Estimations on Persistent Homology

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Introduction

Applied algebraic topology is a branch of algebraic topology which has gained popularity in the last few years. Some topologist such as Gunnar Carlsson, John Harer, etc. and other mathematicians as Herbert Edelsbrunner have worked on this area and have produced some important results as the ones presented on [EH10], [ZC05] [PH15] and [PC14].

The work presented in this text is based on the ideas introduced in [PH15]. There, professors Harer and Perea gave a detailed description of the *Sliding Windows Embedding* for the study of the periodicity of functions using Fourier analysis and persistent homology theory. In that article they shown some analytic bounds for the maximum persistence for the 1 dimensional persistent homology of the sliding window embedding in terms of the Fourier coefficients of the function.

The main tool used in Harer and Perea's article is persistent homology, which is a highly studied method for measuring shapes of spaces and function features. One of the most relevant application of persistent homology is to point cloud data sets, where the result often can be interpreted as some implicit underlying object the data set. A good example of this kind of work is the one presented in [PC14].

Our developments are highly inspired in the work made in [PH15], we are trying to get some analytical bounds for the maximum persistence in the persistence diagram of a filtered complex. In particular if we apply our results to the sliding window embedding we get bounds for the persistence which are slightly more general that some of the results obtained in [PH15], in the sense that our bounds are independent of the dimension and the windows size of the embedding.

It is important to emphasize how difficult is to analytically compute persistent homology of a filtered complex (specially for the Vietoris-Rips complex). Even if you consider one of the most simple examples, as it is to take points over an unitary circle and construct its Vietoris-Rips complex, the task is not easy. A recent article from Adams and Adamaszec [AA15] studies the homotopy type of this complex showing that there we can find odd dimensional spheres until the complex is contractible. Thus having good approximations for the persistence of the Rips complex turns out to be an important and not simple task that deserves our attention.

We begin this document with some preliminaries in linear algebra and some basic definition on persistent homology that will be useful for the reader less used to these concepts. In Chapter 2 we present a result in Theorem 2.2 concerning the persistence of homology classes in the image of a space under a linear transformations. Then we introduce some bounds on eigenvalues of Hermitian matrices developed by Wolcowicz and Stayn in [HP80]. Using this eigenvalue bounds we are able to obtain more bounds on the persistence of the image of a space using similar techniques as the one introduced in Theorem 2.2. Then we include some geometrical information of the data set into the bounds for persistent homology previously obtained using the covariance matrix of the point cloud.

After that we do some analytic computations for the sliding window embedding that allow us to see this embedding as a linear transformation for a special point cloud. Then we apply our results to this sliding window embedding, helping us to obtain some estimations on the maximum persistence in terms of the Fourier coefficients and Fejer Kernels (this last should sound familiar to any one who has studied a little Fourier analysis).

Our results relay on our ability to compute the persistent homology of the original space before applying any linear transformation. For that reason we carry out some computational calculations over the space needed to obtain the bounds for the maximum persistence of sliding window embedding. This calculations where made using the packages JavaPlex and TDATools for Matlab. We encounter some computational restrictions concerning the Java Heap Memory assigned to Matlab. Most of the calculations were made on the computational cluster of the Facultad de Ciencias in the Universidad de los Andes.

Chapter 3 introduces a different approach in the study of the Vietoris-Rips complex used to compute persistent homology of a point cloud. This approach studies such complex for small ϵ's, instead of "big" epsilons as in Chapter 2. In this chapter we give a short description of the so called Mayer-Vietoris spectral sequence for simplicial complexes and mix this construction with some results from Hausmann ([Hau95]) to give some homological equivalence results between the homology of the Vietoris-Rips complex and the homology of the original space. Most of the work done by Hausmann were made concerning Reimannian manifolds; our tools are slightly more general and can be applied to topological spaces in general. In particular if we work over a Reimannian manifolds we can recover some of the results on [Hau95].

Finally we present an Appendix with the detailed results, calculations an coding for most of the computational persistence used in the examples of this document.

Chapter 1

Preliminaries

In this chapter we will cover some important results concerning linear algebra, algebraic topology and persistent homology required to completely understand the work in the following chapters.

1.1 Some results in algebra

Definition 1.1. A linear transformation $T : V \to V$ between vector spaces (over $\mathbb{R}$ or $\mathbb{C}$) is called **self-adjoint** if $T = T^*$.

Lemma 1.2. *If $A : V \to V$ is a self-adjoint linear transformation between finite dimensional vector spaces, we have the following properties:*

(i) *If λ_1, λ_2 are different eigenvalues with eigenvectors v_1, v_2 of A then $\langle v_1, v_2 \rangle = 0$*

(ii) *The eigenvalues of A are real.*

(iii) *The eigenvectors of A form and orthonormal basis for the vector space V.*

Proof. For (i)

$$\begin{aligned} \langle Av_1, v_2 \rangle = \langle v_1, Av_2 \rangle &\iff \langle \lambda_1 v_1, v_2 \rangle = \langle v_1, \lambda_2 v_2 \rangle \\ &\iff \lambda_1 \langle v_1, v_2 \rangle = \overline{\lambda_2} \langle v_1, v_2 \rangle \\ &\iff (\lambda_1 - \overline{\lambda_2}) \langle v_1, v_2 \rangle = 0. \end{aligned}$$

Since $\lambda_1 \neq \lambda_2$ it follows that $\langle v_1, v_2 \rangle = 0$.

To see (ii), in the previous argument, replace $v_2 = v_1$, and since $\|v_1\|^2 \neq 0$, where $\lambda_1 = \overline{\lambda_1}$.

To prove (iii), suppose that A has k different eigenvalues $\{\lambda_1, \dots, \lambda_k\}$ and let V_i be the eigenspace associated to each λ_i for $1 \leq i \leq k$. By (i) we have that $V_i \perp V_j$ whenever $i \neq j$. Using the Gram-Schmidt process we can construct an orthonormal basis for each V_i. If we consider the union of those bases we obtain an orthonormal basis for $\mathbb{R}^n$ which consist of eigenvectors of A. □

Proposition 1.3. *If $T : V \to W$ is an injective linear transformation between finite dimensional vector spaces, then all the eigenvalues of T^*T are positive.*

Proof. If there was any eigenvalue $\lambda \leq 0$ then we will have that for its corresponding eigenvector v the following

$$0 \geq \lambda \|v\|^2 = \lambda \langle v, v \rangle = \langle v, \lambda v \rangle = \langle v, T^*Tv \rangle = \langle Tv, Tv \rangle = \|Tv\|^2$$

This would imply that $\|Tv\| = 0$ so $Tv = 0$, and since T is injective we have $v = 0$; but this is impossible since the zero vector cannot be an eigenvector of any eigenvalue. □

Lemma 1.4. *Let $T : V \to W$ be a linear transformation between finite dimensional vector spaces. If λ_{max} is the greatest eigenvalue of T^*T, then $\|T\| = \sqrt{\lambda_{max}}$.*

Proof. For any $x \in \mathbb{R}^n$ and taking the orthonormal basis for $x \in \mathbb{R}^n$ constructed with the eigenvectors $\{v_i\}_{i=i}^n$ of T^*T (as in (iii) in Lemma 1.2) we have

$$\begin{aligned}
\|Tx\|^2 = \langle Tx, Tx \rangle = \langle x, T^*Tx \rangle &= \langle \sum_{i=1}^n \alpha_i v_i, \sum_{j=1}^n \alpha_j T^*T v_j \rangle \\
&= \sum_{i,j=1}^n \alpha_i \alpha_j \langle v_i, T^*T v_j \rangle = \sum_{i,j=1}^n \alpha_i \alpha_j \langle v_i, \lambda_j v_j \rangle \\
&= \sum_{i=1}^n \alpha_i^2 \lambda_i \langle v_i, v_i \rangle = \sum_{i=1}^n \alpha_i^2 \lambda_i \leq \sum_{i=1}^n \alpha_i^2 \lambda_{max}
\end{aligned}$$

$$= \lambda_{max} \sum_{i=1}^{n} \alpha_i^2 = \lambda_{max} \|x\|^2.$$

Thus, we have $0 \leq \|Tx\|^2 \leq \lambda_{max}\|x\|^2$, so $\lambda_{max} \geq 0$. We now can take the square root to get $\|Tx\| \leq \sqrt{\lambda_{max}}\|x\|$, and taking supremum on both sides of the inequality we get

$$\|T\| = \sup_{\|x\|=1} \|Tx\| \leq \sup_{\|x\|=1} \sqrt{\lambda_{max}}\|x\| = \sqrt{\lambda_{max}}.$$

On the other hand we can take any normal eigenvector v_{max} corresponding to the eigenvalue λ_{max} and we have

$$\|T\| \geq \|Tv_{max}\| = \sqrt{\lambda_{max}}\|v_{max}\| = \sqrt{\lambda_{max}}.$$

This concludes the proof. □

Corollary 1.5. *If $T : V \to W$ is an injective linear transformation between finite dimensional vector spaces, we also have that $\|T^{-1}\| = 1/\sqrt{\lambda_{min}}$, where λ_{min} is the smallest eigenvalue of T^*T.*

Proof. Notice that since T is injective, we can define

$$T^{-1} : \mathrm{Img}(T) \subset \mathbb{R}^m \to \mathbb{R}^n.$$

Since the inverse is also a linear transformation, by lemma 1.4 we have that

$$\|T^{-1}\| = \sqrt{\beta_{max}},$$

where β_{max} is the greatest eigenvalue of $(T^{-1})^*T^{-1} = (TT^*)^{-1}$.

Recall that for any bounded operator A between Hilbert spaces we have that $\|A\|^2 = \|A^*\|^2 = \|A^*A\|$, see [Con94]. This implies that $\|T^*T\| = \|TT^*\|$, so we get $\|(TT^*)^{-1}\| = \|(T^*T)^{-1}\|$.

By Proposition 1.3 we get that T^*T is positive definite. Therefore we can diagonalize it as $T^*T = P^{-1}\Delta P$ so $(T^*T)^{-1} = P^{-1}\Delta^{-1}P$, so the greatest eigenvalue of $(T^*T)^{-1}$ corresponds to the multiplicative inverse of the smallest eigenvalue of T^*T which coincides with the eigenvalues of TT^*. Thus we get that $\beta_{max} = \frac{1}{\lambda_{min}}$. □

Definition 1.6. The **singular values of a compact operator** between Hilbert spaces $T : X \to Y$ are the square roots of the eigenvalues of the non-negative self-adjoint linear operator $T^*T : X \to X$.

Remark. In Lemma 1.4 and Corollary 1.5, if T is an injective linear transformation between finite dimensional vector spaces, then $\|T\| = \sigma_{\max}$ and $\|T^{-1}\| = \sigma_{\min}$ for $\sigma_{\max}$ and $\sigma_{\min}$ the greatest and smallest singular values of T respectively.

The following definitions and results are taken from [DF04] and [ZC05].

Definition 1.7. A **graded ring** is a ring $(R, +, \cdot)$ with a direct sum decomposition into Abelian groups

$$R \cong \bigoplus_{i \in \mathbb{Z}} R_i,$$

so that multiplications is defined by bi-linear pairings $R_i \otimes R_j \to R_{i+j}$.

The elements $x \in R_i$ are called **homogeneous elements** of degree i.

Definition 1.8. A **graded module over a graded ring** A, is a module M over A with a direct sum decomposition

$$M \cong \bigoplus_{i \in \mathbb{Z}} M_i$$

Theorem 1.9 (Structure 1)**.** *If R is a Principal Ideal Domain (PID), then every finitely generated R-module M decomposes uniquely into the form*

$$M \cong R^n \oplus \left(\bigoplus_{j=1}^{m} R/d_j R \right),$$

for $d_j \in R$, $n \in \mathbb{Z}$ and $d_j | d_{j+1}$.

Proof. See Chapter 12, Theorem 5 of [DF04]. □

Remark. For $R = \mathbb{Z}$ the previous theorem gives us the structure of finitely generated Abelian groups.

Theorem 1.10 (Structure 2)**.** *If R is a graded PID, then every finitely generated graded R-module M decomposes uniquely into the form*

$$M \cong \left(\bigoplus_{i=1}^{n} \Sigma^{\alpha_i} R \right) \oplus \left(\bigoplus_{j=1}^{m} \Sigma^{\gamma_j} R/d_j R \right),$$

for $d_j \in R$ homogeneous elements such that $d_j | d_{j+1}$, α_i, $\gamma_j \in \mathbb{Z}$ and Σ^{α} denotes an α-shift upward in grading.

Proof. See Theorem 2.1 on [ZC05]. □

1.2 Persistent homology

We will introduce some basic notions concerning persistent homology, the following are taken from [ZC05] and [EH10].

Given a set X, we shall denote its **power set** or the set of all subsets of X by $\mathbb{P}(X)$.

Definition 1.11. A **simplicial complex** is a set K with a collection $S \subset \mathbb{P}(K)$, called **simplices**, such that:

(i) If $k \in K$ then $\{k\} \in S$,

(ii) If $\tau \subset \sigma \in S$ then $\tau \in S$.

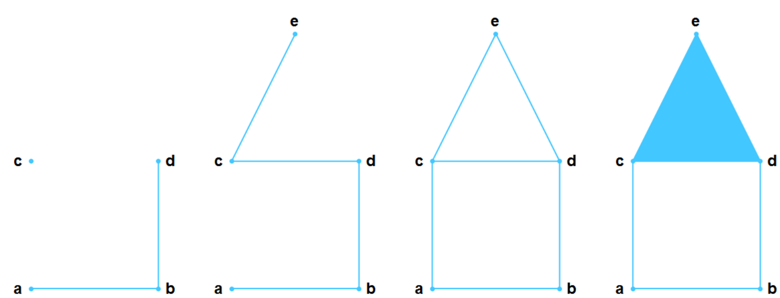
We call the singletons in S the **vertices of** K or 0**-simplices**. Also, we say that $\sigma \in S$ is a k**-simplex** if $|\sigma| = k+1$. Moreover, we will write each k-simplex $\sigma = [x_0, x_1, \ldots, x_k]$ for $\{x_0, x_1, \ldots, x_k\} \in S$. The **dimension of** K is $\sup\{|\sigma| \mid \sigma \in S\}$.

Definition 1.12. A **sub-complex** of a simplicial complex K is a subset $L \subset K$ which is also a simplicial complex.

Definition 1.13. A **filtration** of a simplicial complex K is a nested subsequence of complexes

$$\emptyset = K^0 \subset K^1 \subset \cdots \subset K^m = K.$$

We will define $K^i = K$ for any $i \geq m$.



(a) K^1 (b) K^2 (c) K^3 (d) $K^4 = K$

Figure 1.1: A filtration of the simplicial complex K.

Example 1.14. In Figure 1.1 we present a filtration on a simplicial complex.

Definition 1.15 (Rips complex)**.** Given a metric space X and $\epsilon > 0$. The **Rips complex** or **Vietoris-Rips complex** $\mathrm{R}_\epsilon(X)$ is the simplicial complex whose k-simplices are given by

$$(\mathrm{R}_\epsilon(X))_k := \{[x_0, \dots, x_k] \mid x_i, x_j \in X,\ d(x_i, x_j) \le \epsilon \text{ for every } 1 \le i, j \le k\}.$$

Definition 1.16. We say that to permutations are equivalent if they differ by an even permutation.

An **orientation** on a k-simplex $\sigma = [x_0, \dots, x_k]$ isa choosing of an equivalence class of permutations on the vertices of σ.

Notice that with this definition a simplex only has 2 possible orientations.

Definition 1.17. The **n-th chain group** C_n of K is the free Abelian group on its set of oriented n-simplices. Moreover, an element $c \in C_n$ is called a n-chain and can be written as

$$c = \sum_i n_i \sigma_i$$

where each σ_i is a n-simplex of K and the coefficients $n_i \in \mathbb{Z}$.

Definition 1.18. The **boundary operator** $\partial : C_n \to C_{n-1}$ is a group homomorphism defined linearly on the generators of C_n by

$$\partial\,[x_0, x_1, \dots, x_n] = \sum_{i=0}^{n} (-1)^i\,[x_0, x_1, \dots, \hat{x_i}, \dots, x_k]\,,$$

where $\hat{x_i}$ indicates that x_i is deleted from the sequence.

It is an easy exercise to prove that $\partial^2 := \partial \circ \partial : C_n \to C_{n-2}$ is equal to zero for any $n \ge 2$. This result allow us to define a chain complex C_*:

$$\cdots \xrightarrow{\partial_{k+2}} C_{k+1} \xrightarrow{\partial_{k+1}} C_k \xrightarrow{\partial_k} C_{k-1} \xrightarrow{\partial_{k-1}} \cdots \xrightarrow{\partial_1} C_0 \longrightarrow 0$$

Take a group G and think of it as a $\mathbb{Z}$-module. We can produce a new chain complex tensoring with G in the following way:

$$\cdots \longrightarrow C_k \otimes G \longrightarrow C_{k-1} \otimes G \longrightarrow \cdots \longrightarrow C_0 \otimes G \longrightarrow 0$$

with differentials $\partial_i' := \partial_i \otimes \mathbb{1} : C_i \otimes G \to C_{i-1} \otimes G$.

Definition 1.19. The kth homology group with coefficients over a group G is defined as

$$\mathrm{H}_k(C_*;G) := \frac{Z_k}{B_k} := \frac{\ker(\partial_k')}{\mathrm{Im}(\partial_{k+1}')}.$$

Also the k-th Betti number is defined as $\beta_k := \mathrm{rank}(\mathrm{H}_k(C_*;G)) = \mathrm{rank}(Z_k) - \mathrm{rank}(B_k)$.

To study the structure of this groups we will think of them as $\mathbb{Z}$-modules and apply Theorem 1.9. Notice that we can then change the coefficients group to any PID and we will we still able to use the structure theorem for finitely generated modules and the rank of the free submodule will be the Betti number of the module and d_j are the torsion coefficients of the module.

Moreover, if we replace the previous PID of coefficients with a field, such as $\mathbb{R}$, $\mathbb{Q}$ or $\mathbb{Z}_p$ for any prime p, the torsion submodule disappears.

As explained in [ZC05] (p. 5) the standard method for computing homology groups is to write down the standard matrix representation for each $\partial_k : C_k \to C_{k-1}$ relative to the standard basis of the chain group. This standard matrix representation M_k has entries $\{-1, 0, 1\}$. The null space of M_k corresponds to Z_k and its range-space is B_{k-1}.

To calculate the kernel and image of M_k one must use row and column operations over the PID R. Namely

- exchange row (column) i with row (column) j,
- multiply row (column) i by -1,
- replace row (column) i by row(i)+q·row(j) (column(i)+q·column(j)) with $q \in R$ and $i \neq j$.

This operations reduce M_k to the **Smith normal form**.

Example 1.20. Consider the simplicial complex in Figure 1.1d, then the standard matrix representation M_1 for ∂_1 with coefficients in $\mathbb{Z}$ is

$$M_1 = \left[\begin{array}{c|cccccc} & ab & ac & bd & cd & ce & de \\ \hline a & -1 & -1 & 0 & 0 & 0 & 0 \\ b & 1 & 0 & -1 & 0 & 0 & 0 \\ c & 0 & 1 & 0 & -1 & -1 & 0 \\ d & 0 & 0 & 1 & 1 & 0 & -1 \\ e & 0 & 0 & 0 & 0 & 1 & 1 \end{array}\right].$$

Its Smith normal form is

$$\widetilde{M_1} = \left[\begin{array}{c|cccccc} & ab & ac & bd & ce & z_1 & z_2 \\ \hline b+d & 1 & 0 & 0 & 0 & 0 & 0 \\ c+e & 0 & 1 & 0 & 0 & 0 & 0 \\ d & 0 & 0 & 1 & 0 & 0 & 0 \\ e & 0 & 0 & 0 & 1 & 0 & 0 \\ a+b+c+d+e & 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

where $z_1 = bd - ac + ab - cd$ y $z_2 = ce - bd + ac - ab - de$ are a basis for Z_1 and $\{b+d, c+e, d, e\}$ are a basis for B_0.

We can read the decomposition of $\mathrm{H}_0(K)$ in $\widetilde{M_1}$. If a row has a 1 as a pivot that row does not contributes to the decomposition of $\mathrm{H}_0(K)$, if the row does not have pivot it contributes $\mathbb{Z}$ to $\mathrm{H}_0(K)$. And if the pivot is an element different from one, this is one of the torsion coefficients in Theorem 1.9. In our case we have $\mathrm{H}_0(K) \cong \mathbb{Z}$ with generator the 1-simplex $a+b+c+d+e$.

To compute $\mathrm{H}_1(K)$ we need to write down M_2, which in the standard basis for $C-2$ and C_1 is represented by

$$M_2 = \left[\begin{array}{c|c} & cde \\ \hline ab & 0 \\ ac & 0 \\ bd & 0 \\ cd & 1 \\ ce & -1 \\ de & 1 \end{array}\right].$$

We need to represent this matrix using the basis $\{ab, ac, bd, ce, z_1, z_2\}$ for C_1. It is easy to see that in this basis we get

$$\widetilde{M_2} = \left[\begin{array}{c|c} & cde \\ \hline ab & 0 \\ ac & 0 \\ bd & 0 \\ ce & 0 \\ z_1 & -1 \\ z_2 & -1 \end{array}\right].$$

Since the first entries of the matrix are zeros, we verify that the image of ∂_2

is contained into the kernel of ∂_1. Thus, we are only interested

$$\left[\begin{array}{c|c} & cde \\ \hline z_1 & -1 \\ z_2 & -1 \end{array}\right] \sim \left[\begin{array}{c|c} & cde \\ \hline -z_1 & 1 \\ z_2 - z_1 & 0 \end{array}\right].$$

Then we have that $\mathrm{H}_1(K) \cong \mathbb{Z}$ with generator $z_2 - z_1$.

From now own we will only work with homology with coefficients over fields. These restrictions allow us to use the structure theorems presented at the end of Section 1.1.

We will introduce persistent homology as in Section 2.6 of [ZC05]. They introduce some definitions that allow them to understand persistent homology of a filtered complex as simply the standard homology of a graded module over a polynomial ring, which is a PID when the base ring is a field.

Definition 1.21. A **persistence complex** $\mathcal{C}$ is a family of chain complexes $\left\{C_*^i\right\}_{i\geq 0}$ over a ring R with chain maps $f^i : C_*^i \to C_*^{i+1}$.

We can think of a persistence complex as a commutative diagram

$$\begin{array}{ccccccc} \vdots & & \vdots & & \vdots & & \\ \downarrow{\scriptstyle\partial} & & \downarrow{\scriptstyle\partial} & & \downarrow{\scriptstyle\partial} & & \\ C_2^0 & \xrightarrow{f^0} & C_2^1 & \xrightarrow{f^1} & C_2^2 & \xrightarrow{f^2} & \cdots \\ \downarrow{\scriptstyle\partial} & & \downarrow{\scriptstyle\partial} & & \downarrow{\scriptstyle\partial} & & \\ C_1^0 & \xrightarrow{f^0} & C_1^1 & \xrightarrow{f^1} & C_1^2 & \xrightarrow{f^2} & \cdots \\ \downarrow{\scriptstyle\partial} & & \downarrow{\scriptstyle\partial} & & \downarrow{\scriptstyle\partial} & & \\ C_0^0 & \xrightarrow{f^0} & C_0^1 & \xrightarrow{f^1} & C_0^2 & \xrightarrow{f^2} & \cdots \end{array}$$

Example 1.22. Take a filtered complex $\emptyset = K^0 \subset K^1 \subset \cdots \subset K^m = K$ and consider the chain complex at each level of the filtration. Taking the maps induced at chain level by the inclusions on the filtration we get a persistence complex $\left\{C_*(K^i)\right\}_{i\geq 0}$.

Definition 1.23. A **persistence module** $\mathcal{M}$ is a family of R-modules M^i with homomorphisms $\phi^i : M^i \to M^{i+1}$.

Example 1.24. Let F be the base field of coefficients and K a filtered complex. Since $K^{i-1} \subset K^i$, the inclusion map $f(x) = x$ induces a morphism $f_*^i : \mathrm{H}_*(K^i; F) \to \mathrm{H}_*(K^{i+1}; F)$ between F-modules, this gives us a persistence module (see [ZC05] p. 6). To simplify the notation we will be writing $\mathrm{H}_p^i := \mathrm{H}_p(K^i; F)$. Then we have the following sequence for each natural number p.

$$0 = \mathrm{H}_p^0 \longrightarrow \mathrm{H}_p^1 \longrightarrow \cdots \longrightarrow \mathrm{H}_p^n \longrightarrow \mathrm{H}_p^{n+1} = 0.$$

These morphisms can be composed into maps $f_p^{i,j} : \mathrm{H}_p^i \to \mathrm{H}_p^j$, and the image of $f_p^{i,j}$ are all the p-dimensional homology classes that are born at or before K^i and die after K^j.

Definition 1.25. The **persistent homology groups** of dimension p are the images of the homomorphisms induced by the inclusion $\mathrm{H}_p^{i,j} = \mathrm{Img}(f_p^{i,j})$ for $i < j$. The corresponding p dimensional **persistent Betti numbers** are $\beta_p^{i,j} = \mathrm{rank}(\mathrm{H}_p^{i,j})$.

The persistent Betti numbers count the independent homology classes in K^i , that are still alive and independent in K^j, or we can say that count the homology classes in K^j which are born at or before K^i.

Definition 1.26 (Finite type)**.** A persistence complex $\{C_*^i, f^i\}$ (persistence module $\{M^i, \phi^i\}$) is of **finite type** if:

(i) each component on the complex C_* (module M^i) is a finitely generated R-module,

(ii) the maps f^i (ϕ^i) are isomorphisms for $i \geq m$ for some positive integer m.

Example 1.27. Since we are working with finite complexes and a finite filtration, then they generate a persistence complex of finite type and its homology is a persistence module of finite type.

Theorem 1.28 (Correspondence)**.** *Consider a persistence module $\mathcal{M}$ over a ring R and the standard graduation on $R[t]$. We define a graded module*

$$\alpha(\mathcal{M}) = \bigoplus_{i \in \mathbb{Z}} M^i,$$

with addition as defined on each component and the action of t is defined as

$$t(m^0, m^1, m^2, \dots) = (0, \phi^0(m^0), \phi^1(m^1), \phi^2(m^2), \dots).$$

Then α defines an equivalence between the category of persistence modules of finite type and the category of finitely generated non-negatively graduated $R[t]$-modules.

Proof. See Theorem 3.1 on [ZC05]. □

The previous theorem suggests that whenever we choose a field F since $F[t]$ is a PID, we can easily classify graded modules over PIDs using Theorem 1.10 we can give a complete description for the persistence module obtained from the homology of the persistence complex of a filtered simplicial complex as

$$\mathrm{H}_*(\mathrm{R}_\epsilon(X); F) \cong \left(\bigoplus_{i=1}^{n} \Sigma^{\alpha_i} F[t] \right) \oplus \left(\bigoplus_{j=1}^{m} \Sigma^{\gamma_j} F[t] / (t^{n_j}) \right). \tag{1.1}$$

As in Example 1.20 we can reed this decomposition from the Smith normal form of an adequate matrix, the following result indicates how this decomposition can be interpreted.

Proposition 1.29. *Let $\widetilde{M_k}$ be the column echelon form of ∂_k relative to the basis $\{e_j\}$ and $\{\widehat{e_j}\}$ for C_k and Z_{k-1}, respectively. If row i has a pivot t^n, it contributes $\Sigma^{deg(\widehat{e_i})} F[t]/(t^n)$ to the description of $\mathrm{H}_{k-1}(\mathrm{R}_\epsilon(X); F)$. Otherwise it contributes $\Sigma^{deg(\widehat{e_i})} F[t]$.*

Proof. See Corollary 4.1 in [ZC05]. □

Definition 1.30. We define the multiplicities

$$\mu_p^{i,j} = (\beta_p^{i,j-1} - \beta_p^{i,j}) - (\beta_p^{i-1,j-1} - \beta_p^{i-1,j})$$

whenever $i < j$.

Notice that the first difference counts the homology classes born at or before K^i which dies entering K^j, and the second difference counts the homology classes that are born at or before K^{i-1} that dies entering k^j. Therefore, $\mu_p^{i,j}$ counts the p-dimensional homology classes born at K^i that dies entering K^j.

Definition 1.31. The p-dimensional **persistence diagram** of the filtration is defined as

$$\mathrm{Diag}_p(K) := \left\{(i,j) \in \mathbb{R}^2 \mid \mu_p^{i,j} > 0 \text{ and } i < j\right\}.$$

Each point $(i,j) \in \mathrm{Diag}_p(K)$ has multiplicity $\mu_p^{i,j}$.

For technical reasons we need to add to the diagram the points in the diagonal with countable infinite multiplicity.

The decomposition in equation (1.1) together with Proposition 1.29 give us a description of the persistence diagram for the filtered complex. For each term in the free submodule we get $(\deg(\widehat{e_i}), \infty) \in \mathrm{Diag}_k(X)$ and for each term in the torsion submodule $(\deg(\widehat{e_j}), \deg(\widehat{e_j}) + n_j) \in \mathrm{Diag}_k(X)$.

Example 1.32. We will illustrate how to use the decomposition on equation (1.1) and Proposition 1.29 to calculate the persistent homology of a filtered complex. Consider the filtration in Figure 1.1 and take coefficients over $\mathbb{Z}_2$, then the persistence module corresponds to a $\mathbb{Z}_2[t]$-module under Theorem 1.28.

We can write a matrix representation M_1 for ∂_1 using the filtration in the following manner:

$$M_1 = \left[\begin{array}{c|cccccc} & ab & ac & bd & cd & ce & de \\ \hline a & 1 & t^2 & 0 & 0 & 0 & 0 \\ b & 1 & 0 & 1 & 0 & 0 & 0 \\ c & 0 & t^2 & 0 & t & t & 0 \\ d & 0 & 0 & 1 & t & 0 & t^2 \\ e & 0 & 0 & 0 & 0 & 1 & t \end{array}\right].$$

This can be reduced to the column echelon form

$$\widetilde{M_1} = \left[\begin{array}{c|cccccc} & ab & bd & ce & cd & z_1 & z_2 \\ \hline a & 1 & 0 & 0 & 0 & 0 & 0 \\ b & 1 & 1 & 0 & 0 & 0 & 0 \\ e & 0 & 0 & 1 & 0 & 0 & 0 \\ d & 0 & 1 & 0 & t & 0 & 0 \\ c & 0 & 0 & t & t & 0 & 0 \end{array}\right],$$

with $z_1 = (bd + ab)t^2 + (cd)t + ac$ and $z_2 = (ce + cd)t + de$ a basis for Z_1. Using Proposition 1.29 we have that

$$\mathrm{H}_0(K;\mathbb{Z}_2) \cong \Sigma^0\mathbb{Z}_2[t] \oplus \Sigma^0\mathbb{Z}_2[t]/(t).$$

Moreover we have that $\mathrm{Diag}_0(K) = \{(0,1),(0,\infty)\}$.

To calculate $\mathrm{H}_1(K;\mathbb{Z}_2)$ we only need to write down

$$M_2 = \left[\begin{array}{c|c} & cde \\ \hline ab & 0 \\ ac & 0 \\ bd & 0 \\ cd & t^2 \\ ce & t^2 \\ de & t \end{array}\right].$$

And this matrix can be wrote relative to the basis $\{ab, bd, ce, cd, z_1, z_2\}$ as

$$\widetilde{M_2} = \left[\begin{array}{c|c} & cde \\ \hline ab & 0 \\ bd & 0 \\ ce & 0 \\ cd & 0 \\ z_1 & 0 \\ z_2 & t \end{array}\right],$$

since we are interested on $\mathrm{H}_1(K;\mathbb{Z}_2)$ we only need to see

$$\left[\begin{array}{c|c} & cde \\ \hline z_1 & 0 \\ z_2 & t \end{array}\right].$$

Again by Proposition 1.29 we get

$$\mathrm{H}_1(K;\mathbb{Z}_2) \cong \Sigma^2\mathbb{Z}_2 \oplus \Sigma^2\mathbb{Z}_2/(t).$$

Thus $\mathrm{Diag}_1(K) = \{(2,3),(2,\infty)\}$

Definition 1.33. The **barcode** of the p dimensional persistence diagram of a filtered simplicial complex K is a collection of horizontal line segments, one for each element in $\mathrm{Diag}_p(K)$. These line segments are shown in a plane with the parameter of the filtration on the horizontal axes and the vertical axes is an arbitrary ordering of the elements in de diagram. The line segment corresponding to the point $(i,j) \in \mathrm{Diag}_p(K)$ has length $j-i$ and begins at i and goes to j.

Example 1.34. In the following figure we show the barcode associated to the persistence diagrams for the filtered complex in Figure **??**

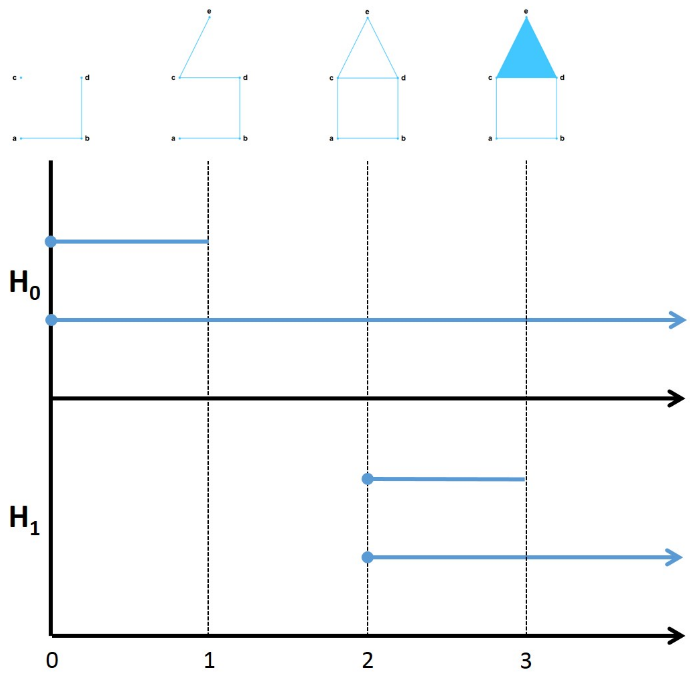


Figure 1.2: Barcode representation of persistent homology of K

For example the point $(2,3) \in \mathrm{Diag}_1(K)$ is represented as a line segment that begins at 2 and ends at 3; this line segment represents and homology class with birth time 2 and death time 3.

Most of the results in this work will we presented in the form of barcodes as graphical representation makes it easy to understand the persistent diagram of a given filtered complex.

Chapter 2

Persistence Estimations

2.1 Estimations using eigenvalues bounds

Lemma 2.1. *Let $X \subset \mathbb{R}^n$ be a set of points. If $T : \mathbb{R}^n \to \mathbb{R}^m$ is a linear transformation, then T induces a homomorphism*

$$T_* : \mathrm{H}_*(\mathrm{R}_\epsilon(X)) \to \mathrm{H}_*(\mathrm{R}_\delta(T(X))),$$

for every $\delta \geq \|T\|\epsilon$.

Proof. First of all recall that $\|Tx - Ty\| \leq \|T\|\|x - y\|$ and that $(\mathrm{R}_\epsilon(X))_k$ is the free module with generating set

$$\{[x_1, \ldots, x_k] \mid x_i, x_j \in X, d(x_i, x_j) \leq \epsilon \text{ for every } 1 \leq i, j \leq k\}.$$

We define a map $T_\# : \mathrm{R}_\epsilon(X) \to \mathrm{R}_\delta(T(X))$ in the generator set as

$$T_\#([x_1, \ldots, x_k]) := [T(x_1), \ldots, T(x_k)].$$

Since $\|T(x_i) - T(x_j)\| \leq \|T\|\epsilon$ this function is well defined for every $\delta \geq \|T\|\epsilon$.

This map is defined at each level of the simplicial complex $\mathrm{R}_\epsilon(X)$, and to make it a chain map, we need to show that it commutes with the differential $\partial : (\mathrm{R}_\epsilon(X))_k \to (\mathrm{R}_\epsilon(X))_{k-1}$ which is also defined on the generating set as

$$\partial\,[x_1, \ldots, x_k] := \sum_{j=1}^{k} (-1)^j\,[x_1, \ldots, \hat{x_j}, \ldots, x_k].$$

To show that $T_{\#}\partial = \partial T_{\#}$, it is enough to verify it on the generators,

$$\begin{aligned} T_{\#}\partial\,[x_1,\dots,x_j,\dots,x_k] &= T_{\#}\left(\sum_{j=1}^{k}(-1)^j\,[x_1,\dots,x_{j-1},x_{j+1},\dots,x_k]\right) \\ &= \sum_{j=1}^{k}(-1)^j T_{\#}\left([x_1,\dots,x_{j-1},x_{j+1},\dots,x_k]\right) \\ &= \sum_{j=1}^{k}(-1)^j\,[T(x_1),\dots,T(x_{j-1}),T(x_{j+1}),\dots,T(x_k)] \\ &= \partial\,[T(x_1),\dots,T(x_j),\dots,T(x_k)] \\ &= \partial T_{\#}\,[x_1,\dots,x_j,\dots,x_k]\,. \end{aligned}$$

This implies that $T_{\#}$ induces a map at the level of homology, namely

$$T_* : \mathrm{H}_*(\mathrm{R}_\epsilon(X)) \to \mathrm{H}_*(\mathrm{R}_\delta(T(X)))$$

for every $\delta \geq \|T\|\epsilon$. □

Theorem 2.2. *Let $X \subset \mathbb{R}^n$ be a set of points and $T : \mathbb{R}^n \to \mathbb{R}^m$ an injective linear transformation with $\sigma_{\max}$ and $\sigma_{\min}$ its maximal and minimal singular values. Moreover, let $(a,b) \in \mathrm{Diag}_k(X)$ be the persistence of some homological class $\alpha \in \mathrm{H}_k(\mathrm{R}_\epsilon(X), \mathbb{F}[t])$ and*

$$\frac{b}{a} > \frac{\sigma_{\max}}{\sigma_{\min}}.$$

Then T_* maps α to a homology class $\beta \in \mathrm{H}_k(\mathrm{R}_\delta(T(X)), \mathbb{F}[t])$ whose persistence $(u,v) \in \mathrm{Diag}_k(T(X))$ is such that

$$u \leq a\sigma_{\max} \leq b\sigma_{\min} \leq v. \tag{2.1}$$

Proof. Since $(a,b) \in \mathrm{Diag}_k(X)$ is the persistence of $\alpha \in \mathrm{H}_k(\mathrm{R}_\epsilon(X), \mathbb{F}[t])$ we have that $\alpha \neq 0$ whenever $a \leq \epsilon < b$ and $\alpha = 0$ for $\epsilon \geq b$.

Since T is a linear injective transformation, there exists an inverse transformation $T^{-1} : \mathrm{Im}(T) \to \mathbb{R}^n$. And, applying the previous lemma to T and T^{-1} we have the following diagram, for every $k \in \mathbb{N}$,

$$\mathrm{H}_k(\mathrm{R}_\epsilon(X)) \xrightarrow{T_*} \mathrm{H}_k(\mathrm{R}_\delta(T(X))) \xrightarrow{i_*} \mathrm{H}_k(\mathrm{R}_{\delta'}(T(X))) \xrightarrow{T_*^{-1}} \mathrm{H}_k(\mathrm{R}_{\epsilon'}(X))$$

for $\delta \geq \|T\|\epsilon$, $\delta \leq \delta'$ and $\epsilon' \geq \|T^{-1}\|\delta'$. By Lemma 1.4 and Corollary 1.5 this is equivalent to $\delta \geq \epsilon\sqrt{\lambda_{\max}}$, $\delta \leq \delta'$ and $\epsilon' \geq {}^{\delta'}/_{\sqrt{\lambda_{\min}}}$.

If $a > 0$ take $\epsilon = a$ and $\delta = a\sqrt{\lambda_{\max}}$. By hypothesis, we have

$$\frac{b}{a} > \frac{\sigma_{\max}}{\sigma_{\min}} = \frac{\sqrt{\lambda_{\max}}}{\sqrt{\lambda_{\min}}}.$$

This implies that $a\sqrt{\lambda_{\max}} < b\sqrt{\lambda_{\min}}$, it is easy to see that there is a sufficiently small $\tilde{\epsilon} > 0$ such that $a\sqrt{\lambda_{\max}} < (b - \tilde{\epsilon})\sqrt{\lambda_{\min}}$. Define $\tilde{b} = b - \tilde{\epsilon}$ and take $\epsilon' = \tilde{b}$ and $\delta' = \tilde{b}\sqrt{\lambda_{\min}}$. So we can write the following commutative diagram:

$$\begin{array}{ccc}
\mathrm{H}_k(\mathrm{R}_{a\sqrt{\lambda_{\max}}}(T(X))) & \xrightarrow{i_*} & \mathrm{H}_k(\mathrm{R}_{\tilde{b}\sqrt{\lambda_{\min}}}(T(X))) \\
\uparrow T_* & & \downarrow T_*^{-1} \\
\mathrm{H}_k(\mathrm{R}_a(X)) & \xrightarrow[i_*]{} & \mathrm{H}_k(\mathrm{R}_{\tilde{b}}(X))
\end{array}$$

Suppose that $T_*(\alpha)$ is zero in $\mathrm{H}_k(\mathrm{R}_t(T(X)))$ for $a\sqrt{\lambda_{\max}} \leq t \leq \tilde{b}\sqrt{\lambda_{\min}}$. That means that $T_*^{-1}i_*T_*(\alpha) = 0$ in $\mathrm{H}_k(\mathrm{R}_{\tilde{b}}(X))$. By definition of i_* and since homology is a covariant functor we have that $\alpha = \mathbb{1}_*(\alpha) = (T^{-1}T)_*(\alpha) = 0$ in $\mathrm{H}_k(\mathrm{R}_{\tilde{b}}(X))$, but this is a contradiction with the persistence of α since $\tilde{b} < b$. This argument shows us that the birth time of $T_*(\alpha)$ is at least $a\sqrt{\lambda_{\max}}$ and its death time must be bigger than $\tilde{b}\sqrt{\lambda_{\min}}$.

Thus taking $\beta = T_*(\alpha)$ we have that its persistence (u, v) is such that $u \leq a\sqrt{\lambda_{\max}} < \tilde{b}\sqrt{\lambda_{\min}} \leq v$. Or equivalently $u \leq a\sqrt{\lambda_{\max}} < (b - \tilde{\epsilon})\sqrt{\lambda_{\min}} \leq v$, taking the limit as $\tilde{\epsilon}$ goes to zero, we get the desired inequality.

On the other hand, if $a = 0$ take a sequence $\{a_n\}_{n\in\mathbb{Z}} \subset \mathbb{R}^+$ such that $\lim_{n\to\infty} a_n = 0$. And for each one of this a_n (since they are all positive) we get

$$u \leq a_n\sqrt{\lambda_{\max}} < b\sqrt{\lambda_{\min}} \leq v$$

for each $n \in \mathbb{Z}$. Then we can take limit as n goes to infinity and obtain that

$$u \leq \left(\lim_{n\to\infty} a_n\right)\sqrt{\lambda_{\max}} < b\sqrt{\lambda_{\min}} \leq v$$

which is equivalent to

$$u \leq 0 < b\sqrt{\lambda_{\min}} \leq v. \qquad \square$$

Example 2.3. One simple example is the linear transformation $T : \mathbb{R}^2 \to \mathbb{R}^2$ defined by the matrix

$$T = \begin{pmatrix} c & 0 \\ 0 & d \end{pmatrix}$$

where c, d are real positive numbers such that $d < c$. This transformation maps the circle of radius r into the ellipse with equation $\frac{x^2}{c^2} + \frac{y^2}{d^2} = r^2$ as illustrated in Figure 2.2.

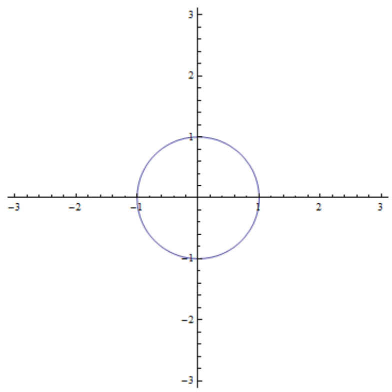


Figure 2.1: $S^1 \subset \mathbb{R}^2$

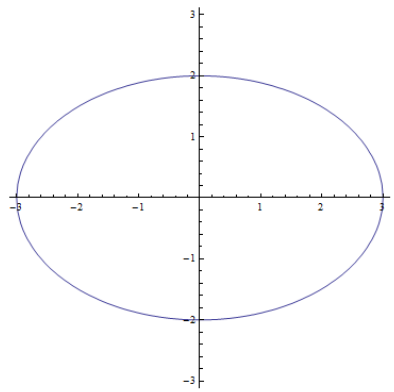


Figure 2.2: $T(S^1) \subset \mathbb{R}^2$

In this example T is injective and we have $T^* = T$, so $T^*T = T^2$. Since the persistence of the 1-dimensional generator of $\mathrm{H}_1(\mathrm{R}_\epsilon(S^1_r))$ is $(0, r\sqrt{3})$, we have that $\frac{0}{r\sqrt{3}} < \sqrt{\frac{d^2}{c^2}}$ and we can conclude, using Theorem 2.2, that there exists a 1-cycle on $\mathrm{H}_1(\mathrm{R}_\delta(T(S^1_r)))$ with persistence (u, v) such that $u \leq 0 < dr\sqrt{3} \leq v$.

We can develop this example further using computational tools to see how this bound behaves with real data. For this simulation we use the TDATools Package for Matlab, we generate 2000 points over S^1 and calculate its 1-dimensional persistence using the code presented in Appendix A.2.1.

We found that there is a homology class in $\mathrm{H}_1(\mathrm{R}_\delta(S^1))$ whose persistence is $(0.024155, 1.7321)$. Applying the linear transformation illustrated in Figure 2.2, which is given by T when $c = 3$ and $d = 2$, we have that

$$\frac{b}{a} = \frac{1.7321}{0.024155} \approx 71.7077 > \sqrt{\frac{9}{4}} = 1.5.$$

Theorem 2.2 shows that there must exist a homology class in $\mathrm{H}_1(\mathrm{R}_\delta(T(S^1)))$

with persistence (u, v) such that

$$u \leq 3(0.024155) < 2(1.7321) \leq v$$

which is

$$u \leq 0.072465 < 3.4642 \leq v.$$

This estimation can be compared to the real persistence obtained from Matlab which is $(0.063524, 3.9597)$.

In the previous example it was easy to calculate the eigenvalues of T^*T, but in most of the cases it is pretty difficult to do so. It will be useful to have a method to estimate those eigenvalues. One of the most well known results concerning bounds for eigenvalues was found by Wolkowicz and Styan in [HP80], there they present some easy to calculate bounds using traces, such as:

Theorem 2.4. *Let A be a $n \times n$ complex matrix with real eigenvalues $\lambda(A)$ and let*

$$m = \frac{\operatorname{tr}(A)}{n} \quad \text{and} \quad s^2 = \frac{\operatorname{tr}(A^2)}{n} - m^2.$$

Then

$$m - s(n-1)^{1/2} \leq \lambda_{\min}(A) \leq m - \frac{s}{(n-1)^{1/2}},$$

$$m + \frac{s}{(n-1)^{1/2}} \leq \lambda_{\max}(A) \leq m + s(n-1)^{1/2}.$$

Proof. See Theorem 2.1 in [HP80]. □

If we apply Theorem 2.4 to our "toy example" 2.3 we find by taking $A = T^*T = T^2$ that:

$$m = \frac{c^2 + d^2}{2} \quad \text{and} \quad s^2 = \frac{c^4 + d^4}{2} - \frac{(c^2 + d^2)^2}{4} = \frac{(c^2 - d^2)^2}{4},$$

and therefore by Theorem 2.4 we get

$$d^2 = \frac{c^2 + d^2}{2} - \frac{c^2 - d^2}{2} \leq \lambda_{\min}(A) \leq \frac{c^2 + d^2}{2} - \frac{c^2 - d^2}{2} = d^2$$

and

$$c^2 = \frac{c^2 + d^2}{2} + \frac{c^2 - d^2}{2} \leq \lambda_{\max}(A) \leq \frac{c^2 + d^2}{2} + \frac{c^2 - d^2}{2} = c^2$$

Wolkowicz and Stain [HP80] also prove the following lemma, which will be used in other the example at the end of this section

Lemma 2.5. *Let A be a $n \times n$ complex matrix such that $\sigma(A) \subset \mathbb{R}$ with precisely k eigenvalues being positive and l negative. Let* $\operatorname{tr}(A^2) > 0$.

(i) *When* $\operatorname{tr}(A) \geq 0$, *then*
$$\frac{(\operatorname{tr}(A))^2}{\operatorname{tr}(A^2)} \geq k,$$

(ii) *When* $\operatorname{tr}(A) \leq 0$, *then*
$$\frac{(\operatorname{tr}(A))^2}{\operatorname{tr}(A^2)} \leq l.$$

Proof. See Corollary 2.2 in [HP80]. □

Remark. If we take a linear transformation $T : \mathbb{R}^n \to \mathbb{R}^m$ and apply the previous lemma to $A = T^*T$ then, due to Proposition 1.3, we get that
$$\frac{(\operatorname{tr}(T^*T))^2}{\operatorname{tr}((T^*T)^2)} \geq n,$$
which also means that $(\operatorname{tr}(T^*T))^2 \geq n \operatorname{tr}((T^*T)^2) > (n-1)\operatorname{tr}((T^*T)^2)$.

This condition is needed to apply some of the following results from [HP80] to T^*T.

Proposition 2.6. *Let A be Hermitian positive definite, and let m and s be defined as in Theorem 2.4. Then*

(i)
$$1 + \frac{2s}{m - s/(n-1)^{1/2}} \leq \frac{\lambda_{\max}(A)}{\lambda_{\min}(A)}.$$

(ii) *If* $\operatorname{tr}(A) > 0$ *and* $(\operatorname{tr}(A))^2 > (n-1)\operatorname{tr}(A^2)$, *then A is positive definite and*
$$\frac{\lambda_{\max}(A)}{\lambda_{\min}(A)} \leq 1 + \frac{(2n)^{1/2}s}{m - s(n-1)^{1/2}}.$$

Proof. In Corollary 2.4 of [HP80]. □

Corollary 2.7. *Let m and s as in Theorem 2.4. If $n \geq 2$,* $\operatorname{tr}(A) > 0$ *and* $(\operatorname{tr}(A))^2 > (n-1)\operatorname{tr}(A^2)$, *then we have*
$$m - s(n-1)^{1/2} > 0 \quad \textit{and} \quad m - s/(n-1)^{1/2} > 0.$$

Proof. Since

$$((n-1)^{1/2})^2 = n-1 \geq 1 = s/s,$$
$$s(n-1)^{1/2} \geq s/(n-1)^{1/2},$$
$$m - s(n-1)^{1/2} \leq m - s/(n-1)^{1/2},$$

and

$$1 \leq \frac{\lambda_{\max}(A)}{\lambda_{\min}(A)} \leq 1 + \frac{(2n)^{1/2}s}{m-s(n-1)^{1/2}},$$
$$0 \leq \frac{(2n)^{1/2}s}{m-s(n-1)^{1/2}},$$

we must have $m - s(n-1)^{1/2} > 0$ and also $m - s/(n-1)^{1/2} > 0$. □

Theorem 2.8. *Let $X \subset \mathbb{R}^n$ be a set of points and let $T : \mathbb{R}^n \to \mathbb{R}^m$ be an injective linear transformation with $\sigma_{\max}$ and $\sigma_{\min}$ its maximal and minimal singular values. Also let $(a, b) \in \mathrm{Diag}_k(X)$ be the persistence of some homological class $\alpha \in \mathrm{H}_k(\mathrm{R}_\epsilon(X), \mathbb{F}[t])$ such that*

$$\frac{b}{a} > \sqrt{\frac{m+s(n-1)^{1/2}}{m-s(n-1)^{1/2}}},$$

and let m and s be defined as in Theorem 2.4. Then we can find a persistent homology class $\beta \in \mathrm{H}_k(\mathrm{R}_\delta(T(X)), \mathbb{F}[t])$ whose persistence $(u, v) \in \mathrm{Diag}_k(T(X))$ is such that

$$u \leq a\sqrt{m+s(n-1)^{1/2}} \leq b\sqrt{m-s(n-1)^{1/2}} \leq v. \tag{2.2}$$

Proof. By the remark before Proposition 2.6 we can apply Corollary 2.7 to $A = T^*T$. Thus, we have that $m - s(n-1)^{1/2} \geq 0$ and we can take square root in all the inequalities of Theorem 2.4.

As in the proof of Theorem 2.2 we only need to find ϵ, ϵ', δ and δ' such that $\delta \geq \|T\|\epsilon$, $\delta \leq \delta'$ and $\epsilon' \geq \|T^{-1}\|\delta'$, to get a well defined sequence:

$$\mathrm{H}_k(\mathrm{R}_\epsilon(X)) \xrightarrow{T_*} \mathrm{H}_k(\mathrm{R}_\delta(T(X))) \xrightarrow{i_*} \mathrm{H}_k(\mathrm{R}_{\delta'}(T(X))) \xrightarrow{T_*^{-1}} \mathrm{H}_k(\mathrm{R}_{\epsilon'}(X)) .$$

As in Theorem 2.2 if $a > 0$ take $\epsilon = a$ and $\epsilon' = b$. We only need to carefully choose δ an δ' for the sequence to be well defined. Notice that

$a\|T\| \leq \delta := a\sqrt{m + s(n-1)^{1/2}}$ and $b\|T^{-1}\| \geq \delta' := b\sqrt{m + s(n-1)^{1/2}}$. Since we need $\delta < \delta'$ this condition is equivalent to

$$\frac{b}{a} > \frac{\sqrt{m + s(n-1)^{1/2}}}{\sqrt{m - s(n-1)^{1/2}}}.$$

Since all the required inequalities are fulfilled by hypothesis it is well defined. We can conclude that $\beta := T_*(\alpha)$ is a homology class with persistence $(u, v) \in \mathrm{Diag}_k(T(X))$ such that

$$u \leq a\sqrt{m + s(n-1)^{1/2}} \leq b\sqrt{m - s(n-1)^{1/2}} \leq v.$$

For $a = 0$ we can take a sequence of positive numbers with limit 0 and apply the previous result and then take the limit to get the desired result (this argument is the same as the one presented at the end of the proof of Theorem 2.2). □

Lemma 2.9. *Let $X \subset \mathbb{R}^n$ be a set of points and let $T : \mathbb{R}^n \to \mathbb{R}^m$ be an injective linear transformation with $\sigma_{\max}$ and $\sigma_{\min}$ its maximal and minimal singular values. Also, let $(a, b) \in \mathrm{Diag}_k(X)$ be the persistence of some homological class $\alpha \in \mathrm{H}_k(\mathrm{R}_\epsilon(X), \mathbb{F}[t])$ and assume that there are some positive constants k, l and h such that*

(i) $$\frac{b}{a} > h \geq \frac{\sigma_{\max}}{\sigma_{\min}},$$

(ii) $$\sigma_{\min} \leq l \quad \textit{and} \quad k \leq \sigma_{\max},$$

(iii) $$\frac{b}{a} > h^2 \frac{l}{k}.$$

Then we can find a persistent homology class $\beta \in \mathrm{H}_k(\mathrm{R}_\delta(T(X)), \mathbb{F}[t])$ whose persistence $(u, v) \in \mathrm{Diag}_k(T(X))$ satisfies

$$u \leq ahl < \frac{bk}{h} \leq v. \tag{2.3}$$

Proof. Similarly as in Theorem 2.2 we have the sequence

$$\mathrm{H}_k(\mathrm{R}_\epsilon(X)) \xrightarrow{T_*} \mathrm{H}_k(\mathrm{R}_\delta(T(X))) \xrightarrow{i_*} \mathrm{H}_k(\mathrm{R}_{\delta'}(T(X))) \xrightarrow{T_*^{-1}} \mathrm{H}_k(\mathrm{R}_{\epsilon'}(X))$$

whenever $\delta \geq \|T\|\epsilon$, $\delta \leq \delta'$ and $\epsilon' \geq \|T^{-1}\|\delta'$.

If $a > 0$ take $\epsilon = a$ and $\epsilon' = b$, we again need to carefully choose δ and δ'. We have $\sigma_{\max} \leq h\sigma_{\min} \leq hl$ and $\sigma_{\min} \geq \sigma_{\max}/h \geq k/h$, if we take $\delta = ahl$ and $\delta' = bk/h$ then $\delta = ahl \geq a\sigma_{\max} = a\|T\|$ and $\delta' = bk/h \leq b\sigma_{\min} = b/\|T^{-1}\|$ which are equivalent to $\delta \geq \|T\|\epsilon$ and $\epsilon' \geq \|T^{-1}\|\delta'$.

Finally we also need that $\delta < \delta'$. This is equivalent to $ahl < bk/h$ which can be restated as $\dfrac{b}{a} > h^2\dfrac{l}{k}$, but this is true by one of the hypothesis. With these inequalities we conclude that $\beta := T(\alpha)$ is a homology class in $\mathrm{H}_k(\mathrm{R}_\delta(T(X)), \mathbb{F}[t])$ with persistence (u, v) satisfying

$$u \leq ahl < \frac{bk}{h} \leq v.$$

This result cam be extended to $a = 0$ taking a positive sequence that converges to 0. □

Theorem 2.10. *Let $X \subset \mathbb{R}^n$ be a set of points and let $T : \mathbb{R}^n \to \mathbb{R}^m$ be an injective linear transformation with $\sigma_{\max}$ and $\sigma_{\min}$ its maximal and minimal singular values. Also, let $(a, b) \in \mathrm{Diag}_k(X)$ be the persistence of some homological class $\alpha \in \mathrm{H}_k(\mathrm{R}_\epsilon(X), \mathbb{F}[t])$ such that*

$$\frac{b}{a} > \sqrt{1 + \frac{(2n)^{1/2}s}{m - s(n-1)^{1/2}}}$$

and

$$\frac{b}{a} > \left(1 + \frac{(2n)^{1/2}s}{m - s(n-1)^{1/2}}\right)\sqrt{\frac{m - s/(n-1)^{1/2}}{m + s/(n-1)^{1/2}}}$$

with m and s as defined as in Theorem 2.4. Then we can find a persistent homology class $\beta \in \mathrm{H}_k(\mathrm{R}_\delta(T(X)), \mathbb{F}[t])$ whose persistence $(u, v) \in \mathrm{Diag}_k(T(X))$ is such that

$$u \leq a\sqrt{m - \frac{s}{(n-1)^{1/2}}}\sqrt{1 + \frac{(2n)^{1/2}s}{m - s(n-1)^{1/2}}} < \frac{b\sqrt{m + \frac{s}{(n-1)^{1/2}}}}{\sqrt{1 + \frac{(2n)^{1/2}s}{m-s(n-1)^{1/2}}}} \leq v \tag{2.4}$$

and also

$$u \leq a\sigma_{\max} \leq b\sigma_{\min} \leq v$$

where $\sigma_{\max}$ and $\sigma_{\min}$ are the maximal and minimal singular values of T.

Proof. The proof of this Theorem is a straightforward application of the previous lemma. Take

$$h = \sqrt{1 + \frac{(2n)^{1/2}s}{m - s(n-1)^{1/2}}},$$

$$l = \sqrt{m - s/(n-1)^{1/2}} \quad \text{and} \quad k = \sqrt{m + s/(n-1)^{1/2}}.$$

With this choice notice that

$$\frac{b}{a} > \sqrt{1 + \frac{(2n)^{1/2}s}{m - s(n-1)^{1/2}}} \geq \frac{\sigma_{\max}}{\sigma_{\min}}$$

holds by hypothesis and Proposition 2.6. Also $\sigma_{\min} \leq \sqrt{m - s/(n-1)^{1/2}}$ and $\sqrt{m + s/(n-1)^{1/2}} \geq \sigma_{\max}$ holds by Theorem 2.4. And

$$\frac{b}{a} > \left(1 + \frac{(2n)^{1/2}s}{m - s(n-1)^{1/2}}\right) \sqrt{\frac{m - s/(n-1)^{1/2}}{m + s/(n-1)^{1/2}}}$$

also holds by hipothesis.

Now we can freely apply the previous lemma and we obtain that there exists a homology class $\beta \in \mathrm{H}_k(\mathrm{R}_\delta(T(X)); \mathbb{F}[t])$ with persistence

$$u \leq a\sqrt{m - \frac{s}{(n-1)^{1/2}}}\sqrt{1 + \frac{(2n)^{1/2}s}{m - s(n-1)^{1/2}}} < \frac{b\sqrt{m + \frac{s}{(n-1)^{1/2}}}}{\sqrt{1 + \frac{(2n)^{1/2}s}{m - s(n-1)^{1/2}}}} \leq v.$$

□

Example 2.11. We now define the injective linear transformation

$$T := \begin{pmatrix} 1/5 & -1/5 & 0 \\ -1 & 1 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 3/2 \end{pmatrix} : \mathbb{R}^3 \to \mathbb{R}^4 \quad \text{and} \quad A = T^*T.$$

We have

$$A = \begin{pmatrix} 51/25 & -1/25 & 0 \\ -1/25 & 51/25 & 0 \\ 0 & 0 & 9/4 \end{pmatrix} \text{ and } A^2 = \begin{pmatrix} 2602/625 & -102/625 & 0 \\ -102/625 & 2602/625 & 0 \\ 0 & 0 & 81/16 \end{pmatrix}.$$

So we get $m = \dfrac{211}{100}$ and $s^2 = \dfrac{163}{15000}$ and therefore

$$\sqrt{m - s(n-1)^{1/2}} = \sqrt{\frac{211}{100} - \frac{\sqrt{163/3}}{50}} \approx 1.40092$$
$$\sqrt{m + s(n-1)^{1/2}} = \sqrt{\frac{211}{100} + \frac{\sqrt{163/3}}{50}} \approx 1.50247.$$

The eigenvalues of A are $\left\{2, \frac{52}{25}, \frac{9}{4}\right\}$. To apply Theorem 2.2 we need an element $(a,b) \in \mathrm{Diag}_k(X)$ such that

$$\frac{b}{a} \geq \sqrt{\frac{9/4}{2}}.$$

Then there exists an element $(u,v) \in \mathrm{Diag}_k(T(X))$ with

$$u \leq a\frac{3}{2} < b\sqrt{2} \leq v.$$

In order to apply the Theorem 2.8 we need

$$\frac{b}{a} \geq \sqrt{\frac{m + s(n-1)^{1/2}}{m - s(n-1)^{1/2}}} \approx \frac{1.50247}{1.40092} \approx 1.07249.$$

Then we can conlcude from Theorem 2.2 that

$$u \leq a\sqrt{\frac{211}{100} + \frac{\sqrt{163/3}}{50}} < b\sqrt{\frac{211}{100} - \frac{\sqrt{163/3}}{50}} \leq v,$$

which is approximately

$$u \leq 1.50247a < 1.40092b \leq v.$$

On the other hand to apply Theorem 2.10 we must have

$$\frac{b}{a} > \sqrt{1 + \frac{(2n)^{1/2}s}{m - s(n-1)^{1/2}}} \approx 1.05178$$

and

$$\frac{b}{a} > \left(1 + \frac{(2n)^{1/2}s}{m - s(n-1)^{1/2}}\right)\sqrt{\frac{m - s/(n-1)^{1/2}}{m + s/(n-1)^{1/2}}} \approx 1.06824$$

to obtain an homology class with persistence approximately

$$u \leq a(1.40092)(1.05178) < \frac{b(1.50247)}{1.05178} \leq v.$$

To complete this example we will apply it to a 2-dimensional sphere. Using Matlab and JavaPlex we generate random points on $S^2 \subset \mathbb{R}^3$ and we calculate its persistent homology. The corresponding barcode is displayed in Figure 2.3

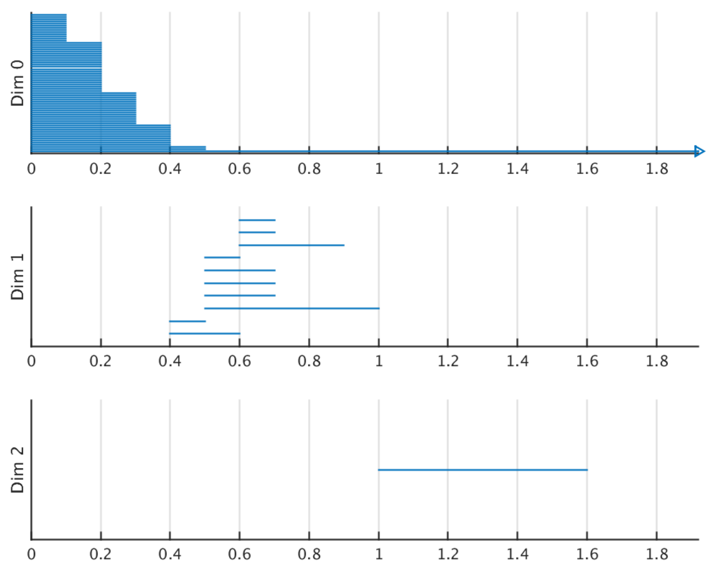


Figure 2.3: Persistent homology for 75 points in S^2

In this example we find that there is a homology class in dimension 2 with persistence $(1, 1.6)$. To apply Theorem 2.2 $\frac{b}{a} > \sqrt{\frac{\lambda_{\max}}{\lambda_{\min}}}$, in this case we get

$$\frac{b}{a} = \frac{1.6}{1} = 1.6 \text{ and } \sqrt{\frac{\lambda_{\max}}{\lambda_{\min}}} \approx \sqrt{\frac{9/4}{2}} \approx 1.06066$$

so the hypothesis is fulfilled and we get that

$$u \leq 1.5 < 2.26274 \leq v.$$

Also to apply Theorem 2.8 we need

$$\frac{b}{a} = 1.6 > 1.07249$$

so we get

$$u \leq 1.50247(1) < 1.40092(1.6) \leq v$$
$$u \leq 1.50247 < 2.24147 \leq v.$$

Finally to use Theorem 2.10 we must have $\frac{b}{a} = 1.6 > 1.05178$ and $\frac{b}{a} = 1.6 > 1.06824$

$$u \leq (1)(1.40092)(1.05178) < \frac{(1.6)(1.50247)}{1.05178} \leq v$$
$$u \leq 1.47346 < 2.2856 \leq v.$$

We need to compare these results with the ones obtained using JavaPlex on Matlab for $T(S^2)$; in the following figure, we show the barcode corresponding to the persistent diagram of $T(S^2)$:

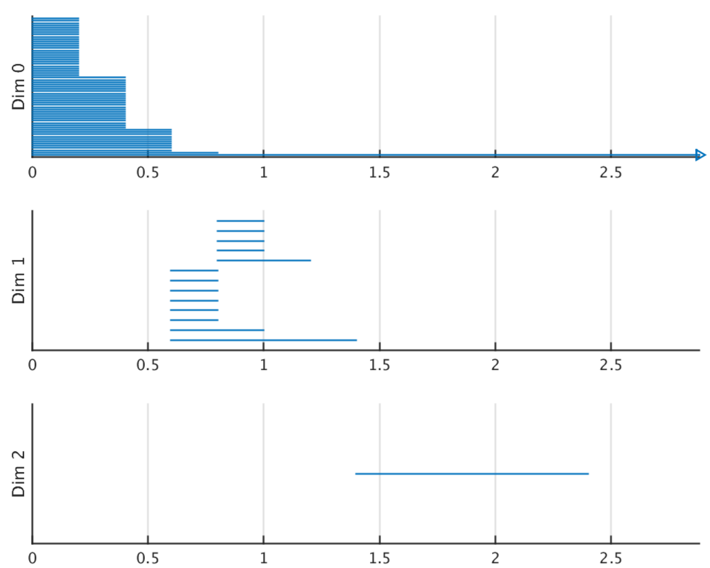


Figure 2.4: Persistent homology for 75 points in $T(S^2)$

In this simulation we get that in dimension 2 we have a class whose persistence is $(1.4, 2.4)$.

The reader can review the extensive results of this calculations on Tables A.1 and A.2. The Matlab code using JavaPlex is available in A.2.2.

2.2 Covariance matrices

It is important to emphasize that in Theorem 2.2 and Corollaries 2.8 and 2.10 we were just using the information provided by the linear transformation T, since its singular value decomposition only give us the information regarding the deformation ratios of T. But those results did not take into account the principal components of the data set X itself. This information can be added to the results using the covariance matrix of X.

Theorem 2.12. *Let $X \subset \mathbb{R}^n$ be a set of points, let $T : \mathbb{R}^n \to \mathbb{R}^m$ be an injective linear transformation,* $\mathrm{Cov}(X)$ *the covariance matrix of X and let $(a, b) \in \mathrm{Diag}_k(X)$ be the persistence of some homological class $\alpha \in \mathrm{H}_k(\mathrm{R}_\epsilon(X), \mathbb{F}[t])$ such that*

$$\frac{b}{a} > \frac{\tilde{\sigma}_{\max}}{\tilde{\sigma}_{\min}}.$$

Let $\tilde{\sigma}_{\max}$ and $\tilde{\sigma}_{\min}$ be the maximal and minimal singular values of $\widetilde{T} := TD^{-1/2}$ and D be the diagonal matrix of principal values of X. Then we can find a persistent homology class $\beta \in \mathrm{H}_k(\mathrm{R}_\delta(TD^{-1/2}Q(X)), \mathbb{F}[t])$ whose persistence $(u, v) \in \mathrm{Diag}_k(TD^{-1/2}Q(X))$ is such that

$$u \leq a\tilde{\sigma}_{\max} \leq b\tilde{\sigma}_{\min} \leq v. \tag{2.5}$$

Proof. First of all we will assume that $\mathrm{Cov}(X)$ is positive definite matrix. If it was not the case some eigenvalues of $\mathrm{Cov}(X)$ will be equal to zero. We can project our data in the principal components corresponding to eigenvalues different from zero. After this dimension reduction our covariance matrix will have eigenvalues all different from zero.

Now, notice that since the covariance matrix is a positive definite symmetric matrix, it can be diagonalized as $\mathrm{Cov}(X) = Q^*DQ$, where Q is an orthogonal matrix and D is diagonal. Thus if we calculate $\mathrm{Cov}(QX) = Q\,\mathrm{Cov}(X)Q^* = QQ^*DQQ^* = D$ it as also diagonal. We can modify our data set X a little more to get a data set with covariance equal to the identity. Consider

$$\begin{aligned}\operatorname{Cov}(D^{-1/2}QX) &= D^{-1/2}Q\operatorname{Cov}(X)(D^{-1/2}Q)^* \\ &= D^{-1/2}Q\operatorname{Cov}(X)Q^*D^{-1/2} = D^{-1/2}DD^{-1/2} = I.\end{aligned}$$

With the linear transformation given by $D^{-1/2}Q$ we capture the principal components into a bijective linear transformation which can be composed with T, into

$$\widetilde{T} := TD^{-1/2}Q : \mathbb{R}^n \supset X \to \mathbb{R}^m.$$

Now we only need to apply Theorem 2.2 to $\widetilde{T}$. Note thtat $\|\widetilde{T}\|^2 = \tilde{\lambda}_{\max}$ where $\tilde{\lambda}_{\max}$ is the greatest eigenvalue of $\tilde{T}^*\tilde{T} = (TD^{-1/2}Q)^*TD^{-1/2}Q = Q^*D^{-1/2}T^*TD^{-1/2}Q$. Since the eigenvalues of a matrix are invariant under change of basis it is enough to calculate the eigenvalues of $D^{-1/2}T^*TD^{-1/2}$. These gives us the singular values of $TD^{-1/2}$, as desired. □

On the previous theorem we exploited the information included in the covariance matrix of X to improve the boundsfor the homologgical persistence of the image $T(X)$. But it is also a difficult problem to compute explicitely the eigenvalues of $D^{-1/2}T^*TD^{-1/2}$ in terms of the eigenvalues of T^*T and the entries of D.

We can also include the covariance information in the following way:

Corollary 2.13. *Let $X \subset \mathbb{R}^n$ be a set of points, let $T : \mathbb{R}^n \to \mathbb{R}^m$ be an injective linear transformation, $\operatorname{Cov}(X)$ the covariance matrix of X, let Δ be the diagonal matrix of principal values of X, δ_1 and δ_n the greatest and smallest entries in Δ respectively, P the matrix of principal components of X and let $(a, b) \in \operatorname{Diag}_k(X)$ be the persistence of some homological class $\alpha \in \mathrm{H}_k(\mathrm{R}_\epsilon(X), \mathbb{F}[t])$ such that*

$$\frac{b}{a} > \frac{\sigma_{\max}\sqrt{\delta_1}}{\sigma_{\min}\sqrt{\delta_n}},$$

with $\sigma_{\max}$ and $\sigma_{\min}$ the maximal and minimal singular values of

$$TP^*\Delta^{1/2}.$$

Then we can find a persistent homology class $\beta \in \mathrm{H}_k(\mathrm{R}_\delta(T(X)), \mathbb{F}[t])$ whose persistence $(u, v) \in \operatorname{Diag}_k(T(X))$ is such that

$$u \leq a\sigma_{\max}\sqrt{\delta_n} \leq b\sigma_{\min}\sqrt{\delta_1} \leq v. \tag{2.6}$$

Proof. As in Theorem 2.12 we can assume that $\mathrm{Cov}(X)$ is a positive definite matrix. Let P be an orthogonal matrix such that $\mathrm{Cov}(X) = P^*\Delta P$ where Δ is a diagonal matrix.

First rewrite $T = TP^*\Delta^{1/2}\Delta^{-1/2}P$ and define $T_1 = TP^*\Delta^{1/2} = \mathbb{R}^n \to \mathbb{R}^m$. Notice that T_1 is injective since P is a change of basis and we have that $\mathrm{Cov}(X)$ is positive definite.

We can apply Theorem 2.2 to $\Delta^{-1/2}P : X \to \mathbb{R}^n$ since the singular values of $\Delta^{-1/2}P$ are the square root of the eigenvalues of $P^*\Delta^{-1/2}\Delta^{-1/2}P = P^*\Delta^{-1}P = \mathrm{Cov}(X)^{-1}$ which are $\lambda_{\max} = 1/\delta_n$ and $\lambda_{\min} = 1/\delta_1$. Therefore

$$\sqrt{\frac{\lambda_{\max}}{\lambda_{\min}}} = \sqrt{\frac{1/\delta_n}{1/\delta_1}} = \sqrt{\frac{\delta_1}{\delta_n}} \leq \frac{\sigma_{\max}\sqrt{\delta_1}}{\sigma_{\min}\sqrt{\delta_n}} < \frac{b}{a}.$$

So there exists an element $(u', v') \in \mathrm{Diag}_k(\Delta^{-1/2}P(X))$ such that

$$u' \leq a\sqrt{\delta_1} < b\sqrt{\delta_n} \leq v'.$$

Now we want to apply Theorem 2.12 to T_1 with $Y := \Delta^{-1/2}PX$. First of all, we need to calculate the covariance of Y. A simple calculatio shows

$$\begin{aligned}\mathrm{Cov}(Y) &= \mathrm{Cov}(\Delta^{-1/2}PX) = \Delta^{-1/2}P\,\mathrm{Cov}(X)(\Delta^{-1/2}P)^* \\ &= \Delta^{-1/2}P\,\mathrm{Cov}(X)P^*\Delta^{-1/2} = \Delta^{-1/2}PP^*\Delta PP^*\Delta^{-1/2} = I,\end{aligned}$$

which is already diagonal. So in Theorem 2.12 we get $D = Q = I$.

We already have an element $(u', v') \in \mathrm{Diag}_k(Y)$ such that

$$\frac{v'}{u'} \geq \frac{b\sqrt{\delta_n}}{a\sqrt{\delta_1}} > \frac{\sigma_{\max}}{\sigma_{\min}}$$

where $\sigma_{\max}$ and $\sigma_{\min}$ are the singular values of $T_1D = T_1$. This last inequality holds by hypothesis.

Notice that the singular values of T_1 are the square roots of the eigenvalues of the operator

$$T_1^*T_1 = (TP^*\Delta^{1/2})^*TP^*\Delta^{1/2} = \Delta^{1/2}PT^*TP^*\Delta^{1/2}.$$

Thus we can conclude that there exists and homology class in $\mathrm{H}_k(\mathrm{R}_\delta(T_1(Y))) = \mathrm{H}_k(\mathrm{R}_\delta(T(X)))$ with persistence (u, v) such that

$$u \leq u'\sigma_{\max} < v'\sigma_{\min} \leq v.$$

Moreover the persistence of such a class also satisfies the inequality

$$u \leq a\sigma_{\max}\sqrt{\delta_1} < b\sigma_{\min}\sqrt{\delta_n} \leq v. \qquad \square$$

Remark. Notice that Corollary 2.13 simplifies if the covariance matrix of X is already diagonal. In this case there is no need to find a diagonalizing matrix P and i suffices to calculate the eigenvalues of

$$\Delta^{1/2}T^*T\Delta^{1/2}.$$

Further more if $\mathrm{Cov}(X)$ is similar to a scalar multiple of the identity, i.e. $\Delta = \alpha I$ for some $\alpha \in \mathbb{R}^+$, then

$$\frac{b}{a} > \frac{\sigma_{\max}\sqrt{\delta_1}}{\sigma_{\min}\sqrt{\delta_n}} = \frac{\sigma_{\max}}{\sigma_{\min}}$$

with $\sigma_{\max}$ and $\sigma_{\min}$ are the singular values of

$$TP^*\Delta^{1/2} = \alpha TP^*.$$

These are calculated as the eigenvalues of $\alpha^2 PT^*TP^*$. Since the spectrum is invariant under a change of basis, it will be enough to calculate the eigenvalues of $\alpha^2 T^*T$. In addition, if λ is an eigenvalue of T^*T then it is easy to see that $\alpha^2\lambda$ is an eigenvalue of $\alpha^2 T^*T$.

Finally notice that for $\alpha = 1$ we have exactly the same result as in Theorem 2.2, so Theorem 2.13 is a generalization of Theorem 2.2.

2.3 Sliding windows

2.3.1 Description

One interesting application of Theorems 2.2, 2.8, 2.10, 2.12 and 2.13 is to the sliding window embedding presented in [PH15].

Definition 2.14. Let f be a real valued function defined on $\mathbb{R}$. Choose $M \in \mathbb{Z}$ and $\tau \in \mathbb{R}$ both positive. We define the **sliding window embedding of** f **based at** $t \in \mathbb{R}$ into $\mathbb{R}^{M+1}$ as

$$SW_{M,\tau}f(t) = \begin{bmatrix} f(t) \\ f(t+\tau) \\ f(t+2\tau) \\ \vdots \\ f(t+M\tau) \end{bmatrix}$$

Remark. If f is a periodic function with period T and Fourier series given by

$$f(x) = \sum_{n=0}^{\infty} a_n \cos(nx) + b_n \sin(nx) = \sum_{n\in\mathbb{Z}} \widehat{f}(n)\, \mathrm{e}^{\mathrm{i}\, nx},$$

where

$$\widehat{f}(n) = \begin{cases} \frac{a_n - \mathrm{i}\, n_b}{2} & n > 0 \\ \frac{a_{-n} + \mathrm{i}\, b_{-n}}{2} & n < 0 \\ a_0 & n = 0 \end{cases}$$

Recall that the coefficients of the Fourier series are $\widehat{f}(n) = \langle f(t), \mathrm{e}^{\mathrm{i}\, nt}\rangle$. If we call $f_{m,\tau}(t) = f(t + m\tau)$, then

$$\begin{aligned}
\widehat{f_{m,\tau}}(n) = \langle f_{m,\tau}(t), \mathrm{e}^{\mathrm{i}\, nt}\rangle &= \int_{t_0}^{t_0+T} f_{m,\tau}(t)\, \mathrm{e}^{-\mathrm{i}\, nt}\, dt \\
&= \int_{t_0}^{t_0+T} f(t + m\tau)\, \mathrm{e}^{-\mathrm{i}\, nt}\, dt = \int_{t_0+m\tau}^{t_0+m\tau+T} f(s)\, \mathrm{e}^{-\mathrm{i}\, n(s-m\tau)}\, ds \\
&= \int_{s_0}^{s_0+T} f(s)\, \mathrm{e}^{-\mathrm{i}\, n(s-m\tau)}\, ds = \mathrm{e}^{\mathrm{i}\, nm\tau} \int_{s_0}^{s_0+T} f(s)\, \mathrm{e}^{-\mathrm{i}\, ns}\, ds \\
&= \mathrm{e}^{\mathrm{i}\, nm\tau} \langle f(s), \mathrm{e}^{\mathrm{i}\, ns}\rangle = \mathrm{e}^{\mathrm{i}\, nm\tau}\, \widehat{f}(n).
\end{aligned}$$

We can use this to get

$$f(t + m\tau) = \sum_{n\in\mathbb{Z}} \widehat{f_{m,\tau}}(n)\, \mathrm{e}^{\mathrm{i}\, nt} = \sum_{n\in\mathbb{Z}} \mathrm{e}^{\mathrm{i}\, nm\tau}\, \widehat{f}(n)\, \mathrm{e}^{\mathrm{i}\, nt}$$

Then we can rewrite the sliding window embedding of f based at t as

$$SW_{M,\tau} f(t) = \begin{bmatrix} f(t) \\ f(t+\tau) \\ f(t+2\tau) \\ \vdots \\ f(t+M\tau) \end{bmatrix} = \begin{bmatrix} \sum_{n\in\mathbb{Z}} \hat{f}(n)\, \mathrm{e}^{\mathrm{i}\, nt} \\ \sum_{n\in\mathbb{Z}} \mathrm{e}^{\mathrm{i}\, n\tau}\, \hat{f}(n)\, \mathrm{e}^{\mathrm{i}\, nt} \\ \sum_{n\in\mathbb{Z}} \mathrm{e}^{\mathrm{i}\, n2\tau}\, \hat{f}(n)\, \mathrm{e}^{\mathrm{i}\, nt} \\ \vdots \\ \sum_{n\in\mathbb{Z}} \mathrm{e}^{\mathrm{i}\, nM\tau}\, \hat{f}(n)\, \mathrm{e}^{\mathrm{i}\, nt} \end{bmatrix}$$

$$= \begin{bmatrix} \cdots & \hat{f}(-1) & \hat{f}(0) & \hat{f}(1) & \cdots \\ \cdots & e^{-i\tau}\,\hat{f}(-1) & \hat{f}(0) & e^{i\tau}\,\hat{f}(1) & \cdots \\ \cdots & e^{-i2\tau}\,\hat{f}(-1) & \hat{f}(0) & e^{i2\tau}\,\hat{f}(1) & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ \cdots & e^{-iM\tau}\,\hat{f}(-1) & \hat{f}(0) & e^{iM\tau}\,\hat{f}(1) & \cdots \end{bmatrix} \begin{bmatrix} \vdots \\ e^{-it} \\ 1 \\ e^{it} \\ \vdots \end{bmatrix} \tag{2.7}$$

$$= \begin{bmatrix} \cdots & 1 & 1 & 1 & \cdots \\ \cdots & e^{-i\tau} & 1 & e^{i\tau} & \cdots \\ \cdots & e^{-i2\tau} & 1 & e^{i2\tau} & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ \cdots & e^{-iM\tau} & 1 & e^{iM\tau} & \cdots \end{bmatrix} \begin{bmatrix} \vdots \\ \hat{f}(-1)\,e^{-it} \\ \hat{f}(0) \\ \hat{f}(1)\,e^{it} \\ \vdots \end{bmatrix}. \tag{2.8}$$

If we consider the truncation of the Fourier series of f at the Nth term we obtain a slight modification of (2.7):

$$SW_{M,\tau}S_N f(t) =$$

$$\begin{bmatrix} \hat{f}(-N) & \cdots & \hat{f}(-1) & \hat{f}(0) & \hat{f}(1) & \cdots & \hat{f}(N) \\ e^{-iN\tau}\,\hat{f}(-N) & \cdots & e^{-i\tau}\,\hat{f}(-1) & \hat{f}(0) & e^{i\tau}\,\hat{f}(1) & \cdots & e^{iN\tau}\,\hat{f}(N) \\ e^{-iN2\tau}\,\hat{f}(-N) & \cdots & e^{-i2\tau}\,\hat{f}(-1) & \hat{f}(0) & e^{i2\tau}\,\hat{f}(1) & \cdots & e^{iN2\tau}\,\hat{f}(N) \\ \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ e^{-iNM\tau}\,\hat{f}(-N) & \cdots & e^{-iM\tau}\,\hat{f}(-1) & \hat{f}(0) & e^{iM\tau}\,\hat{f}(1) & \cdots & e^{iNM\tau}\,\hat{f}(N) \end{bmatrix} \begin{bmatrix} e^{-iNt} \\ \vdots \\ e^{-it} \\ 1 \\ e^{it} \\ \vdots \\ e^{iNt} \end{bmatrix}. \tag{2.9}$$

Definition 2.15. We can think of (2.9) as a linear transformation

$$T : \mathbb{C}^{2N+1} = \mathbb{R}^{4N+2} \supset X \to \mathbb{R}^M,$$

where $X = \left\{ \left[e^{-iNt}, \cdots, e^{-it}, 1, e^{it}, \cdots, e^{iNt} \right] \mid t \in [0, 2\pi] \right\}$.

With this definition we can apply Corollary 2.8, thus we need to calculate the singular values for T. We first need to find T^*T. To do so define each column of T as

$$T_j := \begin{bmatrix} \widehat{f}(j) \\ e^{ij\tau}\,\widehat{f}(j) \\ e^{ij2\tau}\,\widehat{f}(j) \\ \vdots \\ e^{ijM\tau}\,\widehat{f}(j) \end{bmatrix}.$$

First of all notice that $\widehat{f}(j)^* = \widehat{f}(-j)$, so

$$T_j^* = \begin{bmatrix} \widehat{f}(-j) \\ \mathrm{e}^{-\mathrm{i}\, j\tau}\, \widehat{f}(-j) \\ \mathrm{e}^{-\mathrm{i}\, j2\tau}\, \widehat{f}(-j) \\ \vdots \\ \mathrm{e}^{-\mathrm{i}\, jM\tau}\, \widehat{f}(-j) \end{bmatrix} = T_{-j},$$

and therefore we get that

$$[T^*T]_{kl} = T_k^* \cdot T_l = T_{-k} \cdot T_l = \sum_{s=0}^{M} \mathrm{e}^{i(-k+l)s\tau}\, \widehat{f}(-k)\widehat{f}(l).$$

Now we can calculate

$$\begin{aligned} \operatorname{tr}(T^*T) &= \sum_{k=-N}^{N} [T^*T]_{kk} = \sum_{k=-N}^{N} \sum_{s=0}^{M} \mathrm{e}^{i(-k+k)s\tau}\, \widehat{f}(-k)\widehat{f}(k) \\ &= \sum_{k=-N}^{N} \sum_{s=0}^{M} \widehat{f}(k)^* \widehat{f}(k) = \sum_{s=0}^{M} \sum_{k=-N}^{N} \|\widehat{f}(k)\|^2 \\ &= \sum_{s=0}^{M} \|S_N f(t)\|_2^2 = (M+1)\|S_N f(t)\|_2^2. \end{aligned}$$

Moreover

$$\begin{aligned} &\operatorname{tr}((T^*T)^2) = \sum_{k=-N}^{N} \left[(T^*T)^2\right]_{kk} = \sum_{k=-N}^{N} \sum_{l=-N}^{N} [T^*T]_{kl}\, [T^*T]_{lk} \\ &= \sum_{k,l=-N}^{N} \left(\sum_{s=0}^{M} \mathrm{e}^{\mathrm{i}(-k+l)s\tau}\, \hat{f}(-k)\hat{f}(l) \right) \left(\sum_{s=0}^{M} \mathrm{e}^{\mathrm{i}(-l+k)s\tau}\, \hat{f}(-l)\hat{f}(k) \right) \\ &= \sum_{k,l=-N}^{N} \|\hat{f}(k)\|^2 \|\hat{f}(l)\|^2 \left(\sum_{s=0}^{M} \mathrm{e}^{\mathrm{i}(-k+l)s\tau} \right) \left(\sum_{s=0}^{M} \mathrm{e}^{\mathrm{i}(-l+k)s\tau} \right) \\ &= \sum_{\substack{k,l=-N \\ k\neq l}}^{N} \|\hat{f}(k)\|^2 \|\hat{f}(l)\|^2 \left(\sum_{s=0}^{M} \mathrm{e}^{\mathrm{i}(-k+l)s\tau} \right) \left(\sum_{s=0}^{M} \mathrm{e}^{\mathrm{i}(-l+k)s\tau} \right) \\ &\quad + \sum_{k=-N}^{N} \|\hat{f}(k)\|^2 \|\hat{f}(k)\|^2 (M+1)^2 \end{aligned}$$

$$
\begin{aligned}
&= \sum_{\substack{k,l=-N\\k\neq l}}^{N} \|\hat{f}(k)\|^2\|\hat{f}(l)\|^2 \left(\frac{1-\mathrm{e}^{\mathrm{i}(-k+l)\tau(M+1)}}{1-\mathrm{e}^{\mathrm{i}(-k+l)\tau}}\right)\left(\frac{1-\mathrm{e}^{-\,\mathrm{i}(-k+l)\tau(M+1)}}{1-\mathrm{e}^{-\,\mathrm{i}(-k+l)\tau}}\right)\\
&\quad+\sum_{k=-N}^{N} \|\hat{f}(k)\|^2\|\hat{f}(k)\|^2(M+1)^2\\
&= \sum_{\substack{k,l=-N\\k\neq l}}^{N} \|\hat{f}(k)\|^2\|\hat{f}(l)\|^2 \left(\frac{1-\Re(\mathrm{e}^{\mathrm{i}(-k+l)\tau(M+1)})}{1-\Re(\mathrm{e}^{\mathrm{i}(-k+l)\tau})}\right)\\
&\quad+\sum_{k=-N}^{N} \|\hat{f}(k)\|^2\|\hat{f}(k)\|^2(M+1)^2\\
&= \sum_{\substack{k,l=-N\\k\neq l}}^{N} \|\hat{f}(k)\|^2\|\hat{f}(l)\|^2 \left(\frac{1-\cos((-k+l)\tau(M+1))}{1-\cos((-k+l)\tau)}\right)\\
&\quad+\sum_{k=-N}^{N} \|\hat{f}(k)\|^2\|\hat{f}(k)\|^2(M+1)^2\\
&= \sum_{\substack{k,l=-N\\k\neq l}}^{N} \|\hat{f}(k)\|^2\|\hat{f}(l)\|^2 (M+1)F_{M+1}((-k+l)\tau)\\
&\quad+\sum_{k=-N}^{N} \|\hat{f}(k)\|^2\|\hat{f}(k)\|^2(M+1)^2\\
&= \sum_{k,l=-N}^{N} \|\hat{f}(k)\|^2\|\hat{f}(l)\|^2 (M+1)F_{M+1}((-k+l)\tau).
\end{aligned}
$$

Using this calculations we find that

$$
m = \frac{M+1}{2N+1}\|S_N f(t)\|_2^2
$$

and

$$
s^2 = \frac{M+1}{2N+1}\sum_{k,l=-N}^{N} \|\hat{f}(k)\|^2\|\hat{f}(l)\|^2 F_{M+1}((-k+l)\tau) - \left(\frac{M+1}{2N+1}\right)^2 \|S_N f(t)\|_2^4.
$$

Notice that using these two calculations and Theorems 2.8 and 2.10 we obtain bounds for the persistence of the sliding window of $S_N f(t)$ in terms of the Fourier coefficients of f and the Fejer kernel.

An interesting case appears when we take $M = 2N$ as suggested by [PH15]. This assumption simplifies our expressions for m and s^2 in the following way

$$m = \|S_N f(t)\|_2^2$$

and

$$s^2 = \sum_{k,l=-N}^{N} \|\hat{f}(k)\|^2 \|\hat{f}(l)\|^2 F_{2N+1}((-k+l)\tau) - \|S_N f(t)\|_2^4.$$

Since the bounds here obtained are for $SW_{M,\tau} S_n f$ we need some kind of result in stability for persistent homology. For example, [PH15] show a useful result which allow us to understand the sliding windows point cloud of a periodic function $f \in C^k(\mathbb{T}, \mathbb{R})$ in terms of the sliding window of its truncated Fourier series.

Definition 2.16. Let $X \subset \mathbb{R}^n$. The **maximum persistence** of the diagram of X at dimension k is

$$\text{mp}(\text{Diag}_k(X)) = \max \{y - x \mid (x, y) \in \text{Diag}_k(X)\}.$$

Theorem 2.17 (Approximation)**.** *Let $T \subset \mathbb{T} = \mathbb{R}/2\pi\mathbb{Z}$, $f \in C^k(\mathbb{T}, \mathbb{R})$ and $X = SW_{M,\tau} f(T)$, $Y = SW_{M,\tau} S_N f(T)$. Then*

(i) $$d_H(X, Y) \leq \sqrt{4k-2} \left\| R_N f^{(k)} \right\|_2 \frac{\sqrt{M+1}}{(N+1)^{k-1/2}},$$

(ii) $$d_B(\text{Diag}_*(X), \text{Diag}_*(Y)) \leq 2\sqrt{4k-2} \left\| R_N f^{(k)} \right\|_2 \frac{\sqrt{M+1}}{(N+1)^{k-1/2}},$$

(iii) $$|\text{mp}(\text{Diag}_*(X)) - \text{mp}(\text{Diag}_*(Y))| \leq 2 d_B(\text{Diag}_*(X), \text{Diag}_*(Y)).$$

Proof. See Theorem 4.5 in [PH15]. □

From (iii) in the approximation theorem we get that

$$-2 d_B(\text{Diag}_*(X), \text{Diag}_*(Y)) \leq \text{mp}(\text{Diag}_*(X)) - \text{mp}(\text{Diag}_*(Y)).$$

By (ii) we have

$$\begin{aligned}\mathrm{mp}(\mathrm{Diag}_*(X)) &\geq \mathrm{mp}(\mathrm{Diag}_*(Y)) - 2d_B(\mathrm{Diag}_*(X), \mathrm{Diag}_*(Y)) \\ &\geq \mathrm{mp}(\mathrm{Diag}_*(Y)) - 4\sqrt{4k-2}\left\|R_N f^{(k)}\right\|_2 \frac{\sqrt{M+1}}{(N+1)^{k-1/2}}.\end{aligned}$$

We can make this lower bound a little more explicit using Theorem 2.2 for any long enough persistent homology class. First take T and X as defined in Definition 2.15. If $(a,b) \in \mathrm{Diag}_*(X)$ and $b/a \geq \sigma_{\max}/\sigma_{\min}$ for the $\sigma_{\max}$ and $\sigma_{\min}$ the singular values of T, we get that there exists a homology class with persistence $(u,v) \in \mathrm{Diag}_*(T(X)) = \mathrm{Diag}_*(SW_{M,\tau}S_N f(t)) = \mathrm{Diag}_*(Y)$ such that

$$u \leq a\sigma_{\max} < b\sigma_{\min} \leq v.$$

This implies that $\mathrm{mp}(\mathrm{Diag}_*(Y)) \geq b\sigma_{\min} - a\sigma_{\max}$. Therefore

$$\begin{aligned}\mathrm{mp}(\mathrm{Diag}_*(X)) &\geq \mathrm{mp}(\mathrm{Diag}_*(Y)) - 4\sqrt{4k-2}\left\|R_N f^{(k)}\right\|_2 \frac{\sqrt{M+1}}{(N+1)^{k-1/2}} \\ &\geq (b\sigma_{\min} - a\sigma_{\max}) - 4\sqrt{4k-2}\left\|R_N f^{(k)}\right\|_2 \frac{\sqrt{M+1}}{(N+1)^{k-1/2}}.\end{aligned}$$

We obtain similar lower bounds using Theorems 2.8 and 2.10 if their hypothesis are satisfied.

$$\begin{aligned}\mathrm{mp}(\mathrm{Diag}_*(X)) &\geq \mathrm{mp}(\mathrm{Diag}_*(Y)) - 4\sqrt{4k-2}\left\|R_N f^{(k)}\right\|_2 \frac{\sqrt{M+1}}{(N+1)^{k-1/2}} \\ &\geq b\sqrt{m - s(n-1)^{1/2}} - a\sqrt{m + s(n-1)^{1/2}} \\ &\quad - 4\sqrt{4k-2}\left\|R_N f^{(k)}\right\|_2 \frac{\sqrt{M+1}}{(N+1)^{k-1/2}}.\end{aligned}$$

or

$$\begin{aligned}\mathrm{mp}(\mathrm{Diag}_*(X)) &\geq (b\sigma_{\min} - a\sigma_{\max}) - 4\sqrt{4k-2}\left\|R_N f^{(k)}\right\|_2 \frac{\sqrt{M+1}}{(N+1)^{k-1/2}} \\ &\geq \frac{b\sqrt{m + \frac{s}{(n-1)^{1/2}}}}{\sqrt{1 + \frac{(2n)^{1/2}s}{m - s(n-1)^{1/2}}}} - a\sqrt{m - \frac{s}{(n-1)^{1/2}}}\sqrt{1 + \frac{(2n)^{1/2}s}{m - s(n-1)^{1/2}}} \\ &\quad - 4\sqrt{4k-2}\left\|R_N f^{(k)}\right\|_2 \frac{\sqrt{M+1}}{(N+1)^{k-1/2}}.\end{aligned}$$

Since Theorem 2.13 is a generalization of Theorem 2.2 we would also like to apply this result to T and X as in Definition 2.15. To begin we need to calculate the covariance matrix for

$$X = \left\{ \left[\mathrm{e}^{-\mathrm{i}\,Nt}, \cdots, \mathrm{e}^{-\mathrm{i}\,t}, 1, \mathrm{e}^{\mathrm{i}\,t}, \cdots, \mathrm{e}^{\mathrm{i}\,Nt} \right] \mid t \in [0, 2\pi] \right\}.$$

To be able to do this we first need to calculate the expected value of $\mathrm{e}^{\mathrm{i}\,nt}$ for $-N \leq n \leq N$.

Definition 2.18. Given a random variable X, its **characteristic function** is

$$\phi_X(s) = \mathrm{E}\left[\mathrm{e}^{\mathrm{i}\,sX}\right].$$

From now own we will assume that t has an uniform distribution on the interval $(0, 2\pi)$ i.e. $t \sim U(0, 2\pi)$. With this assumption we get that the expected value for $\mathrm{e}^{\mathrm{i}\,nt}$ can be calculated for $n \neq 0$ using the characteristic function for the uniform distribution, so

$$\mathrm{E}\left[\mathrm{e}^{\mathrm{i}\,nt}\right] = \phi_t(n) = \frac{\mathrm{e}^{\imath n 2\pi} - \mathrm{e}^{\imath n 0}}{in(2\pi - 0)} = 0.$$

For $n = 0$ we get $\mathrm{E}[1] = 1$.

Now we can calculate the covariance, recall that it is defined for complex random variables as $\mathrm{Cov}[X, Y] := \mathrm{E}\left[(X - \mathrm{E}\,X)\overline{(Y - \mathrm{E}\,Y)}\right]$. In our particular case we get

$$\mathrm{Cov}\left[\mathrm{e}^{\imath nt}, \mathrm{e}^{\imath mt}\right] = \mathrm{E}\left[\mathrm{e}^{\imath nt}\,\mathrm{e}^{-\imath mt}\right] = \mathrm{E}\left[\mathrm{e}^{\mathrm{i}(n-m)t}\right] = \begin{cases} 0 & n \neq m \\ 1 & n = m \end{cases}.$$

So $\mathrm{Cov}(X) = I$.

In this case Theorem 2.13 is the same as Theorem 2.2. So this covariance matrix did not add any new information to the one we have already calculated.

2.3.2 Computations

The results of the previous section are useful if we are able to calculate the persistent homology of X. Recall from Definition 2.15 that X is the curve in $\mathbb{T}^{2N+1} \subset \mathbb{R}^{4N+2}$ with parametrization

$$r(t) = \begin{bmatrix} e^{-\mathrm{i}\,Nt} \\ \vdots \\ e^{-\mathrm{i}\,t} \\ 1 \\ e^{\mathrm{i}\,t} \\ \vdots \\ e^{\mathrm{i}\,Nt} \end{bmatrix}$$

with $t \in \mathbb{T}$. To calculate its persistent homology first notice that this curve is quasi-isometric to the curve given by

$$r_N(t) = \begin{bmatrix} 1 \\ e^{\mathrm{i}\,t} \\ \vdots \\ e^{\mathrm{i}\,Nt}. \end{bmatrix}$$

The quasi-isometry is given by the projection over the last $N+1$ components of $r(t)$.

Definition 2.19. Given two metric spaces (X, d_X), (Y, d_Y) a function $f : X \to Y$ is called **bi-Lipschitz** if there exists a constant $K \geq 1$ such that for all x_1 and x_2 in X we have:

$$\frac{1}{K} d_X(x_1, x_2) \leq d_y(f(x_1), f(x_2)) \leq K d_X(x_1, x_2).$$

Proposition 2.20. *The projection $p : r(T) \to r_N(T)$ over the last $N+1$ components of $r(T)$ with $T = [0, 2\pi]$ is a bi-Lipschitz map.*

Proof. First notice that

$$\begin{aligned} d(r(t), r(s)) &= \sqrt{|e^{-\mathrm{i}\,Nt} - e^{-\mathrm{i}\,Ns}|^2 + \cdots + |e^{\mathrm{i}\,Nt} - e^{\mathrm{i}\,Ns}|^2} \\ &= \sqrt{2\,|e^{\mathrm{i}\,t} - e^{\mathrm{i}\,s}|^2 + \cdots + 2\,|e^{\mathrm{i}\,Nt} - e^{\mathrm{i}\,Ns}|^2} \\ &= \sqrt{2}\sqrt{|e^{\mathrm{i}\,t} - e^{\mathrm{i}\,s}|^2 + \cdots + |e^{\mathrm{i}\,Nt} - e^{\mathrm{i}\,Ns}|^2}. \end{aligned}$$

On the other hand, we have

Then we have

$$\frac{1}{\sqrt{2}}d(r(t),r(s)) \leq d(p(r(t)),p(r(s))) = \frac{1}{\sqrt{2}}d(r(t),r(s)) < \frac{2}{\sqrt{2}}d(r(t),r(s))$$

so

$$\frac{1}{\sqrt{2}}d(r(t),r(s)) \leq d(p(r(t)),p(r(s))) \leq \sqrt{2}d(r(t),r(s)). \qquad \square$$

In particular this last proposition shows that $r(t)$ and $r_1(t)$ are homeomorphic, so they have the same homotopy type. We have reduced the problem of calculating the persistent homology of $r(t)$ to calculate the persistent homology of $r_N(t)$. Notice that also the constant 1 in the first component of $r_1(t)$ can be dropped for the persistent homology calculations since this "translation" does not affect the homotopy type of $r_N(t)$.

We will present some analytic calculations that are possible for low dimensions, but the problem of determining the persistent homology of $r_N(t)$ for a general N is a pretty hard problem to tackle. We made some computational simulations using the package JavaPlex for Matlab. For all the following Matlab results we use randomly generated values of $t \sim U(0, 2\pi)$.

Using the calculations made in Proposition 2.20 we can define a distance function $f(t) := d(r_N(0), r_N(t))$ for each curve $r_N(t)$. It is interesting and beautiful to see how this distance function behaves. In Figure 2.5 we show the graphs corresponding to the distance functions for some values of N.

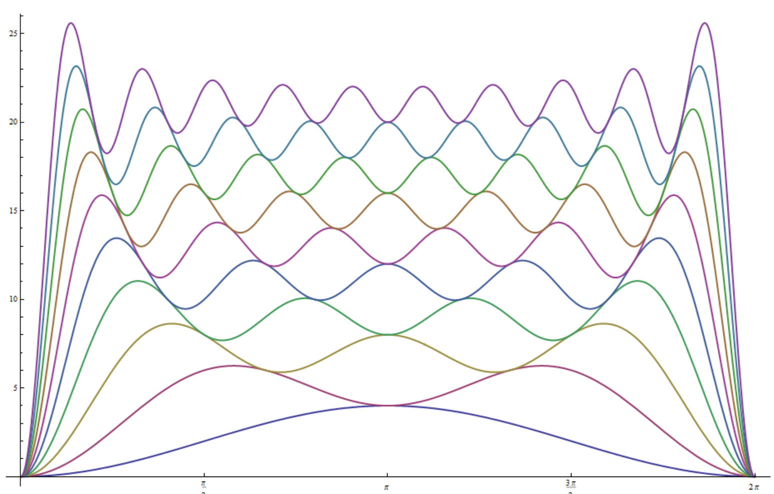
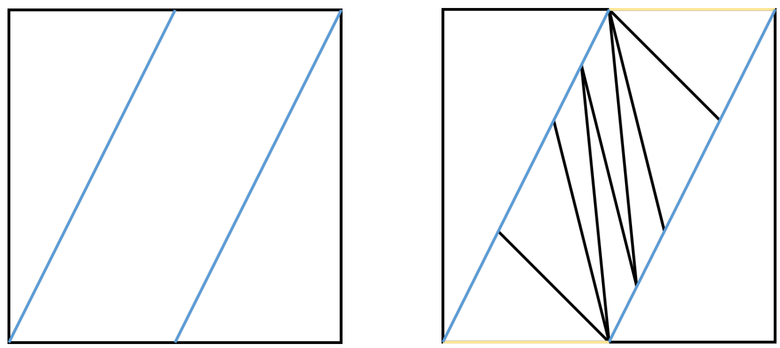


Figure 2.5: Distance functions for $r_N(t)$ with $1 \leq N \leq 10$.

Let us begin with $N = 2$. In this case we are looking at the persistent homology of the curve

$$r_2(t) = (e^{i\,t}, e^{i\,2t}) \subset \mathbb{T}^2.$$

This curve on the 2-dimensional torus can be represented in the square representation of a torus as follows:



(a) Curve $r_2(t)$ (b) Curve $r_2(t)$ as boundary

Figure 2.6: $r_2(t) = (e^{i\,t}, e^{i\,2t})$ in $\mathbb{T}^2$

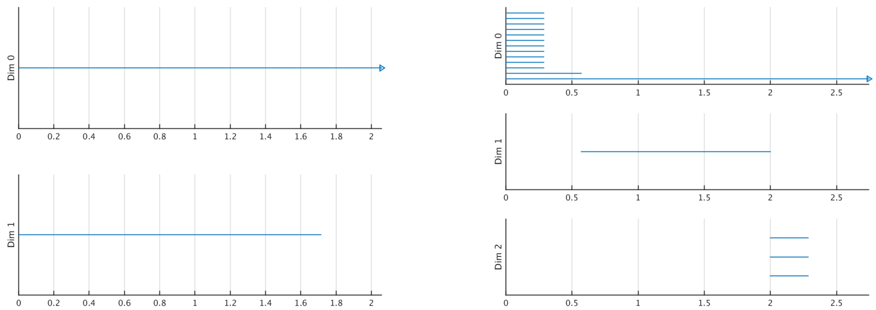
On the Figure 2.6a we have a representation of the curve $r_2(t) = (e^{i\,t}, e^{i\,2t})$ on $\mathbb{T}^2$. This curve forms a 1-cycle in the persistent homology of $r_1(t)$ with birth time equal to zero. We can find a 2-chain with vertices on the points of the curve given by $t \in \{0, \pi/3, 2\pi/3, 5\pi/6, \pi, 7\pi/6, 4\pi/3, 5\pi/3\}$ (see Figure 2.6b), such that its boundary is $r_2(t) \cup S^1$, where S^1 is one of the canonical circles in $\mathbb{T}^2 = S^1 \times S^1$.

Using the fact that the distance between points in $r_2(t)$ can be calculated by the formula

$$\begin{aligned} d(r_2(t), r_2(s)) &= \sqrt{|e^{i\,t} - e^{i\,s}|^2 + |e^{i\,2t} - e^{i\,2s}|^2} \\ &= \sqrt{\left|1 - e^{i(s-t)}\right|^2 + \left|1 - e^{i\,2(s-t)}\right|^2} \\ &= \sqrt{4 - 2\cos(s-t) - 2\cos(2(s-t))} \end{aligned}$$

we easily see that the 2-simplices observed in Figure 2.6b have diameter always less or equal to 2. So we have constructed a 2-chain in $\mathrm{R}_2(r_2(T))$ with boundary $r_2(t) \cup S^1$, and we must have that the persistent 1-cycle represented by $r_2(t)$ must die before $\mathrm{R}_2(r_2(T))$.

This calculation can be computationally verified as shown in Figure 2.7

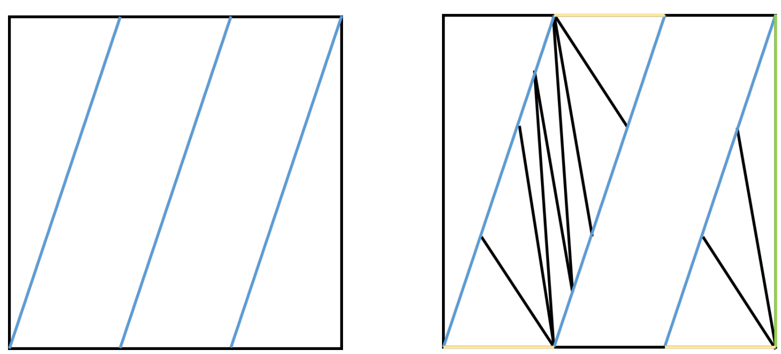


(a) Persis. of $r_1(t) \subset \mathbb{R}^2$ (b) Persis. of $r_2(t) \in \mathbb{R}^4$

Figure 2.7: $r_1(t)$ and $r_2(t)$

Figures 2.7a and 2.7b are the barcodes corresponding to the homological persistence of $r_1(t) \subset \mathbb{R}^2$ and $r_2(t) \subset \mathbb{R}^4$ calculated using JavaPlex on Matlab. Here we see that in fact the is a 1-cycle in $\mathrm{H}_1(\mathrm{R}_\epsilon(r_1(t)))$ which dies at $\epsilon = 2$ as our calculations showed.

We could try to make a similar calculation for $N = 3$ and its corresponding curve $r_3(t) \subset \mathbb{R}^6$, but in this case we cannot represent $\mathbb{T}^3$ as easy as $\mathbb{T}^2$. We try to approximate the desired result working with all the orthogonal projections of $\mathbb{T}^3$ onto $\mathbb{T}^2$. To do so we need to study the following curves: $r_1(t)$, $s_1(T) := (\mathrm{e}^{\mathrm{i}\,t}, \mathrm{e}^{\mathrm{i}\,3t})$ and $s_2(T) := (\mathrm{e}^{\mathrm{i}\,2t}, \mathrm{e}^{\mathrm{i}\,3t})$ on $\mathbb{R}^4$.



(a) Curve $s_1(t)$ (b) Curve $s_1(t)$ as boundary

Figure 2.8: $s_1(t) = (\mathrm{e}^{\mathrm{i}\,t}, \mathrm{e}^{\mathrm{i}\,3t})$ in $\mathbb{T}^2$

We begin with a similar, but a little more difficult calculation for the curve $s_1(T) = (\mathrm{e}^{\mathrm{i}\,t}, \mathrm{e}^{\mathrm{i}\,3t})$ which is illustrated in the Figures 2.8a and 2.8b. In the first one we have a representation of such a curve in $\mathbb{T}^2$ and in the second one we have the representation of a 2-chain with boundary $s(t) \cup S^1 \cup S^1$.

The points used to construct the 2-chain are given by taking

$$t \in \left\{0, \frac{2\pi}{9}, \frac{4\pi}{9}, \frac{6\pi}{9}, \frac{8\pi}{9}, \frac{10\pi}{9}, \frac{12\pi}{9}, \frac{14\pi}{9}, \frac{16\pi}{9}, \frac{10\pi}{18}, \frac{14\pi}{18}\right\}.$$

Using this points and the distance function calculated as

$$d(s_1(t), s_1(s)) = \sqrt{4 - 2\cos(s-t) - 2\cos(3(s-t))}$$

we get that the persistence of $s_1(t)$ must die before or at 2.

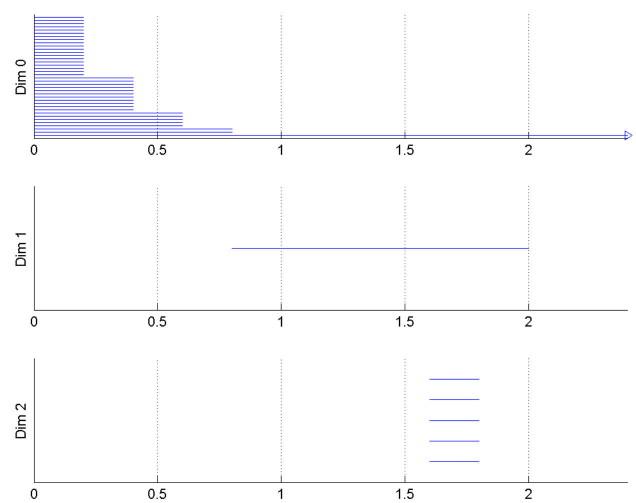


Figure 2.9: Persistence of $s_1(t) \subset \mathbb{R}^4$.

This result can be verified using computational aids such as JavaPlex on Matlab. We found that the persistent 1-dimensional class death time is 2, as seen in the Figure 2.9. Notice that our calculation is a pretty accurate estimation of the computational result.

Furthermore, if we consider now the curve $s_2(T) := (\mathrm{e}^{\mathrm{i}\,2t}, \mathrm{e}^{\mathrm{i}\,3t}) \subset \mathbb{T}^2$ we can make a similar argument using the distance function between points on

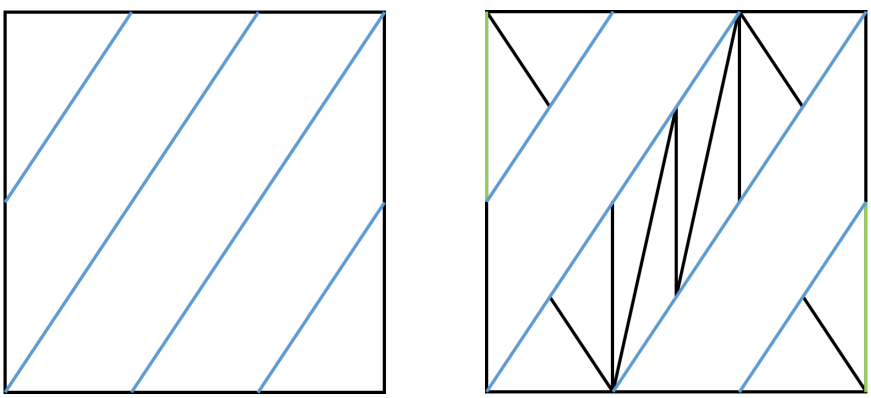
$s_2(t)$ given by

$$d(s_2(t), s_2(s)) = \sqrt{4 - 2\cos(2(s-t)) - 2\cos(3(s-t))}$$

and constructing a 2-chain with boundary $s_2(t) \cup S^1$. This 2-dimensional chain is constructed choosing the points corresponding to

$$t \in \left\{0, \frac{2\pi}{12}, \frac{4\pi}{12}, \frac{6\pi}{12}, \frac{8\pi}{12}, \frac{10\pi}{12}, \frac{12\pi}{12}, \frac{14\pi}{12}, \frac{16\pi}{12}, \frac{18\pi}{12}, \frac{20\pi}{12}, \frac{22\pi}{12}\right\}$$

and considering the 2-simplices shown in Figure 2.10b. Since the maximum diameter of those 2-simplices is 2, we should get that the persistence of the curve $s_2(t)$ must die at or before 2.



(a) Curve $s_2(t)$ (b) Curve $s_2(t)$ as boundary

Figure 2.10: $s_2(t) = (e^{i\,2t}, e^{i\,3t})$ in $\mathbb{T}^2$

This assertion was verified using the JavaPlex package for Matlab. After running the corresponding simulation we get that the persistence of $s_2(t)$ dies exactly at 1.8. This computational calculation confirms that our analytic result is an adequate estimation for the persistence. In Figure 2.11 we present the persistent barcode for the curve $s_2(t)$.

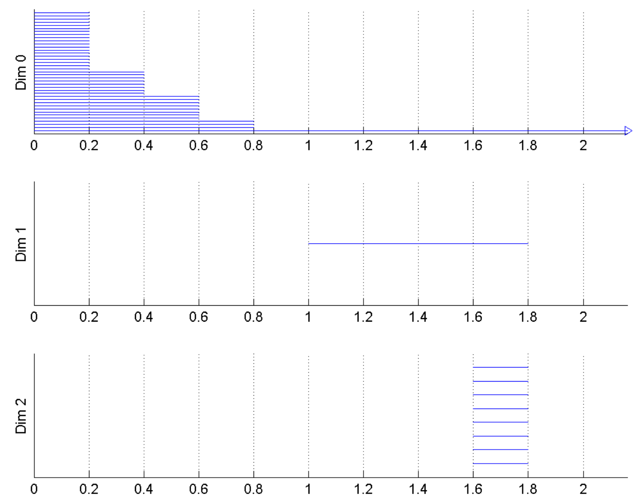
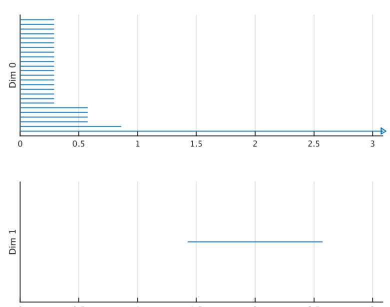
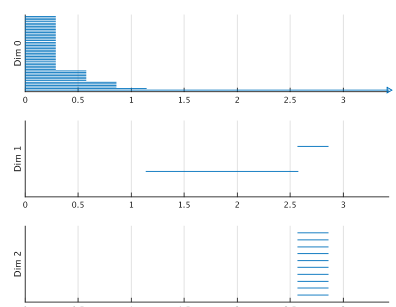


Figure 2.11: Persistence of $s_2(t) \subset \mathbb{R}^4$.

Based on the persistence obtained for the projected curves $r_1(t)$, $s_1(t)$ and $s_2(t)$ we would expect that the persistence for $r_3(t) \subset \mathbb{T}^3$ was not alive after 2. But the simulation conducted showed us that the death time for the 1-cycle was bigger than 2, in fact we obtain that this death time is approximately 2.65 as it is seen on Figure 2.12a.



(a) Persis. of $r_3(t) \subset \mathbb{R}^6$



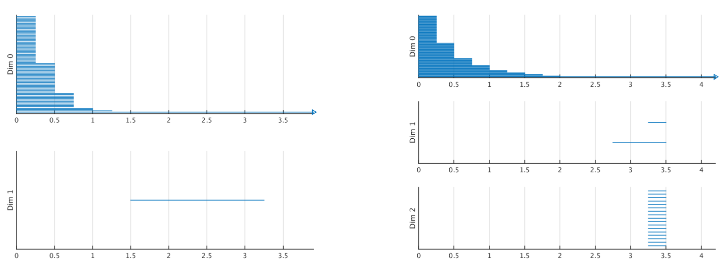
(b) Persis. of $r_4(t) \subset \mathbb{R}^8$

Figure 2.12: $r_3(t)$ and $r_4(t)$

These results show us that there must be a 1-cycle in $r_3(t)$ which cannot be the boundary of any 2-cycle in $\mathrm{H}_2(\mathrm{R}_\delta(r_3(t)))$ for $\delta \leq 2$ and this cycle cannot be easily detected with the techniques described above for $r_1(t)$, $s_1(t)$ and $s_2(t)$.

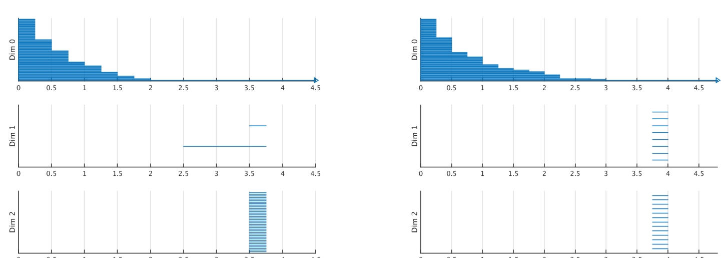
An interesting future problem is the search for adequate analytic techniques that allow us to correctly calculate or even estimate the death time of at least the 1-cycles.

Figures 2.13, 2.14 and 2.15 show the calculations for the persistent homology of the curves $r_N(t) \subset \mathbb{R}^{2N}$ for $5 \leq N \leq 10$. It is interesting to notice how the death time for the longest 1-cycle begins to move forward as the dimension of the curve grows. We still need to find a theoretical explanation for this phenomenon.



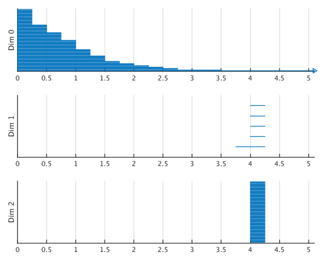
(a) Persis. of $r_5(t) \subset \mathbb{R}^{10}$ (b) Persis. of $r_6(t) \subset \mathbb{R}^{12}$

Figure 2.13: $r_5(t)$ and $r_6(t)$

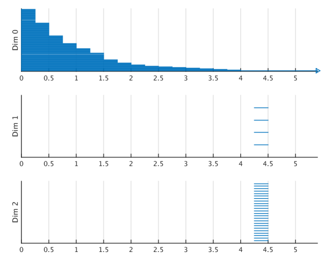


(a) Persis. of $r_7(t) \subset \mathbb{R}^{14}$ (b) Persis. of $r_8(t) \subset \mathbb{R}^{16}$

Figure 2.14: $r_7(t)$ and $r_8(t)$



(a) Persis. of $r_9(t) \subset \mathbb{R}^{18}$



(b) Persis. of $r_{10}(t) \subset \mathbb{R}^{20}$

Figure 2.15: $r_9(t)$ and $r_{10}(t)$

It is of primordial importance to notice that in the previous discussion we were focusing our analysis on the death time of the 1-dimensional persistent homology of the curve defined by $r_N(t)$. We are able to "ignore" the birth time of such homology class since $r_n(t)$ is an embedding of a circle S^1 into a N-dimensional torus $\mathbb{T}^N \subset \mathbb{R}^{2N}$. Consequently the birth time for the largest 1-dimensional persistence homology must be at 0.

The reader should be asking himself "Why is in the previous figures the birth time of the persistence of the 1-cycle of $r_N(t)$ not 0?". The answer is pretty easy and disappointing at he same time. It is impossible for us to tell the computer to work with a continuous object. The best we can do to tackle this problem is to sample randomly "enough" points on the curve.

But how many points are enough? In contrast to previous question this one is pretty interesting, but unresolved. One thing is certain: the bigger the dimension we are working in, the more points we need. This affirmation can easily be checked just by looking at the Figures 2.7a and Figure 2.7b, there we see how for a circle it is enough to take 120 points to make the 1-dimensional cycle to originate at 0. But for $r_2(t) = (e^{\imath t}, e^{\imath 2t})$ the same amount of points is not enough to achieve such feat. Certainly this gap began to increase as the dimensions steadily grew, as can be observed in Figures 2.13, 2.14 and 2.15.

We were not able to take more points on the curve due to the computational hardware restrictions at the moment of the calculations. Even with a Java Heap Memory in Matlab of 16GB, calculations would not be able to finish.

Chapter 3

Tools to study the Vietoris-Rips complex.

3.1 Mayer-Vietoris sequence

Let us begin this chapter with some basic notions in algebraic topology.

Theorem 3.1 (Simplicial Mayer-Vietoris)**.** *Let K be a complex, L and M be sub-complexes such that $K = L \cup M$. Then there is an exact sequence*

$$\dots \longrightarrow \mathrm{H}_p(L \cap M) \longrightarrow \mathrm{H}_p(L) \oplus \mathrm{H}_p(M) \longrightarrow \mathrm{H}_p(K) \longrightarrow$$

$$\mathrm{H}_{p-1}(L \cap M) \longrightarrow \mathrm{H}_{p-1}(L) \oplus \mathrm{H}_{p-1}(M) \longrightarrow \mathrm{H}_{p-1}(K) \longrightarrow \dots$$

Proof. See Theorem 25.1 on [Mun84]. □

We would like to apply the Mayer-Vietoris sequence to the Rips complex of a set $X \subset \mathbb{R}^n$ when a covering of X is given. But you have to be careful when considering how the cover of the set induces a cover on the complex.

Proposition 3.2. *Let $X \subset \mathbb{R}^n$ and $X = U \cup V$ for some open sets U and V. Then for any $\epsilon > 0$*

(i) $$\mathrm{R}_\epsilon(U \cap V) = \mathrm{R}_\epsilon(U) \cap \mathrm{R}_\epsilon(V),$$

(ii) $$\mathrm{R}_\epsilon(X) \supset \mathrm{R}_\epsilon(U) \cup \mathrm{R}_\epsilon(V).$$

Proof. To prove (i) first consider a k-simplex $\sigma \in \mathrm{R}_\epsilon(U \cap V)$. (i) is true if and only if $\sigma = [x_0, \dots, k_k]$ such that $x_i \in U \cap V$ and $d(x_i, x_j) \leq \epsilon$ for $0 \leq i, j \leq k$. This condition is equivalent to $x_i \in U$ and $x_i \in V$ and $d(x_i, x_j) \leq \epsilon$ for $0 \leq i, j \leq k$ which by definition means that $\sigma \in \mathrm{R}_\epsilon(U)$ and $\sigma \in \mathrm{R}_\epsilon(V)$. So $\sigma \in \mathrm{R}_\epsilon(U) \cap \mathrm{R}_\epsilon(V)$.

For (ii) take a k-simplex is such that $\sigma \in \mathrm{R}_\epsilon(V)$. Then $\sigma \in \mathrm{R}_\epsilon(U) \cup \mathrm{R}_\epsilon(V)$ and $\sigma = [x_0, \dots, x_k]$ for $x_i \in U \cup V = X$ and $d(x_i, x_j) \leq \epsilon$ for $0 \leq i, j \leq k$. This is just the definition $\mathrm{R}_\epsilon(X)$. □

Notice that (ii) is not an equality, as can be seen from the next example thtat $\mathrm{R}_\epsilon(X) \subset \mathrm{R}_\epsilon(U) \cup \mathrm{R}_\epsilon(V)$.

Example 3.3. Consider the topological space X in the Figure 3.1.

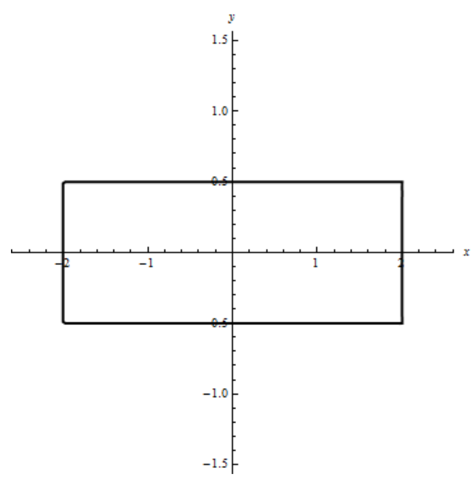


Figure 3.1: $X \subset \mathbb{R}^2$

Define $U := X \backslash \{(x, y) \in \mathbb{R}^2 \mid y = 0.5\}$ and $V := X \backslash \{(x, y) \in \mathbb{R}^2 \mid y = -0.5\}$. Clearly $X = U \cup V$, but notice that for any $\epsilon \geq 1$ we have that the 1-simplex $\sigma := [(0, -0.5), (0, 0.5)]$ is contained in $\mathrm{R}_\epsilon(X)$ but it cannot be contained in $\mathrm{R}_\epsilon(U)$ or $\mathrm{R}_\epsilon(V)$.

Lemma 3.4 (The Lebesgue number lemma)**.** *Given a metric space (X, d) and let $\mathcal{A}$ be an open covering of X. If X is compact, then there is a $\delta > 0$ such that for each subset of X having diameter less than δ, there exists an element of $\mathcal{A}$ containing it.*

The number δ is called a **Lebesgue number for the covering $\mathcal{A}$.**

Proof. See Lemma 27.5 on [Mun75] □

Lemma 3.5. *Let $X \subset \mathbb{R}^n$ be compact, $X = U \cup V$ for some open sets U and V and δ a Lebesgue number for the covering $\{U, V\}$. Then for any $\epsilon < \delta$ we have*

$$\mathrm{R}_\epsilon(X) = \mathrm{R}_\epsilon(U) \cup \mathrm{R}_\epsilon(V).$$

Proof. By (ii) on Proposition 3.2 we have one inclusion. We just need to prove the other one.

Take a k-simplex $\sigma \in \mathrm{R}_\epsilon(X)$. Then $\sigma = [x_0, \ldots, x_k]$ and by definition $d(x_i, x_j) \leq \epsilon$ for any $0 \leq i, j \leq k$. Since $\epsilon < \delta$ we have that $d(x_i, x_j) \leq \delta$ for any $0 \leq i, j \leq k$. This implies that $\mathrm{diam}(\{x_0, \ldots, x_k\}) \leq \delta$ and by the Lebesgue number lemma we get that $\{x_0, \ldots, x_k\} \subset U$ or $\{x_0, \ldots, x_k\} \subset V$. Thus $\sigma \in \mathrm{R}_\epsilon(U)$ or $\sigma \in \mathrm{R}_\epsilon(V)$ as desired. □

Notice that in general a covering of X does not induces a Mayer-Vietoris sequence for the complex $\mathrm{R}_\epsilon(X)$. It can only be achieved whenever the parameter ϵ is less that the Lebesgue number associated with the covering of X.

It is important to notice that the Vietoris-Rips complex of a set X is usually a huge complex and therefore calculating its homology is most of the times difficult and a demanding process (even for a computer, as we saw in Chapter 2).

This is also true for the Mayer-Vietoris sequence. So we would like to have some criteria to decide whenever the cohomology of a subcomplex will be easy or trivial. A first approximation we took is based on the work presented in [Hau95].

Definition 3.6. Let (X, d) be a metric space and A a subspace of X. A **crushing** from X onto A is a continuous map $F : X \times [0, 1] \to X$ such that:

(i) $F(x, 0) = x$, $F(x, 1) \in A$, $F(a, t) = a$ if $a \in A$.

(ii) $d(F(x, s), F(y, s)) \leq d(F(x, t), F(y, t))$ whenever $s \geq t$.

We call the space X **crushable** if there is a crushing from X onto a point.

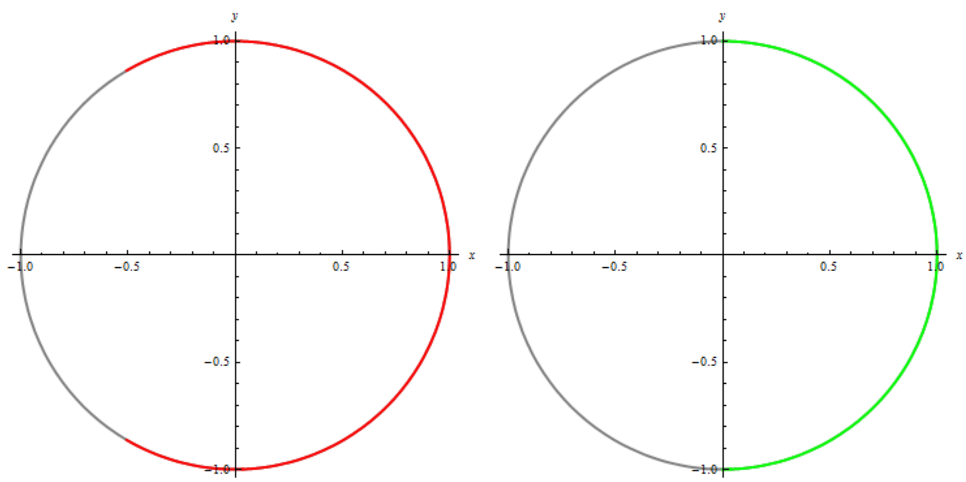
Remark. Notice that a crushing is a retraction by deformation and crushable spaces are contractible, but the converse is not true.

Also the condition (ii) in Definition 3.6 means that $F(x, \_)$ is a distance non increasing function. It means that the distance between the image of the points decreases or reduces as X is deformed onto A.

Proposition 3.7. *Let X be a metric space admitting a crushing into a subspace $A \subset X$. Then, for any $\epsilon > 0$ the map induced by the inclusion in the Rips complex $\mathrm{R}_\epsilon(A) \to \mathrm{R}_\epsilon(X)$ is an homotopy equivalence.*

Proof. See Proposition 2.2 on [Hau95]. □

Example 3.8. Take for example the circle S^1 and the open set shown in Figure 3.2a



(a) Non crushable open set on $S^1 \subset \mathbb{R}^2$. (b) Crushable open set on $S^1 \subset \mathbb{R}^2$.

Figure 3.2: Open sets on $S^1 \subset \mathbb{R}^2$.

This open set can be deformed by a retraction to a point, but there is no crushing into a point, since every function F must take apart the most extreme points in the set. Thus it cannot be crushed into a point.

Notice that the biggest open sets that are crushable on S^1 are semicircles as the one shown on Figure 3.2b.

3.2 Mayer-Vietoris spectral sequence.

The last example illustrates why it is not enough to work with the Mayer-Vietoris sequence for crushable open covers. We need a more general tool, namely the Mayer-Vietoris spectral sequence.

The following constructions and definitions are taken from [Mac12].

Let K be a simplicial complex and takesubcomplexes $U_1, \dots, U_n$ such that

$$K = U_1 \cup \cdots \cup U_n.$$

Let us denote $U_{a_0,\dots,a_p} := U_{a_0} \cap \cdots \cap U_{a_p}$. Then the inclusions

$$i_k : U_{a_0,\dots,a_p} \to U_{a_0,\dots,\hat{a_k},\dots,a_p} \tag{3.1}$$

define a map

$$d : \bigoplus_{a_0<\cdots<a_p} U_{a_0,\dots,a_p} \to \bigoplus_{a_0<\cdots<a_{p-1}} U_{a_0,\dots,a_{p-1}} \tag{3.2}$$

which induces a map in the complex chains

$$\delta : \bigoplus_{a_0<\cdots<a_p} C_*(U_{a_0,\dots,a_p}) \to \bigoplus_{a_0<\cdots<a_{p-1}} C_*(U_{a_0,\dots,a_{p-1}}).$$

This morphisms allow us to construct an exact sequence

$$\cdots\cdots \longrightarrow \bigoplus_{a_0<\cdots<a_p} C_*(U_{a_0,\dots,a_p}) \xrightarrow{\delta} \bigoplus_{a_0<\cdots<a_{p-1}} C_*(U_{a_0,\dots,a_{p-1}}) \longrightarrow \cdots$$

$$\cdots \longrightarrow \bigoplus_{a_0<a_1} C_*(U_{a_0,a_1}) \xrightarrow{\delta} \bigoplus_{a_0} C_*(U_{a_0}) \longrightarrow C_*(K) \longrightarrow 0$$

This one gives rise to a bicomplex.

$$
\begin{array}{ccccccc}
 & & \vdots & & \vdots & & \vdots \\
 & & \downarrow \partial & & \downarrow \partial & & \downarrow \partial \\
0 & \longleftarrow & C_2(K) & \overset{}{\underset{\delta}{\longleftarrow}} & \bigoplus_{a_0} C_2(U_{a_0}) & \underset{\delta}{\longleftarrow} & \bigoplus_{a_0<a_1} C_2(U_{a_0,a_1}) \longleftarrow \cdots \\
 & & \downarrow \partial & & \downarrow \partial & & \downarrow \partial \\
0 & \longleftarrow & C_1(K) & \underset{\delta}{\longleftarrow} & \bigoplus_{a_0} C_1(U_{a_0}) & \underset{\delta}{\longleftarrow} & \bigoplus_{a_0<a_1} C_1(U_{a_0,a_1}) \longleftarrow \cdots \\
 & & \downarrow \partial & & \downarrow \partial & & \downarrow \partial \\
0 & \longleftarrow & C_0(K) & \underset{\delta}{\longleftarrow} & \bigoplus_{a_0} C_0(U_{a_0}) & \underset{\delta}{\longleftarrow} & \bigoplus_{a_0<a_1} C_0(U_{a_0,a_1}) \longleftarrow \cdots \\
 & & \downarrow & & \downarrow & & \downarrow \\
 & & 0 & & 0 & & 0
\end{array}
$$

Here ∂ is the usual differential on the complex chain. It is easy to verify that $\delta\partial + \partial\delta = 0$.

This bicomplex

$$N_{p,q} := \bigoplus_{a_0<\cdots<a_p} C_q(U_{a_0,\ldots,a_p}) \tag{3.3}$$

with differential δ and ∂ of bi-grades $(-1, 0)$ and $(0, -1)$, gives rise to a total complex

$$Tot(N)_k := \bigoplus_{p+q=k} \left(\bigoplus_{a_0<\cdots<a_p} C_q(U_{a_0,\ldots,a_p}) \right)$$

with a differential $D = \delta + \partial$.

Definition 3.9. Let

$$\mathrm{H}^{\delta}_{p,q}(N_{p,q}) := \frac{\ker(\delta : N_{p,q} \to N_{p-1,q})}{\mathrm{Im}(\delta : N_{p+1,q} \to N_{p,q})}.$$

This H^{δ}_p is called the **first homology** of K

The **second homology** of K is defined in a similar way:

$$\mathrm{H}^{\partial}_{p,q}(N_{p,q}) := \frac{\ker(\partial : N_{p,q} \to N_{p,q-1})}{\mathrm{Im}(\delta : N_{p,q+1} \to N_{p,q})}.$$

Notice that the first and second homology of K are just the usual homology groups calculated with respect δ and ∂ respectively.

Definition 3.10. Since the second homology of K is also a bi-graded object with a differential δ induced by the original δ, we can define

$$\mathrm{H}_p^\delta \mathrm{H}_q^\partial(N_{p,q}) := \frac{\ker(\delta : \mathrm{H}_{p,q}^\partial \to \mathrm{H}_{p-1,q}^\partial)}{\mathrm{Im}(\delta : \mathrm{H}_{p+1,q}^\partial \to \mathrm{H}_{p,q}^\partial)},$$

which is also a bi-graded object.

Theorem 3.11. *The first spectral sequence of a double complex N with associated total complex $Tot(N)$ is given by*

$$\mathrm{E}_{p,q}^2 = \mathrm{H}_p^\delta \mathrm{H}_q^\partial(N).$$

If $N_{p,q} = 0$ for $p < 0$, then E^2 converges to the homology of the total complex $Tot(N)$.

Proof. See Chapter XI Theorem 6.1 on [Mac12]. □

The following theorems are fundamental for comparisons between the spectral sequences we are interested in.

Theorem 3.12. *There are isomorphisms of homology groups $\mathrm{H}_*(Tot(N)) \cong \mathrm{H}_*(K)$.*

Proof. See Theorem 5.1 in [Sta15] or Theorem 1 in [DPM11]. □

Definition 3.13. If $X \subset \mathbb{R}^n$ we define an **open good crushable cover** as a collection of open sets $U_i \subset X$ such that

(i) $X = \bigcup_{i \in I} U_i$,

(ii) every finite intersection $U_{a_1,\ldots,a_n}$ is crushable.

Theorem 3.14. *Consider a set $X \subset \mathbb{R}^n$ and an open good crushable cover $\mathcal{U} = \{U_i\}_{i \in I}$ of X. If δ is a Lebesgue number associated to $\mathcal{U}$, then for every $\epsilon < \delta$ we have*

$$\mathrm{H}_*(\mathrm{R}_\epsilon(X)) \cong \mathrm{H}_*(X).$$

Proof. Call $K = \mathrm{R}_\epsilon(X)$. Since $\epsilon < \delta$ by Lemma 3.5 we have

$$\mathrm{R}_\epsilon(X) = \bigcup_{i \in I} \mathrm{R}_\epsilon(U_i).$$

So we a have a simplicial complex and a covering with subcomplexes. Now we can define a bicomplex by (3.3) and therefore we also have a total complex $Tot(N)$. By Theorem 3.12 we know that $\mathrm{H}_*(Tot(N)) \cong \mathrm{H}_*(\mathrm{R}_\epsilon(X))$.

Since every finite intersection $U_{a_1,\ldots,a_n}$ is crushable, then $\mathrm{R}_\epsilon(U_{a_1,\ldots,a_n}) \simeq \mathrm{R}_\epsilon(*) = *$.

Using bi-graded complex for $\mathrm{R}_\epsilon(X)$ of equation (3.3) and the second homology of $\mathrm{R}_\epsilon(X)$ we can see that $E^1_{p,q}(\mathrm{R}_\epsilon(X))$ is

$$\begin{array}{ccccc}
\vdots & & \vdots & & \\
\bigoplus_{a_0} \mathrm{H}_2(\mathrm{R}_\epsilon(U_{a_0})) & \longleftarrow & \bigoplus_{a_0<a_1} \mathrm{H}_2(\mathrm{R}_\epsilon(U_{a_0,a_0})) & \longleftarrow & \cdots \\
\bigoplus_{a_0} \mathrm{H}_1(\mathrm{R}_\epsilon(U_{a_0})) & \longleftarrow & \bigoplus_{a_0<a_1} \mathrm{H}_1(\mathrm{R}_\epsilon(U_{a_0,a_0})) & \longleftarrow & \cdots \\
\bigoplus_{a_0} \mathrm{H}_0(\mathrm{R}_\epsilon(U_{a_0})) & \longleftarrow & \bigoplus_{a_0<a_1} \mathrm{H}_0(\mathrm{R}_\epsilon(U_{a_0,a_0})) & \longleftarrow & \cdots
\end{array}$$

which is equivalent to

$$\begin{array}{ccccc}
0 & \longleftarrow & 0 & \longleftarrow & \cdots \\
0 & \longleftarrow & 0 & \longleftarrow & \cdots \\
\bigoplus_{a_0} \mathbb{Z} & \longleftarrow & \bigoplus_{a_0<a_1} \mathbb{Z} & \longleftarrow & \cdots
\end{array} \tag{3.4}$$

We can construct a similar bi-graded complex for the covering $\mathcal{U}$ of X as the one described on (3.3) and define a spectral sequence in a similar way as in Definitions 3.9 and 3.10. Thus getting that $E^1_{p,q}(X)$ is

$$\begin{array}{ccccc}
\vdots & & \vdots & & \\
\bigoplus_{a_0} \mathrm{H}_2(U_{a_0}) & \longleftarrow & \bigoplus_{a_0<a_1} \mathrm{H}_2(U_{a_0,a_0}) & \longleftarrow & \cdots \\
\bigoplus_{a_0} \mathrm{H}_1(U_{a_0}) & \longleftarrow & \bigoplus_{a_0<a_1} \mathrm{H}_1(U_{a_0,a_0}) & \longleftarrow & \cdots \\
\bigoplus_{a_0} \mathrm{H}_0(U_{a_0}) & \longleftarrow & \bigoplus_{a_0<a_1} \mathrm{H}_0(U_{a_0,a_0}) & \longleftarrow & \cdots
\end{array}$$

which is equivalent to

$$\begin{array}{ccccc}
0 & \longleftarrow & 0 & \longleftarrow & \cdots \\
0 & \longleftarrow & 0 & \longleftarrow & \cdots \\
\bigoplus_{a_0} \mathbb{Z} & \longleftarrow & \bigoplus_{a_0<a_1} \mathbb{Z} & \longleftarrow & \cdots
\end{array} \tag{3.5}$$

Notice that the page $E^1_{p,q}(\mathrm{R}_\epsilon(X))$ is concentrated at $q = 0$. This is also true for $E^1_{p,q}(X)$. Moreover, the differential that appears on the column $q = 0$ comes directly from the map described in equation (3.2) which means that it is a purely combinatorial differential only related to the inclusions of the intersections as described in (3.1).

This implies that for the page 2 of the spectral sequences we have $E^2_{p,q}(\mathrm{R}_\epsilon(X)) = E^2_{p,q}(X)$ and again all collapses on the first column yielding for both complexes

$$\begin{array}{ccccc}
0 & 0 & 0 & 0 & \cdots \\
 & & & & \\
0 & 0 & 0 & 0 & \cdots \\
 & & & & \\
0 & E^2_{0,0} & E^2_{1,0} & E^2_{2,0} & \cdots \\
 & & & & \\
0 & 0 & 0 & 0 & \cdots
\end{array} \tag{3.6}$$

Thus by Definition 5.2.7 in [Wei94] we have that both spectral sequences collapse at E^2 and therefore $H_*(Tot(N)) = H_*(X)$ as desired. □

Chapter 4

Conclusions and future work

Persistent homology has become an extensively used tool for data analysis using the Vietoris-Rips complex. A well known disadvantage of this filtered complex is that it has a large amount of simplices, such amount makes it difficult to compute its persistent homology with the method described in Section 1.2 of Chapter 1. It will be useful to have analytical tools to compute such persistent homology, but even in some of the most simple examples this task is pretty difficult.

A good example of how difficult is to do this calculations is the work of Adams and Adamaszec [AA15], where they calculate the persistent homology for uniformly distributed points over a circle in $\mathbb{R}^2$. They showed that in the Vietoris-Rips complex appears odd dimensional spheres. We do not have any similar results for any other point sets, so it becomes relevant to develop some tools to calculate or estimate the persistent homology.

The main philosophy of this work is to estimate the persistent homology of a given space using the persistence information of a more simple space whose persistence homology is known or easier to study. For example, we wanted to estimate the persistence of a 1-cycle in the Rips complex of an ellipse using the persistence of a 1-cycle in the circle. To be able to do this we captured the “geometry” of the ellipse into a linear transformation which transformed a circle onto an ellipse. Thus we needed to study how the persistence is modified by a linear transformation.

We were able to give bounds on the persistence of the image of a linear transformation in terms of the persistence of the domain and the singular values of the linear transformation. This result is pretty general and can be applied in a good range of cases, but it is usually difficult to calculate the singular values of the linear transformation. We modified this basic result using some well known bounds on eigenvalues that uses traces. Thus we where able to produce more bounds on the persistence, which in general are easier to calculate, in terms of the persistence of the domain and the traces of some specific matrices.

We still need to study the efficiency of such bounds comparing them with the bounds that use the singular values directly. Thus we need to find a tool that tell us how many information we have lost in the persistence estimation with respect to the original bounds or even with the real persistence of the space. In the same direction, it will be useful to set a handful of conditions that help us establish conditions under our bounds gave good approximations to the persistence. All the machinery previously described is part of the future work that must be developed in other to make our results more useful and applicable to real data sets.

However, one interesting application of the results here presented was to apply them to the sliding window embedding studied in [PH15]. In this case we obtained bounds for the persistent homology of the embedding which are independent on the embedding dimension and the window size. The bounds we found are only depend on the Fourier coefficients of the function and some Fejer kernels. But our bounds are useful only if we are able to compute the persistent homology the curve in a torus described in Subsection 2.3.2. But this calculation is pretty hard itself and we do not have analytical approaches to do it. As a continuation of this work we must find a way to calculate or approximate the persistence homology of such curve in an analytical way to make our bounds more robust.

Having in mind the same idea of estimating the persistence of a given space using he persistence of more simple spaces we develop some methods using spectral sequences. More specifically we tried to use the Mayer-Vietoris sequence to the Rips complex using a covering for the complex that emerged from an open covering for the original space. But this technique can only be applied for small values of the parameter in the rips complex, as in some work from Hausmann [Hau95].

The work by Hausmann only described the Rips complex homotopy type for Reimannian manifolds, we where able to extend his results to general topological spaces using spectral sequences whenever the parameter of the rips complex is smaller that the Lebesgue number of the covering of the space. Our result assure us that the homotopy type of the Rips complex is the same than the original space for small vales of the parameter even for general topological spaces.

Here we need to see if it is possible to extend this techniques to understand the homotopy type of the space in terms of the homotopy of a suitable decomposition using the Mayer-Vietoris spectral sequence to capture the behavior of the persistent classes as they are "'included" into the bigger space we are studying.

Appendix A

Appendix

A.1 Tables

A.1.1 Example 2.11

The following tables show the explicit results for the persisitent homology calculated in Example 2.11:

Table A.1: Persistent homology S^2

Dimension: 0					
[0.0, 0.1)	[0.0, 0.1)	[0.0, 0.2)	[0.0, 0.2)	[0.0, 0.3)	[0.0, 0.4)
[0.0, 0.1)	[0.0, 0.1)	[0.0, 0.2)	[0.0, 0.2)	[0.0, 0.3)	[0.0, 0.4)
[0.0, 0.1)	[0.0, 0.2)	[0.0, 0.2)	[0.0, 0.2)	[0.0, 0.3)	[0.0, 0.4)
[0.0, 0.1)	[0.0, 0.2)	[0.0, 0.2)	[0.0, 0.3)	[0.0, 0.3)	[0.0, 0.4)
[0.0, 0.1)	[0.0, 0.2)	[0.0, 0.2)	[0.0, 0.3)	[0.0, 0.3)	[0.0, 0.4)
[0.0, 0.1)	[0.0, 0.2)	[0.0, 0.2)	[0.0, 0.3)	[0.0, 0.3)	[0.0, 0.4)
[0.0, 0.1)	[0.0, 0.2)	[0.0, 0.2)	[0.0, 0.3)	[0.0, 0.3)	[0.0, 0.5)
[0.0, 0.1)	[0.0, 0.2)	[0.0, 0.2)	[0.0, 0.3)	[0.0, 0.4)	[0.0, 0.5)
[0.0, 0.1)	[0.0, 0.2)	[0.0, 0.2)	[0.0, 0.3)	[0.0, 0.4)	[0.0, infinity)
[0.0, 0.1)	[0.0, 0.2)	[0.0, 0.2)	[0.0, 0.3)	[0.0, 0.4)	
[0.0, 0.1)	[0.0, 0.2)	[0.0, 0.2)	[0.0, 0.3)	[0.0, 0.4)	
Dimension: 1					
[0.6, 0.7)	[0.6, 0.9)	[0.5, 0.7)	[0.5, 0.7)	[0.4, 0.5)	
[0.6, 0.7)	[0.5, 0.6)	[0.5, 0.7)	[0.5, 1.0)	[0.4, 0.6)	
Dimension: 2					
[1.0, 1.6)					

Table A.2: Persistent homology $T(S^2)$

Dimension: 0					
[0.0, 0.2)	[0.0, 0.4)	[0.0, 0.4)	[0.0, 0.6)	[0.0, 0.6)	[0.0, 0.8)
[0.0, 0.2)	[0.0, 0.4)	[0.0, 0.4)	[0.0, 0.6)	[0.0, 0.6)	[0.0, 0.8)
[0.0, 0.2)	[0.0, 0.4)	[0.0, 0.4)	[0.0, 0.6)	[0.0, 0.6)	[0.0, 1.0)
[0.0, 0.2)	[0.0, 0.4)	[0.0, 0.4)	[0.0, 0.6)	[0.0, 0.6)	[0.0, 1.0)
[0.0, 0.2)	[0.0, 0.4)	[0.0, 0.4)	[0.0, 0.6)	[0.0, 0.6)	[0.0, 1.0)
[0.0, 0.2)	[0.0, 0.4)	[0.0, 0.4)	[0.0, 0.6)	[0.0, 0.8)	[0.0, 1.0)
[0.0, 0.2)	[0.0, 0.4)	[0.0, 0.4)	[0.0, 0.6)	[0.0, 0.8)	[0.0, 1.0)
[0.0, 0.2)	[0.0, 0.4)	[0.0, 0.4)	[0.0, 0.6)	[0.0, 0.8)	[0.0, 1.0)
[0.0, 0.2)	[0.0, 0.4)	[0.0, 0.4)	[0.0, 0.6)	[0.0, 0.8)	[0.0, 1.2)
[0.0, 0.2)	[0.0, 0.4)	[0.0, 0.6)	[0.0, 0.6)	[0.0, 0.8)	[0.0, infinity)
[0.0, 0.4)	[0.0, 0.4)	[0.0, 0.6)	[0.0, 0.6)	[0.0, 0.8)	
Dimension: 1					
[1.4, 1.6)	[1.2, 1.4)	[1.2, 1.6)	[1.0, 1.2)	[1.0, 1.4)	[0.8, 1.2)
[1.4, 2.2)	[1.2, 1.6)	[1.2, 2.0)	[1.0, 1.2)	[1.0, 1.6)	[0.8, 1.2)
Dimension: 2					
[2.4, 3.4)					

A.1.2 Subsection 2.3.1

Table A.3: Persistence homology $s_1(t)$

Dimension: 0				
[0.0, 0.2)	[0.0, 0.2)	[0.0, 0.2)	[0.0, 0.4)	[0.0, 0.6)
[0.0, 0.2)	[0.0, 0.2)	[0.0, 0.2)	[0.0, 0.4)	[0.0, 0.6)
[0.0, 0.2)	[0.0, 0.2)	[0.0, 0.2)	[0.0, 0.4)	[0.0, 0.6)
[0.0, 0.2)	[0.0, 0.2)	[0.0, 0.4)	[0.0, 0.4)	[0.0, 0.8)
[0.0, 0.2)	[0.0, 0.2)	[0.0, 0.4)	[0.0, 0.4)	[0.0, 0.8)
[0.0, 0.2)	[0.0, 0.2)	[0.0, 0.4)	[0.0, 0.4)	[0.0, infinity)
[0.0, 0.2)	[0.0, 0.2)	[0.0, 0.4)	[0.0, 0.6)	
[0.0, 0.2)	[0.0, 0.2)	[0.0, 0.4)	[0.0, 0.6)	
Dimension: 1				
[0.8, 2.0)				
Dimension: 2				
[1.6, 1.8)	[1.6, 1.8)	[1.6, 1.8)	[1.6, 1.8)	[1.6, 1.8)

Table A.4: Persistence homology $s_2(t)$

Dimension: 0				
[0.0, 0.2)	[0.0, 0.2)	[0.0, 0.2)	[0.0, 0.4)	[0.0, 0.6)
[0.0, 0.2)	[0.0, 0.2)	[0.0, 0.2)	[0.0, 0.4)	[0.0, 0.6)
[0.0, 0.2)	[0.0, 0.2)	[0.0, 0.2)	[0.0, 0.4)	[0.0, 0.6)
[0.0, 0.2)	[0.0, 0.2)	[0.0, 0.4)	[0.0, 0.6)	[0.0, 0.8)
[0.0, 0.2)	[0.0, 0.2)	[0.0, 0.4)	[0.0, 0.6)	[0.0, 0.8)
[0.0, 0.2)	[0.0, 0.2)	[0.0, 0.4)	[0.0, 0.6)	[0.0, 0.8)
[0.0, 0.2)	[0.0, 0.2)	[0.0, 0.4)	[0.0, 0.6)	[0.0, infinity)
[0.0, 0.2)	[0.0, 0.2)	[0.0, 0.4)	[0.0, 0.6)	
Dimension: 1				
[1.0, 1.8)				
Dimension: 2				
[1.6, 1.8)	[1.6, 1.8)	[1.6, 1.8)	[1.6, 1.8)	
[1.6, 1.8)	[1.6, 1.8)	[1.6, 1.8)	[1.6, 1.8)	

A.2 Coding

A.2.1 Code example 2.3

```
clc; clear; close all;
%% Initilize
init
dim = 1;
n=2000;
point_cloud = createPointCloud(n,dim)
%% Calculate the persistence for S^1
[I1,J1] = rca1pc(point_cloud,20);
nameI1 = strcat('X_1_persis_',num2str(dim),'dim','_',num2str(n),
    'points');
nameJ1 = strcat('X_0_persis_',num2str(dim),'dim','_',num2str(n),
    'points');
csvwrite(strcat(nameI1,'.csv'),I1);
csvwrite(strcat(nameJ1,'.csv'),J1);
%% Transform data set
T=[3 0; 0 2];
Img_point_cloud = (T*point_cloud')';
%% Calculate the persistence for T(S^1)
[I2,J2] = rca1pc(Img_point_cloud,20);
nameI2 = strcat('TX_1_persis_',num2str(dim),'dim','_',num2str(n)
    ,'points');
nameJ2 = strcat('TX_0_persis_',num2str(dim),'dim','_',num2str(n)
    ,'points');
```

```
csvwrite(strcat(nameI2,'.csv'),I2);
csvwrite(strcat(nameJ2,'.csv'),J2);
```

A.2.2 Code example 2.11

```
clc; clear; close all;
load_javaplex
%% Parameters for JavaPlex
max_dimension = 3; % Compute up to 2 dim persitent homology
num_divisions = 20;
max_filtration_value = 2;
%% Create point cloud
phi = 2*pi*rand(75,1);
theta = 2*pi*rand(75,1);
x = sin(phi).*cos(theta);
y = sin(phi).*sin(theta);
z = cos(phi);
X = horzcat(x,y,z);
%% Calculate persisitent homology
stream = api.Plex4.createVietorisRipsStream(X, max_dimension,
    max_filtration_value, num_divisions);
persistence = api.Plex4.getModularSimplicialAlgorithm(max_
    dimension, 2);
intervals = persistence.computeIntervals(stream);
barcode_curve = plot_barcodes(intervals);
%% Apply T to point cloud
T=[-1 -1 1; -1 1 -1; 1 1 1; 1 -1 -2];
Img_X = (T*X')';
%% Calculate persisitent homology for podified point cloud
max_filtration_value = 4;
stream_img = api.Plex4.createVietorisRipsStream(Img_X, max_
    dimension, max_filtration_value, num_divisions);
persistence_img = api.Plex4.getModularSimplicialAlgorithm(max_
    dimension, 2);
intervals_img = persistence_img.computeIntervals(stream_img);
barcode_curve_img = plot_barcodes(intervals_img);
```

A.2.3 Subsection 2.3.1

```
clc; clear; close all;
load_javaplex
%% Parameters
max_dimension = 3; % compute up to 2 dim persitent homology
max_filtration_value = 4; % maximo epsilon
num_divisions = 20; %disions of the epsilons
```

```
%% Create point cloud
n = 85;
r = 2*pi*rand(n,1); %angles
com_s1 = exp(i*1*r); %complex points
com_s2 = exp(i*2*r);
com_s3 = exp(i*3*r);
point_cloud13 = [real(com_s1),imag(com_s1),real(com_s3),imag(com_s3)];
point_cloud23 = [real(com_s2),imag(com_s2),real(com_s3),imag(com_s3)];
%% Create complex and calculat epersistence
stream13 = api.Plex4.createVietorisRipsStream(point_cloud13, max_dimension, max_filtration_value, num_divisions); %creates Vietoris-Rips
persistence13 = api.Plex4.getModularSimplicialAlgorithm(max_dimension, 2); %creates persistence element
intervals13 = persistence13.computeIntervals(stream13); %calculates persistence
barcode_curve_13 = plot_barcodes(intervals13); %plotea el codigo de barras
name = strcat('13barcode_',num2str(length(r)),'points','_',num2str(num_divisions),'divisions');
print(barcode_curve_13,name,'-dpng'); %Imprime la imagen anteriro a un archivo .png
%% Create complex and calculat epersistence
stream23 = api.Plex4.createVietorisRipsStream(point_cloud23, max_dimension, max_filtration_value, num_divisions); %creates Vietoris-Rips
persistence23 = api.Plex4.getModularSimplicialAlgorithm(max_dimension, 2); %creates persistence element
intervals23 = persistence23.computeIntervals(stream23); %calculates persistence
barcode_curve_23 = plot_barcodes(intervals23); %plotea el codigo de barras
name = strcat('23barcode_',num2str(length(r)),'points','_',num2str(num_divisions),'divisions');
print(barcode_curve_23,name,'-dpng'); %Imprime la imagen anteriro a un archivo .png
```

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