19]) it is possible to build the basic blocks used to compute the local index formula due to Connes and Moscovici[8] (and revisited by Higson[12]) in noncommutative geometry.

Introduction

The index map associated to a spectral triple $(\mathcal{A}, \mathcal{H}, D)$ in Noncommutative Geometry[6,8,12] is the additive map ind $_D : K_*(\mathcal{A}) \to \mathbb{Z}$ defined as follows. Let us assume that the spectral triple is *even*, which means that there exists a chirality operator $\gamma \in \mathcal{L}(\mathcal{H})$ inducing a $\mathbb{Z}_2$-grading on $\mathcal{H}$ and anticommuting with D, i.e. $\gamma^2 = 1$, $\gamma = \gamma^*$, $\gamma D = -D\gamma$ and $\gamma a = a\gamma$, for all $a \in \mathcal{A}$. With respect to the decomposition $\mathcal{H} = \mathcal{H}^+ \oplus \mathcal{H}^-$, D takes the antidiagonal form

$$D = \begin{pmatrix} 0 & D^- \\ D^+ & 0 \end{pmatrix},$$

with $D^\pm : \mathcal{H}^\mp \to \mathcal{H}^\pm$. For any selfadjoint idempotent $e \in M_q(\mathcal{A})$ (the algebra of $q \times q$ matrices with entries in $\mathcal{A}$), the operator

$$e(D^+ \otimes 1)e : e(\mathcal{H}^+ \otimes \mathbb{C}^q) \to \mathrm{e}(\mathcal{H}^- \otimes \mathbb{C}^q)$$

*Present address: Mathematics Department, Universidad de Los Andes, A.A. 4976, Bogotá, Colombia. E-mail: acardona@uniandes.edu.co.

is Fredholm, and its index depends only on the homotopy class of e. The index is defined by

$$\text{ind}_D[e] = \text{ind}\, e(D^+ \otimes 1)e. \tag{1}$$

It is well known that, e.g. in the case of a compact Riemannian spin manifold (M, g) of even dimension, with D_M the Dirac operator acting on its spin bundle $S = S^+ \oplus S^-$, the index corresponding to the even spectral triple $(C^\infty(M), L^2(M, S), D_M)$ coincides with the classical index. In this case, as shown in the sixties by Atiyah and Singer (see the volumes[1]), the index can be computed through a local formula.

This *locality* property of the index generalizes to the noncommutative case, and the corresponding local index formula was shown in [8] by Connes and Moscovici. Indeed, (under some –spectral– conditions on the spectral triple) the index map (1) can be computed through the pairing of the K-theory $K(\mathcal{A})$ of the algebra with a cyclic cohomology class defined by Connes.[6] The local index theorem states that there exists a cyclic cohomology class $[\varphi_{CM}]$ such that

$$\text{ind}_D(E) = \langle [\varphi_{CM}] | E \rangle, \tag{2}$$

for any $E \in K_0(\mathcal{A})$, and the components of such an even cocycle $\varphi_{CM} = (\varphi_{2p})$ —in the (b, B)-complex of $\mathcal{A}$— are given by local formulas, i.e. by (noncommutative) integrals which generalize the ones found in classical (e.g. Atiyah-Singer) index theorems.

Another feature of classical indices is that it is possible to compute them using ζ-function regularized quantities[2], a method extensively used to define other important invariants of differential manifolds. On the other hand, it is possible to study both algebraic and geometric anomalies appearing in infinite-dimensional geometry through the use of, e.g. trace extensions defined using this kind of regularized quantities (see [17,4]). This approach brings a direct proof of the relation between such anomalies and index-like quantities, as well as a bridge to understand how these quantities appear in the study of related phenomena such as Quantum Field Theory anomalies[5]. In noncommutative geometry [6,10] ζ-type functions are also used to state what would correspond to some classical definitions in the noncommutative case, e.g. what would be the *dimension* of a spectral triple.

Our goal in the following pages is to show how, through the use of a suitable generalization of ζ-function regularized quantities (as the used in[4],[5],[17],[19]) it is possible to build the basic blocks used to compute the local index formula in noncommutative geometry. We follow similar lines to those of Higson[12] and Paycha[18], reviewing some of the results stated in these works. Our approach contains also some common ideas with other works employing regularization by heat-kernel methods, as the work of Ponge[20] on local index formulas. Finally, throughout this paper we consider only *even* spectral triples, although everything applies to the odd case by the usual modifications (see e.g. [6], [8]).

1. Spectral Triples and ζ-functions

1.1. *The weighted algebra of spectral triples*

Let $(\mathcal{A}, \mathcal{H}, D)$ be a spectral triple, i.e. an involutive algebra $\mathcal{A}$ represented in a Hilbert space $\mathcal{H}$, together with a self-adjoint operator D with compact resolvent in $\mathcal{H}$ such that $[D, a]$ is bounded for any $a \in \mathcal{A}$. Given a fixed positive unbounded self-adjoint operator Q acting on $\mathcal{H}$ —which we refer to as the *weight*— such that

$$\mathcal{H}^\infty = \bigcap_{k=1}^{\infty} \mathsf{Dom}\,(Q^k) \subseteq \mathcal{H},$$

and any $a \in \mathcal{A}$ maps $\mathcal{H}^\infty$ into itself, we want to associate to Q and $(\mathcal{A}, \mathcal{H}, D)$ an algebra of operators on $\mathcal{H}^\infty$.

We say that $\mathcal{D}_Q$ is a *q-order weighted algebra of abstract differential operators*, and Q its associated *weight* of order q (see [12] and [18]), if it is an associative algebra of linear operators on $\mathcal{H}^\infty$ which is

(1) *Filtered*, i.e.

$$\mathcal{D}_Q = \bigcup_{p=0}^{\infty} \mathcal{D}_Q^p,$$

and we say $\mathsf{ord}(P) \leq p$ if $P \in \mathcal{D}_Q^p$.

(2) *Closed under commutation with the weight*, i.e. for any $P \in \mathcal{D}_Q^p$,

$$[P, Q] \in \mathcal{D}_Q,$$

and $\mathsf{ord}([P, Q]) \leq p + q - 1$.

(3) *Regular*,[11] i.e. it satisfies the following 'elliptic estimate': For any $P \in \mathcal{D}_Q^p$ there exists a positive constant ϵ such that

$$\| Q^{\frac{p}{q}} X \| + \| X \| \geq \epsilon \| PX \|, \qquad \forall X \in \mathcal{H}^\infty .$$

Notice that, as follows from the definition, if $P \in \mathcal{D}_Q^p$, for any $s \geq 0$ then P extends to a bounded operator from $\mathcal{H}^{s+p}$ to $\mathcal{H}^s$.

As shown by Higson (Section 4 in [12], see also [13]), for any regular spectral triple $(\mathcal{A}, \mathcal{H}, D)$ as above, taking as weight $Q = \Delta = D^2$, it is possible to associate a $2\,\mathsf{ord}(D)$-weighted algebra $\mathcal{D}_\Delta$: the smallest algebra of operators on $\mathcal{H}^\infty$ containing $\mathcal{A}$, the commutator $[\mathcal{A}, D]$ which is closed under commutation with the weight $P \mapsto [Q, P]$. The filtration is, in this case, the one in which elements of $\mathcal{A}$ and $[\mathcal{A}, D]$ have order zero, and the order is raised by (at most) one by commutation with the weight D^2 —it is assumed that the degree of D is 1. An immediate example is the one of the spectral triple $(C^\infty(M), L^2(M, S), D_M)$ associated to a compact Riemannian manifold (M, g), taking $\mathcal{D}_\Delta$ to be the algebra of differential operators acting on spinors, weighted by the Laplacian $\Delta = D_M^2$.

The role of the weight in the context of noncommutative spectral triples will be, as we will see in the next section, the one of parameter for the regularization of the quantities in terms of which the local formula for the index can be written. Indeed, the preliminary definitions introduced in this section are there to ensure that we can define the ζ-functions we will work with in the following.

1.2. *ζ-function regularization, traces and dimension spectrum*

In classical (commutative) geometry the ζ-function regularization method gives rise to important invariants of differential manifolds. It is also used, among other things, for the construction of a —unique in many cases— trace on the algebra of pseudo-differential operators acting on sections of vector fibrations. In this section we recall some of these classical results, as well as the definition of the dimension spectrum of a spectral triple given in [8] using these methods.

1.2.1. *ζ-function regularization and the noncommutative residue*

Let E be a vector bundle over a smooth n-dimensional closed Riemannian manifold M, and let $\mathcal{C}l(E)$ denote the algebra of classical pseudo-differential operators acting on smooth sections of E. Let $Q \in \mathcal{A}d(E)$ be an *admissible operator* —in the sense of references [16] and [5], i.e. a pseudo-differential operator for which arbitrary complex powers can be defined as it was by Seeley[21],

$$Q^{-z} = \frac{1}{2\pi i} \int_\Gamma \lambda^{-z}(Q - \lambda I)^{-1} d\lambda, \tag{3}$$

where Γ denotes the contour of integration, coming from infinity and separating zero from the spectrum of Q. Since, for any $A \in \mathcal{C}l(E)$, $Q \in \mathcal{A}d(E)$, the map $z \mapsto \mathrm{tr}(AQ^{-z})$ extends meromorphically to a map with a simple pole at zero,[16]

$$\mathsf{res}(A) = q \ \mathrm{Res}_{z=0} \left(\mathrm{tr}(AQ^{-z})\right), \tag{4}$$

where $q = \mathsf{ord}(Q)$, defines a quantity which is independent of the reference operator Q. More importantly, this quantity indeed defines a trace (i.e. res([A, B])=0 for any $A, B \in \mathcal{C}l(E)$), the so-called *non-commutative* or *Wodzicki residue*[22] (see also [15] for a review). It also has the remarkable locality property

$$\mathsf{res}(A) = \frac{1}{(2\pi)^n} \int_M \int_{|\xi|=1} \mathrm{tr}_x \left(\sigma_{-n}(x, \xi)\right) d\xi d\mu_M(x), \tag{5}$$

where n is the dimension of M, μ_M the volume measure on M, tr_x the trace on the fibre above x and σ_{-n} the homogeneous component of order $-n$ of the symbol of A. Notice that the Wodzicki residue is *not* an extension of the trace in finite dimensions, as follows from the fact that it vanishes on any finite-rank operator.

Remark 1.1. The meromorphic extension of the map $z \mapsto \mathrm{tr}(AQ^{-z})$ has simple poles located in the set[16] $\left\{\frac{a+m-k}{q}; k \in \mathbb{Z}^+\right\}$, where $a = \mathsf{ord}(A)$, $q = \mathsf{ord}(Q)$ and $m = \dim M$. Notice then that, if $a < -m$ there is no pole at zero, so that $\mathsf{res}(A) = 0$.

1.2.2. *Weighted traces*

The same ζ-function regularization employed to define the Wodzicki residue has been used to define functionals which, although non-tracial, extend the usual trace in finite dimensions[a] (see [4],[5],[17]). Indeed, starting from the same map $z \mapsto \mathrm{tr}(AQ^{-z})$, for A classical pseudo-differential operator and Q admissible, but considering its finite part instead of its residue at the origin, gives rise to *weighted trace functionals* [4] which, although non-tracial and dependent of the reference operator Q, are very useful tools to understand the origin of index-like terms and their appearance in Quantum Field Theory anomalies, which have well known locality features in relation with classical (commutative) index theorems.[5] For $A \in \mathcal{C}l(E)$ and $Q \in \mathcal{A}d(E)$, the Q-weighted trace of A is defined by the expression

$$\mathrm{tr}^Q(A) = \mathrm{f.p.}|_{z=0}\left(\mathrm{tr}(AQ^{-z})\right). \tag{6}$$

It follows from the definition that weighted traces extend usual finite-dimensional traces, i.e. $\mathrm{tr}^Q(A) = \mathrm{tr}(A)$ whenever A is a finite rank operator. Formulae for anomalies associated with weighted trace functionals can be found in references [4] and [5], where their relationship with index theory and field theory anomalies are also discussed.

Remark 1.2. Another very important regularization method used in infinite-dimensional geometry is the *heat kernel method*, in which the map $\mathrm{tr}(e^{-tQ})$, for t real and positive, is used instead of the complex map $\mathrm{tr}(Q^{-z})$ considered above, where Q denotes any admissible reference operator with positive leading symbol. Both methods coincide modulo noncommutative residues. Indeed, notice that, for $A \in \mathcal{C}l(E)$ and an admissible operator Q with positive leading symbol, we can recover the ζ-regularized trace (6) using a heat-kernel expansion, through a *Mellin Transform*. The zeta regularized trace $\mathrm{tr}(AQ^{-z})$ is the Mellin transform $\mathrm{M}[f](z) = \frac{1}{\Gamma(z)}\int_0^\infty f(t)t^{z-1}\,dt$ of the map $f(t) = \mathrm{tr}(Ae^{-tQ}) \in C^\infty(\mathbb{R}^+)$. Since $f(t)$ has an asymptotic expansion for small t of the form $f(t) \sim \sum_{k\geq -n} f_k t^{\frac{k}{q}} + c\log t$, where $q = \mathsf{ord}(Q)$, properties of the Mellin transform [10] imply that

$$\mathrm{f.p.}|_{t=0}\left(\mathrm{tr}(Ae^{-tQ})\right) = \mathrm{f.p.}|_{z=0}\left(\mathrm{tr}(AQ^{-z})\right) + \gamma_E \cdot \mathsf{res}(A) \tag{7}$$

where γ_E is the Euler constant. Thus, if $\mathsf{res}(A) = 0$, the two regularization methods coincide, i.e. $\mathrm{tr}^Q(A) = \mathrm{f.p.}|_{t=0}\left(\mathrm{tr}(Ae^{-tQ})\right)$.

[a] Traces and trace extensions are not the only objects it is possible to build by this method. There are also, among others, the ζ-determinants giving rise to analytic torsion and the eta invariant of Atiyah, Patodi and Singer.

In the more general context of noncommutative spectral triples there is a trace (defined by Dixmier[9] in the 60's on certain ideals of linear operators acting on a Hilbert space) which has been extensively used by Alain Connes as noncommutative integral. [6] Indeed, it coincides with the Wodzicki residue on the algebra of classical pseudodifferential operators acting on smooth sections of a vector bundle over a Riemannian manifold, so that in that case it is given by an integral. The index formulas arising in Connes-Moscovici theorem are local in the sense that they can be written using noncommutative integrals (Dixmier traces), we want to illustrate in the following how they can also be build from ζ-function regularized quantities.

1.2.3. *ζ-functions, analytic dimension and the dimension spectrum*

Let $(\mathcal{A}, \mathcal{H}, D)$ be a spectral triple such that $\mathcal{H}^\infty = \bigcap_{k=1}^{\infty} \mathsf{Dom}\,(\Delta^k) \subseteq \mathcal{H}$, and any $a \in \mathcal{A}$ maps $\mathcal{H}^\infty$ into itself. Let $\mathcal{D}_\Delta$ be the order 2 weighted algebra of abstract differential operators considered in Section 1.1, where $\Delta = D^2$ denotes its associated weight[b] of order 2. An abstraction of the usual properties of the algebra of classical pseudodifferential operators (e.g. the relative to ζ-functions given in Remarks 1.1 and 1.2) suggests what should be the idea of (algebraic and geometric) dimension of a given spectral triple. Let us begin by recalling Higson's definition of analytic dimension for a weighted algebra of abstract differential operators.

Definition 1.1. Let $\mathcal{D}_Q$ be a order q weighted algebra of abstract differential operators. The *analytic dimension* d of $\mathcal{D}_Q$ is the smallest value —provided it exists— $d \geq 0$ for which PQ^{-z} extends to a trace class operator on $\mathcal{H}$, for $P \in \mathcal{D}_Q^p$ and $z \in \mathbb{C}$ such that $Re(z) > \frac{p+d}{q}$. If $\mathcal{D}_Q$ has finite analytic dimension and, for any $P \in \mathcal{D}_Q$, the function $\mathrm{tr}(PQ^{-z})$ extends to a meromorphic function on $\mathbb{C}$, the algebra $\mathcal{D}_Q$ is said to have the *analytic continuation property.*

Now, a spectral triple $(\mathcal{A}, \mathcal{H}, D)$ such that any $a \in \mathcal{A}$ maps $\mathcal{H}^\infty$ into itself is said *p-summable* if [10] for some finite $p \geq 1$

$$a \cdot (1 + D^2)^{-1/2} \in \mathcal{L}^p(\mathcal{H}) \qquad \text{for any } a \in \mathcal{L}^p(\mathcal{H}). \tag{8}$$

Consequently, for any element $P \in \mathcal{D}_\Delta$ there exists some half-plane on which the ζ-function $z \mapsto \mathrm{tr}(P\Delta^{-z})$ is holomorphic. In [8] the following

[b]Here, and in what follows, we assume Δ to be invertible, if it is not the case it should be replaced by $\Delta' = \Delta + \Pi_{\ker \Delta}$.

definition is given for the "dimension" of a spectral triple satisfying the above conditions:

Definition 1.2. The *dimension spectrum* Sd_Δ of a finite summable and regular spectral triple $(\mathcal{A}, \mathcal{H}, D)$, is the subset of $\mathbb{C}$ such that, for any $P \in \mathcal{D}^p_\Delta$, the zeta function

$$\zeta_P(z) = \mathrm{tr}(P\Delta^{-\frac{z}{2}})$$

extends holomorphically to $\mathbb{C} \setminus \mathsf{Sd}_\Delta$.

As follows from Remark 1.1, in the case of a compact Riemannian spin manifold (M, g) of even dimension with its associated even spectral triple $(C^\infty(M), L^2(M, S), D_M)$, the corresponding dimension spectrum is discrete and, moreover, the poles of the corresponding ζ-functions are simple.

Whenever the functions $\zeta_P(z)$, for $P \in \mathcal{D}^p_\Delta$, have a discrete set of poles and such poles are at most simple poles, we call the dimension spectrum *discrete* and *simple*, respectively. Examples of spectral triples with these properties have been discussed in, e.g. Ref. [7].

Under the assumption of simplicity and discreteness for the dimension spectrum associated to a finite summable and regular spectral triple, it is possible to define a pseudodifferential calculus associated to the generalized algebra $\mathcal{D}_\Delta$,[12] and generalize on it the ζ-function machinery used in the analysis of classical pseudodifferential operators. In Section 2.2 we study such objects and we recall how to write local formulas, for the index associated to a noncommutative spectral triple, in terms of such ζ-function quantities.

2. ζ-function regularization and the Connes-Moscovici cocycle

2.1. *The local index formula for even spectral triples*

Recall that a cochain ψ in the spaces $C^k(\mathcal{A})$ of $(k+1)$-linear forms on $\mathcal{A}$, for $k \in \mathbb{N}$, is called *cyclic* if it satisfies

$$\psi(a_1, \cdots, a_k, a_0) = (-1)^k \psi(a_0, a_1, \cdots, a_k) \qquad a_j \in \mathcal{A}, \tag{9}$$

a *cyclic cocycle* is a cyclic cochain ψ such that $b\psi = 0$, and the *cyclic cohomology groups* $HC^*(\mathcal{A})$ of the algebra $\mathcal{A}$ are obtained from by restricting the Hochschild coboundary,

$$b\psi(a_0, \cdots, a_{k+1}) = \sum_{j=0}^{k} (-1)^j \psi(a_0, \cdots, a_j a_{j+1}, \cdots, a_{k+1}) + (-1)^{k+1}\psi(a_{k+1}a_0, \cdots, a_k), \qquad a_j \in \mathcal{A}, \quad (10)$$

to those cochains. It can equivalently be described as the second filtration of the (b, B)-bicomplex of (not necessarily cyclic) cochains, where b is the same as before and $B = AB_0 : C^k(\mathcal{A}) \to C^{k-1}(\mathcal{A})$, with

$$(A\psi)(a_0, \cdots, a_{k-1}) = \sum_{j=0}^{k} (-1)^{(k-1)j} \psi(a_j, \cdots, a_{j-1}),$$

and $B_0\psi(a_0, \cdots, a_{k-1}) = \psi(1, a_0, \cdots, a_{k-1})$, for $a_j \in \mathcal{A}$. The *periodic cyclic cohomology* is the cohomology of the short complex

$$C^{even}(\mathcal{A}) \overset{b+B}{\leftrightarrows} C^{odd}(\mathcal{A}), \qquad C^{even/odd}(\mathcal{A}) = \bigoplus_{k \text{ even/odd}} C^k(\mathcal{A}),$$

whose cohomology groups are denoted $HC^+(\mathcal{A})$ and $HC^-(\mathcal{A})$, respectively.

The pairing between $HC^+(\mathcal{A})$ and $K_0(\mathcal{A})$ used to compute the local formula for the index (1) is given by [6], [10]

$$\langle [\varphi] | [e] \rangle = \sum_{k \geq 0} (-1)^k \frac{(2k)!}{k!} (\varphi_{2k} \# \mathrm{tr})(e, \cdots, e), \qquad (11)$$

for any cocycle $\varphi = (\varphi_{2k})$ in $C^{even}(\mathcal{A})$ and any self-adjoint idempotent e in $M_q(\mathcal{A})$. Here $\varphi_{2k}\#\mathrm{tr}$ denotes the $(2k+1)$-linear map on $M_q(\mathcal{A}) = M_q(\mathbb{C}) \otimes \mathcal{A}$ given by

$$(\varphi_{2k}\#\mathrm{tr})(\mu^0 \otimes a_0, \cdots, \mu^{2k} \otimes a_{2k}) = \mathrm{tr}(\mu^0 \ldots \mu^{2k}) \varphi_{2k}(a_0, \cdots, a_{2k}),$$

for $\mu^j \in M_q(\mathbb{C})$ and $a_j \in \mathcal{A}$.

The following theorem provides a local formula to compute the index in even noncommutative spectral triples:

Theorem 2.1. [8] *Suppose that $(\mathcal{A}, \mathcal{H}, D)$ is even, finitely summable and has a discrete and simple dimension spectrum. Then, for any $E \in K_0(\mathcal{A})$,*

$$\mathrm{ind}_D(E) = \langle [\varphi^+_{CM}] | E \rangle, \qquad (12)$$

where the even cocycle $\varphi^+_{CM} = (\varphi_{2k})$ *in the* (b, B)*-complex of* $\mathcal{A}$ *is given, for* $k = 0$*, by*

$$\varphi_0(a_0) = \mathrm{f.p.}|_{t=0}\mathrm{str}(a_0 e^{-t\Delta}), \tag{13}$$

and for $k > 0$ *by*

$$\varphi_{2k}(a_0, \ldots, a_{2k}) = \sum_{\alpha \geq 0} c_{k,\alpha} \mathrm{Res}_{z=0} Tr\left(\gamma a_0 [D, a_1]^{[\alpha_1]} \ldots [D, a_{2k}]^{[\alpha_{2k}]} \Delta^{-|\alpha|-k-z}\right), \tag{14}$$

where the sum is over the multi-index $\alpha = (\alpha_1, \ldots, \alpha_k)$*, with* $\alpha_j \geq 0$*, and*

$$c_{k,\alpha} = \frac{(-1)^{|\alpha|}\Gamma(|\alpha| + k)}{\alpha!(\alpha_1 + 1)\cdots(\alpha_1 + \cdots + \alpha_{2k} + 2k)}, \tag{15}$$

and the symbol $X^{[j]}$ *denotes the* j*-th iterated commutator of* X *with* $\Delta = D^2$.

2.2. *Weighted cochains and ζ-function cyclic cocicles*

2.2.1. *JLO Functionals*

Let $(\mathcal{A}, \mathcal{H}, D)$ be an even spectral triple, with chirality operator γ (with respect to the decomposition $\mathcal{H} = \mathcal{H}^+ \oplus \mathcal{H}^-$) and Q a weight in the sense of definition given above. Let us assume that the spectral triple is Q-summable, i.e. that for any $t > 0$ the operator e^{-tQ} is trace class [c]. A natural extension of the functional (6) to the context of noncommutative spectral triples can be done by considering the Mellin transform of the following JLO-type multifunctional [14]

$$\phi_p^Q(t)(A_0, \ldots, A_p) = \int_{\Delta_p} \mathrm{tr}(\gamma A_0 e^{-u_0 tQ} A_1 e^{-u_1 tQ} \cdots A_p e^{-u_p tQ}) du, \tag{16}$$

where $t \in \mathbb{R}^+$, $p \in \mathbb{Z}^+$, $A_0, \ldots, A_p \in \mathcal{A}$ and the integration is over the k-simplex $\Delta_p = \{(u_0, \ldots, u_p) : u_0 + \cdots + u_p = 1, u_i \geq 0\}$. This functional, defined using heat kernel regularization methods, was originally used in [14] in order to obtain an infinite-dimensional Chern character for (θ-summable) Fredholm modules. It is also at the origin of the local formula for the index given by Connes and Moscovici.[8]

[c]For $Q = D^2$ the spectral triple is called θ-summable.

Looking at (16) as a function in $C^\infty(\mathbb{R}^+)$ associated to any set of p elements of $\mathcal{A}$, it is natural to consider its Mellin transform

$$\Phi_p^Q(A_0,\ldots,A_p)(z) = \frac{1}{\Gamma(z)}\int_0^\infty \phi_p^Q(t)(A_0,\ldots,A_p)t^{z-1}\,dt, \tag{17}$$

i.e. a complex function associated to any set of p elements of $\mathcal{A}$, defined in a region of the complex plane. As a matter of fact, in [12] it has been shown that this object is well defined whenever $A_0,\ldots,A_p \in \mathcal{D}_Q$, the associated weighted algebra, if it has finite analytic dimension and satisfies the analytic continuation property defined above (see Section 4 in [12] for a careful exposition), properties which we assume in what follows. As prompted by Remark 1.2 —which is only valid in the classical pseudodifferential algebra setting, it is possible to write down a complex power expression for $\Phi_\bullet^Q(z)$.

Lemma 2.1. [12] *Let $(\mathcal{A},\mathcal{H},D)$ be a spectral triple with associated algebra of generalized differential operators $\mathcal{D}_Q$, satisfying the analytic continuation property. Then, for $a_0,\ldots,a_p \in \mathcal{A}$, $\Phi_p^Q(a_0,\ldots,a_p)(z)$ defines a meromorphic function on $\mathbb{C}$, and*

$$\Phi_p^Q(a_0,\ldots,a_p)(z) = \frac{(-1)^p}{2\pi i}\int \lambda^{-z+p}\,\mathrm{tr}(\gamma a_0(\lambda-Q)^{-1}\cdots a_p(\lambda-Q)^{-1})d\lambda, \tag{18}$$

where the integral is performed following a line in the complex plane which separates the spectrum of Q from 0. Moreover, if p is bigger than the analytic dimension of $\mathcal{D}_Q$, and $Q=\Delta=D^2$, $\Phi_p^Q(a_0,\ldots,a_p)(z+\frac{p}{2})$ is holomorphic at zero.

This crucial result follows from some asymptotic expansion properties of commutators between elements of $\mathcal{D}_Q$ and resolvents of the weight (see Lemma 4.20 in [12]), and gives us some clues about the residue generalizing the noncommutative one in the context of classical pseudodifferential algebras. First, notice that in the context of classical pseudodifferential operator algebras, when $p=0$, this map (17) is the one taking $z\in\mathbb{C}$ to

$$\Phi_0^Q(A)(z) = \mathrm{str}(AQ^{-z}), \tag{19}$$

where $\mathrm{str}(A)=\mathrm{tr}(\gamma A)$ denotes the usual supertrace. Thus, it seems natural to consider the different options we know to take traces or residues over these functionals —the result is that doing it, on the more general context of weighted algebras of abstract differential operators associated to a spectral triple, it will be possible to recover the cyclic cohomology cocycles used to give a local formula for the index map (1). Before doing that, let us remark that from (19) it follows that

$$\mathrm{Res}_{z=0}\Phi_0^Q(A)(z) = \frac{1}{q}\mathsf{res}(\gamma A),$$

for any classical pseudodifferential operator acting on sections of a $\mathbb{Z}_2$-graded bundle, where res denotes the Wodzicki residue. In general, [18] for $A_0, \dots, A_p \in \mathcal{C}l(E)$,

$$\mathrm{Res}_{z=0}\Phi_p^Q(A_0, \dots, A_p)(z) = \frac{1}{q}\mathsf{res}(\gamma A_0 \cdots A_p), \tag{20}$$

so that regularized traces built from the multifunctionals $\Phi_\bullet^Q$ are in this context, as expected, proportional to res. Moreover, it follows from Lemma 2.1 that, for $p > d =$ analytic dimension of $\mathcal{D}_Q$, $\Phi_p^Q(a_0, \dots, a_p)(z + \frac{p}{2})$ is holomorphic at zero, therefore

$$\mathrm{Res}_{z=0}\Phi_p^Q(a_0, \dots, a_p)(z + \frac{p}{2}) = 0.$$

Comparing this with Remark 1.1, i.e. $\mathrm{Res}_{z=0}\,\mathrm{tr}(AQ^{-z}) = 0$ if $a < -m$, where $a = \mathsf{ord}(A)$ and $m = \dim M$, we see that the quantity $q \cdot \mathrm{Res}_{z=0}\Phi_p^Q(a_0, \dots, a_p)(z+\frac{p}{2})$ generalizes the Wodzicki residue in the more general context of algebras of abstract differential operators associated to spectral triples.

2.2.2. *Weighted cochains and holomorphic cocycles*

The JLO functional $\phi_\bullet^Q(t)$ defined in (16) gives rise to an improper cocycle (with coefficients in $C^\infty(\mathbb{R}^+)$) in (b, B)-cohomology,

$$(a_0, \dots, a_p) \mapsto \langle a_0, [D, a_1], \dots, [D, a_p]\rangle_t^{JLO}, \tag{21}$$

where

$$\langle a_0, [D, a_1], \dots, [D, a_p]\rangle_t^{JLO} := t^{\frac{p}{2}}\phi_p^Q(t)(a_0, [D, a_1], \dots, [D, a_p]),$$

called *JLO cocycle*.[14] Improper means that it does not vanish in general in any component (i.e. it has an infinite number of components). The same holds true for the functional $\Phi_\bullet^Q(z)$ defined in (17), it gives rise to an improper cocycle (with coefficients in the space of holomorphic functions on a half-plane in $\mathbb{C}$) in (b, B)-cohomology,

$$(a_0, \dots, a_p) \mapsto \langle a_0, [D, a_1], \dots, [D, a_p]\rangle_{z-\frac{p}{2}}^{H}, \tag{22}$$

where

$$\langle a_0, [D, a_1], \ldots, [D, a_p] \rangle_z^H := \Gamma\,(z+p)\, \Phi_p^Q(a_0, [D, a_1], \ldots, [D, a_p])(z+p).$$

This improper cocycle was considered in detail by N. Higson in [12], we will call it *holomorphic cocycle.*

The usual way to find proper cocycles from the improper ones is the computation of traces (residues) which cancel an infinite number of components of these improper cocycles . Indeed, in the classical pseudodifferential setting, when the order of a pseudodifferential operator A is less than $-\dim M$, $\mathsf{res}(A) = 0$. Since this feature appears also for the multifunctional Φ_p^Q in the context of weighted algebras of abstract differential operators associated to noncommutative spectral triples, as follows from Lemma 2.1, a simple degree counting argument shows that —provided that the poles are simple at zero, which is true in the cases we consider— for p big enough, the residues of the components of the holomorphic functional (22) will vanish. This is the way the cyclic cocycles associated to the local index formula can be built in (see [12] and [6]).

Following the lines of the construction of trace extensions for pseudodifferential operator algebras in [4],[17], it is natural to consider the cochains obtained from the holomorphic mappings $\Phi_p^Q(a_0, a_1, \ldots, a_p)(z)$ when, instead of the residues at the origin, the finite part is taken. This was done in [18] by S. Paycha, in the context of the algebra of classical pseudodifferential operators acting on sections of finite rank vector bundles, but can be easily adapted to the case of the weighted algebra of generalized abstract differential operators associated to a regular and finite-summable spectral triple $(\mathcal{A}, \mathcal{H}, D)$.

Definition 2.1. Let us define *weighted cochains* $\chi_\bullet^Q$ as finite parts of the meromorphic functions (with simple poles) $\Phi_\bullet^Q$,

$$\chi_p^Q(a_0, \ldots, a_p) = \mathrm{f.p.}|_{z=0}(\Phi_p^Q(a_0, \ldots, a_p)(z)), \tag{23}$$

where $a_0, \ldots, a_p \in \mathcal{A}$ and Q denotes a reference weight operator.

These weighted cochains have been used to study several types of (algebraic and geometric) anomalies in [18]. The same asymptotic expansions used to show Lemma 2.1 (see Lemma 4.20 and Lemma 4.30 in [12]) can be used to show the following

Proposition 2.1. *Let $a_0, \dots, a_p \in \mathcal{A}$ for a regular and finite-summable spectral triple $(\mathcal{A}, \mathcal{H}, D)$ with associated weighted algebra of generalized differential operators $\mathcal{D}_Q$ with the analytic continuation property. Then*

$$\chi_p^Q(a_0, \dots, a_p) - \text{f.p.}|_{z=0}\Phi_0^Q(a_0 \cdots a_p) = \sum_{|k|\geq 1} \boldsymbol{c}_{|k|} \text{Res}_{z=0} Tr\left(\gamma a_0 a_1^{[k_1]} \cdots a_p^{[k_p]} Q^{-|k|-z}\right), \tag{24}$$

where $\boldsymbol{c}_{|k|} = \frac{(-1)^{|k|}(|k|-1)!}{(|k|+1)!}$ for any multiindex k, $|k| = k_1 + \cdots k_p$ and $q = \mathsf{ord}(Q)$.

Let us come back to the case of the classical pseudodifferential operator algebra. Consider, for example, $A \in \mathcal{C}l(E)$. Then —as follows from (19)—

$$\chi_0^Q(A) = \text{f.p.}|_{z=0}\left(\Phi_0^Q(A)(z)\right) = \text{str}^Q(A). \tag{25}$$

Thus, at the level $p = 0$, weighted cochains and weighted traces are the same (thus, since the weighted traces are non tracial in general, this shows that weighted cochains are not cocycles). However, at higher levels this is no longer true, although the difference between $\chi_p^Q(A_0, \dots, A_p)$ and the Q-weighted (super)trace of the operator $A_0 \cdots A_p \in \mathcal{C}l(E)$ is given by a finite linear combination of noncommutative residues (see [18], Propostion 2): If $A_0, \dots, A_p \in \mathcal{C}l(E)$, whenever Q has scalar leading symbol, then

$$\chi_p^Q(A_0, \dots, A_p) - \text{str}^Q(A_0 \cdots A_p) = \frac{1}{q}\sum_{|k|\geq 1} \frac{(-1)^{|k|}(|k|-1)!}{(|k|+1)!} \mathsf{res}\left(\gamma A_0 A_1^{[k_1]} \cdots A_p^{[k_p]} Q^{-|k|}\right). \tag{26}$$

Notice that, since the order of the operator within the res term in (26) decreases as $|k|$ increases, the residues will cancel from the moment at which the map $z \mapsto \text{tr}(\gamma A_0 A_1^{[k_1]} \cdots A_p^{[k_p]} Q^{-|k|-z})$ has no pole at the origin, so that in the sum there are only a finite number of terms. The same holds true, by Lemma 2.1, for formula (24).

Since weighted cochains are not cocycles, it is clear that it is not possible to build from them the cocycles necessary to write the local index theorem terms. Indeed, formula (26) shows that in the context of pseudodifferential operator algebras weighted cochains are no local in general, but that differences between them and appropriate weighted (super)traces can be local. But actually we have no need of weighted cochains to write down a local expression for the index (1), taking residues of the holomorphic cocycles will be enough to recover the Connes-Moscovici cocycle φ^+_{CM}. After computation of the corresponding residues, the asymptotic expansion giving rise to the results stated in Lemma 2.1 imply the following

Lemma 2.2. *[Higson[12]] Let $(\mathcal{A}, \mathcal{H}, D)$ be a regular and finite-summeble spectral triple, $\Delta = D^2$ and $a_0, \ldots, a_p \in \mathcal{A}$, for $p \geq 0$. Then,*

$$\begin{aligned} & \mathrm{Res}_{z=0} \langle a_0, [D, a_1], \ldots, [D, a_p] \rangle^H_{z-\frac{p}{2}} \\ = \; & \mathrm{Res}_{z=0} \left[\Gamma(z + \frac{p}{2}) \Phi^\Delta_p (a_0, [D, a_1], \ldots, [D, a_p])(z + \frac{p}{2}) \right] \qquad (27) \\ = \; & \sum_{|k| \geq 0} c_{p,k} \mathrm{Res}_{z=0} Tr \left(\gamma a_0 [D, a_1]^{(k_1)} \cdots [D, a_p]^{(k_p)} \Delta^{-(z+\frac{p}{2})-|k|} \right), \end{aligned}$$

where $|k| = k_1 + \cdots k_p$, for the multiindex $k = (k_1, \ldots, k_p)$, and the constant $c_{p,k}$ is given by the formula (15) above.

Once again we observe the (residues of the) multifunctional $\Phi^\Delta_p(z + \frac{p}{2})(a_0, [D, a_1], \ldots, [D, a_p])$ appearing in this expression, for $p > 0$, which –in some sense– generalizes the classical Wodzicki residue.[d] This *a priori* locality is indeed confirmed as taking residues of Higson's holomorphic cocycle (22) gives us back the local index formulas given by Connes-Moscovici in Theorem 2.1.

Theorem 2.2. *[Connes-Moscovici[8], Higson[12]] Let $(\mathcal{A}, \mathcal{H}, D)$ be a regular and finite-summeble spectral triple, with $\Delta = D^2$. If the Δ-weighted algebra of generalized differential operators $\mathcal{D}_\Delta$ has finite analytic dimension and the analytic continuation property then, for any idempotent e,*

$$\langle \mathrm{Res}_{z=0} \Psi^H | e \rangle = \mathrm{ind}\,_D(e),$$

where $\mathrm{Res}_{z=0}\Psi^H$ denotes the residue cocycle $\mathrm{Res}_{z=0}\Psi^H_p(a_0, a_1, \ldots, a_p) = \mathrm{Res}_{z=0}\langle a_0, [D, a_1], \ldots, [D, a_p]\rangle^H_{z-\frac{p}{2}}$.

Thus, the ζ-function regularized objects giving rise to local formulae in the framework of classical pseudodifferential operator theory, such as the formulae for indexes, anomalies, etc., can be defined in the more general setting of noncommutative spectral triples, giving rise to results that could

[d] Notice that, for $p = 0$, since $\mathrm{Res}_{z=0} \left[\Gamma(z) \Phi^\Delta_0(a_0)(z) \right] = \mathrm{f.p.}|_{z=0} \Phi^\Delta_0(a_0)(z)$, the complex residue gives rise to a finete part which generalizes a weighted trace rather than a Wodzicki residue (see equation (25)).

be interpreted as examples of the corresponding locality phenomena for noncommutative spaces. This locality has begun to be discussed on concrete examples recently (see e.g. [7], also [3]), but its possible application to physical models —where the locality features could be fundamental— remains to be done.

Acknowledgments

Part of this work was done while the author was visiting the *Institut des Hautes Etudes Scientifiques*, Bures-sur-Yvette, France, and the author wishes to tank warmly professor Jean-Pierre Bourgignon for his kind invitation. Thanks also goes to all the colleagues with whom I discussed about it, specially to Joseph Varilly, Yoshiaki Maeda and Sylvie Paycha, and to the referee for the careful reading of the text and the remarks which improve its final presentation. Finally, many thanks to the members of the Mathematics Department of Keio University and the organizers of the conference on *Noncommutative Geometry and Physics*, 2004.

References

1. Atiyah, M. *Collected Works. Index Theory (Vols. 3 and 4).* Oxford Science Publications. The Clarendon Press, Oxford University Press, New York, 1988.
2. Atiyah, M., Bott, R., Patodi, V.: *On the heat equation and the index theorem.* Invent. Math. **19**, 279–330, 1973.
3. Benameur, Moulay-Tahar *Noncommutative geometry and abstract integration theory.* Geometric and topological methods for quantum field theory (Villa de Leyva, 2001), 157–227, World Sci. Publishing, River Edge, NJ, 2003.
4. Cardona, A., Ducourtioux, C. Magnot, J-P. and Paycha, S. *Weighted traces on algebras of pseudo-differential operators and geometry on loop groups.* Inf.Dim.Anal. Quant.Prob. and Rel.Top. **5**, 503–540, 2002.
5. Cardona, A., Ducourtioux, C. and Paycha, S. *From tracial anomalies to anomalies in Quantum Field Theory.* Comm.Math.Phys. **242**, 31-65, 2003.
6. Connes, A. *Noncommutative Geometry.* Academic Press, San Diego, 1994.
7. Connes, A. *Cyclic cohomology, quantum group symmetries and the local index formula for* $\mathrm{SU}_q(2)$. J. Inst. Math. Jussieu **3**, no. 1, 17–68, 2004.
8. Connes, A., Moscovici, H.: *The local index formula in noncommutative geometry.* GAFA **5**, 174–243, 1995.
9. Dixmier, J. *Existence de traces non-normales.* C.R. Acad. Sci. Paris A-B **262**, pp. A1107A1108, 1966.
10. Gracia-Bonda, J.; Várilly, J. and Figueroa, H. *Elements of noncommutative geometry.* Birkhäuser, Boston, MA, 2001.
11. Higson, N. *A note on regular spectral triples.* Preprint, 2003.

12. Higson, N. *The residue index theorem of Connes and Moscovici.* Preprint, 2004.
13. Higson, N. *Meromorphic continuation of zeta functions associated to elliptic operators.* Operator algebras, quantization, and noncommutative geometry, *Contemp. Math.*, **365**, Amer. Math. Soc., 129–142, 2004.
14. Jaffe, A., Lesniewski, A. and Osterwalder, K. *Quantum K-theory. I. The Chern character.* Comm. Math. Phys. **118**, no. 1, 1–14, 1988.
15. Kassel, C. *Le résidu non commutatif (d'après M. Wodzicki).* Séminaire Bourbaki, Vol. 1988/89. Astérisque No. 177-178, Exp. No. 708, 199–229, 1989.
16. Kontsevich, M. and Vishik, S. *Determinants of elliptic pseudo-differential operators*, Max Planck Institut Preprint, 1994.
17. Paycha, S. *Renormalized traces as a looking glass into infinite-dimensional geometry*, Inf.Dim.Anal.Quant.Prob. and Rel.Top., Vol **4**, N.2 p.221-266, 2001.
18. Paycha, S. *Weighted Trace Cochains, the geometric setup for anomalies*, U.B.P. Preprint, 2004.
19. Paycha, S. and Rosenberg, S. *Curvature on determinant bundles and first Chern forms*, J. Geom. Phys., Vol **45**, no. 3-4, 393–429, 2003.
20. Ponge, R. *A new short proof of the local index formula and some of its applications.* Comm. Math. Phys. **241** , no. 2-3, 215–234, 2003. Erratum, Comm. Math. Phys. **248** , 639, 2004.
21. Seeley, R. T. *Complex powers of an elliptic operator*, in Proc. Sympos. Pure Math., Vol. 10, pp. 288–307, Amer. Math. Soc., Providence, 1967.
22. Wodzicki, M. *Non commutative residue*, in Lecture Notes in Mathematics **1289**, Springer Verlag, 1987.