Morse Theory and Witten's Proof of the Morse Inequalities

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Chapter 1

Introduction

Morse Theory was introduced by Marston Morse midway through the 20^{th} century as a tool of Differential Topology that enables the study of topological information of manifolds through the study of the critical points of *Morse functions* upon them. In truth, the information associated to the critical points of these functions will be the main tool upon which we develop our work in chapters 2 and 3. John Milnor, in his book *Morse Theory* published in 1963, gives the first united account of the tools developed from the theory.

The first purpose we undertake in this dissertation is to go through the results presented in the first two chapters of Milnor's book. In this sense, this article can truly only be claimed to be an introduction to Morse Theory, and many more results not presented here are attributed to the theory.
In chapter 2, we go through the work of introducing Morse functions, characterizing their behavior through the all-important *Lemma of Morse*, and afterwards proving some existence results. Later on, we undertake the long process required to prove, exclusively through the study of critical points, that every compact manifold M with a Morse function has the homotopy type of a CW-complex. In fact, it can be shown that this holds true for non-compact manifolds as well, but is not done here. This result will allow us to prove *Reeb's Theorem*, which states that any compact manifold with two non-degenerate critical points is homeomorphic to a sphere. We can appreciate how through Morse theory, a hypothesis consisting uniquely of information regarding critical points has a strong topological result. This is what we can expect from Morse Theory.
Chapter 3 of our document is dedicated to proving by Milnor's original account perhaps the most renown result of Morse Theory, the Morse inequalities. The inequalities on one hand, bound the Betti numbers by the number of critical points of certain indexes (this result holds for **any** Morse function, so we can also interpret the Betti numbers as bounding by below the number of critical points a Morse function can have!), and establish the Euler Characteristic as the alternated sum of critical points of different indexes. One more time, Morse Theory shows a surprising and strong relationship between topological informa-

tion of a manifold and the critical points of Morse functions, especially taking into account the strength of the topological information involved.

In chapter 4, we step aside from Milnor's account of Morse theory and take a look at Edward Witten's article *Supersymmetry and Morse Theory*, published in 1982. In it, Witten undertakes a new proof of the Morse inequalities through the connection De Rham theory gives between singular homology and De Rham cohomology and differential forms upon manifolds at large. Through Hodge Theory generalized to elliptic operators, Witten proves that the dimension of $Ker(\Delta)$, with Δ the *Laplace-Beltrami operator*, will also be bounded by the number of critical points of certain indexes.

Beyond giving an independent proof of the Morse inequalities (which makes Witten account quite interesting; in fact, chapter 4 could be read independently from the rest of the document), the techniques and arguments used throughout the chapter should give the reader an idea of how Witten's account of Morse theory makes fascinating ties with other areas of mathematics. For example, John Roe's book *Elliptic operators, topology and asymptotic methods* uses Witten's ideas surrounding De Rham cohomology, Hodge Theory and elliptic operators and applies them to topological index problems and other important results regarding elliptic operators.

Morse theory proves throughout this whole account to be a truly interesting area, reaching out to touch and bound with different branches of mathematics in surprisingly natural ways.

Finally, I would like to recognize and deeply thank the collaboration and guidance during the development of this dissertation of both my advisor, Dr. Andrés Ángel; and my long time professor and mentor Dr. Monika Winklmeier.

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Chapter 2

Morse Functions and Homotopy Types in Terms of their Critical Values

In this chapter we are going to look at some of the initial, somewhat weaker results of Morse Theory, before we get into the profound results later on. The ideas in this chapter are highly intuitive. Mainly, we are going to prove three theorems that ultimately, in descending order of hierarchy, give us a great deal of information regarding the homotopy type of a manifold, using of course the power-tool of Morse Theory, the critical values of Morse functions. Later on we will present motivation for these theorems showing conditions under which Morse functions exist and how they look like, and finally we end exemplifying the information obtained on a well known manifold (the torus) and proving an interesting application, namely *Reeb's Theorem.* Let us begin the chapter by looking at some of the preliminaries needed to deal with the three theorems in question.

2.1 Preliminaries

Let us begin fittingly with the definition of Morse functions and their critical points:

Definition 2.1. Let M be a real manifold, and $f : M \to \mathbb{R}$ a real-valued function. A point $p \in M$ is said to be a *critical point* if $df_p = 0$. Additionally, it is said to be *non-degenerate* if the matrix $[\frac{\partial^2 f}{\partial x_i \partial x_j}(p)]_{i,j}$ is non-singular. Finally, if the function f only has non-degenerate critical points, it is said to be a *Morse function.*

We have said before that these functions will be our power-tool for the rest of the document, so the reader is well advised to always consider a function on a manifold to be Morse, unless it's specifically specified otherwise. It can be checked quite directly, which we shall not do, that none of these definitions are dependant of the coordinate system, like is expected when working on manifolds.

Before we begin our work on Morse Theory, we must take a necessary detour into some concepts of homotopy. The reader must be familiar with some basic concepts of homotopy but it goes beyond the scope of this document to make an extensive introduction on the subject, so we shall restrict ourselves to defining only the working concepts for our purposes. A good secondary source to brush up on the background knowledge is [Hat02].

Definition 2.2. A *k-cell* is a topological space of the form:

$$e^k := \{x \in \mathbb{R}^k : \|x\| \leq 1\},$$

where k determines the dimension of the cell. As a convention, a 0-cell is a point.

Notice that a k-cell is defined to be a solid ball, with 'stuffing' so to speak. In the context of k-cells, we will always denote their boundary by the very expected and standard $S^{k-1} := \{x \in \mathbb{R}^k : \|x\| = 1\}$. This gives us context in which to make the following construction:
Given a a k-cell e^k, a topological space X and a map $g : S^{k-1} \to X$, we refer to the space X *with a k-cell attached by g* (sometimes omitting the "by g"), denoted by $X \sqcup_g e^k$, as the quotient space $X \sqcup e^k / \sim_g$ where $x \sim_g y$ if $y = g(x)$ or $g(y) = x$. With this idea in mind, we will call a *CW-complex* as a space constructed by taking 0-cells, attaching 1-cells and then attach to the resulting space 2-cells, and so on and so forth.
Given a CW-complex X, we will call its *n-skeleton* to the n-th step of the construction process. It will be denoted by X^n.
We will later see that we have sufficient conditions by studying Morse functions on a manifold to verify if it is a CW-complex, which is a strong condition taking the homology of such spaces into account, but we'll get into this later on.
Let us take a look at some needed results regarding this sort of spaces.

Lemma 2.3 (Whitehead)**.** Let φ_0 and φ_1 be homotopic maps from the sphere S^{k-1} to a space X. Then the identity map of X extends to a homotopy equivalence:

$$\Psi : X \sqcup_{\varphi_0} e^k \to X \sqcup_{\varphi_1} e^k$$

Proof. Let us define Ψ by the following formulas:

(1) $\Psi(x) = x$ for $x \in X$

(2) $\Psi(tu) = 2tu$ for $0 \leq t \leq \frac{1}{2}$, $u \in S^{k-1}$

(3) $\Psi(tu) = \varphi_{2-2t}(u)$ for $\frac{1}{2} \leq t \leq 1$, $u \in S^{k-1}$

Here, φ_t denotes the homotopy between φ_0 and φ_1 and tu denotes an element in e^k by seeing u as a unit vector and t as a scalar between 0 and 1. A corresponding map, from $X \sqcup_{\varphi_1} e^k$ to $X \sqcup_{\varphi_0} e^k$ can be defined similarly, and one can directly check that they constitute a homotopy equivalence. □

Lemma 2.4. If a map F has a left homotopy inverse L and a right homotopy inverse R, then F is a homotopy equivalence and both R and L are a two-sided homotopy inverse.

Proof. The statement gives us the relationships $LF \simeq id$ and $FR \simeq id$, so ultimately we have:

$$L \simeq L(FR) \simeq (LF)R \simeq R,$$

and consequently $R \simeq L$ giving us the desired result. □

Lemma 2.5. Let $\varphi : S^{\lambda-1} \to X$ be an attaching map, and $f : X \to Y$ a homotopy equivalence between X and Y two topological spaces. Then it can be extended to a homotopy equivalence $F : X \sqcup_\varphi e^\lambda \to Y \sqcup_{f\varphi} e^\lambda$.

Proof. Define F by the conditions:

(1) $F \mid_X = f$

(2) $F \mid_{e^\lambda} = id$

Let $g : Y \to X$ be a homotopy inverse to f and define

$$G : Y \sqcup_{f_\varphi} e^\lambda \to X \sqcup_{gf\varphi} e^\lambda$$

by the expected conditions:

(1) $G \mid_Y = g$

(2) $G \mid_{e^\lambda} = id$

Since $gf\varphi$ is homotopic to φ, it follows from Whitehead that there is a homotopy equivalence

$$\Psi : X \sqcup_{gf\varphi} e^\lambda \to X \sqcup_\varphi e^\lambda$$

Let us prove that the composition

$$\Psi GF : X \sqcup_\varphi e^\lambda \to X \sqcup_\varphi e^\lambda$$

is homotopic to the identity map. Let h_t be a homotopy between gf and the identity. Using the specific manner in which we defined Ψ (see proof of Whitehead), G and F; note that:

(1) $\Psi GF(x) = gf(x)$ if $x \in X$

(2) $\Psi GF(tu) = 2tu$ for $0 \leq t \leq \frac{1}{2}$, $u \in S^{\lambda-1}$

(3) $\Psi GF(tu) = h_{2-2t}\varphi(u)$ for $\frac{1}{2} \leq t \leq 1$, $u \in S^{\lambda-1}$

We can now directly verify that the homotopy that we are searching for, between the identity map and ΨGF is given explicitly by $q_\tau : X \sqcup_\varphi e^\lambda \to X \sqcup_\varphi e^\lambda$ defined by the formulas:

(1) $q_\tau(x) = h_\tau(x)$ if $x \in X$

(2) $q_\tau(tu) = \frac{2}{1+\tau} tu$ for $0 \leq t \leq \frac{1+\tau}{2}$, $u \in S^{\lambda-1}$

(3) $q_\tau(tu) = h_{2-2t+\tau}\varphi(u)$ for $\frac{1+\tau}{2} \leq t \leq 1$, $u \in S^{\lambda-1}$

Therefore F has a left homotopy inverse and by Lemma 2, it is a homotopy equivalence. □

For now, we are done visiting the murky waters of algebraic topology, but we must push the patience of the reader one last time with a final detour into 1-parameter groups of diffeomorphisms before arriving at the meat of Morse Theory. It will not be long:

Definition 2.6. A *1-parameter group of diffeomorphisms* of a manifold M is a C^∞ map $\varphi : \mathbb{R} \times M \to M$ such that:

(1) For each $t \in \mathbb{R}$, the map $\varphi_t : M \to M$ defined as $\varphi_t(x) = \varphi(t, x)$ is a diffeomorphism

(2) $\forall t, s \in \mathbb{R}$ we have $\varphi_{t+s} = \varphi_t \circ \varphi_s$

Remark 2.7. Notice that given a 1-parameter group of diffeomorphisms on M, $\{\varphi_t\}_\mathbb{R}$ if we fix a point q on M, $\{\varphi_t(q)\}_\mathbb{R}$ defines a curve upon M.

Also notice that given a 1-parameter group of diffeos on M, we can define a vector field X on M as follows: For a smooth function f,

$$X_q(f) = \lim_{h\to 0} \frac{f(\varphi_h(q)) - f(q)}{h}$$

In this sense, the vector field is said to *generate* the group φ. Conversely, a vector field that satisfies this equality, given its uniqueness, is said to generate the 1-parameter group of diffemorphisms.

Lemma 2.8. A smooth vector field on M that vanishes outside of a compact subset $K \subset M$ generates a unique 1-parameter group of diffeomorphisms of M.

The proof is omitted in this document since it relies on the theory of unique solutions of differential equations on manifolds, a subject which we do not treat here. It can be found in [Mil63].

Finally, we arrive at critical points of Morse functions. The first fundamental fact concerning Morse functions is that locally around a critical point, their behavior is completely determined by a single value called the *index*. The specifications of this behavior are given by the Morse Lemma which we will see in a moment, but first let us do some background work:

Lemma 2.9. Let f be a C^∞-function in a convex neighborhood V of 0 in $\mathbb{R}^n$ with $f(0) = 0$. Then f can be expressed in the following way:

$$f(x_1, x_2, ..., x_n) = \textstyle\sum_{i=1}^{n} x_i g_i(x_1, x_2, ..., x_n)$$

for some suitable C^∞ functions g_i defined in V, with $g_i(0) = \frac{\partial f}{\partial x_i}(0)$

Proof. By the fundamental theorem of calculus and the chain rule, $f(x_1, ..., x_n) = \int_0^1 \frac{df(tx_1,tx_2,...,tx_n)}{dt}dt = \int_0^1 \sum_{i=1}^{n} \frac{\partial f(tx_1,tx_2,...,tx_n)}{\partial x_i} x_i dt$. Thus, if we let $g_i(x_1, ..., x_n) = \int_0^1 \frac{\partial f(tx_1,tx_2,...,tx_n)}{\partial x_i}dt$ we have the desired result. □

Definition 2.10. Let M be a manifold, f a Morse function and $p \in M$ a non-degenerate critical point of f. Then the number of negative eigenvalues of $[\frac{\partial^2 f}{\partial x_i \partial x_j}(p)]_{i,j}$ is defined as the *index* of the point p.

As we would expect, this definition can be checked to be independent of the coordinate system. It is a central definition for the theory, so the reader should keep it in mind.
The following result, even though its name suggests it is a simple one, is actually quite a powerful tool. It will be used throughout the document on several occasions, and with it we finish our preliminaries. As we have anticipated, it characterizes the local behavior of a Morse function at a critical point depending only upon the index. The proof is a bit long and technical, but the reader should hang in there since the result is worth it. Let us take a look:

Proposition 2.11. *Lemma of Morse* Let p be a non-degenerate critical point of f. Then there is a local coordinate system $(x_1, x_2, ..., x_n)$, and a neighborhood $\mathcal{U}$ of p with $x_i(p) = 0\ \forall i$ such that throughout $\mathcal{U}$ we can write f as:

🦇$f(q) = f(p) - x_1(q)^2 - x_2(q)^2 - ... - x_{\lambda-1}(q)^2 - x_\lambda(q)^2 + x_{\lambda+1}(q)^2 + ... + x_n(q)^2$,

where λ is the index of p.

Proof. The proof will proceed as follows: first we will prove that if there exists any such local expression for f, then the value λ in the expression must indeed be the index of p. Once we have this fact, it will suffice to show that there is an expression of that sort and we will be done. Let us see: It is clear that if we have an expression like in 🦇for a coordinate system $(y_1, ..., y_n)$, then we can see that:

$$\frac{\partial^2 f}{\partial y_i \partial y_j}(p) = \begin{cases} -2, \text{ if } i = j \leq \lambda \\ 2, \text{ if } i = j > \lambda \\ 0 \text{ otherwise} \end{cases} \quad ,$$

and the matrix $(\frac{\partial^2 f}{\partial x_i \partial x_j}(p))_{i,j}$ has the form:

$$\begin{pmatrix} -2 & & & & & & \\ & -2 & & & & & \\ & & \ddots & & & & \\ & & & -2 & & & \\ & & & & 2 & & \\ & & & & & \ddots & \\ & & & & & & 2 \end{pmatrix},$$

and we have said before that the number of negative eigenvalues of this 'Hessian' matrix is independent of the coordinate system, so in particular, this diagonal form can be seen to have λ negative eigenvalues and so λ must be the index of p. Now we must show that we indeed have such a representation for f. The idea will be constructive:
Assume without loss of generality that $f(p) = f(0)$. By Lemma 2.9, we can write

$$f(x_1, ..., x_n) = \textstyle\sum_{j=1}^{n} x_j g_j(x_1, ..., x_n)$$

in a coordenatized neighborhood of p, and some suitable functions g_j. Since p is assumed to be a critical point, then $\forall i$, $\frac{\partial f}{\partial x_i} = 0$, and referring ourselves to the proof of Lemma 2.9, we can see that the form of the g_i's is such that $g_i(0) = 0$. Hence, each function g_i fulfils the hypothesis of Lemma 2.9. Reapplying the lemma, we obtain the following form for f:

🦇🦇 $$f(x_1, ..., x_n) = \textstyle\sum_{i,j} x_i x_j h_{ij}(x_1, ..., x_n).$$

Take notice that without loss of generality, we can assume that $h_{ij} = h_{ji}$, because if this were not true then we could just redefine $\hat{h}_{ij} = \frac{1}{2}(h_{ij} + h_{ji})$ and it would still be true that $f = \sum_{i,j} x_i x_j \hat{h}_{ij}$.
Now, let us observe that the matrix:

$$(\hat{h}_{i,j}(0))_{i,j} = (\tfrac{\partial^2 f}{\partial x_i \partial x_j}(0))_{i,j} \text{ (again, see the proof of Lemma 4)}$$

is non-singular since p is non-degenerate.
What we are going to do now is a sort of induction. Suppose we have the following expression for some neighborhood $\mathfrak{A}_1$ of p, coordenatized by $(u_1, ..., u_n)$:

$$f = \pm u_1^2 \pm ... \pm u_{r-1}^2 + \textstyle\sum_{i,j \geq r} u_i u_j h_{i,j},$$

where the functions h_{ij} are symmetrical, smooth and the matrix $[h_{(r-1+i)(r-1+j)}(0)]_{i,j}$ is non-singular (all these are suppositions).
Now, through a linear change in the last $n-r+1$ coordinates, since $[h_{(r-1+i)(r-1+j)}(0)]_{i,j}$ is non-singular, we may assume that $h_{rr}(0) \neq 0$. Therefore we have that for a smaller neighborhood of 0, $\mathfrak{A}_2 \subseteq \mathfrak{A}_1$, $h_{rr}(u_1, ..., u_n)$ is a smooth non-zero function.
We are going to introduce then new coordinates as follows:

(1) $v_i = u_i$ for $i \neq r$

(2) $v_r(u_1, ..., u_n) = \sqrt{\mid h_{rr}(u_1, ..., u_n) \mid}[u_r + \sum_{i>r} u_i \frac{h_{ir}(u_1,...,u_n)}{h_{rr}(u_1,...,u_n)}]$

It follows from the inverse function theorem (clearly the matrix $(\frac{\partial v_i}{\partial u_j})_{i,j}$ is invertible, since in particular it is an upper-triangular matrix with a non-zero diagonal) that $(v_1, ..., v_n)$ will serve as a coordinate system in a sufficiently small neighborhood of 0, $\mathfrak{A}_3 \subseteq \mathfrak{A}_2$, and that throughout this neighborhood we have that $u_i = v_i$ for $i \neq r$, and $u_r = g(v_1, ..., v_n)$ for some smooth function g. What we want to do then is show that:

$$f = \pm v_1^2 \pm ... \pm v_r^2 + \textstyle\sum_{i,j>r} v_i v_j \overline{h_{ij}}$$

and that these new functions $\overline{h}_{ij}$ still satisfy the assumptions we made at the beginning of this 'induction', that is to say they will still be symmetric, smooth and the matrix $(\overline{h_{(r+i)(r+j)}}(0))_{i,j}$ will be non-singular. Since these assumptions are true for $r = 0$ in 🦇🦇, then by induction on r we will have finished the proof. This part is truly very technical, and an extremely detailed account seems pointless, so instead let us skim through it a bit. A more detailed account can be found in [Ria92] or [Pie07]. Let us see:
Calculation reveals that:

$$v_r^2 = u_r^2 h_{rr} + 2\sum_{i>r} u_r u_i h_{ir} + \left(\sum_{i>r} u_i^2 \frac{h_{ir}}{h_{rr}}\right)^2$$

Therefore, we can define our new functions $\overline{h}_{ij}$ such that $[\overline{h}_{ij}(p)]_{ij}$ is non-singular and

$$f(v_1, ..., v_n) = \pm v_1^2 \pm ... \pm v_r^2 + \sum_{i,j>r} v_i v_j \overline{h_{ij}}$$

like we wanted. □

Remark 2.12. Notice that the argument is in no way damaged by multiplying by a positive constant, so withouth loss of generality, we can just as well define local coordinates such that the local behavior of f around a critical point of index λ is:

$$f(q) = f(p) - \frac{1}{2}(x_1(q)^2 - x_2(q)^2 - ... - x_{\lambda-1}(q)^2 - x_\lambda(q)^2 + x_{\lambda+1}(q)^2 + ... + x_n(q)^2)$$

This will come in handy later on.

Remark 2.13. Notice that the Lemma of Morse gives us complete information regarding the local behavior of Morse functions. If we are near a critical point, we just saw the behavior. Outside these neighborhoods, recalling our Differential Geometry and Linear Algebra courses, we can choose local coordinates so that $f = x_1$ since the differential of f in these neighborhoods is surjective.

As important corollaries, we have that:

Corollary 2.14. The critical points of a Morse function are isolated

The reasoning is obvious, given the explicit expression for f, around a critical point its differential will only be 0 at the point itself.

Corollary 2.15. A Morse function f on a compact manifold M only has finitely many critical points.

Again the reasoning is clear, since M is compact, $f(M)$ is compact and if we had infinitely many critical points, they would have a cluster point contradicting the previous corollary.

2.2 Existence of Morse Functions

As a motivation to our work that relies on the existence of Morse functions over manifolds, in this section we present not only the existence of such functions, but also some characteristics that they posses which will ease out our work in later sections. We will omit the proofs in this section focusing on the results, but the work has been drawn primarily from two sources; [Mat02] and [Mil65]. They can both be referenced for further detail.

Let us begin by establishing a convention of *closeness* of functions. At first, let K be a compact subset of $\mathbb{R}^n$ and $\epsilon > 0$. We will say two C^2 functions f, g are *(C^2, ϵ)-close* if $\forall x \in K$:

(1) $|f(x) - g(x)| < \epsilon$

(2) $|\frac{\partial f}{\partial x_i}(x) - \frac{\partial g}{\partial x_i}(x)| < \epsilon$, $\forall i$

(3) $|\frac{\partial^2 f}{\partial x_i \partial x_j}(x) - \frac{\partial^2 g}{\partial x_i \partial x_j}(x)| < \epsilon$, $\forall i, j$

Now on a compact manifold M, we know we have a finite covering of neighborhoods diffeomorphic to $\mathbb{R}^n$, namely $\{U_i\}_I$, and hence what we will call a *compact refinement* of this covering, i.e. a collection $\{K_i\}_I$ with K_i compact and $K_i \subset U_i$ $\forall i$, such that M is covered by the K_i's. This work can be found in [Mun84].
With this in mind, we define *(C^2, ϵ)-closeness* on M as *(C^2, ϵ)-closeness* on each K_i.

Now, remember that our goal for this section is to provide some sort of *existence* assurance of Morse functions on compact manifolds. This fact can be formulated in several ways. We choose the following formulation taken from [Mat02]:

Theorem 2.16. *Existence of Morse Functions* Let M be a compact manifold, and $g : M \to \mathbb{R}$ a *smooth* function. Then there exists a Morse function $f : M \to \mathbb{R}$ arbitrarily C^2-close to g.

This formulation has several direct consequences which are of interest to us. First of all, the existence is clear, since there are obviously many smooth functions on any manifold. Second of all, it gives a sort of *dense-like* quality to Morse functions which is certainly appealing.
We will get a bit more into this characteristic in a moment, but first, let us present some of the previous results to Theorem 2.16 so the techniques used can be somewhat presented to our reader. The proof of it relies strongly on some interesting lemmas:

Lemma 2.17. Let $K \subseteq U \subseteq \mathbb{R}^n$ with K compact and U open, and $f : U \to \mathbb{R}$ a C^2 function such that it only has non-degenerate critical points in K. Then, there exists $\delta > 0$ such that if for any other C^2 function $g : U \to \mathbb{R}^n$ satisfying that $\forall x \in K$:

(1) $|\frac{\partial f}{\partial x_i}(x) - \frac{\partial g}{\partial x_i}(x)| < \epsilon$, $\forall i$

(2) $|\frac{\partial^2 f}{\partial x_i \partial x_j}(x) - \frac{\partial^2 g}{\partial x_i \partial x_j}(x)| < \epsilon$, $\forall i, j$

we have that g also has only non-degenerate critical points within K.

Lemma 2.18. Suppose $\varphi : U \to U'$ is a diffeomorphsim between open sets of $\mathbb{R}^n$, such that it carries a compact set $K \subseteq U$ onto the compact set $K' \subseteq U'$. Suppose we have a function $f : U' \to \mathbb{R}$ and $\epsilon > 0$. Then there exists $\delta > 0$ such that if within K':

(1) $|f(x)| < \delta$

(2) $|\frac{\partial f}{\partial x_i}(x)| < \delta$, $\forall i$

(3) $|\frac{\partial^2 f}{\partial x_i \partial x_j}(x)| < \delta$, $\forall i, j$

then for all points in K:

(1) $|f \circ \varphi(x)| < \epsilon$

(2) $|\frac{\partial f \circ \varphi}{\partial x_i}(x)| < \epsilon$, $\forall i$

(3) $|\frac{\partial^2 f \circ \varphi}{\partial x_i \partial x_j}(x)| < \epsilon$, $\forall i, j$

For the proofs of these lemmas refer to [Mil65].

We can at least present to the reader an idea of the proofs; with lemma 2.17 we attempt to do our work locally, and with lemma 2.18 we attempt to both gain the global result with the transition maps of our atlas, as well as proving independence of coordinate system.

We finish this section off with an interesting proposition which will come very much in handy during Chapter 3. The proof can be found in [Mil65]:

Proposition 2.19. Let $f : M \to \mathbb{R}$ be a Morse function over a compact manifold M with critical points $p_1, ..., p_k$. Then f can arbitrarily C^2-approximated by a Morse function $g : M \to \mathbb{R}$ such that $g(p_i) \neq g(p_j)$ whenever $i \neq j$.

Remark 2.20. What this result means is that without loss of generality, we always have Morse functions such that each critical *value* corresponds to exactly one critical *point*. This fact will come in handy to ease out the proofs of later results.

2.3 First Theorem

From now on, we are going to use the following convention: Let M be a manifold, f a Morse function and $a \in \mathbb{R}$. We define:

$$M^a := \{p \in M : f(p) \leq a\}$$

The result of this first theorem is very self explanatory:

Theorem 2.21. Let f be a smooth real-valued function on a manifold M. Let $a < b$ and suppose that the set $f^{-1}[a, b]$ is compact and contains no critical points of f. Then, M^a is diffeomorphic to M^b. Furthermore, M^a is a deformation retract of M^b, so that the inclusion map $M^a \to M^b$ is a homotopy equivalence.

Proof. Begin by choosing a Riemannian metric on M, and let $\langle x, y\rangle_p$ denote the inner product of tangent vectors at $p \in M$.
Notice that we can characterize the gradient of f, ∇f, as the vector field on M which fulfils that for any vector X:

$$\langle X, \nabla f\rangle = X(f)$$

Thus such a vector field will vanish at the critical points of f.
Now define a map $\rho : M \to \mathbb{R}$ such that it is equal to $\frac{1}{\|\nabla f\|}$ in the compact set $f^{-1}[a, b]$, and vanishes outside a compact subset of M. The capacity of being able to do this is left for a Differential Topology course.
Now, we define the following vector field:

$$X_q := \rho(q)(\nabla f)_q$$

We can see that it fulfils the requirements of lemma 2.8, and so we have a one-parameter group $\{\varphi_t\}_{\mathbb{R}}$. Considering Remark 2.7 we have that if $\varphi_t(q)$ is in $f^{-1}[a, b]$, then:

$$\frac{d(f \circ \varphi_t(q))}{dt} = \langle \frac{d(\varphi_t(q))}{dt}, \nabla f\rangle = \langle X, \nabla f\rangle = 1$$

where by $\frac{d(\varphi_t(q))}{dt}$ we mean the velocity vector field of the curve $\{\varphi_t(q)\}_{\mathbb{R}}$.
Let us now consider the diffeomorphism $\varphi_{b-a} : M \to M$. We want to see that

this map carries M^a into M^b diffeomorphically.
Consider $p \in M$ such that $f(p) \leq a$. We want to see that $f(\varphi_{b-a}(p)) \leq b$. This is evident since we proved that $\frac{d(f\circ\varphi_t(q))}{dt} = 1$ and $f(p) = f(\varphi_0(p))$, so $f(\varphi_{b-a}(p)) \leq f(\varphi_0(p)) + \frac{d(f\circ\varphi_t(q))}{dt}(b-a) \leq a + (b-a) = b$. The fact that for every $y \in M^b$, $\exists x \in M^a$ such that $\varphi_{b-a}(x) = y$ is direct from the fact that since we have a 1-parameter group parametrized by $\mathbb{R}$, then φ_{a-b} is an inverse that obeys the exact same argument.
We need only see now that M^a is a deformation retract of M^b, where the identity induces a homotopy equivalence. This is pretty straightforward, we define $h_t : M^b \to M^b$ by:

$$h_t(q) = \begin{cases} q \text{ if } f(q) \leq a \\ \varphi_{t(a-f(q))}(q) \text{ if } a \leq f(q) \leq b \end{cases}$$

This is quite a simple procedure, it is just a sort of “re-scaling” the diffeomorphism we already had to make it into a retraction. Clearly, h_0 is the identity and h_1 is the desired retraction. □

2.4 Second Theorem

The First Theorem showed that as long as we do not pass through a critical value, M^a is homotopically equivalent to M^b, but what happens when we do pass through a critical value? This next theorem gives us the answer:

Theorem 2.22. Let $f : M \to \mathbb{R}$ be a smooth function, and let p be a non-degenerate critical point with index k. Setting $f(p) = c$, suppose that $f^{-1}[c-\epsilon, c+\epsilon]$ is compact, and contains no critical point of f other than p, for some $\epsilon > 0$. Then, for all sufficiently small ϵ, the set $M^{c+\epsilon}$ has the homotopy type of $M^{c-\epsilon}$ with a k-cell attached.

We advise the reader to read the proof in parallel with the example of the torus in section 2.6 so he may illustrate the constructive process and not get lost in the technical details.

Proof. Let us begin by using the Lemma of Morse to choose a coordinate system $x_1, x_2, ..., x_n$ in a neighborhood U of p such that in U, p is parametrized as 0, and f behaves thus:

$$f = c - x_1^2 - ... - x_k^2 + x_{k+1}^2 + ... + x_n^2$$

Now, let us choose $\epsilon > 0$ small enough so that:

(1) $f^{-1}[c-\epsilon, c+\epsilon]$ is compact and contains no other critical points of f (recalling the results of section 2.2, this can without loss of generality be the case)

(2) $\psi(U)$ contains the closed ball $\{(x_1, ..., x_n \mid \sum(x_i)^2 \leq 2\epsilon\}$ where ψ corresponds to the atlas around U. (♡)

Now define e^λ as the points in U such that:

$$x_1^2 + ... + x_\lambda^2 \leq \epsilon \text{ and } x_{\lambda+1} = ... = x_n = 0$$

Milnor's book attempts to illustrate the construction in the following diagram, which is pretty clear keeping in mind the Lemma of Morse:

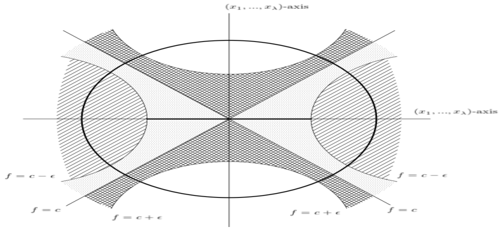

Figure 2.1:

Notice that given the local form of f, any point in what we just defined as e^λ is such that it is in $f^{-1}[c-\epsilon, c]$. Additionally, realize that $e^\lambda \cap M^{c-\epsilon} = S^{\lambda-1}$, so that we can view $M^{c-\epsilon} \cup e^\lambda$ as $M^{c-\epsilon}$ with a λ-cell attached. What we want to do now is prove that $M^{c-\epsilon} \cup e^\lambda$ has the homotopy type of $M^{c+\epsilon}$ and the statement is proved. Let us do this:

Define a new smooth map $F : M \to \mathbb{R}$ as follows:
First, let $\nu : \mathbb{R} \to \mathbb{R}$ be a C^∞ function that satisfies the following conditions:

(1) $\nu(0) > \epsilon$

(2) $\nu(t) = 0$ for $t \geq 2\epsilon$

(3) $-1 < \nu'(t) \leq 0 \; \forall t$

Now define F as a map coinciding with f outside of U, and inside of U being determined by:

$$F = f - \nu(x_1^2 + ... + x_\lambda^2 + 2x_{\lambda+1}^2 + ... + 2x_n^2)$$

The reader can readily verify that F is smooth, considering (♡).

Applying this construction to the torus example, the corresponding situation would look something like Figure 2.2:
CLAIM 1: $F^{-1}(-\infty, c+\epsilon] = M^{c+\epsilon} = f^{-1}(-\infty, c+\epsilon]$
Whenever we are outside of U, F and f coincide. Now, inside of U, if we are outside of $x_1^2 + ... + x_\lambda^2 + 2x_{\lambda+1}^2 + ... + 2x_n^2 \leq 2\epsilon$, again they coincide. Let us concern ourselves then for the points within this region. Here, $F \leq f =$

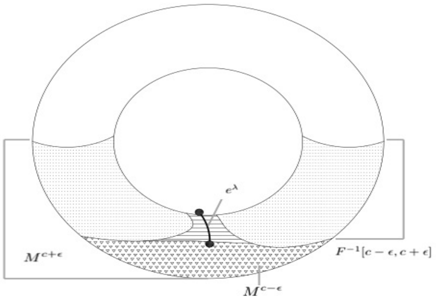

Figure 2.2: Torus

$c - x_1^2 - ... - x_\lambda^2 + x_{\lambda+1}^2 + ... + x_n^2 \leq c + \frac{1}{2}[x_1^2 + ... + x_\lambda^2] + x_{\lambda+1}^2 + ... + x_n^2 \leq c + \epsilon$, completing the proof of our claim.

CLAIM 2: F and f have the same critical points.
Let us establish the following convention: Let $\xi := x_1^2 + x_2^2 + ... + x_\lambda^2$. Then:

$$\frac{\partial F}{\partial \xi} = -1 - \nu'(x_1^2 + ... + x_\lambda^2 + 2x_{\lambda+1}^2 + ... + 2x_n^2)$$

and by condition (3) of the definition of ν, $\frac{\partial F}{\partial \xi} < 0$.
On the other hand, define $\eta := x_{\lambda+1}^2 + ... + x_n^2$. Then:

$$\frac{\partial F}{\partial \eta} = 1 - 2\nu'(x_1^2 + ... + x_\lambda^2 + 2x_{\lambda+1}^2 + ... + 2x_n^2)$$

and similarly, $\frac{\partial F}{\partial \eta} \geq 1$.
Now by the chain rule, we have that $dF = \frac{\partial F}{\partial \eta} d\eta + \frac{\partial F}{\partial \xi} d\xi$, and a moment's thought reveals that this expression is only 0 at the origin.
Additionally, notice that $F(p) = c - \nu(0) < c - \epsilon$ by condition (1) of the definition of ν. Together with last statement, this means that $F^{-1}[c-\epsilon, c+\epsilon]$ is compact and contains no critical points of F. Therefore by the First Theorem and Claim 1, we have that:

$F^{-1}[-\infty, c-\epsilon]$ *has the homotopy type of* $F^{-1}[-\infty, c+\epsilon] = M^{c+\epsilon}$.

To finish, we need to prove that $M^{c-\epsilon} \cup e^\lambda$ is a deformation retract of $F^{-1}[-\infty, c-\epsilon]$ and by the transitivity of *same homotopy type*, we will be done. This part of the proof consists in constructing explicitly the homotopy map, and hence somewhat technical. Let us see:
We are going to define our retraction explicitly: Let $h_t : F^{-1}[-\infty, c-\epsilon] \to F^{-1}[-\infty, c-\epsilon]$ be defined as the identity outside of U, and inside of U by the following conditions:

(1) Within the region $x_1^2 + ... + x_\lambda^2 \leq \epsilon$, we establish:

$$(x_1, ..., x_n) \mapsto (x_1, ..., x_\lambda, tx_{\lambda+1}, ..., tx_n)$$

We can easily notice then that h_1 is the identity and h_0 maps the entire region into e^λ. Additionally, the explicit form of F and ν shows that these maps h_t are well defined.

(2) Within the region $\epsilon \leq x_1^2 + ... + x_\lambda^2 \leq x_{\lambda+1}^2 + ... + x_n^2 + \epsilon$ we set:

$$(x_1, ..., x_n) \mapsto (x_1, ..., x_\lambda, c_t x_{\lambda+1}, ..., c_t x_n)$$

where the numbers c_t are defined by:

$$c_t = t + (1-t)\left(\frac{x_1^2 + ... + x_\lambda^2 - \epsilon}{x_{\lambda+1}^2 + ... + x_n^2}\right)^{\frac{1}{2}}$$

This formula is indeed a bit of a knot, so let us notice a couple of things: First of all, when $x_1^2 + ... + x_\lambda^2 = \epsilon$, we can directly see that the definition of case (2) coincides with that of case (1), and moreover that the transition between both cases is smooth. In fact, since we are precisely in the region $\epsilon \leq x_1^2 + ... + x_\lambda^2 \leq x_{\lambda+1}^2 + ... + x_n^2 + \epsilon$, when $x_{\lambda+1}^2 + ... + x_n^2 \to 0$, simultaneously $x_1^2 + ... + x_\lambda^2 \to \epsilon$ and the smoothness is preserved. Also notice that h_1 is the identity. Finally, we need to see that h_0 falls entirely inside $M^{c-\epsilon} \cup e^\lambda$.
The explicit formula for h_0 here is:

$$(x_1, ..., x_n) \mapsto \left(x_1, ..., x_\lambda, \left(\frac{x_1^2 + ... + x_\lambda^2 - \epsilon}{x_{\lambda+1}^2 + ... + x_n^2}\right)^{\frac{1}{2}} x_{\lambda+1}, ..., \left(\frac{x_1^2 + ... + x_\lambda^2 - \epsilon}{x_{\lambda+1}^2 + ... + x_n^2}\right)^{\frac{1}{2}} x_n\right)$$

Calculating f of these points, the reader can verify its value is exactly $c - \epsilon$ (proving additionally that our map is surjective).

(3) Finally, within the region $x_{\lambda+1}^2 + ... + x_n^2 + \epsilon \leq x_1^2 + ... + x_\lambda^2$ notice that we are precisely inside $M^{c-\epsilon}$, so we leave it as it is. It is convenient because at $x_{\lambda+1}^2 + ... + x_n^2 + \epsilon = x_1^2 + ... + x_\lambda^2$, this definition coincides with case (2) and the transition is again smooth.

We can present the work done schematically in the following diagram:
In this illustration, the striped part is $M^{c-\epsilon}$, the line joining both striped parts alon the 'x'-axis is e^λ, and $M^{c-\epsilon} \cup e^\lambda$ along with the white sort of handle joining both sections of $M^{c-\epsilon}$ is $F^{-1}[-\infty, c-\epsilon]$. The heavily shaded section is where $F > c + \epsilon$. As the arrows show, we are trying to retract $F^{-1}[-\infty, c-\epsilon]$ onto $M^{c-\epsilon}$. Figure 2.4 provides a closer look.
Again, refer to the example of the torus to clarify.

The smoothness of $h : [0,1] \times F^{-1}[-\infty, c-\epsilon] \to F^{-1}[-\infty, c-\epsilon]$ proves that $M^{c-\epsilon} \cup e^\lambda$ is a deformation retract of $F^{-1}[-\infty, c-\epsilon]$ and hence of $M^{c+\epsilon}$ like we wanted to prove. □

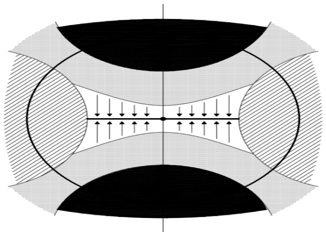

Figure 2.3: Retraction

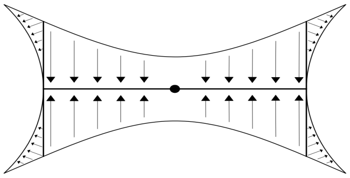

Figure 2.4: Retraction

Remark 2.23. We can generalize this result as follows: Let c be a critical *value* of f which has several critical *points* $p_1, ..., p_k$ such that $f(p_i) = c$. Suppose the points have indexes $\lambda_1, ..., \lambda_k$ respectively. Then for a sufficiently small ϵ, $M^{c+\epsilon}$ has the homotopy type of $M^{c-\epsilon} \cup e^{\lambda_1} \cup ... \cup e^{\lambda_k}$.

2.5 Third Theorem

In this last theorem of the initial chapter of our document, we combine the results of the First and Second theorems to characterize some manifolds as CW-complexes. The result here proven is of tremendous use since a look at the cellular homology of CW-complexes will of course allow us to dive into homological results in the next chapter. With this theorem we finish these first results of Morse Theory, and hope that the gist in which it works has become clear so that we may explore the more profound results in the chapters to come.

Theorem 2.24. If f is a differentiable Morse function on a compact manifold M such that each M^a is compact, then M has the homotopy type of a CW-complex, with one cell of dimension k for each critical point of index k.

Proof. Recall section 2.2. We can then without loss of generality choose $c_1 < c_2 < ...$ to be the critical values of our Morse function f.
First off, notice that for $a < c_1$, M^a is empty and trivially has the homotopy type of a CW-complex. Now, suppose we have a point a such that $a \neq c_i$ for al i, and such that M^a has the homotopy type of a CW-complex. Now let c_{i_0} be the smallest critical value such that $a < c_{i_0}$ (Notice that this can always be

done: if $\forall i$, $c_i < a$; then $M^a = M$ since M is compact, and we are done. If this doesn't happen, then given that the sequence $\{c_i\}_I$ has no cluster points given the compacity of M^a, we can again choose such a c_{i_0}).
Then by the Second Theorem along with remark 2.23, along with the fact that the critical values are isolated we have that for a sufficiently small ϵ, $M^{c_{i_0}+\epsilon}$ has the homotopy type of

$$M^{c_{i_0}+\epsilon} \cup_{\varphi_1} e^{\lambda_1} \cup_{\varphi_2} ... \cup_{\varphi_k} e^{\lambda_k}$$

for some integers $\lambda_1, ..., \lambda_k$ and some attaching maps $\varphi_1, ..., \varphi_k$. Now by the First Theorem, we have a homotopy equivalence $\psi : M^{c-\epsilon} \to M^a$, and by hypothesis another homotopy equivalence $\phi : M^a \to K$ where K is a CW-complex.
Then, defining $\gamma_i := \phi \circ \psi \circ \varphi_i$, we gain an attaching map:

$$\gamma_i : S^{\lambda_i - 1} \to K^{\lambda_i - 1}$$

where $K^{\lambda_i - 1}$ is the $(\lambda_i - 1)$-skeleton of K.

Remark 2.25. We have a couple of inconvenients with this work. For starters, when we defined CW-complexes we established them as the attaching of k-cells but all the time *increasing* the dimensions of the cells being attached. This is in fact not necessary and not even fully standardized. However, to make sue our work is clean and sound, we can simply make sure our Morse function is *self-indexed* so that we do indeed attach cells in order. We can read this work in [Mil65]. For now, let us accept the attaching in disarray. Furthermore, notice that by

$$M^{c_{i_0}+\epsilon} \cup_{\varphi_1} e^{\lambda_1} \cup_{\varphi_2} ... \cup_{\varphi_k} e^{\lambda_k}$$

we are not referring to

$$((...(M^{c_{i_0}+\epsilon} \cup_{\varphi_1} e^{\lambda_1}) \cup_{\varphi_2} ...) \cup_{\varphi_k} e^{\lambda_k}$$

but attaching each cell to the skeleton preceding the whole process. This can complicate the argument but ultimately, since we can not only isolate critical points but critical *values* again, this doesn't affect the validity of the proof

Therefore, by applying *Whitehead* and lemma 2.5, $M^{c+\epsilon}$ has the homotopy type of

$$K \cup_{\gamma_1} e^{\lambda_1} \cup_{\gamma_2} ... \cup_{\gamma_k} e^{\lambda_k}$$

which is a CW-complex.
With the work done it is clear that for any choice of a, M^a has the homotopy type of a CW-complex. Since M is compact, we can choose a such that $M^a = M$ and we obtain the desired result. □

2.6 Applications and Example

We can illustrate all the work we have done with a manageable example taken from the Milnor book [Mil63] itself:

The example shows explicitly how the 2-dimensional torus is constructed as a CW-complex. We begin by putting the torus on its tip and establishing the height function as our Morse function. It can be seen, as we illustrate in the Figure 2.5, that this function has 4 critical points.

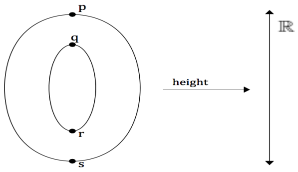

Figure 2.5: Height Function

Let us follow step by step its cellular construction:
We begin by attaching to the empty set a 0-cell, a point. However, the point is homotopically equivalent to a disk which is diffeomorphic to the sort of vessel-shaped surface we would get if we sliced the torus horizontally near the bottom. This step takes place after reaching s, before reaching r. To use our previous terminology, we are at the M^x step for $s < x < r$. We can illustrate this with a diagram:

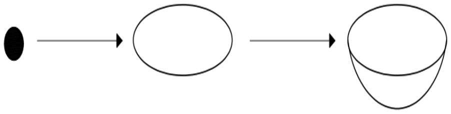

Figure 2.6: First 0-cell attached

The next step would be after reaching r, before reaching q. In this step, we attach a 1-cell (an interval) to our vessel, and this is homotopically equivalent to attaching a wider sort of handle to our vessel. We can deform this into the "magnet-shaped" figure we would associate to the torus sliced in half.

Let us advance. The following step is now between q and p. For this step, we again attach a 1-cell to join the two halves of our magnet, and this is equivalent to attaching a wider ribbon-type handle and we obtained our almost-torus, that is only missing what we called before a vessel on the top to be completed.

To finish, we attach our final 2-cell (disk), and end up with the entire torus.

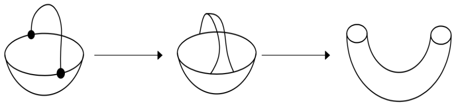

Figure 2.7: First 1-cell attached

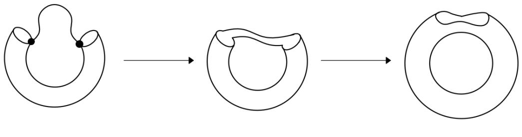

Figure 2.8: Second 1-cell attached

This simple example was quite easy to illustrate and understand, and shows in a very satisfyingly explicit way what we have worked towards.

A recognizable, down to earth application to the work we have done can be presented in the following result known as the Reeb's Theorem.

Theorem 2.26 (Reeb)**.** If M is a compact n-dimensional manifold and f a Morse function on it with only two critical points, then M is homeomorphic to the n-sphere.

Proof. Since M is compact, we know that the critical points of f (let us refer to them as p and q) must be where the maximum and minimum are reached. By convention, set $f(p) = 1$ and $f(q) = 0$. Now take $0 < \epsilon < 1/4$ (we will give an additional restraint for ϵ in a moment). By our Three Theorems, M^ϵ is a k-cell, and a bit more extravagantly this also means $f^{-1}[1-\epsilon, 1]$ is an k-cell (This argument may seem weird but it is actually exactly equivalent to the statement that M^ϵ is a k-cell, because the fact that we defined $M^a := \{p \in M : f(p) \leq a\}$ was rather arbitrary, we might have just as well defined $M^a := \{p \in M : f(p) \geq a\}$. Furthermore, since 0 and 1 are the maximum and minimum of our function, then we obtain M^ϵ by gluing a k-cell to $M^{-\epsilon}$ by our theorems, but $M^{-\epsilon} = \emptyset$, and the same can be said for $f^{-1}[1-\epsilon, 1]$). Notice additionally that since our two critical points achieve maximum and minimum values, then for the maximum value particularly, taking the Lemma of Morse into consideration, the critical point must have index n where n is the dimension of M since given the local expression of f, we cannot have any positive terms that would contradict the maximality of f at the point in question. So at this end we have an n-disk. On the other end, a similar argument now not permitting negative terms shows us that the minimum critical point has index 0. Intuitively, we think of warping

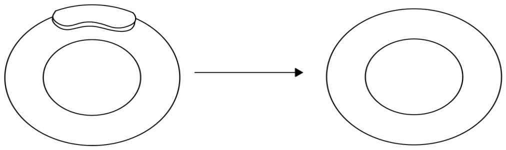

Figure 2.9: Final 2-cell attached

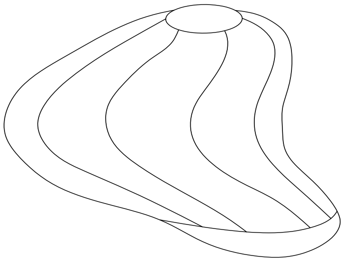

Figure 2.10: Sphere

our n-disk and then attaching a point "stereographically" to obtain our sphere. It can be shown that this is in fact the *one-point compactification* of $\mathbb{R}^n$ which is in turn homeomorphic to $\mathbb{S}^n$.
A different rather technical proof can be found in [Alt08]. A sketch of what she does depends on this lemma:

Lemma 2.27. Let M be a manifold and f a Morse function upon it. Suppose $a, b \in \mathbb{R}$ are such that $a > b$, neither are critical values of f and there are no critical values in between them. Then $f^{-1}(b) \times [b, a]$ is diffeomorphic to $f^{-1}[b, a]$.

With this result, Altman proves that M is in fact homeomorphic to an n-cylinder, with the extremes being n-disks and constituting the "lids"of the cylinder, and the space in-between joining the disks through the lemma. The n-cylinder will in turn be homeomorphic to the n-sphere. $\square$

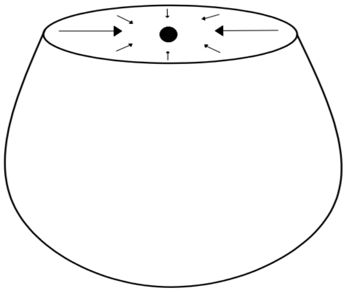

Figure 2.11: Warped Disk with Boundary Identified

Chapter 3

Morse Inequalities: Milnor

In this chapter, we dedicate our attention to the original treatment Milnor's book gives towards proving the Morse inequalities. At first, we will introduce the basic concepts of homology needed to undertake the proofs, so we take some time treating the homology of CW-complexes before proceeding to the proof of the inequalities, which relies strongly on the work undertaken in the previous chapter. As we commented before, the result of the Morse inequalities exemplifies beautifully the strength of the relationship that Morse theory has developed between topology and algebraic topology, and critical points of real-valued functions. With it we finish our account of Morse theory from Milnor's viewpoint, but hope the reader realizes its potential.

3.1 Homology of CW-Complexes

Let us quickly recall the basic construction of homology groups.

Definition 3.1. A *chain complex* of a space X is defined as a an indexed pair $(C_n, \delta_n)_{n\in\mathbb{N}}$ where the C_n's are groups encoding information of X and the δ_n's are homomorphisms (usually called the boundary homomorphisms) organized as:

$$... \overset{\delta_{n+1}}{\to} C_n \overset{\delta_n}{\to} C_{n-1} \overset{\delta_{n-1}}{\to} C_{n-2} \overset{\delta_{n-2}}{\to} ...$$

fulfilling that $\delta_n\delta_{n+1} = 0 \ \forall n$.
This obviously means that $Im(\delta_{n+1}) \subseteq Ker(\delta_n)$, and we may readily define what we call the the n^{th} *homology group of the chain* $(C_n, \delta_n)_{n\in\mathbb{N}}$ as the quotient $H_n(X) := Ker(\delta_n)/Im(\delta_{n+1})$.

Now as anybody with working knowledge in algebraic topology would know, on a given space (generally manifolds) we can define a variety of chain complexes along with their boundary maps to get different homology groups. Generally, different definitions for chain complexes will yield isomorphic groups, and provide the same topological information. The majority of this work will not be undertaken here, and the results and definitions are rapidly skimmed through, so any reader hoping for a more profound treatment on the subject can refer to [Hat02].

Recalling the characterization of manifolds with Morse functions as CW-complexes, and our goal of proving the Morse inequalities, we are going to be interested in results of *singular* and *relative* homologies, so let us quickly review these constructions:

We begin by defining the *singular* homology of a space X: We can view a *singular n-simplex* as a continuous map $\sigma : \Delta^n \to X$ where Δ^n is the standard n-simplex. We define $C_n(X)$ as the free abelian group generated by the n-simplexes on X, and the boundary maps as $\delta_n(\sigma) := \sum_i^n (-1)^i \sigma \mid [v_1, ..., v_{i-1}, v_{i+1}, ..., v_n]$ (if we restrict a map such as σ to the boundary of Δ^n we get $n-1$ simplexes of X). It can be seen that (C_n, δ_n) is a chain complex, the one that precisely corresponds to *singular* homology. We denote the n^{th} singular homology group by $H_n(X)$.

Now, we can define an additional homology on X called the *relative (singular)* homology, which is constructed simply enough: given $A \subset X$ a subspace, we define the chain complex $C_n(X, A) := C_n(X)/C_n(A)$ along with the quotient map $\hat{\delta}_n$ which is well defined since δ_n carries $C_n(A)$ to $C_{n-1}(A)$. We denote the induced homology groups by $H_n(X, A)$. As a convention, if $A = \emptyset$, we denote $H_n(X)$, which of course coincides with the *singular* homology groups, so the notation is adequate.

Observation 3.2. A pertinent commentary on the construction undertaken of the relative homology groups is that for $C \subset B \subset A$ we have a natural exact sequence of the form:

$$... \to H_n(B, C) \to H_n(A, C) \to H_n(A, B) \to H_{n-1}(B, C) \to ...$$

We do not specify the form of the boundary maps here, but the need for such a sequence will come up later. The precise construction is shown in [Hat02].

Proposition 3.3. If X is a CW-complex, then:

(1) $H_k(X^n, X^{n-1})$ is zero for $k \neq n$, and is a free abelian group whose generators are in correspondence with the n-cells of X when $k = n$.

(2) $H_k(X^n) = 0$ for $k > n$.

(3) The inclusion map, $i : X^n \hookrightarrow X$ induces an isomorphism $i_* : H_k(X^n) \to H_k(X)$ if $k < n$.

Again, for the proof of this proposition, along with the other results of this section, we encourage the reader to refer to [Hat02]. We promote special attention to result (1), which begins to clear up the relationship between homology and critical points of Morse functions taking the Three Theorems into account.

Theorem 3.4 (Excision)**.** Given subspaces $Z \subseteq Y \subseteq X$ such that $\overline{Z} \subseteq Int(Y)$. Then:

$$H_n(X - Z, Y - Z) \approx H_n(X, Y)$$

Observation 3.5. The importance of this theorem is evident when taking into account the form of CW-complexes. We may readily observe that if we have a space X with a k-cell attached by a map f, then:

$$H_n(X \sqcup_f e^k, X) = H_n(e^k, S^{k-1})$$

Theorem 3.6. If X,Y are homotopy equivalent spaces, and $V \subseteq X$,$W \subseteq Y$ are also homotopically equivalent, then $\forall n$:

(1) $H_n(X) \approx H_n(Y)$

(2) $H_n(X, V) \approx H_n(Y, W)$

These results on homology will be enough to undertake our proof of the Morse inequalities, so let us proceed.

3.2 The Inequalities

We start off seeing a rather technical result:

Definition 3.7. If X is a set, a map $S : \mathbb{P}(X) \times \mathbb{P}(X) \to \mathbb{Z}$ is defined to be *subadditive* if whenever $Z \subseteq Y \subseteq X$, then:

$$S(X, Z) \leq S(X, Y) + S(Y, Z)$$

If equality holds, we call it *additive*. Moreover, we are going to use the convention $S(X) := S(X, \emptyset)$.

Lemma 3.8. Let S be subadditive over a set X, and a sequence of subsets $X_0 \subseteq X_1 \subseteq ...X_n$. Then $S(X_n, X_0) \leq \sum_{i=1}^{n} S(X_i, X_i - 1)$. If S is additive then the equality holds.

Proof. We are going to prove this by induction on n. The cases $n = 1$ and $n = 2$ hold trivially and by definition. Now if we suppose the result to be true for $n - 1$, then by definition and induction hypothesis:

$$S(X_n, X_0) \leq S(X_{n-1}, X_0) + S(X_n, X_{n-1}) \leq \sum_{i=1}^{n} S(X_i, X_i - 1)$$

□

Definition 3.9. For a space X, and $A \subseteq B \subseteq X$:

(1) We define the n-th **Betti Number relative to B and A** as the rank of the n-th relative homology group $H_n(B, A)$. We denote it by $B_n(B, A)$.

(2) We define the **Euler Characteristic relative to B and A** as:

$$\chi(B, A) := \sum_i (-1)^i B_i(B, A)$$

.

Proposition 3.10. For a CW-complex X, we have:

(1) The Betti numbers relative to n-skeletons of X is subadditive, that is to say that whenever $m < p < n$:

$$B_k(X^n, X^m) \leq B_k(X^n, X^p) + B_k(X^p, X^m)$$

(2) The *Euler Characteristic* χ is additive over the n-skeletons of X; that is t say:

$$\chi(X^n, X^m) = \chi(X^n, X^p) + \chi(X^p, X^m)$$

The proof can be found in [Alt08] or [Hat02].

We have arrived, at last and with a sigh of relief, to the Morse Inequalities. We begin with the weak form: Let M be a compact manifold, and f a Morse function with isolated and finite k critical points. Then recalling section 2.2, we can without loss of generality choose $k+1$ points on $\mathbb{R}$ organized by "<" as: $a_0 < a_1 < ... < a_k$ such that M^{a_i} contains exactly i critical points; and we set the indexes of the critical points contained in $M^{a_{i+1}} - M^{a_i}$ as λ_i. Also, we fix a_k, a_0 so that $M^{a_k} = M$ and $M^{a_0} = \emptyset$ (remember M is compact).

Theorem 3.11 (Weak Morse Inequalities)**.** If F_λ denotes the number of critical points of index λ on a compact manifold M; then:

(1) $B_\lambda \leq F_\lambda$

(2) $\chi(M) = \sum(-1)^\lambda B_\lambda(M) = \sum(-1)^\lambda F_\lambda$

Proof. To prove (1), fix μ a positive integer. Then by Proposition 3.10 and Lemma 3.8;

$$B_\mu(M) \leq \sum_{i=1}^{k} B_\mu(M^{a_i}, M^{a_{i-1}}),$$

and using the Second Theorem along with Excision and Theorem 3.6, we obtain that:

$$B_\mu(M^{a_i}, M^{a_{i-1}}) = B_\mu(M^{a_{i-1}} \sqcup e^{\lambda_i}, M^{a_{i-1}}) = B_\mu(e^{\lambda_i}, S^{\lambda_i - 1})$$

Now, using part (1) of proposition 3.3 we know that

$$B_\mu(e^{\lambda_i}, S^{\lambda_i-1}) = \begin{cases} 1 \text{ if } \mu = \lambda_i \\ 0 \text{ otherwise} \end{cases},$$

and one can readily reflect that this means that $B_\mu \leq F_\mu$.

For the proof of (2), we again use proposition 3.10 to see that:

$$\chi(M) = \sum_{i=1}^{k} \chi(M^{a_i}, M^{a_{i-1}}) = \sum_{i=1}^{k} \sum_{j} (-1)^j B_j(M^{a_i}, M^{a_{i-1}})$$

Again using the same argument as before and by Proposition 3.10, $B_j(M^{a_i}, M^{a_{i-1}}) \neq 0$ only when $j = \lambda_i$, in which case it is equal to 1. Since this happens F_{λ_i} times, and this is true for all i, we obtain $\chi(M) = \sum(-1)^\lambda B_\lambda(M) = \sum(-1)^\lambda F_\lambda$. $\square$

However, as we have anticipated, we can obtain a slightly stronger form of these inequalities. To prove this form, we need the following proposition:

Proposition 3.12. The application $S_\lambda(X,Y) := B_\lambda(X,Y) - B_{\lambda-1}(X,Y) + B_{\lambda-2}(X,Y) - ... \pm B_1(X,Y) \pm B_0(X,Y)$ is subadditive.

Remark 3.13. Notice that the application S_λ is similar to the Euler Characteristic χ when $\lambda = n$, where n is the dimension of the space in question. The main difference holds in the *signs* of the sums of the B_i's, and that no matter how we choose λ, the term B_λ is positive and then we alternate the signs. After the proof of the strong form of the inequalities we will comment on how this makes these stronger than the weak forms.

Proof. Consider an exact sequence of the form:

$$... \xrightarrow{h} A \xrightarrow{i} B \xrightarrow{j} C \xrightarrow{k} ... \xrightarrow{l} D \xrightarrow{m} 0$$

Then direct reflection on the nature of exact sequences prove that:

$$\begin{aligned} rank(h) &= rank(A) - rank(i) \\ &= rank(A) - rank(B) + rank(j) \\ &= rank(A) - rank(B) + rank(C) - rank(k) \\ &... \\ &= rank(A) - rank(B) + rank(C) - ... \pm rank(D) \end{aligned}$$

Now recalling that we have an exact sequence

$$... \to H_{\lambda+1}(X,Y) \xrightarrow{\delta} H_\lambda(Y,Z) \to H_\lambda(X,Z) \to H_\lambda(X,Y) \to ...$$

hence applying the previous argument to this sequence, we obtain that

$$rank(\delta) = B_\lambda(Y,Z) - B_\lambda(X,Z) + B_\lambda(X,Y) - B_{\lambda-1}(Y,Z) + ... \pm B_0(X,Y)$$

If we group terms intelligently, we get that $rank(\delta) = S_\lambda(Y,Z) - S_\lambda(X,Z) + S_\lambda(X,Y)$ and since obviously $rank(\delta) \geq 0$, we obtain that $S_\lambda(Y,Z) + S_\lambda(X,y) \geq S_\lambda(X,Z)$. ☐

Theorem 3.14. *Morse Inequalities* Using the same notation as before, for any λ:

$$B_\lambda(M) - B_{\lambda-1}(M) + ... \pm B_0(M) \leq F_\lambda - F_{\lambda-1} + ... \pm F_0$$

Proof. The proof is exactly equivalent to the one we used for the weak form of the inequalities. Retaking the previous notation, by Proposition 3.12:

$$B_\lambda(M) - B_{\lambda-1}(M) + ... \pm B_0(M) = S_\lambda(M) \leq \sum_{i=1}^{k} S_\lambda(M^{a_i}, M^{a_{i-1}})$$

Now, we defined $S_\lambda(M^{a_i}, M^{a_{i-1}}) = B_\lambda(M^{a_i}, M^{a_{i-1}}) - B_{\lambda-1}(M^{a_i}, M^{a_{i-1}}) + ... \pm B_0(M^{a_i}, M^{a_{i-1}})$, and hence a moment's thought reveals that:

$$\sum_{i=1}^{k} S_\lambda(M^{a_i}, M^{a_{i-1}}) = \sum_{i=1}^{k} B_\lambda(M^{a_i}, M^{a_{i-1}}) - \sum_{i=1}^{k} B_{\lambda-1}(M^{a_i}, M^{a_{i-1}}) + ... \pm \sum_{i=1}^{k} B_0(M^{a_i}, M^{a_{i-1}})$$

Revisiting the proof of the weak forms, we see that $\sum_{i=1}^{k} B_\mu(M^{a_i}, M^{a_{i-1}}) = F_\mu$ for any μ and we obtain our result. □

Remark 3.15. As we anticipated before, the essence of why this form of the inequalities is stronger than the previous forms lies in the fact that no matter the choice of λ, the term $B_\lambda(M)$ is positive in the sumation. So for example taking the inequality for λ and $\lambda - 1$, and adding them, we obtain $B_\lambda(M) \leq F_\lambda$. On the other hand, if we choose $\lambda > n + 1$ where n is the dimension of M (by dimension we refer to the maximum m such that we have m-cells), so that $H_\lambda(M) \approx 0$, and such that $\lambda > \max_i(\lambda_i) + 1$ then:

$$\begin{aligned} B_\lambda(M) - B_{\lambda-1}(M) + ... \pm B_0(M) &= 0 - 0 + 0 - ... \pm B_n(M) \pm ... \pm B_0(M) \\ &\leq \pm F_{max(\lambda_i)} \pm ... \pm F_0 \end{aligned}$$

and

$$\begin{aligned} B_{\lambda-1}(M) - B_{\lambda-2}(M) + ... \pm B_0(M) &= 0 - 0 + 0 - ... \pm B_n(M) \pm ... \pm B_0(M) \\ &\leq \pm F_{max(\lambda_i)} \pm ... \pm F_0 \end{aligned}$$

where the signs ± between these two expressions are interchanged. Therefore we have both sides of the inequalities and gain the second weak form; $\chi(M) = \sum (-1)^\lambda B_\lambda(M) = \sum (-1)^\lambda F_\lambda$.

Chapter 4

Morse Inequalities: Witten

As we said before, the approach this chapter takes on the subject is quite independent from the rest of the chapters of the book, but its technique is also quite interesting. It can be read independently from the rest of the theory (with the possible exception of the section on cohomology) which is also appealing. Ultimately, we will give a new proof for the weak form of the Morse inequalities, independent from the approach we have been working on. A proof for the strong form can also be attempted through this technique, but it requires machinery regarding the asymptotic behavior of eigenvalues; and involves work we will not do here. Witten's original article does make an argument for the strong form, so an interested reader could reference [Wit82].

The focus of this chapter to prove the inequalities is the study of the eigenvalues of operators over the exterior algebra of sections of the cotangent bundle of the manifold in question, so we shall begin the chapter treating some of these concepts:

4.1 Differential Forms and Usual Operators

Let us remember the definition of differential forms:

Definition 4.1. Let M be a manifold. For k a positive integer, we call *differential k-form* a smooth (albeit, not necessarily; in fact we will broaden the types of forms we accept later on) section of the kth exterior algebra of the cotangent bundle of M. In other words, given ω a differential k-form, and a point $p \in M$, what we will call ω_p, that is to say the section ω evaluated on p, is an anti-symmetric bilinear map from $\underbrace{T_pM \times T_pM \times ... \times T_pM}_{k-\text{times}}$ to $\mathbb{R}$.

Let us notice several things. A standard technique is to refer to the tangent space T_pM as the vector space with basis $\{\frac{\partial}{\partial x_1} |_p, \frac{\partial}{\partial x_2} |_p, ..., \frac{\partial}{\partial x_n} |_p\}$, where n is the dimension of M. Now let us establish a working convention for the cotangent space like this: We will refer as $\{dx_1 \mid_p, dx_2 \mid_p, ..., dx_n \mid_p\}$ to the dual basis of $\{\frac{\partial}{\partial x_1} \mid_p, \frac{\partial}{\partial x_2} \mid_p, ..., \frac{\partial}{\partial x_n} \mid_p\}$, that is to say:

$$dx_i(\frac{\partial}{\partial x_j}) = \delta_{ij}, \forall j, i$$

Now, since for example a 1-form is precisely an element of $(T_pM)^*$, we can express any 1-form ω evaluated at p as a linear combination of this basis, something like:

$$\omega_p = \textstyle\sum_{i=1}^n \alpha_{i,p} dx_i \mid_p$$

Now, if we define a function $\hat{\omega}_i$ over the manifold as $\hat{\omega}_i(p) := \alpha_{i,p}$, then we can generalize to write the form as:

$$\omega = \textstyle\sum_{i=1}^n \hat{\omega}_i dx_i$$

The generalization to k-forms obeys the same idea. The natural orthonormal basis (we will talk about an internal product of k-forms in a moment) is of course $\{dx_{i_1} \wedge dx_{i_2} \wedge, ..., \wedge dx_{i_k}\}_{i_1<i_2<...<i_n}$ with $i_j \in \{1, ..., n\}$, and so a k-form can be expressed as:

$$\omega = \sum_{i_1<i_2<...<i_k} \hat{\omega}_{i_1,i_2,...,i_k} dx_{i_1} \wedge dx_{i_2} \wedge ... \wedge dx_{i_k}$$

Definition 4.2. The space of all k-forms upon the manifold M is referred to as $\Omega_k(M)$. Additionally, $\bigoplus_{k\in\mathbb{N}} \Omega_k(M)$ is called the *exterior algebra of forms.*

Now as our aim is to calculate eigenvalues of some operators, we are going to need to calculate adjoints, and for this we need to specify an internal product upon differential forms.
We begin our work pointwise, so fix $p \in M$.
Given k a positive integer, we want to calculate an inner product of two k-forms on p. Once we have this, we can generalize to an inner product between forms of different degree with the usual inner product of direct sums, but this really will not be necessary for our purposes, so we'll only consider inner products between two k-forms. So how do we manage this?
First of all, we are going to work over Riemannian manifolds, so our tangent spaces are going to have an inner product, and we can set our usual basis $\{\frac{\partial}{\partial x_1} \mid_p, \frac{\partial}{\partial x_2} \mid_p, ..., \frac{\partial}{\partial x_n} \mid_p\}$ as an *orthonormal* basis, and gain an inner product through bilinearity. The idea is to do the same with any exterior power of the cotangent spaces, so in general, we set the basis $\{dx_{i_1} \wedge dx_{i_2} \wedge, ..., \wedge dx_{i_k}\}_{i_1<i_2<...<i_n}$ with $i_j \in \{1, ..., n\}$ as our orthonormal basis and define the rest by bilinearity. Let us look at a specific formula for an inner product then:

Let $\varphi = \sum_{i_1<i_2<...<i_k} \hat{\varphi}_{i_1,i_2,...,i_k} dx_{i_1} \wedge dx_{i_2} \wedge ... \wedge dx_{i_k}$ and
$\psi = \sum_{j_1<j_2<...<j_k} \hat{\psi}_{j_1,j_2,...j_k} dx_{j_1} \wedge dx_{j_2} \wedge ... \wedge dx_{j_k}$ be two k-forms. Then:

$$\begin{aligned}\langle\varphi,\psi\rangle_p &= \sum_{i_1<\ldots<i_k}\hat{\varphi}_{i_1,\ldots,i_k}(p)\sum_{j_1<\ldots<j_k}\hat{\psi}_{j_1,\ldots,j_k}(p)\langle dx_{i_1}\wedge\ldots\wedge dx_{i_k}, dx_{j_1}\wedge\ldots\wedge dx_{j_k}\rangle\\ &= \sum_{i_1<\ldots<i_k}\hat{\varphi}_{i_1,\ldots,i_k}(p)\hat{\psi}_{i_1,i_2,\ldots,i_k}(p)\end{aligned}$$

Now to generalize over the manifold, as seems natural, we integrate and hence define:

$$\langle\varphi,\psi\rangle := \int_M \langle\varphi,\psi\rangle_p$$

Observation 4.3. Notice that to check if an operator is the adjoint of another globally, it suffices to check that it is so point by point because then the integral in the previous definition is obviously true, but the converse is not upheld generally.

Remark 4.4. Notice that if $f: M\to\mathbb{R}$; and L is any given operator on forms; then $(fL)^*$=fL^* since for any given forms φ and ψ; we have that pointwise $\langle(fL)\varphi,\psi\rangle_p = f(p)\langle L\varphi,\psi\rangle_p = f(p)\langle\varphi, L^*\psi\rangle_p = \langle\varphi,(fL^*)\psi\rangle_p$

Now notice that with the definition of this inner product, we gain a space of forms in which to work, namely, the natural definition of $L^2(M)$ gained by this inner product. Notice however that since the manifold is compact, this definition establishes no real restriction upon our working space of forms.
In fact, we are actually permitting more forms than those expected, because an L^2 form over M need not be necessarily smooth.
This fact of choosing domains of forms on which to work is actually one of the more problematic and shunned parts of the theory in the literature.

Notice also that given the definition of our inner product, what matters when deciding whether a given form is or not L^2 is the “companion function”. Therefore, when we look at $L^2(M)$ forms locally, we will always be dealing with $L^2(\mathbb{R}^n)$-companion functions.
A converse reasoning is **very** difficult. If we want to work with certain types of functions *locally*, it is not at all clear how we define a *global* domain so that when looked at locally, it fulfils our original requirement. This is a problem which will accompany us throughout the whole chapter. In fact, the operators which we will define in the following sections are presented without a domain, and some preliminary restrictions upon the forms over which they are permitted to act will be given naturally through the course of the chapter. This problem is commented a bit more thoroughly in the Conclusions, but for now we push the understanding of the reader and simply move on.

4.1.1 Some Operators on Forms and their Adjoints

In this section, we want to introduce some of the operators we are going to be using further on.

Definition 4.5. Exterior or Wedge Product Given φ and ψ a k and l form respectively, their *exterior product* denoted by $\varphi \wedge \psi$ is a $(k+l)$-form defined pointwise as follows:

$$(\varphi \wedge \psi)_p(v_1, ..., v_{k+l}) =$$
$$\tfrac{1}{k!l!} \textstyle\sum_{\sigma \in S_{k+l}} (-1)^{sgn(\sigma)} \varphi_p(v_{\sigma(1)}, ..., v_{\sigma(k)}) \psi_p(v_{\sigma(k+1)}, ..., v_{\sigma(k+l)})$$

A particular operator on forms which we want to consider is the exterior multiplication on the left by a fixed 1-form dx_i, and we are going to denote it by E_{dx_i}; so $E_{dx_i}(\psi) := dx_i \wedge \psi$.

Definition 4.6. Interior Multiplication Given X a vector field over M, and ψ a k-form, the *interior multiplication of* ψ *by* X, denoted by $i_X \psi$ is a $(k-1)$-form defined pointwise as:

$$(i_X \psi)_p(v_1, ..., v_{k-1}) = \psi_p(X_p, v_1, ..., v_{k-1})$$

Proposition 4.7. For X a vector field over M and ω, η any given forms, the following identity holds:

$$i_X(\omega \wedge \eta) = i_X(\omega) \wedge \eta + (-1)^{|\omega|} \omega \wedge i_X(\eta)$$

where by $|\omega|$ we refer to the *degree* of ω.

As with the exterior multiplication, we are going to consider particularly the interior multiplication setting $X = \frac{\partial}{\partial x_i}$, and we are going to denote it by I_{dx_i}.

Observation 4.8. One may readily verify that both the exterior and interior multiplication are linear operators.

Proposition 4.9. Setting $f \in \{1, 2, ..., n\}$, exterior multiplication by dx_f is the adjoint operation to interior multiplication by $\frac{\partial}{\partial x_f}$. That is to say, $(E_{dx_f})^* = I_{dx_f}$.

Proof. Let $\varphi \in \Omega_k(M)$ and $\psi \in \Omega_{k-1}(M)$. It will be enough to check that pointwise, $\langle I_{dx_f}(\varphi), \psi \rangle = \langle \varphi, E_{dx_f}(\psi) \rangle$.
Now let $\{i_1, ..., i_{k-1})\} \subseteq \{1, ..., n\}$. Let us evaluate $I_{dx_f}(\varphi)(\frac{\partial}{\partial x_{i_1}}, ..., \frac{\partial}{\partial x_{i_{k-1}}})$:
$I_{dx_f}(\varphi)(\frac{\partial}{\partial x_{i_1}}, ..., \frac{\partial}{\partial x_{i_{k-1}}}) = \varphi(\frac{\partial}{\partial x_f}, \frac{\partial}{\partial x_{i_1}}, ..., \frac{\partial}{\partial x_{i_{k-1}}}) = \hat{\varphi}_{i_1, ..., i_{j-1}, f, i_j, ..., i_{k-1}} (-1)^{\textit{f}}$, where $\textit{f}$ is the number of transpositions it took to place f in its place ordered by "<" in the set $\{i_1, ..., i_{k-1}\}$.

Now let $\{j_1, ..., j_k\} \subseteq \{1, ..., n\}$. Let us evaluate $E_{dx_f}(\psi)(\frac{\partial}{\partial x_{j_1}}, ..., \frac{\partial}{\partial x_{j_k}})$:
$E_{dx_f}(\psi)(\frac{\partial}{\partial x_{j_1}}, ..., \frac{\partial}{\partial x_{j_k}}) = \sum_{m=1}^{k} (-1)^{m+1} dx_f(\frac{\partial}{\partial x_{j_m}}) \hat{\psi}_{j_1, ..., j_{m-1}, j_{m+1}, ..., j_k}$.
Since $dx_f(\frac{\partial}{\partial x_{j_m}})$ is only 1 when $f = j_m$ and 0 otherwise, this can be written as: $(-1)^{m+1} \hat{\psi}_{j_1, ..., j_{m-1}, j_{m+1}, ..., j_k}$ when $j_m = f$, 0 otherwise.

All in all, taking into account the explicit expression we gave for inner products, the equation we want to verify is:

$$\begin{aligned} &\sum_{i_1<\ldots<i_{k-1}} (-1)^f \hat{\varphi}_{i_1,\ldots,i_{m-1},f,i_{m+1},\ldots,i_{k-1}} \hat{\psi}_{i_1,\ldots,i_{k-1}} \\ = &\sum_{i_1<\ldots<i_k \, f=i_m} \hat{\psi}_{i_1,\ldots,i_{m-1},i_{m+1},\ldots,i_k} \hat{\varphi}_{i_1,\ldots,i_k} (-1)^f. \end{aligned}$$

We leave it to the reader to verify this equality, but a moment's thought, as tangled up as it seems to be, will reveal it to be correct. □

Remark 4.10. The linearity of adjoints gives us a more generalized version of this proposition. Consider any function f over M. Then its usual exterior derivative, the linear map $df\mid_p : T_pM \to \mathbb{R}$ (well known from differential geometry courses, but we will speak of its generalizations in a moment), is in fact a 1-form, given as a linear combination of the basis like this:

$$df\mid_p = \textstyle\sum_{i=1}^n \frac{\partial f}{\partial x_i}(p)dx_i,$$

so if we denote the exterior multiplication by df by E_{df}, we may readily calculate its adjoint:
$(E_{df})^* = (E_{\sum_{i=1}^n \frac{\partial f}{\partial x_i} dx_i})^* = (\sum_{i=1}^n \frac{\partial f}{\partial xi} E_{dx_i})^* = \sum_{i=1}^n \frac{\partial f}{\partial x_i} I_{dx_i} = i_{\sum_{i=1}^n \frac{\partial f}{\partial x_i}\frac{\partial}{\partial x_i}} = i_{\nabla f}$,
where ∇f represents the gradient of f. Again, we are going to establish the convention $i_{\nabla f} = I_{df}$, so we have established that $(E_{df})^* = I_{df}$.

Proposition 4.11. $E_{df}I_{df} + I_{df}E_{df} = \|df\|^2$, where by $\|df\|$ we mean the operator which simply multiplies a form by that scalar.

Proof. Let us calculate on a form ω:

$$\begin{aligned} (E_{df}I_{df} + I_{df}E_{df})\omega &= E_{df}I_{df}\omega + I_{df}E_{df}\omega \\ &= df \wedge i_{\nabla f}\omega + i_{\nabla f}(df \wedge \omega) \\ &= df \wedge i_{\nabla f}\omega + i_{\nabla f}(df) \wedge \omega - df \wedge i_{\nabla f}\omega \text{ by Proposition 4.7} \\ &= i_{\nabla f}(df) \wedge \omega \\ &= \|df\|^2\omega \end{aligned}$$

□

4.1.2 The Exterior Derivative

An essential operator for our purposes is going to be an extension of the regular derivative of smooth functions on manifolds. In this section we are going to look at the definition and some of the essential properties:

Definition 4.12. Exterior Derivative Given ω a k-form, $\omega = \sum_{i_1<i_2<\ldots<i_k} \hat{\omega}_{i_1,\ldots,i_k} dx_{i_1} \wedge \ldots \wedge dx_{i_k}$, its *exterior derivative*, denoted by $d\omega$, is the $k+1$-form defined by:
$d\omega = \sum_{i_1<i_2<\ldots<i_k} \sum_{j=1}^n \frac{\partial \hat{\omega}_{i_1,\ldots,i_k}}{\partial x_j} dx_j \wedge dx_{i_1} \wedge \ldots \wedge dx_{i_k}$.

Proposition 4.13. Regarding the exterior derivative, the following properties hold to be true:

(1) $d(d\omega) = 0$ for any k-form ω

(2) $d(\varphi \wedge \psi) = d\varphi \wedge \psi + (-1)^k(\varphi + d\psi)$ where k is the degree of φ, for any k-form φ and l-form ψ.

Proof. Let us prove (1) and leave (2) for the reader. These are straightforward proofs, and simply require writing out the definitions.
Let us write out $d^2(\psi dx_{i_1} \wedge ... \wedge dx_{i_k}$:

$$= d(\sum_j \frac{\partial \psi}{\partial x_j} dx_j \wedge dx_{i_1} \wedge ... \wedge dx_{i_k})$$

$$= \sum_k \sum_j \frac{\partial^2 \psi}{\partial x_k \partial x_j} dx_k \wedge dx_j \wedge dx_{i_1} \wedge ... \wedge dx_{i_k}$$

Now, choose a pair of numbers, a, b. Notice that the member of our double sum when $j = a$ and $k = b$ is minus the term when $j = b$ and $k = a$. When we complete the double sum, we are left with 0. □

4.2 De Rham Cohomology and Theorem

Now let us present how this path will take us towards the Morse Inequalities. Recall that the inequalities give us information on the rank of the homology groups of a manifold. An analogous concept to homology is that of *cohomology*, which is just a dual concept in the sense that our boundary maps in *cohomology* increase dimension instead of decreasing it as in homology. In short, *cohomology* groups correspond to *cochain complexes* that are of the form:

$$... \overset{\delta_{n-1}}{\to} C_n \overset{\delta_n}{\to} C_{n+1} \overset{\delta_{n+1}}{\to} C_{n+2} \overset{\delta_{n+2}}{\to} ...$$

such that $\forall n\ \delta_{n+1}\delta_n = 0$. As expected, we define the nth cohomology group as $Ker(\delta_{n+1})/Im(\delta_n)$.

Now, notice that by part (1) of proposition 4.13, we can establish a cochain complex of the form:

$$... \overset{d_{n-1}}{\to} \Omega_n(M) \overset{d_n}{\to} \Omega_{n+1}(M) \overset{d_{n+1}}{\to} \Omega_{n+2}(M) \overset{d_{n+2}}{\to} ...$$

where d_n is the exterior derivative restricted to n-forms.

Definition 4.14. We define the *n-th de Rham cohomology group* as $Ker(d_{n+1})/Im(d_n)$, and denote it by $H^n_{dR}(M)$.

The following theorem shows us why this will be of our interest:

Theorem 4.15 (De Rham Theorem)**.** There is an isomorphism $H^n_{dR}(M) \cong H^n(M)$ where $H^n(M)$ denotes the n-th singular cohomology group.

Remark 4.16. Notice that since all of our spaces are finite dimensional, then $H^n(M) \cong H_n(M)$. This means that the dimension of the latter, the Betti numbers, are equally characterized by the dimension of the former.

The proof of this theorem is long and requires a large machinery of theory to be developed previously, but a very pertinent paper can be referenced [Mar]. However, notice that we can now characterize the *Betti numbers* of our manifold M as the dimension of the space of forms of a given degree that are cancelled under exterior derivation, but are not the derivation of a form of minor degree. To uncover these forms we can calculate the eigenvalues of the Laplacean Δ as we will see in a moment, and the strategy adopted by Witten becomes evident.

4.3 The Laplacean and Modified Laplacean

Recall the result we want to prove. We want to bound the pth Betti number, $B_p(M)$ by the number of critical points of index p, which we have called F_p. We just saw that $B_p(M)$ can be characterized precisely as the number of linear independent p-forms such that $d\psi = 0$, but $\psi \neq d\varphi$ for any $(p-1)$-form φ.

To find these forms define the Laplacean operator by:

$$\Delta = dd^* + d^*d,$$

acting on $\Omega_p(M)$ for any given p.
Also, let f be a Morse function over M. Define $d_t := e^{-tf}de^{tf}$, and obviously we analogously have that $d_t^* = e^{tf}d^*e^{-tf}$.
The operator

$$\Delta_t = d_t d_t^* + d_t^* d_t$$

is going to be our modified Laplacean operator.

Remark 4.17. Notice trivially that $d_t^2 = 0$

Now, consider the following map: $T(\omega) \mapsto e^{-tf}\omega$. It is obviously an invertible mapping, so establishes a one-to-one correspondence(a permutation if you will) between k-forms. Let us observe the following:

(1) Suppose we have a form ω such that $d\omega = 0$. Then $d_t(T\omega) = e^{-tf}de^{tf}e^{-tf}\omega = e^{-tf}d\omega = 0$

(2) We can easily verify that:

$$\begin{aligned} d_t T\omega &= e^{-tf}de^{tf}T\omega \\ &= e^{-tf}d\omega \\ &= Td\omega \end{aligned}$$

hence, if φ is such that $\neg\exists\psi$ such that $d\psi = \varphi$; then it is direct to see by the previous equality that $\neg\exists\gamma$ such that $d_t\gamma = T\varphi$

What this reflection shows is that we have a one-to-one correspondence between forms such that $d\varphi = 0$ and $\neg\exists\psi$ such that $d\psi = \varphi$, and forms such that $d_t\varphi = 0$ and $\neg\exists\psi$ such that $d_t\psi = \varphi$.
If we define $H^n_{dRt}(M)$ as the nth cohomology group corresponding to the cochain complex:

$$\ldots \overset{d_{t_{n-1}}}{\to} \Omega_n(M) \overset{d_{t_n}}{\to} \Omega_{n+1}(M) \overset{d_{t_{n+1}}}{\to} \Omega_{n+2}(M) \overset{d_{t_{n+2}}}{\to} \ldots$$

where d_{t_n} is d_t restricted to n-forms, then we have established thus far the following fact:

$$H^n_{dRt}(M) \cong H^n_{dR}(M) \cong H_n(M)$$

Remark 4.18. Notice the invariance of the dimension of $H^n_{dRt}(M)$ with resect to t.

Through Hodge Theory, we will discover how to calculate the dimension of $H^n_{dRt}(M)$ and $H^n_{dR}(M)$, hence calculating the Betti numbers like we wanted.

4.3.1 Hodge Theory

Let us introduce Hodge Theory by working towards his original decomposition theorem.

Definition 4.19. Given p a positive integer, we define a p-form φ to be *harmonic* if $\Delta(\varphi) = 0$. The space of harmonic p-forms is denoted by $\mathcal{H}^p(M)$.

Remark 4.20. The notation $\mathcal{H}^p(M)$ of the harmonic p-forms is not to be confused with $H^p(M)$, the p-th singular cohomology group.

Let us take a look at an interesting lemma:

Lemma 4.21. On a Riemannian manifold M, for a p-form ω we have that

$$\Delta\omega = 0 \iff d\omega = 0 \text{ and } d^*\omega = 0$$

Proof. One of the implications is clear. Namely, if we suppose $d\omega = 0$ and $d^*\omega = 0$, then $\Delta\omega = dd^*\omega + d^*d\omega = d0 + d^*0 = 0$.
Now for the converse, suppose $\Delta\omega = 0$. Now we calculate:

$$\langle\Delta\omega, \omega\rangle = \langle(dd^* + d^*d)\omega, \omega\rangle = \langle dd^*\omega, \omega\rangle + \langle d^*d\omega, \omega\rangle = \langle d^*\omega, d^*\omega\rangle + \langle d\omega, d\omega\rangle = 0$$

and this equality can only hold if $d\omega = 0$ and $d^*\omega = 0$. □

Remark 4.22. Notice that the only property of the Laplacean used for the proof of lemma 4.21 is the fact that it is the alternated sum of a map and its adjoint. In particular, this lemma holds true for Δ_t, d_t and d^*_t respectively as well..

The following theorem we present without proof due to its length and complexity. A proof can be read in [AMR83].

Theorem 4.23 (Hodge's Decomposition Theorem)**.** Let M be a Riemannian manifold. Then for each positive integer p, $\mathcal{H}^p(M)$ is finite dimensional and:

$$\Omega_p(M) = \Delta(\Omega_p(M)) \oplus \mathcal{H}^p(M) = d(\Omega_{p-1}(M)) \oplus d^*(\Omega_{p+1}(M)) \oplus \mathcal{H}^p(M)$$

Corollary 4.24. $\mathcal{H}^p(M) \cong H^p_{dR}(M)$

Proof. Let us construct the isomorphism. Notice first of all that if $\omega \in \mathcal{H}^p(M)$, then it must be true that $d\omega = 0$. Let $i : \mathcal{H}^p(M) \hookrightarrow Ker(d_p)$ be the inclusion map and $\Pi : Ker(d_p) \to H^p_{dR}(M)$ be the projection. We will prove that $\Pi \circ i$ is an isomorphism.
Suppose we have a form $\gamma \in \mathcal{H}^p(M)$ such that $\Pi \circ i(\gamma) = 0$. Then by definition we must have a form β such that $d\beta = \gamma$. However, since $\gamma \in \mathcal{H}^p(M)$, then γ is orthogonal to $d\beta$, hence orthogonal to itself and hence $\gamma = 0$. On the other hand, to check surjectivity, suppose $[w] \in H^p_{dR}(M)$. By Hodge's Decomposition, we can write $w = d\alpha + d^*\beta + \gamma$ with $\gamma \in \mathcal{H}^p(M)$. Now, since $d\omega = 0$ by hypothesis, $dd\alpha + dd^*\beta + d\gamma = dd^*\beta = 0$, and so $0 = \langle \beta, dd^*\beta \rangle = \langle d^*\beta, d^*\beta \rangle$ and we conclude that $d^*\beta = 0$; and therefore $\omega = d\alpha + \gamma$ which means that $\Pi \circ i(\gamma) = [\omega]$, proving it has a preimage like we wanted. ☐

We now realize that calculating the geometric multiplicity of the eigenvalue 0 of Δ on p-forms is equal to calculating the p-th Betti number $B_p(M)$.
However, Hodge Theory has since gone further than this. To present the applications we need for this document, we need to go through Index Theory and Elliptic Operators Theory, but this is a subject we will not elaborate on. The depth of the theory can be found in [AMR83] or [LM89].

Definition 4.25. Let M be an n-dimensional manifold, m a positive integer and E a k-dimensional vector bundle over M. As expected, we denote by $\Gamma(E)$ the smooth sections of E. For $\gamma \in \Gamma(E)$, notice that since we are over an n-dimensional manifold M, the vector bundle looks locally like $\mathbb{R}^n \times \mathbb{R}^k$ and γ as a map $\gamma : \mathbb{R}^n \to \mathbb{R}^k$. Therefore it makes sense to speak of $\frac{\partial \gamma}{\partial x_i}(\mathbf{x})$, defined as:

$$\lim_{t \to 0} \frac{\gamma(\mathbf{x} + t x_i) - \gamma(\mathbf{x})}{t}$$

as constituting a new smooth section, and hence we can generalize to $\frac{\partial^2 \gamma}{\partial x_i \partial x_j}(\mathbf{x})$, $\frac{\partial^3 \gamma}{\partial x_i \partial x_j \partial x_k}(\mathbf{x})$, etc.
In this sense, we say γ is *vanishing of m-th order* if for any multi-index α with $|\alpha| = m$, we have that:

$$\frac{\partial^{|\alpha|}\gamma}{\partial x_{\alpha_1} \partial x_{\alpha_2} ... \partial x_{\alpha_m}} = \mathbf{0}$$

and $\forall h < m$ there exists at least one multi-index β with $|\beta| = h$ such that:

$$\frac{\partial^{|\beta|}\gamma}{\partial x_{\beta_1} \partial x_{\beta_2} ... \partial x_{\beta_m}} \neq \mathbf{0}$$

It can easily be shown that this definition, just as with critical points, is independent of coordinate system.

Definition 4.26. Let E, F be two vector bundles over M. We call a linear map $D : \Gamma(E) \to \Gamma(F)$ a *differential operator of degree* m if for any $\gamma \in \Gamma(E)$ such that γ is vanishing of m-th order, we have that $D(\gamma) = 0$.

It is not difficult to see, as Marsden points out, that a differential operator can be written out locally as:

$$D(\gamma) = \sum_{|\alpha| \leq m} A_\alpha \frac{\partial^{|\alpha|}\gamma}{\partial x_{\alpha_1} \partial x_{\alpha_2} ... \partial x_{\alpha_m}}$$

where A_α are matrices corresponding to bundle maps between E and F.

Definition 4.27. Fix $p \in M$, $\xi \in T_p^*M$ and a local coordinate system $(x_1, ..., x_n)$. Naturally, we have an expansion:

$$\xi = \sum_i \xi_i dx_i$$

For a differential operator $D = \sum_{|\alpha| \leq m} A_\alpha \frac{\partial^{|\alpha|}}{\partial x_{\alpha_1} \partial x_{\alpha_2} ... \partial x_{\alpha_m}}$ we define its *principal symbol* at ξ as:

$$\sum_{|\alpha| = m} \xi_{\alpha_1} \xi_{\alpha_2} ... \xi_{\alpha_m} A_\alpha$$

Definition 4.28. A differential operator P is said to be *elliptic* if for every $p \in M$ and $\xi \in T_p^*M - \{0\}$, the principal symbol $\sigma_\xi(P) : E_p \to F_p$ is invertible.

It can be found in [AMR83] that both Δ and Δ_t are self-adjoint elliptic operators. A sketch is somewhat as follows:

(1) The symbol of d is E_ξ

(2) The symbol of d^* is $I_{\xi^@}$ where by $\xi^@$ we mean the vector in $T_p(M)$ defined by $\sum_i \xi_i \frac{\partial}{\partial x_i}$

(3) The principal symbol obeys a multiplicative property

With these facts, together with proposition 4.11, the symbol of Δ is $\|\xi\|^2$, invertible as long as $\xi \neq 0$.
The interest of doing this is the following theorem, which can be found in [LM89]:

Theorem 4.29. Let E be a tangent bundle over M, a compact Riemannian manifold, and $P : \Gamma(E) \to \Gamma(E)$ be an elliptic self-adjoint differential operator. Then there is an L^2-orthogonal decomposition of the form:

$$\Gamma(E) = Ker(P) \oplus Img(P)$$

This result is much stronger than the Hodge Decomposition Theorem, which is in fact a particular case of this statement taking $E = \Omega_p(M)$ and $P = \Delta$; we obtain that any k-form may be written as $\omega = \Delta\alpha + \gamma = dd^*\alpha + d^*d\alpha + \gamma$ with $\gamma \in \mathcal{H}^k(M)$. By taking $\rho = d^*\alpha$ and $\beta = d\alpha$, and noticing that trivially $Im(dd^*) \perp Im(d^*d)$, we obtain Hodge's Decomposition.
Now, since this same analysis can be made for Δ_t, we obtain an equivalent decomposition for Δ_t, d_t and d_t^*; and hence an analogue result to that of corollary 4.24. Thus, if we denote $\mathcal{H}_t^p(M) := \{\omega \in \Omega_p(M) \mid \Delta_t\omega = 0\}$ (the t-harmonic forms), so far we have achieved:

$$\mathcal{H}_t^p(M) \cong H_{dRt}^p(M) \cong H_{dR}^p(M) \cong \mathcal{H}^p(M) \cong H_n(M)$$

and hence we may very well calculate Betti numbers by calculating the dimension of $Ker(\Delta_t \mid_{\Omega_p(M)})$, which is precisely the strategy Witten employs. The rest of our document is aimed at this purpose.

4.3.2 The Local Behavior of Δ_t

We are going to attempt to obtain an explicit formula for Δ_t, at least locally, so we may calculate its eigenvalues. From this point on we are always gong to assume our work to be local around a critical point, and as conventions let us accept that p will be the critical point in question, and λ will be its index. Let us begin by some simple calculations:

Proposition 4.30. $d_t = d + tE_{df}$

Proof. Let $\omega \in \Omega_p(M)$. Then we can calculate that:
$d_t(\omega) = e^{-tf}de^{tf}(\omega)$. By the Liebniz rule, this is equal to
$e^{-tf}(de^{tf} \wedge \omega + e^{tf} \wedge d\omega) = tdf \wedge \omega + d\omega = (d + tE_{df})(\omega)$ □

Corollary 4.31. $d_t^* = d^* + tI_{df}$

This result gives us a first thread on which to tug to calculate an explicit form for Δ_t. Let us calculate:

$$\begin{aligned}&\Delta_t = d_td_t^* + d_t^*d_t = (d + tE_{df})(d^* + tI_{df}) + (d^* + tI_{df})(d + tE_{df})\\ &= dd^* + tdI_{df} + tE_{df}d^* + t^2E_{df}I_{df} + d^*d + tI_{df}d + td^*E_{df} + t^2I_{df}E_{df}\\ &= \Delta + [t\underbrace{(dI_{df} + E_{df}d^* + I_{df}d + d^*E_{df})}_{\clubsuit}] + t^2[\underbrace{E_{df}I_{df} + I_{df}E_{df}}_{\heartsuit}]\end{aligned}$$

By Proposition 4.11, we can see that $\heartsuit$ is in fact $\|df\|$. Now we want to calculate $\clubsuit$:
To begin, we need the following definitions:

Definition 4.32. Let $f \in C^\infty(\mathbb{R}^n)$. We say that f is *rapidly decreasing* if:

$$\lim_{|x|\to\infty} |x|^m D^\alpha f(x) = 0 \ \forall m, \forall\alpha \in \mathbb{N}^n$$

where by D^α we mean $D_1^{\alpha_1}D_2^{\alpha_2}...D_n^{\alpha_n}$ with $D_i := \frac{\partial}{\partial x_i}$.

Definition 4.33. The space:

$$\mathcal{S}(\mathbb{R}^n) := \{f \in C^\infty(\mathbb{R}^n) \ \mid \ f \text{ is rapidly decreasing}\}$$

is defined as the *Schwart Space*.

Let us now look at the following result.

Proposition 4.34. For functions $f, g \in \mathcal{S}(\mathbb{R}^n)$, we have that $\forall i$:

$$\int_{\mathbb{R}^n} \frac{\partial f}{\partial x_i} g = - \int_{\mathbb{R}^n} \frac{\partial g}{\partial x_i} f$$

This is a well known fact from an analysis course.
Let us now introduce a new operator on forms over $\mathbb{R}^n$:

Definition 4.35. For a fixed $n \in \mathbb{N}^*$ and $1 \leq i \leq n$, we define an operator on forms over $\mathbb{R}^n$ which we will call *partial* i and denote by ∂_i, pointwise as follows:

$$\partial_i(\psi(x_1, ..., x_n) dx_{i_1} \wedge ... \wedge dx_{i_k}) = \frac{\partial \psi}{\partial x_i} dx_{i_1} \wedge ... \wedge dx_{i_k}$$

which we extend by linearity naturally.

Proposition 4.36. We have that $\partial_i^* = -\partial_i$

Remark 4.37. Notice that for ∂_i to be defined we need some sort of derivability condition on the "companion function" of a form. Now, for this theorem to hold, we also need to locally have companion functions in the Schwartz Space for proposition 4.34 to be used, so since we want to use this calculation of the adjoint freely, we impose this additional condition upon our domain and show that it does not interfere with our upcoming work. In the Localization Theorem section, we will see that this approximation is indeed appropriate for our work.

Proof. The result is quite simply a consequence of proposition 4.34, and directly checking that both ∂_i and ∂_i^* are defined over the same domain. ☐

With this tool let us notice the following fact:

Remark 4.38. Take a form ω, and apply d. By definition, using the same notation we used at the beginning of this chapter, this is equal to:

$$\sum_i \frac{\partial \hat{\omega}}{\partial x_i} dx_i \wedge \omega$$

and using our recent definition, this is equivalent to:

$$(\sum_i E_{dx_i} \partial_i)(\omega)$$

or by commutativity

$$(\sum_i \partial_i E_{dx_i})(\omega)$$

Taking adjoints we may also calculate that:

$$d^* = \sum_i -\partial_i I_{dx_i}$$

With our reflections so far, we have obtained the following form of ♣:

$$(\sum_i \partial_i E_{dx_i}) I_{df} + I_{df}(\sum_i \partial_i E_{dx_i}) + (\sum_i -\partial_i I_{dx_i}) E_{df} + E_{df}(\sum_i -\partial_i I_{dx_i})$$

Recall additionally that $E_{df} = \sum_j \frac{\partial f}{\partial x_j} E_{dx_j}$ and $I_{df} = \sum_j \frac{\partial f}{\partial x_j} I_{dx_j}$. Inserting in our formula we have

$$\begin{gathered}(\sum_i \partial_i E_{dx_i})(\sum_j \frac{\partial f}{\partial x_j} I_{dx_j}) + (\sum_j \frac{\partial f}{\partial x_j} I_{dx_j})(\sum_i \partial_i E_{dx_i}) \\ +(\sum_i -\partial_i I_{dx_i})(\sum_j \frac{\partial f}{\partial x_j} E_{dx_j}) + (\sum_j \frac{\partial f}{\partial x_j} E_{dx_j})(\sum_i -\partial_i I_{dx_i})\end{gathered}$$

Consider than when acting on a function ψ, $\partial_i \frac{\partial f}{\partial x_j}\psi = \frac{\partial^2 f}{\partial x_i \partial x_j}\psi + \frac{\partial f}{\partial x_j}\partial_i \psi$; and since when acting on forms this operator "only sees the companion function", we see that $\partial_i \frac{\partial f}{\partial x_j}$ as an operator on forms can be written as $\frac{\partial^2 f}{\partial x_i \partial x_j} + \frac{\partial f}{\partial x_j}\partial_i$. Additionally with the fact that we can commute ∂i and $\frac{\partial f}{\partial x_j}$ with I_{dx_i} and E_{dx_j} freely, inserting this information in our previous formula slowly and carefully we obtain the following expression:

$$\begin{gathered}\sum_i \sum_j (\frac{\partial^2 f}{\partial x_i \partial x_j} E_{dx_i} I_{dx_j}) + \sum_i \sum_j (\frac{\partial f}{\partial x_j} \partial_i E_{dx_i} I_{dx_j}) \\ +\sum_i \sum_j (\frac{\partial f}{\partial x_j} \partial_i I_{dx_j} E_{dx_i}) - \sum_i \sum_j (\frac{\partial^2 f}{\partial x_i \partial x_j} I_{dx_i} E_{dx_j}) \\ -\sum_i \sum_j (\frac{\partial f}{\partial x_j} \partial_i E_{dx_i} I_{dx_j}) - \sum_i \sum_j (\frac{\partial f}{\partial x_j} \partial_i I_{dx_j} E_{dx_i})\end{gathered}$$

Cancelling out the terms we obtain that ♣ is:

$$\sum_i \sum_j \frac{\partial^2 f}{\partial x_i \partial x_j}(E_{dx_i} I_{dx_j} - I_{dx_i} E_{dx_j})$$

The formula we obtain for Δ_t, locally when we choose coordinates $(x_1, ..., x_n)$ is:

$$\Delta_t = \Delta + t^2 \|df\|^2 + t \sum_i \sum_j \frac{\partial^2 f}{\partial x \partial y}(E_{dx_i} I_{dx_j} - I_{dx_j} E_{dx_i})$$

Now recalling the Lemma of Morse and remark 2.12, we can express the Morse function f locally around a critical point p of index λ as:

$f(x) = f(0) + \frac{1}{2}(\sum_{i=1}^{\lambda} -x_i^2 + \sum_{i=\lambda+1}^{n} x_i^2)$,
so recalling that the Laplacean in $\mathbb{R}^n$ is $\sum_{i=1}^{n} -\frac{\partial^2}{\partial x_i^2}$, and additionally calculating that $\|df\| = \sum_i x_i$ and $\frac{\partial^2 f}{\partial x_i \partial x_j} = \begin{cases} 0 \text{ if } i \neq j \\ -1 \text{ if } i = j \leq \lambda \\ 1 \text{ if } i = j > \lambda \end{cases}$ the local form for Δ_t becomes approximately:

$$\Delta_t = \sum_{i=1}^{n}(-\frac{\partial^2}{\partial x_i^2} + t^2 x_i^2) + t\sum_{i=1}^{\lambda} -(E_{dx_i} I_{dx_i} - I_{dx_i} E_{dx_i}) + t\sum_{i=\lambda+1}^{n} (E_{dx_i} I_{dx_i} - I_{dx_i} E_{dx_i})$$

.

4.4 Weak Form of the Morse Inequalities

We enter the home stretch of our proof of the Morse Inequalities. At this point we diverge slightly from Witten's article in the hope of filling in some gaps his article omits. The divergence is taken through the Localization Theorem presented in the following section. Witten does not use this tool, but instead attempts to obtain an expansion for the eigenvalues of Δ_t of the form:

$$\lambda_n(t) = t\left(A_n + \frac{B_n}{t} + \frac{C_n}{t^2} + ...\right),$$

but we found it healthier to take this approach.

4.4.1 The Localization Theorem

The proofs of this section are omitted.

We begin looking at the Localization Theorem for the case of $\mathbb{R}^n$.
Suppose we have an operator parametrized by λ of the form:

$$H(\lambda) = -\Delta + \lambda^2 h + \lambda g,$$

where $h, g \in C^\infty(\mathbb{R}^n)$, g is bounded, $h \geq 0$ and vanishes only at finite points $(x_\alpha)_{\alpha=1}^{k}$.
Notice that locally around the critical points of h for large λ, we begin to have an approximation of the sort:

$$H(\lambda)(x) \approx \Delta + \frac{\lambda^2}{2}\sum_{ij} \frac{\partial^2 h}{\partial x_i \partial x_j}(x - x_\alpha)_i (x - x_\alpha)_j + \lambda g(x_\alpha)$$

Now, defining new operators of the form:

$$L^\alpha(x) = \Delta + \frac{1}{2}\sum_{ij} \frac{\partial^2 h}{\partial x_i \partial x_j}(x - x_\alpha)_i (x - x_\alpha)_j + g(x_\alpha),$$

we can consider

$$\sigma(L) := \bigcup_{\alpha} \sigma(L^{\alpha})$$

where by $\sigma(L^{\alpha})$ we refer to the eigenvalues of L^{α} (counting multiplicity).

Theorem 4.39 (Localization Theorem)**.** Let $H(\lambda)$ and $\sigma(L)$ be defined as above. Let $E_n(\lambda)$, e_n be the n-th eigenvalues (when they are ordered) of $H(\lambda)$ and $\sigma(L)$ respectively counting multiplicity. Then:

$$\lim_{\lambda \to \infty} \frac{E_n(\lambda)}{\lambda} = e_n.$$

The result of this theorem can be generalized to manifolds, with the result remaining pretty much the same.
For the generalization, we consider functions h and g in $C^{\infty}(M)$, and the result remains the same as long as we define L^{α} using local coordinates around critical points such that we have the local neighborhood diffeomorphic to the *whole* of $\mathbb{R}^n$ and acting on forms with $L^2(\mathbb{R}^n)$ companion functions.

Now, returning to our local calculations for the local behavior of Δ_t, the operators with which we could apply the technique of the Localization Theorem would be:

$$L^{\alpha} = \sum_{i=1}^{n}(-\frac{\partial^2}{\partial x_i^2} + x_i^2) + \sum_{i=1}^{\lambda} -(E_{dx_i}I_{dx_i} - I_{dx_i}E_{dx_i}) + \sum_{i=\lambda+1}^{n}(E_{dx_i}I_{dx_i} - I_{dx_i}E_{dx_i})$$

We can write these operators using the following conventions:
if we write $H_i = -\frac{\partial^2}{\partial x_i^2} + x_i^2$, $K_i = \pm(E_{dx_i}Idx_i - I_{dx_i}E_{dx_i})$, we can write:
$L^{\alpha} = \sum_i H_i + K_i$,
and we may also want to consider the convention $L^{\alpha} = H + K$ if we define $H := \sum_i H_i$ and $K := \sum_i K_i$.

4.4.2 The Eigenvalues of L^{α}

Now, with the formulation the preceding section just gave us, let us calculate the eigenvalues of L^{α} for a given critical point p_{α}:

The operators H_i are the Hamiltonian operators.
Recall that the Localization Theorem explicitly tells us that we can view the operators L^{α} as acting locally on L^2-forms (with companion functions in $L^2(\mathbb{R}^n)$).
Let us comment on what this means for our calculations of eigenvalues:

(1) The Schwartz Space is dense in $L^2(\mathbb{R}$ in the L^2-norm, so our previous comment allowing $\partial_i^* = -\partial_i$ was indeed appropriate.

(2) For $L^2(\mathbb{R})$, we have an orthonormal basis of eigenvectors of H_i $\forall i$. This can be found in [Sau] or [Tes09] for a slightly different albeit similar domain, but ultimately the work is very similar.

(3) If we consider the harmonic oscillator acting on a large domain like $C^\infty(\mathbb{R}^n)$, we have that each eigenvalue occurs with multiplicity two. One of the dimension of eigen-functions consists of functions that are in now way decreasing ad infinitum, which seems intuitively clear is not desirable. The other eigen-functions are nicely behaved and in $L^2(\mathbb{R})$.

(4) In $L^2(\mathbb{R}^n)$, the harmonic oscillator is essentially self-adjoint and hence closable. This fact will come in handy later on, and can be found in [Sau].

Now, it is well known (check [Sau]) that when acting on $L^2(\mathbb{R})$, their eigenvalues are:

$$\{2k+1 : k \in \mathbb{N}\},$$

each one appearing with geometric multiplicity one and forming an orthogonal basis as we just commented.
Now, suppose that $i \neq j$, and let $\varphi_1(x_1), \varphi_2(x_2) \in L^2(\mathbb{R})$ such that $H_1(\varphi_1) = \lambda_1\varphi_1$ and $H_2(\varphi_2) = \lambda_2\varphi_2$.
Now consider H_1+H_2 acting on $\mathbb{R}^2$. Obviously, defining $\psi(x_1,x_2) := \varphi_1(x_1)\varphi_2(x_2)$, we have that:

$$H_1 + H_2(\psi) = (H_1(\varphi_1))\varphi_2 + \varphi_1(H_2(\varphi_2))$$

since H_i only takes derivatives with respect to x_i. Hence we have that:

$$H_1 + H_2(\psi) = (\lambda_1 + \lambda_2)\psi$$

We want to prove a converse statement, that is, if $\exists\beta$ such that $H_1+H_2(\psi) = \beta\psi$, then $\exists\lambda_1, \lambda_2$ such that λ_1 is an eigenvalue of H_1; λ_2 is an eigenvalue of H_2, and $\lambda_1 + \lambda_2 = \beta$.
This will need a great deal of machinery. First of all, we will strongly use that $L^2(\mathbb{R}^2) = L^2(\mathbb{R}) \otimes L^2(\mathbb{R})$. Additionally, we can also argue that we have an orthogonal basis for $L^2(\mathbb{R})$ of eigenvectors of H_i. This work can be found in [Tes09].
Therefore, we have that for any $\psi \in L^2(\mathbb{R}^2)$, it can be written uniquely as:

$$\psi(x_1,x_2) = \sum_{j,k} \alpha_{j,k}\varphi_1^j(x_1)\varphi_2^k(x_2)$$

with φ_1^j and φ_2^k are eigenfunctions of H_1, H_2 respectively, using the notation $H_i(\varphi_i^j) = t(2j+1)\varphi_i^j$.
So calculating $H_1 + H_2(\psi)$, we obtain:

$$\begin{aligned}\sum_{j,k} \alpha_{j,k}(H_1(\varphi_1^j))\varphi_2^k + \varphi_1^j(H_2(\varphi_2^k)) &= \sum_{j,k} \alpha_{j,k}t(2j+1)\varphi_1^j\varphi_2^k + t(2k+1)\varphi_1^j\varphi_2^k \\ &= \sum_{j,k} t[2(k+j)+2]\alpha_{j,k}\varphi_1^j\varphi_2^k\end{aligned}$$

(Read remark 4.42)
Given the linear independence of $\{\varphi_1^j \varphi_2^k\}_{j,k}$; if we were to have that $H_1 + H_2(\psi) = \beta\psi$, then we must have that:

$$\beta\alpha_{j,k} = t[2(k+j)+2]\alpha_{j,k}$$

for any choice of j, k. Since 0 is not an eigenvalue, then at least one of the coefficients $\alpha_{j,k}$ must be different from 0, and we conclude that $\beta = t[2(k+j)+2]$ like we wanted. Extending this argument naturally and using the generalized fact that $L^2(\mathbb{R}^k) = \underbrace{L^2(\mathbb{R}) \otimes ... \otimes L^2(\mathbb{R})}_{k \text{ times}}$, we obtain that the eigenvalues of H acting on $L^2(\mathbb{R}^n)$ are:

$$\{2(k_1 + k_2 + ... + k_n) + n \ | \ k_1, ..., k_n \in \mathbb{Z}^+\}$$

Now, notice that by our definition of $L^2(M)$ of forms, locally a k-form will be precisely of the form $\psi(x_1, ..., x_n)dx_{i_1} \wedge ... \wedge dx_{i_k}$ where $\psi \in L^2(\mathbb{R})$. Given the way in which H acts on forms (acting only on the companion function ψ), if ψ is such that $H(\psi) = \lambda\psi$ when H acts on it as a function, then $H(\psi dx_{i_1} \wedge ... \wedge dx_{i_k}) = \lambda\psi dx_{i_1} \wedge ... \wedge dx_{i_k}$ acting on it as a form.

Let us now concern ourselves with the eigenvalues of the K_i's:

Let $\{i_1, ..., i_k\}$ be any choice of k numbers between 1 and n, and λ be the index of the critical point around which we are expanding.
Using the simpler notation $x_i = \frac{\partial}{\partial x_i}$, one may readily calculate that:

(1) $(*)\ I_{dx_i} E_{dx_i}(dx_{i_1} \wedge ... \wedge dx_{i_k})(x_{i_1}, ..., x_{i_k}) = E_{dx_i}(dx_{i_1} \wedge ... \wedge dx_{i_k})(x_i, x_{i_1}, ..., x_{i_k}) = \frac{1}{k!}\sum_{\sigma \in S_{k+1}} (-1)^{sgn \sigma} dx_i(x_{j_{\sigma(1)}})(dx_{i_1} \wedge ... \wedge dx_{i_k})(x_{j_{\sigma(2)}}, ..., x_{j_{\sigma(p+1)}})$, where we use the convention $j_1 := i$, and $j_l := i_{l-1}$ for $l \geq 2$.
Now consider two cases:

   (i) $i \in \{i_1, ..., i_k\}$: It is not so difficult to see that $(*)$ is equal to 0, since $dx_i(x_{j_{\sigma(1)}}) \neq 0$ only when $j_{\sigma(1)} = i$, and this happens in two scenarios for each permutation σ, when $\sigma(1) = 1$ and when $\sigma(1) = h+1$ where h is the index that fulfils that $i = i_h$. However, if for $\sigma \in S_{k+1}$ we define $\hat{\sigma} \in S_{k+1}$ by leaving it exactly the same except that we switch around the preimages of 1 and $h+1$ with respect to σ, then we will have that:

$$dx_i(x_{j_{\sigma(1)}})(dx_{i_1} \wedge ... \wedge dx_{i_k})(x_{j_{\sigma(2)}}, ..., x_{j_{\sigma p+1)}}) = \\ dx_i(x_{j_{\hat{\sigma}(1)}})(dx_{i_1} \wedge ... \wedge dx_{i_k})(x_{j_{\hat{\sigma}(2)}}, ..., x_{j_{\hat{\sigma} p+1)}})$$

   except that now $(-1)^{sgn\sigma} = -(-1)^{sgn\hat{\sigma}}$, and so when we add over all the permutations, these pairs, σ and $\hat{\sigma}$ will cancel themselves out.

   (ii) $i \notin \{i_1, ..., i_k\}$: In this case, $(*)$ is going to be 1, because the value of $dx_i(x_{j_{\sigma(1)}})$ is 1 only when $\sigma(1) = 1$, and so we truly add only over the permutations of the other k values and in each permutation of this sort, $(-1)^{sgn\sigma}(dx_{i_1} \wedge ... \wedge dx_{i_k})(x_{j_{\sigma(2)}}, ..., x_{j_{\sigma p+1)}}) = 1$, and we then divide by $k!$.

(2) $(**)$ $E_{dx_i} I_{dx_i}(dx_{i_1} \wedge ... \wedge dx_{i_k})(x_{i_1}, ..., x_{i_k}) = \frac{1}{(k-1)!} \sum_{\sigma \in S_k} (-1)^{sgn\sigma} dx_i(x_{i_{\sigma(1)}})[I_{dx_i}(dx_{i_1} \wedge ... \wedge dx_{i_k})](x_{i_{\sigma(2)}}, ..., x_{i_{\sigma(k)}}) = \frac{1}{(k-1)!} \sum_{\sigma \in S_k} (-1)^{sgn\sigma} dx_i(x_{i_{\sigma(1)}})(dx_{i_1} \wedge ... \wedge dx_{i_k})(x_i, x_{i_{\sigma(2)}}, ..., x_{i_{\sigma(k)}})$
Again, we consider the two cases:

   (i) $i \in \{i_1, ..., i_k\}$: We can verify that $(**)$ is equal to 1 because, using the same convention for h as before, $dx_i(x_{i_{\sigma(1)}})$ will only be different from zero when $\sigma(1) = h$, in which case it will equal 1. In this case, $(dx_{i_1} \wedge ... \wedge dx_{i_k})(x_i, x_{i_{\sigma(2)}}, ..., x_{i_{\sigma(k)}})$ will be precisely $(-1)^{sgn\sigma}$ by the "ant-symmetric" property, so then $[(-1)^{sgn\sigma}]^2 = 1$, and we sum over all the permutations such that $\sigma(1) = h$, that is to say$(k-1)!$ permutations, and then divide precisely by $(k-1)!$.

   (ii) $i \notin \{i_1, ..., i_k\}$: In this case, $(**)$ is going to be 0 because $dx_i(x_{i_{\sigma(1)}})$ will be 0 for any σ we choose.

(3) Additionally, we may calculate that $K_i(dx_{i_1} \wedge ... \wedge dx_{i_k})(x_{j_1}, ..., x_{j_k}) = 0$ where $\{i_1, ..., i_k\} \neq \{j_i, ..., j_k\}$, proving that $K_i(dx_{i_1} \wedge ... \wedge dx_{i_k})$ is a multiple of $dx_{i_1} \wedge ... \wedge dx_{i_k}$ since it is only non-zero when evaluated on the member $(x_{i_1}, ..., x_{i_k})$ of the basis.

Now, recalling the definition of the K_i's, the previous calculations reveal that:

$$K_i(dx_{i_1} \wedge ... \wedge dx_{i_k}) = \begin{cases} -dx_{i_1} \wedge ... \wedge dx_{i_k} \text{ if } i \in \{i_1, ..., i_k\}, \text{ and } i \in \{1, ..., \lambda\} \\ dx_{i_1} \wedge ... \wedge dx_{i_k} \text{ if } i \in \{i_1, ..., i_k\}, \text{ and } i \notin \{1, ..., \lambda\} \\ dx_{i_1} \wedge ... \wedge dx_{i_k} \text{ if } i \notin \{i_1, ..., i_k\}, \text{ and } i \in \{1, ..., \lambda\} \\ -dx_{i_1} \wedge ... \wedge dx_{i_k} \text{ if } i \notin \{i_1, ..., i_k\}, \text{ and } i \notin \{1, ..., \lambda\} \end{cases}$$
(🦇)

Lemma 4.40. Suppose that $K(dx_{i_1} \wedge ... \wedge dx_{i_k}) = \alpha dx_{i_1} \wedge ... \wedge dx_{i_k}$ for some integer α and some choice of $i_1, ..., i_k$. Then $\alpha = 2\lambda + 2k - n - 4s$ where s is the cardinal of $\{i_1, ..., i_k\} \cap \{1, ..., \lambda\}$.

Proof. Reviewing our most recent reflection, the number of positive $K_i(dx_{i_1} \wedge ... \wedge dx_{i_k})$ is precisely $(k-s)+(\lambda-s)$, obtained from cases 2 and 3 in (🦇).Similarly, the number of negative terms is $s + [(n-k) - (\lambda - s)]$ obtained from cases 1 and 4. In total, we have that $\alpha = (k-s) + (\lambda - s) - s - [(n-k) - (\lambda - s)] = k + \lambda - 2s - 2s - n + k + \lambda = 2k + 2\lambda - n - 4s$. □

Observation 4.41. Notice that $s \leq k$ and $s \leq \lambda$, so then any such α fulfils that $\alpha \geq -n$.

Additionally notice that, in a way, K only "sees the wedges" of forms upon which it acts, similarly to how H only "sees the companion function". What we mean precisely is that if $dx_{i_1} \wedge dx_{i_2} \wedge ... \wedge dx_{i_k}$ is a form such that

$K(dx_{i_1}\wedge dx_{i_2}\wedge ...\wedge dx_{i_k} = \alpha dx_{i_1}\wedge dx_{i_2}\wedge ...\wedge dx_{i_k}$, then or any "companion function" $\psi(x_1,...,x_n)$; $K(\psi(x_1,...,x_n)dx_{i_1}\wedge dx_{i_2}\wedge ...\wedge dx_{i_k}) = \alpha\psi(x_1,...,x_n)dx_{i_1}\wedge dx_{i_2}\wedge ...\wedge dx_{i_k}$. This can be checked by the reader directly from the definitions of E_{dx_i} and I_{dx_i}.

What we wish to do now is similar to what we did with the sum of the H_i's, arguing that every eigenvalue of $H+K$ comes from the sum of two eigenvalues of H and K respectively; and conversely that the sum of an eigenvalue of H with one of K is an eigenvalue of $H+K$.
This last statement is quite direct. Suppose λ is an eigenvalue of H and μ is so of K. Now recalling our discussion on the eigenvalues of H, we must have $\psi\in L^2(\mathbb{R}^n)$ with $H(\psi)=\lambda\psi$ acting on ψ as a *function*. Equivalently, if μ is an eigenvalue of K, we must have that there exists an eigen-form with "companion function" 1, that is to say $dx_{1_1}\wedge ...\wedge dx_{i_k}$ such that $K(dx_{1_1}\wedge ...\wedge dx_{i_k}) = \mu dx_{1_1}\wedge ...\wedge dx_{i_k}$. It can be easily checked that $H+K(\psi dx_{1_1}\wedge ...\wedge dx_{i_k}) = (\lambda+\mu)\psi dx_{1_1}\wedge ...\wedge dx_{i_k}$.
We now search for a converse statement.
The argument works similarly to the one used previously. Suppose we have a form $\psi(x_1,...,x_n)dx_{{}_1}\wedge ...\wedge dx_{i_k}$ such that $(H+K)(\psi(x_1,...,x_n)dx_{{}_1}\wedge ...\wedge dx_{i_k}) = \beta(\psi(x_1,...,x_n)dx_{{}_1}\wedge ...\wedge dx_{i_k})$. Then we can write:

$$\psi(x_1,...,x_n) = \sum_{j_1,...,j_n}\alpha_{j_1,...,j_n}\varphi_1^{j_1}(x_1)\varphi_2^{j_2}(x_2)...\varphi_n^{j_n}(x_n)$$

and then H will act on it as follows:

$$(\lambda_1^{j_1}+...+\lambda_1 n^{j_n})(\psi(x_1,...,x_n)dx_{{}_1}\wedge ...\wedge dx_{i_k})$$

with the $\lambda_i^{j_i}$'s eigenvalues of each H_i. Since K acts on the form in the way presented in lemma 4.40, then by the same linear independence argument of before, we have that $\beta=\lambda_1^{j_1}+...+\lambda_n^{j_n}+\alpha$.

Remark 4.42. Notice that we are inserting H freely inside the converging sum

$$\sum_{j_1,...,j_n}\alpha_{j_1,...,j_n}\varphi_1^{j_1}(x_1)\varphi_2^{j_2}(x_2)...\varphi_n^{j_n}(x_n)$$

without truly knowing this to be possible. This gives us an opportunity to comment further on a reason for our choice of $\mathcal{S}(\mathbb{R}^n)$:
Notice that since the formulation of such an eigen-form $\psi(x_1,...,x_n)dx_{{}_1}\wedge ...\wedge dx_{i_k}$, we implicitly assumed that $\psi(x_1,...,x_n)dx_{{}_1}\wedge ...\wedge dx_{i_k}$ was in the domain of H. Now, since H, is closable, then $\psi(x_1,...,x_n)dx_{{}_1}\wedge ...\wedge dx_{i_k}$ is also in the domain of $\bar{H}$. Now, since $\bar{H}$ is closed, and all our $\varphi_i^{j_i}$ are in the domain of H,

the following holds:

$$\begin{aligned}
&H(\sum_{j_1,\ldots,j_n} \alpha_{j_1,\ldots,j_n} \varphi_1^{j_1}(x_1)\varphi_2^{j_2}(x_2)...\varphi_n^{j_n}(x_n)) \\
=\ &\bar{H}(\sum_{j_1,\ldots,j_n} \alpha_{j_1,\ldots,j_n} \varphi_1^{j_1}(x_1)\varphi_2^{j_2}(x_2)...\varphi_n^{j_n}(x_n)) \\
=\ &\sum_{j_1,\ldots,j_n} \alpha_{j_1,\ldots,j_n} \bar{H}(\varphi_1^{j_1}(x_1)\varphi_2^{j_2}(x_2)...\varphi_n^{j_n}(x_n)) \\
=\ &\sum_{j_1,\ldots,j_n} \alpha_{j_1,\ldots,j_n} H(\varphi_1^{j_1}(x_1)\varphi_2^{j_2}(x_2)...\varphi_n^{j_n}(x_n))
\end{aligned}$$

and the argument used is correct.
Now, notice as well that since the arguments we have given for the eigenvalues and eigenvectors of the harmonic oscillator have been given for the *closure* of H according to standard literature. However, since all the eigenvectors are in the domain of H, the eigenvectors of H and its closure are the same, which is all we care about for now.

Considering the known forms for the $\lambda_i^{j_i}$'s, we've worked towards this fundamental fact:

Proposition 4.43. *Eigenvalues of L^α:* The eigenvalues are of the following form:

$$n + \alpha + 2(m_1 + .. + m_n),$$

where α is of the form exposed in lemma 4.40, and the m_i's take any value in $\mathbb{N}^0$.

4.4.3 The Inequalities

Theorem 4.44. If we restrict ourselves to k-forms, then the kernel of L^α where p_α has index λ has dimension one only if $k = \lambda$, and has dimension zero in any other case.

Proof. We already saw that any eigenvalue has the form:

$$\mu = n + \alpha + 2(m_1 + .. + m_n)$$

where α is of the form seen in Proposition 4.43, and $m_1, ..., m_n$ are any choice of positive integers. Now for μ to be equal, we need $m_1 = m_2 = ... = m_n = 0$ and $\alpha = -n$. Now retaking our past reflections and using the same terminology, we need $2s = \lambda + k$, but since $\lambda \leq s$ and $k \leq s$, this requires $s = k = \lambda$, proving that only λ-forms can be 0 under Δ_t around a critical point of index λ.
The fact that this is of dimension 1 stems from the fact that for each H_i acting on functions, the eigenvalue corresponding to $m_i = 0$ occurs with multiplicity 1, so there is a unique function $\varphi_i(x_i)$ such that $H_i(\varphi_i(x_i) = t(2m_i + 1)\varphi_i(x_i) =$

$t\varphi(x_i)$.
Recalling the discussion on what the eigenfunctions of H are, the only one corresponding to $2(m_1 + ... + m_n) + n$ with $m_i = 0$ $\forall i$ is $\psi(x_1, ..., x_n) = \varphi_1(x_1)\varphi_2(x_2)...\varphi_n(x_n)$. Additionally, since we want $\alpha = -n$, according to what we just commented we need the form $dx_1 \wedge ... \wedge dx_\lambda$. Therefore, the reader may readily notice that the eigenspace of 0 is:

$$span\{\varphi_1(x_1)...\varphi_n(x_n)dx_1 \wedge ... \wedge dx_\lambda\}$$

□

We can see ourselves edging towards our desired result.
The result we just proved establishes that the number of 0's in $\sigma(L)$ is equal to F_k (when acting on k-forms). Now, we want to calculate the number of 0 eigenvalues of Δ_t, which we saw before is independent of t. This means that ordering the eigenvalues of each Δ_t, there is a constant number N such that $\forall n \leq N$:

$$E_n(t) = 0$$

The keen observer might have realized that $N = B_k(M)$.
Now, using the Localization Theorem, we know that sing the same terminology we established previously, we have that $\forall n \leq N$:

$$0 = \lim_{t\to\infty} \frac{E_n(t)}{t} = e_n,$$

establishing the inequality

$$B_k(M) \leq F_k$$

The other inequality is not upheld because it is possible for limits to become 0 without their terms ever being 0.

Chapter 5

Conclusions

The results we have proven are very self-explanatory, and have been thoroughly commented in due moment. Alternatively, we will use this "Conclusions" chapter to comment some difficulties encountered during the subject study, maybe as a motivation for further study either by myself or any third reader.

The biggest and most obvious patchy area of our work is the simultaneous choice of coordinates that both obey the result of the Morse Lemma, and parametrize the neighborhoods around critical points as the whole of $\mathbb{R}^n$. This is not only not assured by the Morse Lemma which speaks of arbitrary coordinates that fulfil its result, but seems downright impossible since the function would diverge inside the manifold if local coordinates obeyed both conditions.

The literature regarding Witten's technique is surprisingly oblivious of these difficulties, so they actually constitute an appealing field of study for any interested reader.

A possible solution which I propose as an investigation topic for both myself and any interested reader is a reformulation of the Localization Theorem which ceases to require the local coordinates to parametrize as the whole of $\mathbb{R}^n$, adopting to any open neighborhood of $\mathbb{R}^n$, in particular the one provided by the Morse Lemma, hence getting rid of the problem.

I suspect this is possible because it does not cease to be true that the solutions of $\Delta_t \psi = 0$ concentrate around critical points of f.

However, it seems like long involved work. Not only would we need to revisit the proof of the Localization Theorem which is in itself quite a chore, but we would need to reformulate the ideas behind the calculus of the eigenvalues of L^α, since most of this work requires us to be in $L^2(\mathbb{R}^n)$. I infer it could be done in a space of functions decreasing towards the border of any open neighborhood in a sense similar to L^2, but this, the reader must agree, seems like long and involved investigative work.

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