Universidad de Los Andes

Masters thesis

Aleksandrov-Fenchel's Inequality and Intrinsic volumes

by

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Abstract

We study in this thesis the Brunn–Minkowski inequality in the euclidean space and the Aleksandrov – Fenchel inequality for convex bodies. We do this in order to get a better comprehension of the intrinsic volumes (euclidean and spherical) and their properties. We get as a consequence from the Brunn–Minkowski inequality the isoperimetric inequality for convex bodies. Subsequently, we prove the Aleksandrov–Fenchel inequality using mixed volumes and the reproduction of Aleksandrov's proof found in [Sch13] which corresponds to the first proof of the inequality.

We talk later about euclidean intrinsic volumes. We see there that the sequence of intrinsic volumes for any convex body is log-concave. Whether the spherical intrinsic volumes are log-concave remains unknown. Our main contribution was to find explicit formulas for the intrinsic volumes of a spherical polygon. Also we proved that this particular sequence of intrinsic volumes is log-concave using the isoperimetric inequality on the sphere.

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Chapter 1

Introduction

The main nature of this thesis is related to convex geometry. The purpose of convex geometry is to study convex bodies: compact and convex subsets of the euclidean space that have nonempty interior. Convex sets can be found in areas like linear programming, probability theory, computational and discrete geometry, functional analysis, partial diferential equations, information theory, and the geometry of numbers, etc.

We became interested in convex bodies and their properties because of the work in [ALMT14] which provided a rigorous analysis explaining why phase transitions are ubiquitous in random convex optimization problems. The problem of calculating the probability that a randomly rotated cone shares a ray with a fixed cone was improved there by finding accurate bounds. The role played by the sequence of the intrinsic volumes of the cone was very important to solve this problem (the key fact was to prove that this sequence concentrates strongly around the statistical dimension). The more we understand such a sequence (their behavior, how to compute them explicitly), the more we can apply to get better bounds and even solve problems in other fields, such as linear inverse problems with convex constraints and bounds for normal approximations of the lengths of projections of Gaussian vectors on - closed - convex sets. See for example [GNP14]

One important question regarding intrinsic volumes in general and their behavior is whether or not they are log-concave. Indeed, the answer is unknown yet for the spherical case, existing just a bit of evidence in favour of a positive answer (products of circular cones [Ame11]). In this thesis, we focused in the conjecture of log-concavity of the intrinsic volumes and how to compute them for basic cones and spherical polygons. Our approach was different than the one used by Amelunxen and we could actually prove that the intrinsic volumes of any spherical polygon on $\mathbb{R}^3$ are log-concave by exhibiting explicit formulas for them and using the isoperimetric inequality on the sphere, which we consider is new in attempts to solve this conjecture. The intrinsic volumes in the euclidean case are defined by the Steiner formula to compute the volume of the Minkowski's sum of a convex body and the n-dimensional ball. The proof that these volumes are actually log-concave is deduced from a deep inequality: the Aleksandrov-Fenchel inequality and is reproduced in this paper. It has been applied to differential geometric uniqueness theorems, to extremal problems for geometric probabilities, and to combinatorial questions, in particular to showing that certain sequences of combinatorial interest are log-concave [Sta81].

We develop the theory historically. Chapter 3 and 4 present the Brunn- Minkowski's inequality and some of the mixed volumes theory. We included two important inequalities due to Minkowski in chapter 4. One of these inequalities (the second) is a particular case of the Aleksandrov-Fenchel's inequality (chapter 5), but is relevant for the proof of the latter. Finally, chapter 6 is devoted to the theory of intrinsic volumes in the euclidean and spherical case.

Chapter 2

Preliminaries

Here we set up the most important notation used along the thesis. We also define some mathematical objects; the main concepts needed are related to convex geometry and some of differential geometry.

We will use $\mathbb{R}^n$ for the n-dimensional Euclidean space and the double bars $||.||$ to indicate the euclidean norm of any vector x in $\mathbb{R}^n$ defined as $||x|| = \sqrt{x_1^2 + x_2^2 + \ldots + x_n^2}$. The unit sphere in $\mathbb{R}^n$ will be denoted by $S^{n-1} := \{x \in \mathbb{R}^n : ||x|| = 1\}$ and the corresponding closed unit ball by B^n.

Definition 2.1. The *Minkowski sum* of two subsets $X, Y \subseteq \mathbb{R}^n$ is the set $X + Y := \{x + y : x \in X,\ y \in Y\}$

Definition 2.2. The euclidean distance in $\mathbb{R}^n$, denoted by $d^e : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$ is defined by

$$d^e(x, y) = ||x - y||$$

and the *spherical distance* $d(p, q) : S^{n-1} \times S^{n-1} \to \mathbb{R}$ between two points p and q (up to normalization) is the angle between them, i.e:

$$d(p, q) = \arccos(\langle p, q \rangle).$$

A set $K^e \subseteq \mathbb{R}^n$ is *convex* if for every $x, y \in K^e$ the line segment joining x and y, $(1 - t)x + ty$ is contained in K^e for every $t \in [0, 1]$. Analogously, a subset $K \subseteq S^{n-1}$ is **convex** if for every $p, q \in K$ with $p \neq -q$, the unique shortest arc between p and q is contained in K.

Definition 2.3. A set C is called a **cone** if for every $\lambda \geq 0$ and $c \in C$, $\lambda c \in C$. So, say that a set $K \subseteq S^{n-1}$ is convex is equivalent to say that the set $\text{cone}(K) := \{\lambda x : \lambda \geq 0, x \in K\}$ is convex.

A convex cone is *polyhedral* if it is a finite intersection of closed half-spaces, where a half-space is one of the two regions $H^+_{u,a}, H^-_{u,a}$ determined by a hyperplane

$$H_{u,a} := \{x \in \mathbb{R}^n : \langle x, u \rangle = a\}$$

and

$$H^+_{u,a} := \{x \in \mathbb{R}^n : \langle x, u \rangle \geq a\},$$
$$H^-_{u,a} := \{x \in \mathbb{R}^n : \langle x, u \rangle \leq a\}.$$

So, linear subspaces are polyhedral, and polyhedral cones are closed.

We will be interested mainly in convex cones, so, given a convex cone C, a *supporting hyperplane* is a hyperplane H such that C lies entirely in one of the closed half-spaces induced by H. A *face* of C is a set of the form $F = C \cap H$, where H is a supporting hyperplane. The linear span $\text{lin}(C)$ of a cone C is the smallest linear subspace containing C and is given by $\text{lin}(C) = C + (-C)$. The *dimension* of a face F is $\dim(F) := \dim(\text{lin}(F))$, similarly we define the dimension of C. A cone is pointed if the origin 0 is a zero-dimensional face, or equivalently, if it does not contain a linear subspace of dimension greater than zero. If C is not pointed, then it contains a nontrivial linear subspace of maximal dimension $k > 0$, given by $L = C \cap -C$.

Definition 2.4. The set of nonempty compact convex sets in $\mathbb{R}^n$

$$\mathcal{K}(\mathbb{R}^n) := \{K^e \subseteq \mathbb{R}^n : K^e \text{ is a nonempty compact convex set}\}$$

is called the set of *convex bodies* in $\mathbb{R}^n$. Likewise, in the unit sphere, we call a set *spherical convex* if it is closed, convex, and neither the empty set nor the whole sphere:

$$\mathcal{K}(S^{n-1}) := \{K \subseteq \mathbb{R}^{n-1} : K \text{ is closed, convex}, K \neq \emptyset, K \neq S^{n-1}\}.$$

A spherical convex set K is called *polyhedral* if $\text{cone}(K)$ is the intersection of finitely many n-dimensional half-spaces. Denoting by $\mathcal{K}^p(S^{n-1})$ the set of the polyhedral spherical sets, we have

$$\mathcal{K}^p(S^{n-1}) := \{K \in \mathcal{K}(S^{n-1}) : \text{cone}(K) = H_1 \cap \ldots \cap H_k,\ H_i \text{ a half-space}\}$$

A *cap* K is a convex body on the sphere S^{n-1} which is not a subsphere. We denote the set of the caps in $\mathcal{K}(S^{n-1})$ by $\mathcal{K}^c(S^{n-1})$. Note that K is a cap if and only if there is $p \in K$ such that $-p \notin K$, or equivalently, if and only if K° is a cap.

Definition 2.5. The *support function* $h(K, *) : \mathbb{R}^n \to \mathbb{R}$ of a convex body $K \subseteq \mathbb{R}^n$ for any $u \in \mathbb{R}^n$ is defined as

$$h(K, u) = \sup\{\langle x, u\rangle : x \in K\}$$

Definition 2.6. Let $K, L \in \mathcal{K}(S^{n-1})$. One way to measure the distance between K and L is by using the *Hausdorff distance* defined by

$$\delta(K, L) := \min\{\lambda \geq 0 : K \subseteq L + \lambda B^n,\ L \subseteq K + \lambda B^n\}.$$

Lemma 2.1. *Let $K_1, \ldots, K_r \in \mathcal{K}(\mathbb{R}^n)$ and $\lambda_1, \ldots, \lambda_r \geq 0$, then the support function satisfies the following equality*

$$h\left(\sum_i \lambda_i K_i, u\right) = \sum_i \lambda_i h(K_i, u)$$

for any $u \in \mathbb{R}^n$

Let $u \in \mathbb{R}^n \setminus \{0\}$ such that $h(K, u)$ is finite. Let

$$\begin{aligned} H(K, u) &:= \{x \in \mathbb{R}^n : \langle x, u\rangle = h(K, u)\} \\ H^-(K, u) &:= \{x \in \mathbb{R}^n : \langle x, u\rangle \leq h(K, u)\} \\ F(K, u) &:= H(K, u) \cap K \end{aligned}$$

$H(K, u)$, $H(K^-, u)$, $F(K, u)$ are, respectively, the support plane, supporting halfspace and support set of K, each with outer normal vector u. Note that these definitions of support plane and supporting halfspace extend the notions mentioned before.

Definition 2.7 (Duality)**.** Given a convex cone C, we define the *Dual* of C as:

$$C^* := \{x \in C : \langle x, y\rangle \geq 0\}.$$

Likewise, the **Normal Cone** of C is

$$C^\circ := -C^*.$$

Therefore, for $K \in \mathcal{K}(S^{n-1})$ and $D = \text{cone}(K)$ we may define

$$K^\circ := D^\circ \cap S^{n-1}$$

Remark 2.1.

- The polar map is an involution on the set of spherical convex sets, i.e: $(K^\circ)^\circ = K$ for $K \in \mathcal{K}(S^{n-1})$
- There is an intrinsic characterization of the normal cone via $K^\circ = \{x \in S^{n-1} : d(K, v) \geq \pi/2\}$ where $d(K, x) := \min\{d(p, x) : p \in K\}$.
- The boundary of K° is given by $\partial K := \{x \in S^{n-1} : d(K, x) = \pi/2\}$

Definition 2.8. Let C be a closed convex set and $\widehat{x} \in C$. The *normal cone* of C at $\widehat{x}$ is

$$N_{\widehat{x}}(C) := \{y \in \mathbb{R}^n : \langle y, x - \widehat{x}\rangle \leq 0, \forall x \in C\} = \{x \in \mathbb{R}^n : \textstyle\prod_C(x + \widehat{x}) = \widehat{x}\}$$

Where $\prod_C$ is the euclidean projection map onto the convex set C:

$$\textstyle\prod_C(x) := \arg\min\{||x - y||^2 : y \in C\}$$

1. We denote by $N^S_{\widehat{x}}(C)$ the set $N_{\widehat{x}}(C) \cap S^{n-1}$.
2. For a spherical convex set $K \in \mathcal{K}(S^{n-1})$ and $D = \text{cone}(K)$, we define for $p \in K$

$$N_p(K) := N_p(D) \quad \text{and} \quad N^S_p(K) := N_p(K) \cap S^{n-1}$$

Remark 2.2.

- Note that by the first equality and the definition of the standard scalar product in $\mathbb{R}^n$, $N_{\widehat{x}}(C)$ is in fact a cone. This motivates the name of $N_{\widehat{x}}(C)$.
- We have that $N_{\widehat{x}}(C) = \{0\}$ if and only if $x \in \text{int}(C)$, equivalently, $N_{\widehat{x}}(C) \neq \{0\}$ if and only if $x \in \partial(C)$.
- The Moreau's decomposition theorem ([CR13]) asserts that for any $x \in \mathbb{R}^n$, $x = \prod_C(x) + \prod_{C^\circ}(x)$. So, we have the following decomposition of $\mathbb{R}^n$:

$$\mathbb{R}^n = \bigcup_{x \in C}(x + N_x(C))$$

Given a convex closed set C in $\mathbb{R}^n$ and a face F of C, we have that

$$N_x(C) = N_y(C)$$

for any $x, y \in \text{relint}(F)$, here $\text{relint}(F)$ denotes the relative interior of F. Accordingly, we may thus define

$$N_F(C) := N_p(C) \quad \text{and} \quad N^S_F(C) := N^S_p(C)$$

Where $p \in \text{relint}(F)$. Similarly, given $K \in \mathcal{K}(S^{n-1})$ and F a face of K,

$$N_F(K) := N_p(K) \text{ and } N^S_F(K) = N^S_p(K),$$

Where $p \in \text{relint}(F^e) \cap S^{n-1}$, and $F^e = \text{cone}(F)$.

Proposition 2.1. *Let $C \subseteq \mathbb{R}^n$ be a closed convex cone and $x \in C$. Then,*

$$N_x(C) = x^\perp \cap C^\circ,$$

where $x^\perp = \{y \in \mathbb{R}^n : \langle x, y\rangle = 0\}$. Note that in particular, $N_0(C) = C^\perp$

Proof. "$\subseteq$" Sea $v \in N_x(C) \Rightarrow \langle v, y - x\rangle = \langle v, y\rangle - \langle v, x\rangle \leq 0$ for all $y \in C$. If we set $y = 0$ then we conclude that $\langle v, x\rangle \geq 0$, and if we put $y = 2x \in C$ then $\langle v, x\rangle \leq 0$. So, we get that $\langle v, x\rangle = 0$ and therefore $v \in x^\perp$.

From $\langle v, y\rangle - \langle v, x\rangle \leq 0$ and $\langle v, x\rangle = 0$ we get that $\langle v, y\rangle \leq 0$, so $y \in C^\circ$.
" $\supseteq$ " Suppose that $\langle y, x\rangle = 0$ and $\langle y, v\rangle \leq 0$ for all $v \in C$. Then, $\langle y, v - x\rangle = \langle y, v\rangle - \langle y, x\rangle = \langle y, v\rangle \leq 0$. Thus $y \in N_x(C)$. □

Chapter 3

The Brunn-Minkowski inequality

One important inequality known by many mathematicians is the *isoperimetric inequality* in the plane:

$$L^2 \geq 4\pi A,$$

which relates for a simple closed curve γ, its length L and the area A enclosed by it. Equality occurs only if γ is a circle. In general, the isoperimetric inequality for convex bodies relates the n-dimensional volume of a body K to the $(n-1)$-dimensional volume of its boundary:

$$\left(\frac{\mathrm{vol}_n(K)}{\mathrm{vol}_n(B^n)}\right)^{1/n} \leq \left(\frac{S_n(K)}{S_n(B)}\right)^{1/n-1} \tag{3.1}$$

Here $S_n(k)$ is the surface area of K, equivalently, the $(n-1)$-dimensional volume of ∂K.

The Brunn-Minkowski inequality relates the volume of the Minkowski sum of two convex sets in the Euclidean space to the individual volume of each set. We will state it and prove it for convex bodies. The proof that we will give is sketched in [Sch13] and reproduced in [LAR] as well, and is due to Kneser and Süss (in 1932). One very important application that We will see later is that the general isoperimetric inequality 3.1 is a consequence of the Brunn-Minkowski inequality.

3.1 Brunn-Minkowski inequality

Theorem 3.1 (**Brunn-Minkowski Inequality**). *Let $K, L \in \mathcal{K}(\mathbb{R}^n)$ and $\lambda \in [0,1]$, then*

$$\mathrm{vol}_n((1-\lambda)K + \lambda L)^{1/n} \geq (1-\lambda)\mathrm{vol}_n(K)^{1/n} + \lambda \mathrm{vol}_n(L)^{1/n} \tag{3.2}$$

And the equality holds for some $\lambda \in (0,1)$ if and only if K and L are homothetic or lie in parallel planes.

Notation. Let $K_0, K_1 \in \mathcal{K}(\mathbb{R}^n)$ and $\lambda \in [0,1]$, we set

$$K_\lambda = (1-\lambda)K_0 + \lambda K_1.$$

According to this notation, we get that for $\sigma, \tau, \lambda \in [0,1]$

$$\begin{aligned}
(1-\lambda)K_\sigma + \lambda K_\tau &= (1-\lambda)[(1-\sigma)K_0 + \sigma K_1] + \lambda[(1-\tau)K_0 + \tau K_1] \\
&= (1-\sigma-\lambda[1-\sigma-1+\tau])K_0 + [(1-\lambda)\sigma + \lambda\tau]K_1 \\
&= (1-\sigma+\sigma\lambda-\lambda\tau)K_0 + [(1-\lambda)\sigma+\lambda\tau]K_1 \\
&= (1-[(1-\lambda)\sigma+\lambda\tau])K_0 + [(1-\lambda)\sigma+\lambda\tau]K_1 \\
&= (1-\alpha)K_0 + \alpha K_1,
\end{aligned}$$

Where $\alpha = (1-\lambda)\sigma + \lambda\tau$.

Proof. We can rewrite the Brunn-Minkowski inequality as

$$\mathrm{vol}_n(K_\lambda)^{1/n} \geq (1-\lambda)\mathrm{vol}_n(K_0)^{1/n} + \lambda \mathrm{vol}_n(K_1)^{1/n}.$$

Case 1 Suppose that $\dim(K_0) < n$ and $\dim(K_1) < n$ then $\mathrm{vol}_n(K_0) = \mathrm{vol}_n(K_1) = 0$ and the inequality hods.

Case 2 Suppose that $\dim(K_0) < n$ and $\dim(K_1) = n$. For any $x \in K_0$, we have that $(1-\lambda)x + \lambda K_1 \subseteq K_\lambda$, so

$$\mathrm{vol}_n(K_\lambda) \geq \mathrm{vol}_n((1-\lambda)x + \lambda K_1) = \mathrm{vol}_n(\lambda K_1) = \lambda^n \mathrm{vol}_n(K_1)$$

So, $\mathrm{vol}_n(K_\lambda)^{1/n} \geq \lambda \mathrm{vol}_n(K_1)^{1/n} = (1-\lambda)\mathrm{vol}_n(K_0)^{1/n} + \lambda \mathrm{vol}_n(K_1)^{1/n}$

Case 3 Suppose that $\dim(K_0) = \dim(K_1) = n$. We can assume that both K_0 and K_1 have volume equal to one. This is because if the statement is true for such bodies, then for general L_0, L_1 convex bodies, we can associate

$$\tilde{L}_0 = \mathrm{vol}_n(L_0)^{-1/n} L_0 \;\text{ and } \tilde{L}_1 = \mathrm{vol}_n(L_1)^{-1/n} L_1$$

and

$$\tilde{\lambda} = \frac{\lambda \mathrm{vol}_n(L_1)^{1/n}}{(1-\lambda)\mathrm{vol}_n(L_0)^{1/n} + \lambda \mathrm{vol}_n(L_1)^{1/n}}$$

And as we are assuming Brunn-Minkowski holds for $\tilde{L}_0, \tilde{L}_1, \tilde{\lambda}$ with

$$\begin{aligned}
\mathrm{vol}_n((1-\tilde{\lambda})\tilde{L}_0 + \tilde{\lambda} L_1)^{1/n} &= \frac{\mathrm{vol}_n((1-\lambda)L_0 + \lambda L_1)^{1/n}}{(1-\lambda)\mathrm{vol}_n(L_0)^{1/n} + \lambda \mathrm{vol}_n(L_1)^{1/n}} \\
(1-\tilde{\lambda})\mathrm{vol}_n(\tilde{L}_0) &= \frac{(1-\lambda)\mathrm{vol}_n(L_0)^{1/n}}{(1-\lambda)\mathrm{vol}_n(L_0)^{1/n} + \lambda \mathrm{vol}_n(L_1)^{1/n}} \\
\tilde{\lambda}\mathrm{vol}_n(\tilde{L}_1)^{1/n} &= \frac{\lambda \mathrm{vol}_n(L_1)^{1/n}}{(1-\lambda)\mathrm{vol}_n(L_0)^{1/n} + \lambda \mathrm{vol}_n(L_1)^{1/n}}
\end{aligned}$$

then, we can see that $\mathrm{vol}_n(\tilde{L}_{\tilde{\lambda}})^{1/n} \geq (1-\tilde{\lambda})\mathrm{vol}_n(\tilde{L}_0)^{1/n} + \tilde{\lambda}\mathrm{vol}_n(\tilde{L}_1)^{1/n}$ if and only if $\mathrm{vol}_n(L_\lambda)^{1/n} \geq (1-\lambda)\mathrm{vol}_n(L_0)^{1/n} + \lambda \mathrm{vol}_n(L_1)^{1/n}$.

We will prove the theorem using induction over n. The case $n = 1$ we have that the compact and convex sets of $\mathbb{R}$ are the closed intervals, so the assertion follows.

Assume that the statement is true for $n-1$, where $n \geq 2$, then we need to prove it for n.

Let $u \in S^{n-1}$ and for $\zeta \in \mathbb{R}$ we set $H(\zeta) := H_{u,\zeta},\; H^-(\zeta) := H^-_{u,\zeta}$. We also define $\alpha_\lambda := -h(K_\lambda, -u)$ and $\beta_\lambda := h(K_\lambda, u)$. Here, $H_{u,\zeta}, H^-_{u.\zeta}$ and h correspond to the definitions of hyperplane, halfspace and support function we made at the beginning.
Besides, in order to ease the notation, let

$$\begin{aligned}
v_0(\zeta) &= \mathrm{vol}_{n-1}(K_0 \cap H(\zeta)) \\
w_0(\zeta) &= \mathrm{vol}_n(K_0 \cap H^-(\zeta))
\end{aligned}$$

And likewise

$$\begin{aligned}
v_1(\zeta) &= \mathrm{vol}_{n-1}(K_1 \cap H(\zeta)) \\
w_1(\zeta) &= \mathrm{vol}_n(K_1 \cap H^-(\zeta))
\end{aligned}$$

So, according to these notation, we have

$$w_0(\zeta) = \int_{\alpha_0}^{\zeta} v_0(t)dt, \text{ and } w_1(\zeta) = \int_{\alpha_1}^{\zeta} v_1(t)$$

The functions v_0 and v_1 are continuous on (α_0, β_0) and (α_1, β_1) respectively, so the function $w_i(\zeta)$ is differentiable there for $i = \{0, 1\}$ and

$$\frac{d}{d\zeta}[w_i(\zeta)] = v_i(\zeta) > 0.$$

Thus, we can use the inverse function theorem to find an inverse function $z_i : (0, 1) \to (\alpha_i, \beta_i)$ of w_i such that (note that the domain is $(0, 1)$ by the assumption $\mathrm{vol}_n(K_i) = 1$).

$$z_i'(\tau) = \frac{1}{v_i(z_i(\tau))}, \text{ with } 0 < \tau < 1$$

Define the following:

$$k_i(\tau) = K_i \cap H(z_i(\tau))$$
$$z_\lambda(\tau) = (1 - \lambda)z_0(\tau) + \lambda z_1(\tau)$$

In consequence, for $\lambda, \tau \in (0, 1)$

$$K_\lambda \cap H(z_\lambda(\tau)) \supset (1 - \lambda)k_0(\tau) + \lambda k_1(\tau)$$

To see this, take $x = (1-\lambda)x_0 + \lambda x_1 \in (1-\lambda)k_0(\tau) + \lambda k_1(\tau)$. Clearly, $x \in K_\lambda$, moreover, by definition, $\langle u, (1-\lambda)x_0\rangle = (1-\lambda)z_0(\tau)$ and $\langle u, \lambda x_1\rangle = \lambda z_1(\tau)$, so $\langle u, (1-\lambda)x_0 + \lambda x_1\rangle = (1 - \lambda)z_0(\tau) + \lambda z_1(\tau) = z_\lambda(\tau)$ and then $x \in H(z_\lambda(\tau))$.
We note that $z_\lambda(0) = (1 - \lambda)z_0(0) + \lambda z_1(0) = (1 - \lambda)\alpha_0 + \lambda\alpha_1 = \alpha_\lambda$ (because $w_0(\alpha_0) = w_1(\alpha_1) = 0$), similarly, $z_\lambda(1) = \beta_\lambda$. Now, using this inclusion and the induction hypothesis we have

$$\begin{aligned}
\mathrm{vol}_n(K_\lambda) &= \int_{\alpha_\lambda}^{\beta_\lambda} \mathrm{vol}_{n-1}(K_\lambda \cap H(\zeta))d\zeta \\
&= \int_0^1 \mathrm{vol}_{n-1}(K_\lambda \cap H(z_\lambda(\tau)))z_\lambda'(\tau)d\tau \\
&\geq \int_0^1 \mathrm{vol}_{n-1}((1 - \lambda)k_0(\tau) + \lambda k_1(\tau)) \left[\frac{1-\lambda}{v_0(z_0(\tau))} + \frac{\lambda}{v_1(z_1(\tau))}\right] d\tau \\
&\geq \int_0^1 \left[(1 - \lambda)v_0(z_0(\tau))^{1/(n-1)} + \lambda v_1(z_1(\tau))^{1/(n-1)}\right]^{n-1} \left[\frac{1-\lambda}{v_0(z_0(\tau))} + \frac{\lambda}{v_1(z_1(\tau))}\right] d\tau \\
&\geq 1 \\
&= (1 - \lambda)\mathrm{vol}_n(K_0)^{1/n} + \lambda \mathrm{vol}_n(K_1)^{1/n}
\end{aligned}$$

The last inequality follows from

Lemma 3.1. *Let $a, b, p > 0$ and $\lambda \in (0, 1)$, then*

$$[(1 - \lambda)a^p + \lambda b^p]^{1/p} \left[\frac{1-\lambda}{a} + \frac{\lambda}{b}\right] \geq 1.$$

Proof. Using that the logarithm is a concave function we have that:

$$\begin{aligned}
&\log\left([(1-\lambda)a^p+\lambda b^p]^{1/p}\left[\frac{1-\lambda}{a}+\frac{\lambda}{b}\right]\right) = \frac{1}{p}\log\left([(1-\lambda)a^p+\lambda b^p]\left[\frac{1-\lambda}{a}+\frac{\lambda}{b}\right]^p\right)\\
&\geq \frac{1}{p}(1-\lambda)\log\left(a^p\left[\frac{1-\lambda}{a}+\frac{\lambda}{b}\right]^p\right)+\frac{1}{p}\lambda\log\left(b^p\left[\frac{1-\lambda}{a}+\frac{\lambda}{b}\right]^p\right)\\
&= (1-\lambda)\log\left((1-\lambda)+\frac{a\lambda}{b}\right)+\lambda\log\left(\frac{b(1-\lambda)}{a}+\lambda\right)\\
&\geq (1-\lambda)\lambda\log\left(\frac{a}{b}\right)+\lambda(1-\lambda)\log\left(\frac{b}{a}\right)\\
&= (1-\lambda)\lambda\log\left(\frac{a}{b}\frac{b}{a}\right)\\
&= 0.
\end{aligned}$$

Note that given that the logarithm is an increasing function, we have equality in this series of inequalities if and only if $a = b$. Hence, in our original case, we will have equality in the last inequality if and only if $v_0(z_0(\tau)) = v_1(z_1(\tau))$. □

Now, we will focus on the conditions of equality. If the bodies K_0, K_1 lie in two parallel hyperplanes, then K_λ lies in some hyperplane as well, thus we have equality in 3.2.

If K_0, K_1 are homothetic we have that $K_1 = rk_0 + c$ for some $c \in \mathbb{R}$ and $c \in \mathbb{R}^n$. The Minkowski addition is distributive for convex sets, as is seen in [Sch13, remark 1.1.1], then

$$\begin{aligned}
(1-\lambda)K_0+\lambda K_1 &= (1-\lambda)K_0+\lambda(rK_0+c)\\
&= (1-\lambda+\lambda r)K_0+\lambda c
\end{aligned}$$

And being vol_n transitive invariant we also have

$$\begin{aligned}
&\mathrm{vol}_n((1-\lambda)K_0+\lambda K_1)^{1/n} = \mathrm{vol}_n((1-\lambda+\lambda r)K_0+\lambda c)^{1/n}\\
&= \mathrm{vol}_n((1-\lambda+\lambda r)K_0)^{1/n}\\
&= (1-\lambda+\lambda r)\mathrm{vol}_n(K_0)^{1/n}\\
&= (1-\lambda)\mathrm{vol}_n(K_0)^{1/n}+\lambda r\mathrm{vol}_n(K_0)^{1/n}\\
&= (1-\lambda)\mathrm{vol}_n(K_0)^{1/n}+\lambda\mathrm{vol}_n(rK_0+c)^{1/n}\\
&= (1-\lambda)\mathrm{vol}_n(K_0)^{1/n}+\lambda\mathrm{vol}_n(K_1)^{1/n}
\end{aligned}$$

Again, we get equality in 3.2.

Suppose now that equality holds in 3.2, we will consider three cases as before:

Case 1 If $\dim K_0 < n$ and $\dim K_1 < n$ then K_λ is contained in some hyperplane, therefore, K_0 and K_1 must be in parallel hyperplanes.

Case 2 If $\dim K_0 < n$ and $\dim K_1 = n$ then we must have equality in

$$\mathrm{vol}_(K_\lambda) \geq \mathrm{vol}_n((1-\lambda)x+\lambda K_1) = \lambda^n\mathrm{vol}_n(K_1)$$

if and only if $K_0 = \{x\}$, so, in this case, K_0 and K_1 are homothetic.

Case 3 If we had that $\dim(K_0) = \dim(K_1) = n$, and suppose as before that K_0, K_1 have volume equal to one, then we would have equality in all the inequalities in the proof of the case 3 above and equality in 3.1 as well, with $a = v_0(z_0(\tau)) = b = v_1(z_1(\tau))$, which implies that $z_0'(\tau) = z_1'(\tau)$ for all $\tau \in [0,1]$. So, $z_1(\tau) - z_0(\tau)$ is a constant. We can assume without loss of generality by the translation invariance of vol_n that K_0 and K_1 have their center of mass at the origin. Then

$$0 = \int_{K_i} \langle x, u \rangle dx = \int_{\alpha_i}^{\beta_i} \mathrm{vol}_{n-1}(K_i \cap H(\zeta))\zeta d\zeta \tag{3.3}$$

$$= \int_0^1 z_i(\tau) d\tau \tag{3.4}$$

For this reason, $z_0(\tau) = z_1(\tau)$ for all $\tau \in [0,1]$, in particular, we get that $\beta_0 = \beta_1$, and by the definition of β_i this implies that $h(K_0, u) = h(K_1, u)$. But u is an arbitrary vector in $\mathbb{R}^n$, then we must have that $K_0 = K_1$. However, $K_0 = K_1$ was after the translation and normalization of the bodies, which means that they are homothetic and we finish the proof.

□

3.2 Isoperimetric Inequality for convex bodies

Now, we present one important application of the Brunn-Minkowski inequality as we said at the beginning.

Definition 3.1. Let $K \in \mathcal{K}(\mathbb{R}^n)$, the **surface area** of K is defined as

$$S_n(K) = \mathrm{vol}_{n-1}(\partial K) := \lim_{\varepsilon \to 0} \frac{\mathrm{vol}_n(K + \varepsilon B^n) - \mathrm{vol}_n(K)}{\varepsilon}$$

Corollary 3.1 (**Isoperimetric Inequality for convex bodies**). *Let K be in $\mathcal{K}(\mathbb{R}^n)$, then for $n \geq 1$*

$$\left(\frac{\mathrm{vol}_n(K)}{\mathrm{vol}_n(B^n)}\right)^{1/n} \leq \left(\frac{S_n(K)}{S_n(B^n)}\right)^{1/(n-1)}$$

Proof. The idea is to use the previous definition and 3.2. Let $K \in \mathcal{K}(\mathbb{R}^n)$ and $\epsilon = \dfrac{t}{1-t} > 0$, then,

$$\begin{aligned}
\mathrm{vol}_n(K + \epsilon B^n) &= \mathrm{vol}_n\left(\frac{1}{1-t}((1-t)K + tB^n)\right) = \left(\frac{1}{1-t}\right)^n \mathrm{vol}_n((1-t)K + tB^n) \\
&\geq \left(\frac{(1-t)\mathrm{vol}_n(K)^{1/n} + t\mathrm{vol}_n(B^n)^{1/n}}{1-t}\right)^n \\
&= \left(\mathrm{vol}_n(K)^{1/n} + \epsilon \mathrm{vol}_n(B^n)^{1/n}\right)^n \\
&= \mathrm{vol}_n(K)\left(1 + \epsilon\left(\frac{\mathrm{vol}_n(B^n)}{\mathrm{vol}_n(K)}\right)^{1/n}\right)^n \\
&\geq \mathrm{vol}_n(K)\left(1 + n\epsilon\left(\frac{\mathrm{vol}_n(B^n)}{\mathrm{vol}_n(K)}\right)^{1/n}\right)
\end{aligned}$$

And the last inequality follows by expanding $(1+x)^n = 1 + \binom{n}{1}x + C$ where $x = \epsilon\left(\dfrac{\mathrm{vol}_n(B^n)}{\mathrm{vol}_n(K)}\right)^{1/n}$ and noticing that $C \geq 0$.

Now, by the definition of surface area we have that

$$\begin{aligned}
S_n(K) = \mathrm{vol}_{n-1}(\partial K) &= \lim_{\epsilon \to 0} \frac{\mathrm{vol}_n(K + \epsilon B^n) - \mathrm{vol}_n(K)}{\epsilon} \\
&\geq \lim_{\epsilon \to 0} \frac{\mathrm{vol}_n(K) + n\epsilon \mathrm{vol}_n(K) \left(\frac{\mathrm{vol}_n(B^n)}{\mathrm{vol}_n(K)} \right)^{1/n} - \mathrm{vol}_n(K)}{\epsilon} \\
&= n\mathrm{vol}_n(K) \left(\frac{\mathrm{vol}_n(B^n)}{\mathrm{vol}_n(K)} \right)^{1/n} \\
&= n\mathrm{vol}_n(K)^{(n-1)/n} \mathrm{vol}_n(B^n)^{1/n}
\end{aligned}$$

In particular, if $K = B^n$ we get equality in all the previous inequalities, so $S_n(B^n) = n\mathrm{vol}_n(B^n)$. Moreover, $\mathrm{vol}_n(B^n + \epsilon B^n) = (1+\epsilon)^n \mathrm{vol}_n(B^n)$. Then

$$\begin{aligned}
\left(\frac{S_n(K)}{S_n(B^n)} \right)^{1/(n-1)} &\geq \left(\frac{n\mathrm{vol}_n(K)^{(n-1)/n} \mathrm{vol}_n(B^n)^{1/n}}{n\mathrm{vol}_n(B^n)} \right)^{1/(n-1)} \\
&= \left(\frac{\mathrm{vol}_n(K)}{\mathrm{vol}_n(B^n)} \right)^{1/n}
\end{aligned}$$

And the isoperimetric inequality follows. $\square$

Chapter 4

Mixed volumes

In the following chapters, we will focus on mixed volumes and the Aleksandrov–Fenchel inequality. The mixed volumes arise when we combine the concepts of Minkowski sum and volume. We will see a special type of mixed volumes for a convex body K, sometimes called *quermassintegrals* [Sch13, Chapter 4], resulting from the (euclidean) Steiner polynomial to compute the volume of the Minkowski sum of K and a ball or radius r. This will be the be starting point to talk about the euclidean intrinsic volumes which are relevant to us. The importance of the Aleksandrov–Fenchel inequality in this particular case relies on the fact that the log-concavity of these quermassintegrals can be inferred from it.

Definition 4.1 ([LAR])**.** Let $r \in \mathbb{N}$ and $\lambda_1, \dots, \lambda_r \geq 0$. The **mixed volume** of the convex bodies $K_1, \dots, K_n$ is the unique, symmetric and multilinear form $V : (\mathcal{K}(\mathbb{R}^n))^n \to \mathbb{R}^+$ such that:

$$\mathrm{vol}_n\left(\sum_{i=1}^{r} \lambda_i K_i\right) = \sum_{i_1=1}^{r} \dots \sum_{i_n=1}^{r} V(K_{i_1}, \dots, K_{i_n})\lambda_{i_1} \cdots \lambda_{i_n}$$

We will see later two important inequalities due to Minkowski relating the Mixed volume of two bodies to their volumes. As a matter of fact, the second inequality will be very important in the future in order to prove the Aleksandrov-Fenchel inequality. Keeping this in mind, we introduce the following notation:

$$V_m(K_1, K_2) = V(\underbrace{K_1, \dots, K_1}_{n-m}, \underbrace{K_2, \dots, K_2}_{m}) \text{ for all } 0 \leq m \leq n. \tag{4.1}$$

Remark 4.1. Note that $V_0(K_1, K_2) = V(K_1, \dots, K_1) = \mathrm{vol}_n(K_1)$ by the definition of V. In the same way, $V_n(K_1, K_2) = W(K_2, \dots, K_2) = V_n(K_2)$

Example 4.1. If we use the notation just mentioned, then

- For $K_1, K_2 \in \mathcal{K}(\mathbb{R}^n)$ and $\lambda_1, \lambda_2 \geq 0$ we have

$$\mathrm{vol}_n(\lambda_1 K_1 + \lambda_2 K_2) = \sum_{m=0}^{n} \binom{n}{m} \lambda_1^{n-m} \lambda_2^m V_m(K_1, K_2).$$

- Let $K \in \mathcal{K}(\mathbb{R}^n)$

$$\begin{aligned}
S_n(K) = \mathrm{vol}_{n-1}(\partial K) &= \lim_{\varepsilon \to 0} \frac{\mathrm{vol}_n(K + \varepsilon B^n) - \mathrm{vol}_n(K)}{\varepsilon} \\
&= \lim_{\varepsilon \to 0} \frac{\displaystyle\sum_{m=0}^{n} \binom{n}{m} \varepsilon^m V_m(K, B^n) - V_0(K, B^n)}{\varepsilon} \\
&= \lim_{\varepsilon \to 0} \sum_{m=1}^{n} \binom{n}{m} \varepsilon^{m-1} V_m(K, B^n) \\
&= \lim_{\varepsilon \to 0} \left[\binom{n}{1} V_1(K, B^n) + \sum_{m=2}^{n} \binom{n}{m} \varepsilon^{m-1} V_m(K, B^n) \right] \\
&= n V_1(K, B^n)
\end{aligned}$$

So, the surface area is a particular mixed volume.

4.1 Mixed volumes are polynomials

The fact that for $K_1, \ldots, K_r$ nonempty convex bodies and $\lambda_1, \ldots, \lambda_r \geq 0$ the volume $\mathrm{vol}_n(\lambda_1 K_1 + \ldots + \lambda_r K_r)$ is a polynomial is not a trivial fact. Actually, we are using implicitly this in the definition of mixed volume given above and is what gives sense to it. Moreover, knowing this is what will allow us to define the euclidean intrinsic volumes using the Steiner formula. Indeed, we can say a bit more: $\mathrm{vol}_n(\lambda_1 K_1 + \ldots + \lambda_r K_r)$ is a homogeneous polynomial and we state it as a theorem.

Theorem 4.1. *Let $\lambda_1, \ldots, \lambda_r > 0$ and $K_1, \ldots, K_r$ strictly convex sets in $\mathbb{R}^n$ with C^∞ boundary. Then $\mathrm{vol}_n(\lambda_1 K_1 + \ldots + \lambda_r K_r)$ is a homogeneous polynomial of degree n.*

Proof. We denote by $\widehat{dx_i}$ the form $(-1)^{i-1} dx_1 \wedge \ldots \wedge dx_{i-1} \wedge dx_{i+1} \wedge \ldots \wedge dx_n$. Then

$$d\left(\sum_{i=1}^{r} x_i \widehat{dx_i}\right) = n dx_1 \wedge \ldots \wedge dx_n = n dx$$

Then, by the Stokes theorem, we get that

$$\mathrm{vol}_n(K) = \int_K dx = \frac{1}{n} \int_{\partial K} \sum_{i=1}^{r} x_i \widehat{dx_i}$$

The gradient $\nabla h_k : S^{n-1} \to \partial K$ is a parametrization of the boundary of K. So, we can pullback the integral to S^{n-1}:

$$\mathrm{vol}_n(K) = \frac{1}{n} \int_{S^{n-1}} \sum_{i=1}^{n} \frac{\partial h_k}{\partial x_i} \widehat{d \frac{\partial h_k}{\partial x_i}}.$$

Let's denote the support function of K_i by H_i. Using that $h_{\sum \lambda_i K_i} = \sum \lambda_i h_{K_i}$ we get

$$\int_{\sum_j \lambda_j K_j} dx = \frac{1}{n} \int_{S^{n-1}} \sum_{i=1}^{n} \left(\sum_j \lambda_j \frac{\partial H_j}{\partial x_i} \right) \left(d \sum_j \lambda_j \frac{\partial H_j}{\partial x_i} \right)^{\wedge}$$

Where $(*)^\wedge$ is used to denote $\widehat{(*)}$. So, $\mathrm{vol}_n(\lambda_1 K_1 + \ldots + \lambda_r K_r)$ is a homogeneous polynomial of degree n in the variables $\lambda_1, \ldots, \lambda_r \geq 0$. Now, if we want to move from the space of strictly convex bodies with smooth boundary to the space of non-empty convex bodies, we use the fact that any general convex set can be approximated by strictly convex sets. □

4.2 Properties of Mixed volumes

From the theorem 4.1 we see that for convex bodies $K_1, \dots, K_n$ the mixed volume $V : (\mathcal{K}(\mathbb{R}^n))^n \to \mathbb{R}^+$ is a symmetric form. We list now some more properties of V.

Theorem 4.2. *Let $K_1', K_1, \dots, K_n \in \mathcal{K}(\mathbb{R}^n)$ with non-empty interior and $a, b \geq 0$, then*

$$V(ak_1' + bK_2, K_2, \dots, K_n) = aV(K_1', K_2, \dots, K_n) + bV(K_1, \dots, K_n)$$

The following property follows from the translation invariance of vol_n and equating the coefficients of $\lambda_1 \cdots \lambda_n$. Note also the fact that a translation of some K_i results in a translation of $\sum \lambda_i K_i$.

Theorem 4.3 (Translation invariance of mixed volumes)**.** *Let $K_1', K_1, K_2, \dots, K_n \in \mathcal{K}(\mathbb{R}^n)$ such that $K_1' = K_1 + a$ for some $a \in \mathbb{R}^n$, then*

$$V(K_1', K_2, \dots, K_n) = V(K_1, K_2, \dots, K_n)$$

Theorem 4.4 (Monotonicity)**.** *Let $K_1', K_1, K_2, \dots, K_n$ convex bodies such that $K_1' \subseteq K_1$, then*

$$V(K_1', K_2, \dots, K_n) \leq V(K_1, K_2, \dots, K_n)$$

One important consequence of this theorem is that the mixed volumes are non-negative: By the translation invariance, we can assume that all the convex bodies $K_1, \dots, K_n$ contain the origin, then $0 = V(0, \dots, 0) \leq V(K_1, \dots, K_n)$.

Theorem 4.5 (Continuity of mixed volumes)**.** *For every $j = 1, \dots, n$ let $\{K_j^i\}_{i=1}^{\infty}$ be a sequence of sets in $\mathcal{K}(S^{n-1})$ such that $K_j^i \to K_j \in \mathcal{K}(S^{n-1})$ with respect to the Hausdorff metric on $\mathcal{K}(S^{n-1})$. Then*

$$V(K_1^i, \dots, K_n^i) \to V(K_1, \dots, K_n) \text{ as } i \to \infty.$$

4.3 Minkowski's inequalities for mixed volumes

We present now two inequalities discovered by Minkowski. The second inequality is a particular case of something more general, which is the Aleksandrov-Fenchel inequality. However, we present it to show again the relevance of the Brunn-Minkowski and Aleksandrov-Fenchel's inequalities.

Theorem 4.6. *Let K, L convex bodies in $\mathbb{R}^n$, then*

$$V_1(K, L) \geq \mathrm{vol}_n(K)^{(n-1)/n} \mathrm{vol}_n(L)^{1/n}$$

Proof. For $0 \leq \lambda \leq 1$, define the function

$$f(\lambda) = \mathrm{vol}_n(K_\lambda)^{1/n}$$

where $K_\lambda = (1-\lambda)K_0 + \lambda K_1$ for convex bodies K_0 and K_1. By the Brunn-Minkowski inequality, *f is a concave function on* $(0, 1)$: To see this, let $x, y, \lambda \in [0, 1]$, then

$$\begin{aligned} f((1-\lambda)x + \lambda y) &= \mathrm{vol}_n\big((1 - [(1-\lambda)x + \lambda y])K_0 + [(1-\lambda)x + \lambda y]K_1\big)^{1/n} \\ &= \mathrm{vol}_n\big((1 - x + \lambda x - \lambda y)K_0 + (x - \lambda x + \lambda y)K_1\big)^{1/n} \\ &= \mathrm{vol}_n\big((1 - x - \lambda + \lambda x)K_0 + (x - \lambda x)K_1 + (\lambda - \lambda y)K_0 + \lambda y K_1\big)^{1/n} \\ &= \mathrm{vol}_n\big((1-\lambda)(1-x)K_0 + (1-\lambda)xK_1 + \lambda[(1-y)K_0 + yK_1]\big)^{1/n} \\ &= \mathrm{vol}_n\big((1-\lambda)K_x + \lambda K_y\big)^{1/n} \end{aligned}$$

And using the Brunn-Minkowski inequality we get

$$\geq (1-\lambda)f(x) + \lambda f(y).$$

Thus, as f is concave, this implies that $f'(0) \geq \dfrac{f(1)-f(0)}{1-0} = f(1) - f(0)$. Now,

$$f'(0) = \frac{1}{n}\mathrm{vol}_n(K)^{(1-n)/n}\frac{\partial}{\partial\lambda}\left[\mathrm{vol}_n(K_\lambda)\right](0).$$

Let's focus on $\frac{\partial}{\partial\lambda}\left[\mathrm{vol}_n(K_\lambda)\right](0)$:

$$\frac{\partial}{\partial\lambda}\left[\mathrm{vol}_n(K_\lambda)\right](0) = \lim_{\varepsilon\to 0}\frac{\mathrm{vol}_n(K_\varepsilon) - \mathrm{vol}_n(K_0)}{\varepsilon}$$

Now, as we did in example 4.1,

$$= \lim_{\varepsilon\to 0}\frac{\displaystyle\sum_{i=0}^{n}\binom{n}{i}(1-\varepsilon)^{n-i}\varepsilon^i V_i(K_0,K_1) - \mathrm{vol}_n(K_0)}{\varepsilon}$$

Recall that $V_0(K_0,K_1) = \mathrm{vol}_n(K_0)$, so

$$= \lim_{\varepsilon\to 0}\sum_{i=1}^{n}\binom{n}{i}(1-\varepsilon)^{n-i}\varepsilon^{i-1}V_i(K_0,K_1) - \frac{(1-\varepsilon)^n - 1}{\varepsilon}\mathrm{vol}_n K_0$$

Note that $(1-\varepsilon)^n - 1 = (-\varepsilon)[(1-\varepsilon)^{n-1} + \ldots + (1-\varepsilon) + 1]$, then

$$= nV_1(K_0,K_1) - n\mathrm{vol}_n(K_0)$$

therefore,

$$f'(0) = \frac{V_1(K_0,K_1) - \mathrm{vol}_n(K_0)}{\mathrm{vol}_n(K_0)^{(n-1)/n}}$$

and the inequality $f'(0) \geq f(1) - f(0)$ can be restated as

$$V_1(K_0,K_1) - \mathrm{vol}_n(K_0) \geq \mathrm{vol}_n(K_0)^{(n-1)/n}\left(\mathrm{vol}_n(K_1)^{1/n} - \mathrm{vol}_n(K_0)^{1/n}\right)$$

which after simplification is equivalent to

$$V_1(K_0,K_1) \geq \mathrm{vol}_n(K_0)^{(n-1)/n}\mathrm{vol}_n(K_1)^{1/n}$$

and the proof is complete. tails here. □

For the next inequality, the proof is based on an idea found in [LAR, Theorem 3.9] but we give the complete the details here.

Theorem 4.7 (Minkowski's second inequality)**.** *Let K, L convex bodies in $\mathbb{R}^n$, then*

$$V_1(K,L)^2 \geq V_0(K,L)V_2(K,L) = \mathrm{vol}_n(K)V_2(K,L)$$

Proof. Let $f(\lambda)$ and K_λ as before. Being f concave on $(0,1)$ we have that $f''(0) \leq 0$. If we use a similiar procedure to prove the Minkoski's first inequality, then we find that

$$f''(\lambda) = \frac{1-n}{n^2}\mathrm{vol}_n(K_\lambda)^{(1-2n)/n}\left[\frac{\partial}{\partial\lambda}[\mathrm{vol}_n(K_\lambda)]\right]^2 + \frac{1}{n}\mathrm{vol}_n(K_\lambda)^{(1-n)/n}\frac{\partial^2}{\partial\lambda^2}[\mathrm{vol}_n(K_\lambda)]$$

Using the previous computation in the first inequality

$$f''(0) = (1-n)\mathrm{vol}_n(K_0)^{(1-2n)/n}[V_1(K_0,K_1) - \mathrm{vol}_n(K_0)]^2 + \frac{1}{n}\mathrm{vol}_n(K_0)^{(1-n)/n}\frac{\partial^2}{\partial\lambda^2}[\mathrm{vol}_n(K_\lambda)](0).$$

Now,

$$\begin{aligned}\frac{\partial^2}{\partial\lambda^2}[\mathrm{vol}_n(K_\lambda)](0) &= \lim_{\varepsilon\to 0}\frac{\frac{\partial}{\partial\lambda}[\mathrm{vol}_n(K_\lambda)](\varepsilon) - \frac{\partial}{\partial\lambda}[\mathrm{vol}_n(K_\lambda)](0)}{\varepsilon}\\ &= \lim_{\varepsilon\to 0}\frac{\frac{\partial}{\partial\lambda}[\mathrm{vol}_n(K_\lambda)](\varepsilon) - nV_1(K_0,K_1) + n\mathrm{vol}_n(K_0)}{\varepsilon}\end{aligned}$$

So, we need to compute $\frac{\partial}{\partial\lambda}[\mathrm{vol}_n(K_\lambda)](\varepsilon)$. We use the expression obtained in 4.1 to do that:

$$\begin{aligned}\frac{\partial}{\partial\lambda}[\mathrm{vol}_n(K_\lambda)] = -n(1-\lambda)^{n-1}V_0(K_0,K_1) + \sum_{m=1}^{n-1}\binom{n}{m}(1-\lambda)^{n-m-1}\lambda^{m-1}(m-n\lambda)V_m(K_0,K_1)\\ +n\lambda^{n-1}V_n(K_0,K_1)\end{aligned}$$

Thus,

$$\begin{aligned}\frac{\frac{\partial}{\partial\lambda}[\mathrm{vol}_n(K_\lambda)](\varepsilon) - nV_1(K_0,K_1) + n\mathrm{vol}_n(K_0)}{\varepsilon} &= -n\frac{[(1-\varepsilon)^{n-1}-1]}{\varepsilon}V_0(K_0,K_1)\\ +n\frac{[-\varepsilon n(1-\varepsilon)^{n-2} + \big((1-\varepsilon)^{n-2}-1\big)]}{\varepsilon}V_1(K_0,K_1) &+ \frac{\binom{n}{2}(1-\varepsilon)^{n-3}\varepsilon(2-n\varepsilon)V_2(K_0,K_1)}{\varepsilon}\\ +\frac{\sum_{m=3}^{n-1}\binom{n}{m}(1-\varepsilon)^{n-m-1}\varepsilon^{m-1}(m-n\varepsilon)V_m(K_0,K_1) + n\varepsilon^{n-1}V_n(K_0,K_1)}{\varepsilon}\end{aligned}$$

And if we make $\varepsilon \to 0$ we get

$$\begin{aligned}&n(n-1)V_0(K_0,K_1) + n[-n-(n-2)]V_1(K_0,K_1) + n(n-1)V_2(K_0,K_1)\\ &= n(n-1)V_0(K_0,K_1) - 2n(n-1)V_1(K_0,K_1) + n(n-1)V_2(K_0,K_1)\end{aligned}$$

Putting this expression in $f''(0)$ we get

$$\begin{aligned}&= (1-n)\mathrm{vol}_n(K_0)^{(1-2n)/n}V_1(K_0,K_1)^2 - 2(1-n)\mathrm{vol}_n(K_0)^{(1-n)/n}V_1(K_0,K_1) + (1-n)\mathrm{vol}_n(K_0)^{1/n}\\ &- 2(n-1)\mathrm{vol}_n(K_0)^{(1-n)/n}V_1(K_0,K_1)^+(n-1)\mathrm{vol}_n(K_0)^{(1-n)/n}V_2(K_0,K_1) + (n-1)\mathrm{vol}_n(K_0)^{1/n}\\ &= (1-n)\mathrm{vol}_n(K_0)^{(1-2n)/n}[V_1(K_0,K_1)^2 - V_2(K_0,K_1)\mathrm{vol}_n(K_0)]\end{aligned}$$

Recalling that $f''(0)$ must be less or equal than cero

$$f''(0) = -(n-1)\mathrm{vol}_n(K_0)^{(1-2n)/n}\Big(V_1(K_0,K_1)^2 - V_2(K_0,K_1)V_0(K_0,K_1)\Big) \leq 0.$$

We deduce that what is between the big parenthesis must be negative, so the inequality follows. □

Chapter 5

The Aleksandrov-Fenchel Inequality

In the next chapter, we will talk about intrinsic volumes. In particular, the euclidean intrinsic volumes will be discussed. However, in order to do that, and most importantly, to prove that they form a log-concave sequence is crucial for us to use the Aleksandrov-Fenchel inequality. In this line, we saw in the previous chapter the second inequality of Minkowski and its proof based on the Brunn- Minkowski inequality. Now, we will present a more general result relating mixed volumes and considered as the most important in its nature.

Theorem 5.1 (**Aleksandrov-Fenchel Inequality**). *Let $K_1, \dots, K_n \in \mathcal{K}(\mathbb{R}^n)$, then we have the following inequality*

$$V^2(K_1, \dots, K_n) \geq V(K_1, K_1, K_3, \dots, K_n)V(K_2, K_2, K_3, \dots, K_n) \tag{5.1}$$

If K_1 and K_2 are homothetic equality holds, but conditions of equality in general are unknown.

The proof we shall give of the theorem 5.1 is due to Aleksandrov [Ale96], which historically was the first proof of the inequality and uses strongly isomorphic polytopes and approximation. This proof is reproduced in [Sch13] and here as well. The second proof given by Aleksandrov of this inequality uses the same idea of Hilbert's proof of Brunn- Minkowski's inequality. In [LAR] we can find a proof of 5.1 based on the concept of positive differential forms and similar to the second one given by Aleksandrov.

5.1 Definitions and notations

Definition 5.1. A polytope P is the convex hull of a finite set of points in $\mathbb{R}^n$. Each polytope is the intersection of a finite set of (closed) half-spaces [Sch13, Thm. 2.4.3]. In other words, every polytope is polyhedral.

A polytope is called simplicial if all its proper faces (equivalently, all its facets) are simplices. An polytope P of dimension n (n-polytope) is called simple if each of its vertices is contained in exactly n facets.

Definition 5.2. Let P_1 and P_2 polytopes in $\mathbb{R}^n$. Then, P_1 is *strongly isomorphic* to P_2 if

$$\dim F(P_1, u) = \dim F(P_2, u)$$

for every $u \in S^{n-1}$. This notion gives rise to an equivalence relation on the set of polytopes. The corresponding equivalence class of a polytope P will be called its *a-type*.

Let $\mathcal{A}$ be a given a-type of strongly isomorphic simple n-dimensional polytopes. Let $u_1, \dots, u_n$ be the normal vectors of the facets of any $P \in \mathcal{A}$. The N-tuple

$$\overline{P} := (h_1, \dots, h_N) \in \mathbb{R}^n \text{ with } h_i := h_i(P, u_i)$$

is called the *support vector* of P. Moreover, let $F_i := F(P, u_i)$ and $F_{ij} = F_i \cap F_j$. Define

$$J := \{(i,j) : i, j \in \{1, \dots, N\},\ \dim F_{ij} = n-2\}$$

So, J depends only on the a-type $\mathcal{A}$. Furthermore, for $(i,j) \in J$ let θ_{ij} be the angle between u_i and u_j.

We will need the following lemma and the proof is omitted here ([Sch13, Lemma 5.1.3])

Lemma 5.1. *Given $P \in \mathcal{A}$. The volume of P can be represented as*

$$\operatorname{vol}_n(P) = \sum a_{j_1 \dots j_n} h_{j_1} \dots h_{j_n}$$

Where the sum extends over $j_1, \dots, j_n \in \{1, \dots, N\}$ and the coefficients $a_{j_1 \dots j_n}$ depend only on the a-type $\mathcal{A}$.

Now, suppose we have polytopes $P_1, \dots, P_n$. Let $h_i^{(r)} := h_i(P_r, u_i)$ and $F_i^{(r)} := F(P_r, u_i)$ then we introduce the *mixed volume* of $P_1, \dots, P_n$ by

$$V(P_1, \dots, P_n) = \sum a_{j_1 \dots j_n} h_{j_1}^{(1)} \dots h_{j_n}^{(n)}$$
$$v(F_i^{(1)}, \dots, F_i^{(n-1)}) = \sum a_{k_1 \dots k_{n-1}}^{(i)} h_{ik_1}^{(1)} \dots h_{ik_{n-1}}^{(n-1)}$$

Where the coefficients are those of the previous lemma. Thus, $V(P_1, \dots, P_n)$ is symmentric in its arguments and

$$V(P, \dots, P) = \operatorname{vol}_n(P)$$

Now, we want to extend this notion to form the *mixed volume* of N-tuples $X_1, \dots, X_n$ where $X_i = (x_1^{(r)}, \dots, x_N^{(r)})$ by letting

$$V(X_1, \dots, X_n) := \sum a_{j_1 \dots j_n} x_{j_1}^{(1)} \dots x_{j_n}^{(n)}$$

Then V is an N-linear function on $\mathbb{R}^N$. Given $X = (x_1, \dots, x_N) \in \mathbb{R}^n$ we define

$$x_{ij} = \begin{cases} x_j \csc \theta_{ij} - x_i \cot \theta_{ij}, & \text{if } (i,j) \in J \\ 0, & \text{if } (i,j) \notin J \end{cases}$$

And

$$\Lambda_i X := (x_{i1}, \dots, x_{iN})$$

So that $\Lambda_i : \mathbb{R}^N \to \mathbb{R}^N$ is a linear map. Then we put

$$v(\Lambda_i X_1, \dots, \Lambda_i X_{n-1}) := \sum a_{k_1 \dots k_{n-1}}^{(i)} x_{ik_1}^{(1)} \dots x_{ik_{n-1}}^{(n-1)}.$$

We shall often identify $P \in A$ with its support vector $\overline{P}$; that is, if in $V(X_1, \dots, X_n)$ or $v(\Lambda_i X_1, \dots, \Lambda_i X_{n-1})$ one of the arguments X_r is a support vector $\overline{P}_r$, we will replace X_r by P_r and $\Lambda_i X_r$ by $F_i^{(r)} := F(P_r, u_i)$. Further we say that $Z = (\zeta_1, \dots, \zeta_N)$ is the *support vector* of $z \in \mathbb{R}^n$ if $\zeta_i = h(\{z\}, u_i)$ for all $i = 1, \dots, N$. In other words, Z is the support vector of z if

$$Z = (\langle z, u_1 \rangle, \dots, \langle z, u_N \rangle).$$

Again, we need another lemma to generalize the results to N-tuples in $\mathbb{R}^N$ [Sch13, Lemma 5.1.5].

Lemma 5.2. $V(P_1, \dots, P_n) = \frac{1}{n} \sum_{i=1}^{N} h_i^{(1)} v(F_i^{(2)}, \dots, F_i^{(n)})$.

This lemma extends to

$$v(\Lambda_i X_1, \dots, \Lambda_i X_{n-1}) := \frac{1}{n} \sum_{i=1}^{N} x_i^{(1)} v(\Lambda_i X_2, \dots, \Lambda_i X_n)$$

5.2 Proof of the inequality

The following theorem is a sharper version of a special case of Theorem 5.1, and the latter can be deduced from it.

Theorem 5.2. *Let $P, P_3, \dots, P_n$ be strongly isomorphic polytopes of the simple a-type $\mathcal{A}$ and let $Z \in \mathbb{R}^N$, then*

$$V(Z, P, P_3, \dots, P_n)^2 \geq V(Z, Z, P_3, \dots, P_n) V(P, P, P_3, \dots, P_n).$$

The equality holds if and only if $Z = \lambda P + A$, where $\lambda \in \mathbb{R}$ and A is the support vector of a point.

The general case of the Aleksandrov-Fenchel inequality follows from this theorem, the continuity of the mixed volumes and the following approximation theorem [Sch13, Thm. 2.4.15]:

Theorem 5.3. *Let $K_1, \dots, K_m$ be convex bodies in $\mathbb{R}^n$. For each $\varepsilon > 0$ there exist simple strongly isomorphic polytopes $P_1, \dots, P_m$ of dimension n satisfying $\delta(K_i, P_i) < \varepsilon$ for $i = 1, \dots, m$.*

proof of the main theorem. We introduce a symmetric bilinear form Φ on $\mathbb{R}^N$ by

$$\Phi(X, Y) := V(X, Y, P_3, \dots, P_n) \text{ for } X, Y \in \mathbb{R}^N.$$

Proposition 5.1 (Claim 1). *If $\Phi(Z, P) = 0$ then*

$$\Phi(Z, Z) \leq 0$$

, and equality holds if and only if Z is the support vector of a point.

In fact, suppose that claim 1 is true. If $Z \in \mathbb{R}^N$ is given, define

$$\lambda := \frac{\Phi(Z, P)}{\Phi(P, P)} \text{ and } Z' = Z - \lambda \overline{P}.$$

Note that $\Phi(P, P) = V(P, P, P_3, \dots, P_n) > 0$. Then $\Phi(Z', P) = 0$ and therefore $\Phi(Z', Z') \leq 0$ with equality if and only if Z' is the support vector of a point. From

$$\Phi(Z', Z') = \Phi(Z, Z) - \frac{\Phi(Z, P)^2}{\Phi(P, P)}$$

The assertion of the theorem follows.

To prove claim 1, we make induction on n. The case $n = 2$ is the inequality

$$V(Z, P, \dots, P) \geq V(Z, Z, P, \dots, P) V(P, P, \dots, P)$$

which is the Minkowski's second inequality.

Suppose that the assertion is true for $2 \leq k < n$. For each $i = 1, 2, \dots, N$ we define a symmetric bilinear form ϕ on $\mathbb{R}^N$ by

$$\phi_i(X, Y) := v(\Lambda_i X, \Lambda_i Y, F_i^{(4)}, \dots, F_i^{(n)}).$$

Proposition 5.2 (Claim 2)**.** *Let $Z \in \mathbb{R}^N$. Then Z is an eigenvector of Φ with eigenvalue 0 if and only if Z is the support vector of a point.*

Proof. Note that

$$\Phi(X, Y) = \frac{1}{n} \sum_{i=1}^{N} x_i \phi(Y, P_3).$$

Since $\phi(., P_3)$ is linear, it is of the form

$$\phi_i(Y, P_3) = \sum_{j=1}^{N} b_{ij} y_j$$

Thus,

$$\Phi(X, Y) = \frac{1}{n} \sum_{i,j=1}^{N} b_{ij} x_i y_j.$$

Here, $b_{ij} = b_{ji}$ because Φ is symmetric. Say that $Z = (\zeta_1 \ldots, \zeta_N)$ is an eigenvector of Φ associated to the eigenvalue 0 means that

$$\sum_{j=1}^{N} b_{ij} \zeta_j = 0$$

for every i. Equivalently,

$$\phi(Z, P_3) = 0 \text{ for } i = 1, \ldots, N.$$

If Z is the support vector of the point z, then

$$\phi(Z, P_3) = v(\{z\}, F_i^{(3)}, \ldots, F_i^{(n)}) = 0.$$

Conversely, suppose that $\phi_i(Y, P_3) = \sum_{j=1}^{N} b_{ij} y_j$. By the induction hypothesis, this implies $\phi_i(Z, Z) \leq 0$. Without loss of generality, we may assume that $h(P_3, u_i) > 0$; then

$$\begin{aligned} 0 &= \frac{1}{n} \sum_{i=1}^{N} \zeta_i \phi_i(Z, P_3) = \Phi(Z, Z) \\ &= V(Z, Z, P_3, \ldots, P_n) \\ &= V(P_3, Z, Z, P_4, \ldots, P_n) \\ &= \frac{1}{n} \sum_{i=1}^{N} h(P_3, u_i) \phi_i(Z, Z) \leq 0. \end{aligned}$$

And hence $\phi(Z, Z) = 0$. By the induction hypothesis this implies that $\Lambda_i Z$ is the support vector, relative to the a-type of F_i, of a point z_i. In other words, this means that

$$\Lambda_i Z = (\langle z_i, v_{i1} \rangle, \ldots, \langle z_i, v_{iN} \rangle),$$

with $v_{ij} = 0$ if $(i, j) \notin J$. Pick $\varepsilon > 0$ such that $\overline{P}_3 + \varepsilon Z$ is a support vector of some polytope $Q \in \mathcal{A}$. The fact that $\Lambda_i(\overline{P}_3 + \varepsilon Z) = \Lambda_i \overline{P}_3 + \varepsilon \Lambda_i Z$ yields

$$\begin{aligned} (F(Q, u_i), v_{ij}) &= h(F_i^{(3)}, v_{ij}) + \varepsilon \langle z_i, v_{ij} \rangle \\ &= h(F_i^{(3)} + \varepsilon z_i, v_{ij}). \end{aligned}$$

And thus $F(Q, u_i) = F_i^{(3)} + t_i$ with a vector $t_i = \varepsilon z_i + a_i u_i$ for some $a_i \in \mathbb{R}$. Then, for $(i, j) \in J$ we conclude that the $(n-2)$-face $G := F(Q, u_i) \cap F(Q, u_j)$ satisfies $G = F_{ij}^{(3)} + t_i$ and $G = F_{ij}^{(3)} + t_j$. So we have $t_i = t_j$. Since any two facets of P_3 can be joined by a chain of facets such that any two consecutive facets in the chain have an $(n-2)$-dimensional intersection, we conclude that $t_i = t_j$ for all i, j. Thus Q is a translate of P_3 and, hence, Z is the support vector of a point. This completes the proof of claim 2. □

We now introduce another symmetric bilinear form Ψ on $\mathbb{R}^N$. Let $X, Y \in \mathbb{R}^N$:

$$\Psi(X, Y) = \frac{1}{n} \sum_{i=1}^{N} \frac{\phi_i(P, P_3)}{h(P, u_i)} \, x_i y_i$$

Here we assume, without loss of generality, that $h(P, u_i) > 0$ for $i = 1, \ldots, N$. Since $\phi_i(P, P_3) > 0$, the form Ψ is positive definite.

We consider the eigenvalues $\lambda_1 > \lambda_2 > ...$ of Φ relative to Ψ and make use of the fact that

$$\lambda_1 = \max\{\Phi(X, X) : \Psi(X, X) = 1\}$$
$$\lambda_2 = \max\{\Phi(X, X) : \Psi(X, X) = 1 \text{ and } \Psi(X, Y) = 0 \text{ for all } Y \text{ in the } \lambda_1 - \text{eigenspace}\}$$

Analogously, we write

$$\Psi(X, Y) = \frac{1}{n} \sum_{i,j=1}^{N} c_{ij} x_i y_j$$

Where

$$c_{ij} := \begin{cases} \dfrac{\phi_i(P, P_3)}{h(P, u_i)} & \text{if } i = j \\ 0 & \text{if } i \neq j \end{cases}$$

Then $Z = (\zeta_1, \ldots, \zeta_N) \in \mathbb{R}^N$ is an eigenvector of Φ relative to Ψ with eigenvalue λ if and only if

$$\sum_{j=1}^{N} (b_{ij} - \lambda c_{ij}) \zeta_j = 0 \text{ for all } i = 1, \ldots, N$$

Or, equivalently, if

$$\phi_i(Z, P_3) = \lambda \frac{\phi_i(P, P_3)}{h(P, u_i)} \zeta_i \text{ for all } i = 1, \ldots, N$$

So, in particular, $\lambda = 1$ is an eigenvalue with associated eigenvector $Z = \overline{P}$

Proposition 5.3 (Claim 3). *The only positive eigenvalue of Φ relative to Ψ is 1, and it is simple.*

For the proof, we first assume that $P = P_3 = \ldots = P_n$. Suppose claim 3 were false in this case. If there is a positive eigenvalue $\mu \neq 1$, then there exists $Z \in \mathbb{R}^N$ with $\Psi_(Z, P) = 0$ and $\Phi(Z, Z) = \mu \Psi(Z, Z) > 0$. If 1 is a multiple eigenvalue, the corresponding eigenspace is at least two-dimensional and hence contains a vector Z with $\Psi(Z, P) = 0$ and $\Phi(Z, Z) = \Psi(Z, Z) > 0$. Thus, in either case we conclude from

$$\Psi(Z, P) = \frac{1}{n} \sum_{i=1}^{N} \frac{\phi_i(P, P)}{h(P, u_i)} \zeta_i h(P, u_i) = V(Z, P, \ldots, P)$$

that $V(Z, P, \ldots, P) = 0$. But $V(Z, Z, P, \ldots, P)$, which contradicts the Minkowski's second inequality.

Now let $P_3, \dots, P_n \in \mathcal{A}$ be arbitrary. For $\vartheta \in [0,1]$ let $P_r(\vartheta) := (1-\vartheta)P + \vartheta P_r$, $r = 3, \dots, n$. The coefficients of the corresponding forms Φ, Ψ depend continuously on ϑ, hence the same is true for the relative eigenvalues. By Claim 2, the number 0 is always an eigenvalue with multiplicity n. It follows that the sum of the multiplicities of the positive eigenvalues is independent of ϑ. Since it is equal to 1 for $\vartheta = 0$, it must be equal to 1 for $\vartheta = 1$, proving the claim 3.

Claim 3 implies that the eigenspace corresponding to the eigenvalue 1 coincides with $\mathrm{lin}\overline{P}$ and that the second eigenvalue is not positive, and hence that $\Phi(Z,Z) \leq 0$ for all Z satisfying $\Psi(Z,P) = 0$; the latter is equivalent to $\Phi(Z,P) = 0$. Thus $\Phi(Z,P) = 0$ implies $\Phi(Z,Z) \leq 0$. Suppose that we have equality for some $Z \neq 0$. Since at Z the maximum in λ_2 is attained, Z is an eigenvector with eigenvalue 0. By Claim 2, Z is the support vector of a point, completing the proof of the theorem. □

Chapter 6

Intrinsic Volumes

6.1 Euclidean intrinsic volumes

The euclidean intrinsic volumes are a particular case resulting from the mixed volumes. They are very important for us because this is what motivated the study of the spherical intrinsic volumes and the properties satisfied by them. We will take this introduction toward the proof of the log-concavity of the euclidean intrinsic volumes and motivate in this way the study of the spherical ones.

Let $M^e \subseteq \mathbb{R}^n$ and $r \geq 0$. The *tube or radius* r around M^e is defined as

$$T^e(M^e, r) := \{x \in \mathbb{R}^n : d^e(x, y) \leq r \text{ for some } y \in M^e\} = M^e + B^n(r).$$

Where $B^n(r) := \{x \in \mathbb{R}^n : ||x|| \leq r\}$ is the ball of radius r centered at the origin.
If $M \subset S^{n-1}$ then we can make an analogous definition, the *tube of radius* α in S^{n-1} is

$$T(M, \alpha) := \{x \in S^{n-1} : d(x, y) \leq \alpha \text{ for some } y \in M\}.$$

Remark 6.1. If we have the case where $M^e \subseteq \mathbb{R}^n$ and $M \subseteq S^{n-1}$ are closed sets, we can write

$$T^e(M^e, r) = \{x \in \mathbb{R}^n : \min\{d^e(x, y) : y \in M^e\} \leq r\}$$
$$T(M, \alpha) = \{x \in S^{n-1} : \min\{d(x, y) : y \in M\} \leq \alpha\}$$

In the early 1840's, Steiner found that the volume of the tube of radius r around a convex body $K \subset \mathbb{R}^n$ is justly a polynomial in r. We have seen this before, this is a consequence of the theorem 4.1.

Definition 6.1 (Section 4.2)**.** [[Sch13]] Let K be a convex body in $\mathbb{R}^n$, ***the intrinsic volumes*** $V_i^e(K)$ are defined as scaled versions of the coefficients of the Steiner formula for $T(K, r)$:

$$\text{vol}_n T^e(K, r) = \text{vol}_n(K + rB^n) = \sum_{i=0}^{n} \binom{n}{i} V_i(K, B^n) r^i = \sum_{i=0}^{n} \omega_i V_{n-i}^e(K) r^i,$$

Where $\omega_i = \text{vol}_i(B^i) = \dfrac{\pi^{i/2}}{\Gamma(\frac{i+2}{2})}$ is the volume of the ball $B^k \subset \mathbb{R}^k$ and $\Gamma(z) = \displaystyle\int_0^\infty x^{z-1} e^{-x} dx$ is the gamma function.

The numbers V_{n-i}^e are called intrinsic volumes of K because they do not depend on the embedding of K in $\mathbb{R}^n$, in other words, considering the convex body $K \subseteq \mathbb{R}^m$ with $m \geq n$ will give us the same intrinsic volumes that if we consider K embedded in $\mathbb{R}^n$.

Remark 6.2. Let K be a convex body in $\mathbb{R}^n$. From the definition above, we have that $\binom{n}{i}V_i(K,B^n) = \omega_i V^e_{n-i}(K)$

$$V^e_n(K) = \mathrm{vol}_n(K), \quad V^e_{n-1}(K) = \frac{nV_1(K,B)}{2} = \frac{\mathrm{vol}_{n-1}(\partial K)}{2}, \quad V^e_0(K) = 1.$$

Moreover, in the case where K is a polytope, the intrinsic can be computed through the formula

$$V^e_i(K) = \sum_F \mathrm{vol}_i(F)\frac{\mathrm{vol}_{n-i-1}(N^S_F)}{\mathcal{O}_{n-i-1}} \tag{6.1}$$

Where

$$\mathcal{O}_k := \mathrm{vol}_n S^k = \frac{2\pi^{(k+1)/2}}{\Gamma(\frac{k+1}{2})}$$

Example 6.1. Consider $K = B^n$ the unit ball in $\mathbb{R}^n$. As we said before, $B^n(r) := \{x \in \mathrm{R}^n : ||x|| \leq r\} = rB^n$ and therefore, $T(B^n,r) = B^n + rB^n = (1+r)B^n$. So,

$$\mathrm{vol}_n T(B^n,r) = \mathrm{vol}_n\big((1+r)B^n\big) = (1+r)^n \mathrm{vol}_n(B^n) = \sum_{i=0}^{n}\binom{n}{i}\mathrm{vol}_n(B^n)r^i$$
$$= \sum_{i=0}^{n}\binom{n}{i}\omega_n r^i.$$

So,

$$V^e_{n-i}(B^n) = \binom{n}{i}\frac{\omega_n}{\omega_i}.$$

Or, equivalently

$$V^e_i(B^n) = \binom{n}{i}\frac{\omega_n}{\omega_{n-i}}.$$

The following theorem is the most relevant for us, and it says that for any convex body its sequence of intrinsic volumes is log-concave.

Theorem 6.1 ([Ame11])**.** *Let K a convex body in $\mathbb{R}^n$. Then the sequence $V^e_0(K),\ldots,V^e_n(K)$ is log-concave, i.e*

$$V^e_i(K)^2 \geq V^e_{i-1}(K)V^e_{i+1}(K).$$

For all $i = 1,\ldots,n-1$.

Proof. From the definition 6.1 we have that

$$\binom{n}{i}V_i(K,B^n) = \omega_i V^e_{n-i}(K) \text{ or } V^e_i(K) = \binom{n}{i}\frac{V_{n-i}(K,B^n)}{\omega_{n-i}}.$$

Now, if we apply the Aleksandrov-Fenchel inequality to $V_{n-i}(K,B^n)$ we obtain

$$V^2_{n-i}(K,B^n) \geq V_{n-i+1}(K,B^n)V_{n-i+1}(K,B^n)$$

And, correspondingly

$$\frac{V^e_i(K)^2}{V^e_{i+1}(K)V^e_{i-1}(K)} \geq \frac{\omega_{n-i-1}\ .\ \omega_{n-i+1}}{\omega^2_{n-i}}\frac{\binom{n}{i}^2}{\binom{n}{i+1}\binom{n}{i-1}} \geq 1$$

The last inequality follows from the following lemma.

Lemma 6.1. *Let n be a positive integer.*

1. *The sequence $\binom{n}{0}, \dots, \binom{n}{n}$ is log-concave*
2. *The sequence $\omega_0, \omega_1, \dots, \omega_n$ is log-concave.*

1. Let $0 < k < n$. Then

$$\frac{\binom{n}{k}^2}{\binom{n}{k+1}\binom{n}{k-1}} = \frac{(k+1)(n-k+1)}{k(n-k)} \geq 1$$

2. Again, let $0 < k < n$. Here, we use the fact that the Gamma function is a log-convex function, so

$$\frac{\omega_k^2}{\omega_{k-1}\omega_{k+1}} = \frac{\frac{\pi^k}{\Gamma(\frac{k+2}{2})^2}}{\frac{\pi^k}{\Gamma(\frac{k-1}{2})\Gamma(\frac{k+1}{2})}} \geq 1$$

Thus, the theorem follows. □

6.2 Spherical intrinsic volumes

We consider now spherical intrinsic volumes. Central to the definition of them is the following fact:

Theorem 6.2. *Let $K \in \mathcal{K}(S^{n-1})$ and $0 \leq \alpha \leq \pi/2$. The volume of the tube of radius α around K is given by*

$$\mathrm{vol}_{n-1} T(K, \alpha) = \mathrm{vol}_{n-1}(K) + \sum_{j=0}^{n-2} V_j(K) \mathcal{O}_{n-1,j}(\alpha),$$

For some continous functions $V_j : \mathcal{K}(S^{n-1}) \to \mathbb{R}$ and $0 \leq j \leq n-2$. We use the notation $\mathcal{O}_{n-1,j}(\alpha) := \mathrm{vol}_{n-1} T(S, \alpha)$ and $S \in S^k(S^{n-1}) := \{S \subseteq S^{n-1} : S \text{ is a } k\text{-dimensional subsphere}\}$.

Definition 6.2. Let $-1 \leq j \leq n-1$. The $j-$th **spherical intrinsic volume** is a function

$$V_j : \mathcal{K}(S^{n-1}) \cup \{\emptyset, S^{n-1}\} \to \mathbb{R},$$

such that for $K \in \mathcal{K}(S^{n-1})$ and $0 \leq j \leq n-2$ the value of $V_j(K)$ is precisely the quantity $V_j(K)$ in the theorem 6.2. Besides, for $j \in \{-1, n-1\}$,

$$V_{n-1}(K) := \frac{\mathrm{vol}_{n-1}(K)}{\mathcal{O}_{n-1}}, \quad V_{-1}(K) := \frac{\mathrm{vol}_{n-1}(K^\circ)}{\mathcal{O}_{n-1}}.$$

Finally, we define V_j on $\{\emptyset, S^{n-1}\}$ as

$$V_j(\emptyset) := \begin{cases} 1 & \text{if } j = -1 \\ 0 & \text{else} \end{cases}, \quad V_j(S^{n-1}) := \begin{cases} 1 & \text{if } j = -n-1 \\ 0 & \text{else} \end{cases}$$

The following result will be very useful for future computations.

Theorem 6.3. *Let $K \in \mathcal{K}(S^{n-1})$ and $0 \leq j \leq n-2$. If K is a polyhedral cap with j-dimensional faces F_j, then*

$$V_j(K) = \sum_{F \in F_j} \frac{\mathrm{vol}_j(F)}{\mathcal{O}_j} \cdot \frac{\mathrm{vol}_{n-2-j}(N_F^S)}{\mathcal{O}_{n-2-j}}.$$

We can decompose a polyhedral cone C into the disjoint union of the relative interiors of its faces [Sch13, Thm. 2.1.1]:

$$C = \bigsqcup_{F \text{ a face of C}} \mathrm{relint}(F)$$

So, we can make the following definition

Definition 6.3. Let $x \in C$, then the *face* of x is

$$\mathrm{face}(x) := \begin{cases} C & \text{if } x \in \mathrm{int}(C) \\ F & \text{if } x \in \mathrm{relint}(F) \text{ and } F \text{ is a face of } C. \end{cases}$$

The following proposition shows why intrinsic volumes are important in probability and some optimization problems. This characterization is the most frequently used in the literature. The proof is based on [Ame11, Proposition 4.4.6]

Proposition 6.1. *Let K be a spherical polyhedron in S^{n-1}, i.e, $K \in \mathcal{K}(S^{n-1})$ and $\mathrm{cone}(K) = H_1 \cap \ldots \cap H_k$ for some $H_1, \ldots, H_k$ half-spaces. Let $C = \mathrm{cone}(K)$ denote the corresponding polyhedral cone. We define the function*

$$d_C : \mathbb{R}^n \to \{0, 1, \ldots, n\} \text{ such that } d_C(x) = \dim\big(\mathrm{face}(\textstyle\prod_C(x))\big).$$

Let p be a vector drawn uniformly at random on S^{n-1}. Then, the j-th spherical intrinsic volume of K, V_j, is given by

$$V_j(K) = \mathbb{P}\big(d_C(p) = j + 1\big) \tag{6.2}$$

Proof. The assertion is true for $j = -1, n-1$ because

$$d_C(p) = n \iff \dim(C) = n \text{ and } p \in C$$
$$d_C(p) = 0 \iff \dim(C^\circ) = n \text{ and } p \in C^\circ$$

So, let's assume that $0 \geq j \geq n - 2$.

Let F^e be a face of C such that $\dim F = j + 1$, and let $N = N_{F^e}(C)$ be the normal cone of C in F^e as we mentioned before. We can suppose without loss of generality that $\mathrm{lin}(F^e) = \mathbb{R}^{j+1} \times \{0\}$, then

$$F^e = \tilde{F}^e \times \{0\} \text{ and } N = \{0\} \times \tilde{N},$$

with $\tilde{F}^e \subseteq \mathbb{R}^{j+1}$ and $\tilde{N} \subseteq \mathbb{R}^{n-j-1}$. Keeping this in mind, we can see also that $\prod_C^{-1}(F^e) = \tilde{F}^e \times \tilde{N}$.

Let $x = (x_1, x_2) \in \mathbb{R}^{j+1} \times \mathbb{R}^{n-j-1}$ be a standard normal vector in $\mathbb{R}^n$ (i.e $x \in \mathcal{N}(0, I_{nxn})$). Consequently,

$$\begin{aligned}\mathbb{P}(x \in \textstyle\prod_C^{-1}(F^e)) &= \mathbb{P}(x_1 \in \tilde{F}^e)\mathbb{P}(x_2 \in \tilde{N}) \\ &= \frac{\mathrm{vol}_j(\tilde{F}^e \cap S^j)}{\mathcal{O}_j} \cdot \frac{\mathrm{vol}_{n-2-j}(\tilde{N} \cap S^{n-2-j})}{\mathcal{O}_{n-2-j}}\end{aligned}$$

So, summing over all the probabilities for each face of face of C, we get

$$\begin{aligned}\underset{p\in S^{n-1}}{\mathbb{P}}(d_C(p)=j+1) &= \underset{x\in\mathcal{N}(0,I_n)}{\mathbb{P}}(d_C(x)=j+1)\\ &= \sum_{F\in F_j(K)} \frac{\mathrm{vol}_j(F)}{\mathcal{O}_j}\cdot\frac{\mathrm{vol}_{n-2-j}(N_F^S)}{\mathcal{O}_{n-2-j}}\\ &\underset{\text{Thm. 6.3}}{=} V_j(K)\end{aligned}$$

□

Example 6.2. Let $C=\mathbb{R}^n_+$ and $K=C\cap S^{n-1}$. If we pick an element $x\in\mathbb{R}^n$ then $\prod_C(x)=\overline{x}=(\overline{x}_1,\dots,\overline{x}_n)$ where

$$\overline{x}_i=\begin{cases}x_i & \text{if } x_i\geq 0,\\ 0, & \text{else.}\end{cases}$$

Then, the function d_C is given by

$$d_C(x)=|\{i : x_i>0\}|.$$

If we have a point p uniformly distribute on S^{n-1}, then the probability that its i-th component is positive is 1/2.

$$V_j(K)=\frac{\binom{n}{j+1}}{2^n}.$$

So, this distribution of probability coincides with the binomial distribution of the random variable $Y=\prod_C(g)$ lies in the relative interior of a $j+1$-dimensional face of C for a random standard normal vector g.

Now, we list some properties of the spherical intrinsic volumes. The proof is omitted here, but for more details see [Ame11, Prop. 4.4.10]

Theorem 6.4.

1. *The intrinsic volumes are nonnegative. That is to say* $V_j(K)\geq 0$ *for all* $-1\leq j\leq n-1$ *and for all* $K\in\mathcal{K}(S^{n-1})\cup\{\emptyset,S^{n-1}\}$.
2. *If* $S\subset S^{n-1}$ *is an* i*-dimensional sphere, then*

$$V_j(S)=\begin{cases}1 & \text{if } j=i\\ 0 & \text{else}\end{cases}$$

3. *Let* $K\in\mathcal{K}(S^{n-1})$. *Then the intrinsic volumes of* K° *are given by*

$$V_j(K)=V_{n-2-j}(K^\circ).$$

4. *For every* $K\in\mathcal{K}(S^{n-1})\cup\{\emptyset,S^{n-1}\}$ *we have the equality*

$$\sum_{j=-1}^{n-1}V_j(K)=1.$$

In particular, $V_j(K)\leq 1$ *and the spherical intrinsic volumes of* K *form a distribution probability.*

5. *For any convex cap K, the following equality holds*

$$\sum_{\substack{j=-1 \\ j\equiv 0 \pmod 2}} V_j(K) = \sum_{\substack{j=-1 \\ j\equiv 1 \pmod 2}} V_j(K) = \frac{1}{2}.$$

So, we have $V_j(K) \leq \frac{1}{2}$ for any cap K.

In the past section we proved that the sequence of euclidean intrinsic volumes is log-concave for any convex set K. The spherical analog is just a conjecture, and in case of being true, it would provide a lot of applications. However, there is relation connecting euclidean intrinsic volumes and the spherical for a special kind of convex sets on the sphere and in $\mathbb{R}^n$ due to Amelunxen [Ame11, Prop. 4.4.18].

Theorem 6.5 (Transformation formula)**.** *Let $C \subseteq \mathbb{R}^n$ be a closed convex cone, and define $K = C \cap S^{n-1}$ and $K^e = C \cap B^n$, then*

$$V_i^e(K^e) = \sum_{j=i}^{n} V_i^e(B^j)V_{j-1}(K) = \sum_{j=i}^{n} \frac{\omega_j}{\omega_{j-i}} V_{j-i}(K)$$

We could attempt to use this transformation formula to attack the problem of the log-concavity of the spherical intrinsic volumes. Unfortunately, the transformation formula does not help to get a proof of such conjecture. Even if we know that the euclidean intrinsic volumes are log-concave. To see this, suppose we have $a_0. \ldots, a_n$ a log-concave sequence of positive numbers. If we define $c_0, \ldots, c_n$ as the sequence that satisfy

$$a_i = \sum_{j=i}^{n} \binom{j}{i} \frac{\omega_j}{\omega_{j-i}} c_j$$

Then, $c_0, \ldots, c_n$ is not necessarily log-concave. For example, if we take the sequence of a_i's as $a = (e^3, e^3, e^{2.5}, e)$ which is log-concave, the associated sequence $c \approx (6.57, 4.05, 6.75, 2.72)$ is not log-concave.

In the following example we show that the intrinsic volumes of the spherical triangle are log-concave. This result validates the general conjecture for the intrinsic volumes of spherical convex bodies and gives a glimpse to a general proof in $\mathbb{R}^n$.

Example 6.3. Let's compute the spherical intrinsic volumes for a spherical triangle T (lying on the sphere of radius one, S^2) with sides a, b, c (which correspond to the dihedral angles between the rays of the cone generated by the spherical triangle) and internal angles A, B, C, the idea is to use theorem 6.2.

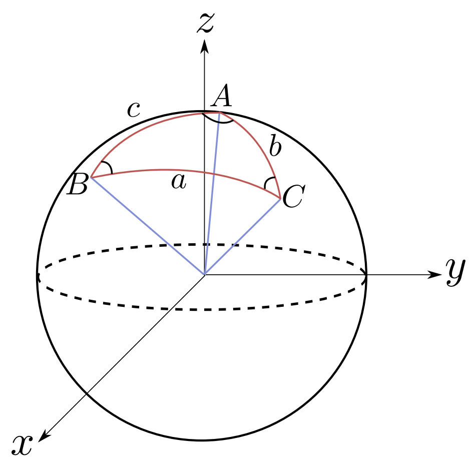

Figure 6.1: Spherical triangle T

If A', B', C' denote the internal angles of K° and a', b', c' its corresponding sides, then $A' = \pi - a,\ B' = \pi - b,\ C' = \pi - c$. Likewise, $a' = \pi - A,\ ,b' = \pi - B,\ c' = \pi - C$. See [Tod63]. By definition,

$$V_{-1}(T) = \frac{\mathrm{vol}_2(T^\circ)}{\mathcal{O}_2} = \frac{\text{Area of } T^\circ}{4\pi} = \frac{2\pi - a - b - c}{4\pi}, \tag{6.3}$$

And

$$V_2(T) = \frac{\mathrm{vol}_2(T)}{\mathcal{O}_2} = \frac{\text{Area of } T}{4\pi} = \frac{A + B + C - \pi}{4\pi} \tag{6.4}$$

Now, we move to $V_0(T)$ and $V_1(T)$. Given any side F of the triangle T, $N_F^S = \{*\}$ is just a point. So $V_0(N_F^S) = 1$. Then,

$$V_1(T) = \frac{\text{Perimeter of } T}{4\pi} = \frac{a + b + c}{4\pi}, \tag{6.5}$$

And analogously

$$V_0(T) = \frac{\text{Perimeter of } T^\circ}{4\pi} = \frac{3\pi - A - B - C}{4\pi}. \tag{6.6}$$

What if we ask about the log-concavity of $V_{-1}(T), V_0(T), V_1(T), V_2(T)$ in this case? The answer lies on the *isoperimetric inequality on the sphere.*

Proposition 6.2 (Isoperimetric inequality on the sphere)**.** *Let C be a simple closed curve on S^2. If we denote by L the length of C and by A the area enclosed by C, then*

$$L^2 \leq A(4\pi - A)$$

More references about this inequality can be found in [Oss78]. Note now that by taking $C = T$, we can deduce easily that $V_1(T)^2 \geq V_2(T)V_0(T)$. In the same way, if we make $C = T^\circ$ we get $V_0(T) \geq V_1(T)V_{-1}(T)$.

Remark 6.3.

- Note that these formulas do not depend directly of the coordinates of the rays generating the cone of the triangle T. So, the intrinsic volumes of the spherical triangle are invariant under rotation and translation on the sphere which extends the notion of *intrinsic* that we saw in the euclidean case.

- If we have any spherical polygon on the sphere, the formulas for the intrinsic volumes of K will depend only of the area and perimeter of K and K°. Nonetheless, verify the isoperimetric inequality in this case will require more work because we do not have the relations between angles and sides of K and K° that we did have in the case of the spherical triangle.

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