
On Hilbert's fifth problem

Luis Germán Polanco Contreras

Advisor: Andrés Angel



Universidad de los Andes
Facultad de Ciencias
Departamento de Matemáticas

Diciembre 2013

Contents

1	Introduction	3
2	Basic notions	6
2.1	Lie groups	6
2.2	1-parameter subgroups	9
2.3	Campbell-Baker-Hausdorff formula	12
3	Gleason metrics and the construction of a Lie structure	20
3.1	Topological vector space	20
3.2	Strong Gleason metric	22
3.3	Construction of a Lie structure from a strong Gleason metric	26
3.4	Construction of an exponential map	36
4	Gleason-Yamabe theorem for compact groups	42
4.1	Haar Measure	42
4.2	Gleason-Yamabe theorem for compact groups	48
5	Building metrics on groups	54
5.1	Construction of metrics from functions	54
5.2	Groups with NSS property	59

Chapter 1

Introduction

In 1900, Hilbert published a set of problems in the Bulletin of the American Mathematical Society [Hil02]. One of these problems, which today is known as Hilbert's fifth problem, asks for the conditions a given group must satisfy in order to assure the differentiability of the group operations. Since its formulation many mathematicians have searched for answer to this natural question.

The earliest answer to this problem was given in 1933 for the compact case by von Neumann using the Peter-Weyl theorem for representations of compact groups. Later on Iwasawa asserted that if the quotient of a group over a normal Lie group is a Lie group, then the whole group was originally a Lie group. Some years after that conjecture was proved, Gleason showed that every locally compact finite dimensional group with no small subgroups is a Lie group using some results given by Montgomery and Zippin. Afterwards, Yamabe was able to remove the finite dimensional hypothesis.

The main purpose of this dissertation is to make a compilation of some of the main results previously mentioned based in the work of Tao [Tao12].

In chapter 1 we introduce some basic notions concerning Lie groups and Lie algebras and their exponential map. Furthermore, we study how the linear structure of the Lie algebra gives us control over the group operations through the Campbell-Baker-Hausdorff theorem. Finally we prove Cartan's theorem which will play an important role later on.

In chapter 2 we develop the tools needed to construct an exponential map based on the existence of a strong Gleason metric in a locally compact group. This exponential map will allow us to define the local differentiable structure of the given topological group. It is important to notice that we introduce the concept of strong Gleason metric in order to simplify some of the arguments presented in [Tao12].

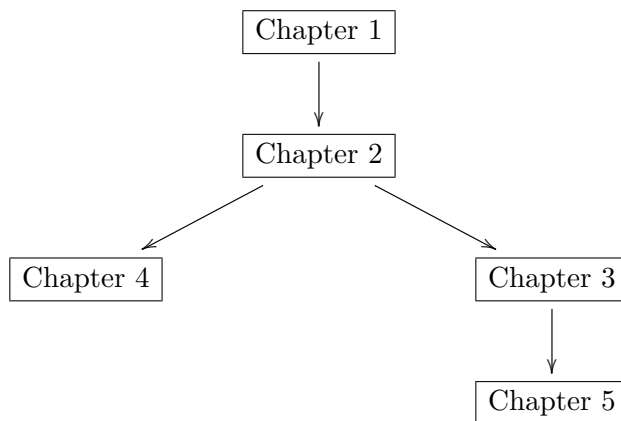
Chapter 3 is dedicated to the existence of Haar measures on locally compact groups and the development of the machinery necessary to prove that every compact group can be made a Lie group, after considering its quotient by some normal closed (well selected) subgroup. This result is achieved after making a strong use of the existence of a Haar

measure and the Peter-Weyl theorem for representations of compact Hausdorff groups.

The last chapter is dedicated to the construction of strong Gleason metrics on locally compact groups. We begin by defining a weakly strong Gleason metric, which at first sight may appear weaker than a strong Gleason metric. But we show that they are actually equivalent. Now that we have reduced the requirements for a metric to become a strong Gleason metric, we only need to find some necessary conditions for the existence of such weak Gleason metrics. The required condition turns out to be that the group has the *no small subgroup* property.

From the no small subgroups property and by an argument, which in spirit is similar to the proof of the Birkhoff-Kakutani theorem, we are able to construct a weak Gleason metric. The proof of existence of such metrics will be constructive, but not necessarily enlightening. This final construction will use most of the tools developed through the whole text.

Due to the organization of the document and the huge amount of technical details, sometime the reading can be quite challenging. We understand that the reader could lose sight of the goals proposed. In order to give some help on the understanding we provide a little diagram which could help the exploration of the document



The previous flowchart gives a slight insight on the structure of the text. For example, someone who is interested in the answer to Hilbert's fifth problem for compact groups can go directly from chapter 2 to chapter 4. But if the reader is interested in the locally compact case of the Hilbert's fifth problem and is familiar with some of the most important tools in functional analysis, he could avoid chapter 4 and go to chapter 5. However, the most inexperienced reader should follow the given organization of the text, since most of the ideas developed in chapter 4 can be highly illustrative in chapter 5. Furthermore, the technical issue addressed in chapter 4 may serve as an interesting "warm up" to the proofs presented in chapter 5.

A little summary of the main results presented in this text concerning Hilbert's fifth problem for locally compact groups can be made in the following diagram

$$NSS \implies \begin{array}{c} \text{weak strong} \\ \text{Gleason metric} \end{array} \implies \begin{array}{c} \text{strong} \\ \text{Gleason} \\ \text{metric} \end{array} \implies \begin{array}{c} L(G) \text{ is} \\ \text{a t.v.s} \end{array} \implies \begin{array}{c} \text{Exp map} \\ \text{is a local} \\ \text{homeomorphism} \end{array} \implies \begin{array}{c} G \text{ is a} \\ \text{Lie group.} \end{array}$$

The previous diagram shows us the path of theorems that leads us from the NSS property to the construction of a differentiable structure on locally compact groups. It can be translated into: every locally compact group with the NSS property has a weak strong Gleason metric, which is in fact a strong Gleason metric. Every locally compact group with a strong Gleason metric is locally homeomorphic to a finite dimensional topological vector space named $L(G)$ with local homeomorphism given by the Exp map.

Keeping this in mind will help the reader to navigate through the text and not drown into the technical proofs, which are important and beautiful themselves, but can distract from the main target of this document.

Finally, I would like to acknowledge the inspiration and constant support given to me by my advisor and role model Professor Andrés Angel, and the assistance of Professor Monika Winklmeier, who help me unceasingly through the construction of this text.

Chapter 2

Basic notions

In this chapter we will introduce some of the most basic concepts which will be needed for the development of this work.

2.1 Lie groups

Definition 2.1 (Lie group). A *Lie group* G is a topological group such that:

- (i) G is a differentiable manifold.
- (ii) The group operations are C^∞ , which means that $\cdot : G \times G \rightarrow G$ and $(\cdot)^{-1} : G \rightarrow G$ are differentiable.

We will denote the identity element of the group with the letter e hereinafter.

Some of the most basic examples of Lie groups are $(\mathbb{R}^n, +)$, (\mathbb{C}^*, \cdot) (the non-zero complex numbers) with complex multiplication, S^1 with the induced multiplication by \mathbb{C}^* and $M_n(\mathbb{R}) = \{\text{matrices } n \times n\}$ with matrices multiplication.

Definition 2.2 (Lie algebra). A *Lie algebra* \mathfrak{g} is a \mathbb{K} -vector space together with a bilinear operation called Lie bracket

$$[\cdot, \cdot] : \mathfrak{g} \times \mathfrak{g} \rightarrow \mathfrak{g}$$

which fulfills the following properties:

- (i) $[x, y] = -[y, x]$,
- (ii) $[[x, y], z] + [[y, z], x] + [[z, x], y] = 0$ (Jacobi identity)

Definition 2.3. Let G be a Lie group.

- (i) For $g \in G$ the left and right translations by g are the respective diffeomorphisms $l_g : G \rightarrow G$, and $r_g : G \rightarrow G$ defined by

$$\begin{aligned} l_g(x) &= gx, \\ r_g(x) &= xg. \end{aligned}$$

- (ii) Let X be a vector field over G . We say that X is left invariant if $\forall g \in G$ we have $X \circ l_g = dl_g \circ X$ or

$$\begin{array}{ccc} TG & \xrightarrow{dl_g} & TG \\ X \uparrow & & \uparrow X \\ G & \xrightarrow{l_g} & G \end{array}$$

is a commutative diagram.

This definition can be restated in terms of φ -related vector fields as follows.

Definition 2.4. Let $\varphi : M \rightarrow N$ be a smooth function. Then we say that some vector fields X on M and Y on N are φ -related if we have that the following is a commutative diagram

$$\begin{array}{ccc} TM & \xrightarrow{d\varphi} & TN \\ X \uparrow & & \uparrow Y \\ M & \xrightarrow{\varphi} & N \end{array}$$

Clearly since l_g is a diffeomorphism of G into itself we can say that a left invariant vector field is l_g -related to itself, which coincides with the previous definition.

Proposition 2.5. Let $\varphi : M \rightarrow N$ be C^∞ . Let X and X_1 be smooth vector fields on M , and let Y and Y_1 be smooth vector fields on N . If X is φ -related to Y and X_1 is φ -related to Y_1 then $[X, X_1]$ is φ -related to $[Y, Y_1]$.

Proof. We need to show that $d\varphi[X, X_1] = [Y, Y_1] \circ \varphi$. Let $m \in M$ and $f \in C^\infty(N)$. We need to verify that $d\varphi([X, X_1])_m(f) = [Y, Y_1]_{\varphi(m)}(f)$. Applying the definition we obtain

$$\begin{aligned} d\varphi([X, X_1])_m(f) &= [X, X_1]_m(f \circ \varphi) \\ &= X_m(X_1(f \circ \varphi)) - (X_1)_m(X(f \circ \varphi)) \\ &= X_m((d\varphi \circ X_1)(f)) - (X_1)_m((d\varphi \circ X)(f)) \\ &= X_m(Y_1(f) \circ \varphi) - (X_1)_m(Y(f) \circ \varphi) \\ &= d\varphi(X_m)(Y_1(f)) - d\varphi((X_1)_m)(Y(f)) \\ &= Y_{\varphi(m)}(Y_1(f)) - (Y_1)_{\varphi(m)}(Y(f)) \\ &= [Y, Y_1]_{\varphi(m)}(f). \end{aligned}$$

□

Proposition 2.6. *Let G be a Lie group and \mathfrak{g} the set of left-invariant vector fields on G . Then:*

- (i) \mathfrak{g} is an \mathbb{R} -vector field, and $\alpha : \mathfrak{g} \rightarrow T_e G$, $\alpha(X) = X(e)$ is a natural isomorphism.
- (ii) The bracket of two invariant vector fields is itself an invariant vector field.
- (iii) \mathfrak{g} forms a Lie algebra with the Lie bracket operation of vector fields.

Proof. It is obvious that \mathfrak{g} is an \mathbb{R} -vector space and the linearity of α comes from the linearity of the vector fields. Now let us verify the injectivity of α . If $\alpha(X) = \alpha(Y)$, then for all $g \in G$

$$X(g) = dl_g(X(e)) = dl_g(Y(e)) = Y(g).$$

Hence $X = Y$. Moreover, take $x \in T_e G$ and define the vector field $X_x : G \rightarrow TG$ as $X_x(h) = dl_h(x)$. Clearly by construction X_x is left-invariant since $X_x(l_g(h)) = X_x(gh) = dl_{gh}(x) = dl_g(dl_h(x)) = dl_g(X_x(h))$ and $\alpha(X_x) = X_x(e) = dl_e(x) = x$, so α is onto. Showing that α is an isomorphism.

Note that X and Y are l_g -related to themselves. Hence by proposition 2.5 $[X, Y]$ is l_g -related to itself, which means that $dl_g[X, Y] = [X, Y]l_g$ as desired.

by (i) and (ii) in this proposition it follows that \mathfrak{g} is a Lie algebra which concludes the proof of (iii). \square

Definition 2.7. We define the *Lie algebra of a Lie group G* as the Lie algebra \mathfrak{g} of smooth left invariant vector fields on G . By numeral (i) in proposition 2.6 we can identify the Lie algebra of the Lie group G as the tangent space of G at the identity $T_e G$.

Definition 2.8. Immersion Let M and N be two differentiable manifolds and $f : M \rightarrow N$ be a differentiable map. We say that f is an *immersion* if and only if

$$d_p f : T_p M \rightarrow T_{f(p)} N$$

is an injective map for every $p \in M$.

Definition 2.9 (Lie subgroup). (H, i) is a Lie subgroup of a Lie group G if and only if:

- (i) H is itself a Lie group,
- (ii) (H, i) is a sub-manifold of G with injective immersion i ,
- (iii) $i : H \rightarrow G$ is a group homomorphism.

Here i is called the inclusion map.

Definition 2.10 (Lie sub-algebra). Let \mathfrak{g} be a Lie algebra. A subspace $\mathfrak{h} \subset \mathfrak{g}$ is called a *Lie sub-algebra* if $[X, Y] \in \mathfrak{h}$ whenever $X, Y \in \mathfrak{h}$.

Remark. A Lie sub-algebra $\mathfrak{h} \subset \mathfrak{g}$ clearly forms a Lie algebra under the induced bracket operation of \mathfrak{g} .

If (H, i) is a Lie subgroup, then i is an injective immersion. Moreover, if \mathfrak{h} is the Lie algebra of H and \mathfrak{g} is the Lie algebra of G then $di : TH \rightarrow TG$ gives an isomorphism of \mathfrak{h} with a Lie sub algebra $di(\mathfrak{h}) \subset \mathfrak{g}$.

Definition 2.11. Let (H_1, i_1) and (H_2, i_2) be two Lie subgroups of G . We call them *equivalent* if there exists a Lie group isomorphism $\alpha : H_1 \rightarrow H_2$ such that $i_2 \circ \alpha = i_1$. Or equivalently if the following diagram

$$\begin{array}{ccc} G & \xrightarrow{id} & G \\ i_1 \uparrow & & \uparrow i_2 \\ H_1 & \xrightarrow{\alpha} & H_2 \end{array}$$

is commutative.

Taking into account this equivalence relation, we will consider a Lie subgroup H of G dropping any reference to its inclusion map. The same will be done with their corresponding Lie algebras.

Now we are interested in the relation between the Lie group G and its Lie algebra \mathfrak{g} . In order to do such, we will introduce the notion of the *exponential map*, which will help us to understand the local behavior of group operations in terms of the Lie algebra operations.

This will be useful since a Lie group is highly non-linear. Therefore, its operations are really difficult to understand, but the exponential map will allow us to “approximate” them with some linear operations.

2.2 1-parameter subgroups

Definition 2.12. A continuous homomorphism $\phi : \mathbb{R} \rightarrow G$ is called a *1-parameter* subgroup of G .

Definition 2.13 (Exponential map). Let G be a Lie group, \mathfrak{g} its Lie algebra. And let $X \in \mathfrak{g}$. Consider the homomorphism

$$\lambda \frac{d}{dt} \mapsto \lambda X$$

of the Lie algebra of \mathbb{R} into \mathfrak{g} . \mathbb{R} is simply connected, thus by [War71] there exists a unique 1-parameter subgroup

$$\exp_X : \mathbb{R} \rightarrow G$$

such that

$$d(\exp_X)\left(\lambda \frac{d}{dt}\right) = \lambda X.$$

Note that $t \mapsto \exp_X(t)$ is the unique 1-parameter subgroup of G whose tangent vector at 0 is $X(e)$. Now we can define the *exponential map* as

$$\text{Exp} : \mathfrak{g} \rightarrow G, \text{Exp}(X) = \exp_X(1).$$

The following theorem explains why this function is called “exponential” map.

Theorem 2.14. *Let G be a Lie group with Lie algebra \mathfrak{g} . Then for every $X \in \mathfrak{g}$*

- (i) $\text{Exp}(tX) = \exp_X(t)$ for all $t \in \mathbb{R}$,
- (ii) $\text{Exp}((t+s)X) = \text{Exp}(tX)\text{Exp}(sX)$ for all $t, s \in \mathbb{R}$,
- (iii) $\text{Exp}(-tX) = (\text{Exp}(tX))^{-1}$ for all $t \in \mathbb{R}$.

Proof. Let $t \in \mathbb{R}$ and define the homomorphisms $\varphi, \psi : \mathbb{R} \rightarrow G$ as $\varphi(s) = \exp_{tX}(s)$ and $\psi(s) = \exp_X(ts)$. We want to see that $\varphi(s) = \psi(s)$. Recall that $d_s\psi : T_s\mathbb{R} \rightarrow T_{\psi(s)}G$ and the basic element of $T_s\mathbb{R}$ is called $\frac{d}{dr}$. Moreover, by definition of \exp_{tX} we have that φ is the unique homomorphism such that

$$d_s\varphi \left(\frac{d}{dr} \right) = d_s(\exp_{tX}) \left(\frac{d}{dr} \right) = tX.$$

On the other hand we have that

$$d_s\psi \left(t \frac{d}{dr} \right) = d_s(\exp_X) \left(t \frac{d}{dr} \right) = tX$$

Then by the uniqueness of the homomorphism we have that $\varphi = \psi$ then $\exp_{tX}(s) = \exp_X(ts)$ for all $s \in \mathbb{R}$ and if we make $s = 1$ then we obtain that $\text{Exp}(tX) = \exp_{tX}(1) = \exp_X(t)$ as desired in (i).

Part (ii) follows from (i) and the fact that \exp_X is a homomorphism, because

$$\text{Exp}((t+s)X) = \exp_X(t+s) = \exp_X(t)\exp_X(s) = \text{Exp}(tX)\text{Exp}(sX).$$

In an analogous way we can get (i)

$$\text{Exp}(-tX) = \exp_X(-t) = (\exp_X(t))^{-1} = (\text{Exp}(tX))^{-1}. \quad \square$$

Proposition 2.15. *Let $\phi : H \rightarrow G$ be a Lie group homomorphism. Then the following diagram is commutative:*

$$\begin{array}{ccc} H & \xrightarrow{\phi} & G \\ \text{Exp} \uparrow & & \uparrow \text{Exp} \\ \mathfrak{h} & \xrightarrow{d\phi} & \mathfrak{g} \end{array}$$

Proof. Let $X \in \mathfrak{h}$. Then $t \mapsto \text{Exp}(tX)$ is a 1-parameter subgroup in H and if we apply the homomorphism ϕ , we obtain that $t \mapsto \phi(\text{Exp}(tX))$ is a 1-parameter subgroup in G . Now, remember that the tangent at 0 of $t \mapsto \text{Exp}(tX)$ is $X(e)$ so the tangent at 0 of $t \mapsto \phi(\text{Exp}(tX))$ is $d\phi(X(e))$.

On the other hand we have that $d\phi(X) \in \mathfrak{g}$, hence by construction of Exp we know that $t \mapsto \text{Exp}(t(d\phi(X)))$ is the unique 1-parameter subgroup of G such that the tangent at 0 is $(d\phi(X))(e)$. Thus by the uniqueness of the exponential map

$$\phi(\text{Exp}(tX)) = \text{Exp}(t(d\phi(X))),$$

and in particular

$$\phi(\text{Exp}(X)) = \text{Exp}(d\phi(X)). \quad \square$$

Lemma 2.16. *Let M, N be differential manifolds and let $f : M \rightarrow N$ be a differentiable function such that $f(M)$ is contained in a sub-manifold P . If the map $F : M \rightarrow P$ is continuous, then it is also differentiable.*

Proof. In the accompanying commutative diagram

$$\begin{array}{ccc} M & \xrightarrow{f} & N \\ & \searrow F & \uparrow i \\ & & P \end{array}$$

f and F are functions given in the hypothesis, and i is the immersion of P in N .

Since i is an immersion, for every $p \in P$ there exist neighborhoods U of p and V of $i(p)$ in N and a smooth map $g : V \rightarrow U$ such that $g \circ i = \text{id}|_U$.

$$\begin{array}{ccc} M & \xrightarrow{f} & V \\ & \searrow F & \uparrow i \\ & & U \end{array} \quad \begin{array}{c} \downarrow g \\ \downarrow \end{array}$$

Thus we have that locally $F = \text{id} \circ F = (g \circ i) \circ F = g \circ f$. Since the right side of the previous expression is a composition of diffeomorphisms we obtain that F is also a diffeomorphism as desired. \square

Proposition 2.17. *Let G be a Lie group with Lie algebra \mathfrak{g} and let H be a Lie subgroup of G with Lie algebra \mathfrak{h} . Then*

$$h = \{X \in \mathfrak{g} \mid \forall t \in \mathbb{R} : \text{Exp}(tX) \in H \text{ and } t \mapsto \text{Exp}(tX) \text{ is continuous}\}. \quad (2.1)$$

Proof. Let $\tilde{\mathfrak{h}}$ be the right side in equation (2.1). We will show a double inclusion.

We begin showing that $\mathfrak{h} \subset \tilde{\mathfrak{h}}$. For $X \in \mathfrak{h}$ by theorem 2.14 for all $t \in \mathbb{R}$ we have $\text{Exp}(tX) = \text{exp}_X(t)$ and since for all $X \in \mathfrak{h}$ we define $\text{exp}_X : \mathbb{R} \rightarrow H$ then $\text{Exp}(tX) \in H$. Moreover, recall that $t \rightarrow \text{Exp}(tX)$ is a 1-parameter subgroup then it is continuous.

To see that $\tilde{\mathfrak{h}} \subset \mathfrak{h}$, suppose for $X \in \mathfrak{g}$ that the map $\alpha(t) := \text{Exp}(tX)$ is continuous. By lemma 2.16 if we take $f = \text{Exp}_G$, $M = \mathbb{R}$, $N = G$ and $P = H$

$$\begin{array}{ccc} \mathbb{R} & \xrightarrow{\text{exp}_G} & G \\ & \searrow F & \uparrow i \\ & & H \end{array}$$

we get that $F : \mathbb{R} \rightarrow H$, which by definition is $F(t) = \text{Exp}(tX)$, is differentiable. Furthermore, since $F(0) = e$ so $X = dF(0) \in T_e(H) = \mathfrak{h}$ as desired. \square

Corollary 2.18. *If H_1 and H_2 are two Lie subgroups of the Lie group G such that they are equal as topological groups, then, they are equivalent as Lie groups.*

Proof. By Proposition 2.17 we get that the Lie algebras \mathfrak{h}_1 and \mathfrak{h}_2 are equal. Since Exp is a local diffeomorphism near the origin, by Proposition 2.15 we get that H_1 and H_2 are the same. \square

2.3 Campbell-Baker-Hausdorff formula

In the following section we will demonstrate the *Campbell-Baker-Hausdorff formula* which we will use to prove Cartan's theorem (theorem 2.25). The latter asserts that every topologically closed subgroup of a Lie group is a Lie subgroup.

Take into account that in order to prove Cartan's theorem we will need to have control over group operations. But, we have already noticed that group operations can be difficult to understand in general. To solve this issue, we will use the Campbell-Baker-Hausdorff formula. This formula will determine the group operation in terms of the Lie bracket and the vector space operations of the Lie algebra associated to the group.

Once we have complete understanding of the group operation we will be able to construct a Lie structure over a closed subgroup. This result is the main goal of this section (see theorem 2.25).

Definition 2.19. Let G be a Lie group, let f be a real valued analytic function at $p \in G$ and let \hat{X} be a vector field on G such that $X \in T_e(G)$. The *Taylor series* expansion for f is defined by the formula if there exists $\epsilon > 0$ such that, for $|t| < \epsilon$,

$$f(p\text{Exp}(tX)) := \sum_{n=0}^{\infty} \frac{t^n}{n!} (\hat{X}^n f)(p).$$

Theorem 2.20. *Let G be a Lie group and let $X, Y \in T_e(G)$. Then there exists $\epsilon > 0$ such that for all $|t| < \epsilon$ we get*

$$\text{Exp}(tX)\text{Exp}(tY) = \text{Exp}(tX + tY + \frac{t^2}{2}[X, Y] + o(t^3)).$$

where $o(t^3)$ is a vector contained in T_eG such that for $|t| < \epsilon$, $(1/t^3)o(t^3)$ is bounded and analytic.

Proof. Let f be an analytic function at $p \in G$. By the formula given in proposition 2.5 of [SW73]

$$(\hat{X}^n f)(p) := \left[\left(\frac{d}{dt} \right)^n f(p\text{Exp}(tX)) \right]_{t=0}$$

applying twice this formula f at $e \in G$ we get

$$(\hat{X}^n \hat{Y}^m f)(e) = \left[\left(\frac{d}{dt} \right)^n \left(\frac{d}{ds} \right)^m f(\text{Exp}(tX)\text{Exp}(sY)) \right]_{t=0, s=0}.$$

Moreover, the Taylor series expansion of f is

$$f(\text{Exp}(tX)\text{Exp}(tY)) = \sum_{n \geq 0, m \geq 0} \frac{t^{n+m}}{n!m!} (\hat{X}^n \hat{Y}^m f)(e)$$

where the coefficient of t is $(\hat{X}f)(e) + (\hat{Y}f)(e)$ and the coefficient of t^2 is $\frac{1}{2}(\hat{X}^2 f)(e) + (\hat{X}\hat{Y}f)(e) + \frac{1}{2}(\hat{Y}^2 f)(e)$.

However, since $\text{Exp} : U \rightarrow V$ is a diffeomorphism between a neighborhood U of $0 \in \mathfrak{g}$ and a neighborhood V of the identity in G , we can construct a differentiable inverse $\hat{F} : V \rightarrow U$ such that we can pre-compose with a differentiable function $\mathbb{R} \supset I \rightarrow V$ which will give us a differentiable function $F : \mathbb{R} \supset I \rightarrow U \subset T_e(G)$. Moreover, this F will allow us for each $t \in \mathbb{R}$ to write the product of elements in G as follows.

$$\text{Exp}(tX)\text{Exp}(tY) = \text{Exp}(F(t)).$$

Note that F has the Taylor series expansion

$$F(t) = \sum_{n \geq 0} t^n F_n$$

with $F_n \in T_e(G)$.

Since f is analytic, we can calculate $\text{Exp}(F(0)) = \text{Exp}(0)\text{Exp}(0) = e$ thus $F(0) = 0$.

$$\begin{aligned} f(\text{Exp}(tX)\text{Exp}(tY)) &= f(\text{Exp}(F(t))) \\ &= f(\text{Exp}(tF_1 + t^2F_2 + o(t^3))) \\ &= f(\text{Exp}(tF_1 + t^2F_2)) + o'(t^3) \\ &= \sum_{n \geq 0} \frac{1}{n!} ((t\hat{F}_1 + t^2\hat{F}_2)^n f)(e) + o'(t^3) \end{aligned}$$

where the last equality came from the Taylor series formula for $tF_1 + t^2F_2$.

We only need to compare the coefficients for t and t^2 in the last expression with those found at the beginning of the proof. We will that

$$\begin{aligned} (\hat{F}_1 f)(e) &= (\hat{X}g)(e) + (\hat{Y}g)(e) \\ \text{and } (\hat{F}_2 f)(e) + \frac{1}{2}(\hat{F}_1^2 f)(e) &= \frac{1}{2}(\hat{X}^2 f)(e) + (\hat{X}\hat{Y}f)(e) + \frac{1}{2}(\hat{Y}^2 f)(e), \end{aligned}$$

which leads us to the following result.

$$\begin{aligned} (\hat{F}_1 f)(e) &= (\hat{X}g)(e) + (\hat{Y}g)(e) \\ \text{and } (\hat{F}_2 f)(e) &= \frac{1}{2} \left([\hat{X}, \hat{Y}] \right) (e). \end{aligned} \quad \square$$

Corollary 2.21. *Under the same conditions as in the previous theorem, the following formulas hold:*

- (i) $\text{Exp}(t(X + Y)) = \lim_{n \rightarrow \infty} (\text{Exp}((t/n)X)\text{Exp}((t/n)Y))^n$,
- (ii) $\text{Exp}(t^2[X, Y]) = \lim_{n \rightarrow \infty} (\text{Exp}((-t/n)X)\text{Exp}((-t/n)Y)\text{Exp}((t/n)X)\text{Exp}((t/n)Y))^{n^2}$.

Proof. To show (i) fix an $t \in \mathbb{R}$ and sufficiently large n . By theorem 2.20 we get

$$\text{Exp}\left(\frac{tX}{n}\right)\text{Exp}\left(\frac{tY}{n}\right) = \text{Exp}\left(\frac{t(X + Y)}{n} + \frac{t^2[X, Y]}{2n^2} + o\left(\frac{1}{n^3}\right)\right),$$

so

$$\left(\text{Exp}\left(\frac{tX}{n}\right)\text{Exp}\left(\frac{tY}{n}\right)\right)^n = \text{Exp}\left(t(X + Y) + \frac{t^2[X, Y]}{2n} + o\left(\frac{1}{n^2}\right)\right)$$

which by the continuity of Exp give us the desired result.

To see (ii) we just need to apply the previous result and theorem 2.20.

$$\begin{aligned} &\text{Exp}\left(\frac{-tX}{n}\right)\text{Exp}\left(\frac{-tY}{n}\right)\text{Exp}\left(\frac{tX}{n}\right)\text{Exp}\left(\frac{tY}{n}\right) \\ &= \text{Exp}\left(-\frac{t(X + Y)}{n} + \frac{t^2[X, Y]}{2n^2} + o\left(\frac{1}{n^3}\right)\right)\text{Exp}\left(\frac{t(X + Y)}{n} + \frac{t^2[X, Y]}{2n^2} + o\left(\frac{1}{n^3}\right)\right) \\ &= \text{Exp}\left(\frac{t^2[X, Y]}{n^2} + o\left(\frac{1}{n^3}\right)\right). \end{aligned}$$

Thus,

$$\left(\text{Exp}\left(\frac{-tX}{n}\right)\text{Exp}\left(\frac{-tY}{n}\right)\text{Exp}\left(\frac{tX}{n}\right)\text{Exp}\left(\frac{tY}{n}\right)\right)^{n^2} = \text{Exp}\left(t^2[X, Y] + o\left(\frac{1}{n}\right)\right)$$

which give us the result when consider $n \rightarrow \infty$. □

Remark. The formula in ?? in the previous corollary should remind us of the Trotter-Kato formula [Var84] which is a generalization to certain unbounded linear operators.

Definition 2.22. We shall call $F(X, Y, \dots, Z)$ a *Lie polynomial* if it lies in the Lie algebra generated by X, Y, \dots, Z .

For example, $F(X, Y)$ is a Lie polynomial if it is a linear combination of X, Y and $[X, Y]$ and possible nested Lie brackets.

Definition 2.23. We say that two polynomials P, Q are *Lie related* or $P \equiv_{Lie} Q$ if and only if $P - Q$ is a Lie polynomial.

Theorem 2.24 (Campbell-Baker-Hausdorff theorem). *Let G be a Lie group with Lie algebra \mathfrak{g} . Let $X, Y \in \mathfrak{g}$ and let s be sufficiently close to $0 \in \mathbb{R}$. Then the multiplication can be expressed as*

$$\text{Exp}(sX)\text{Exp}(sY) = \text{Exp}(F(sX, sY)),$$

where each homogeneous term $F_n(sX, sY)$ of $F(sX, sY)$ is a Lie polynomial.

Proof. We will proceed by induction on n . Recall that each $F_n(sX, sY)$ is the homogeneous term of degree n in the series $F(sX, sY)$, which means that $F(sX, sY) = \sum_{n \geq 0} F_n(sX, sY)$. By Theorem 2.20 we have the base cases for $n = 1, 2$. As induction hypothesis we will assume that we have the result for all $k \leq n$.

From the associative law and the Taylor series of the exponential map we get that

$$(\text{Exp}(sX)\text{Exp}(sY))\text{Exp}(sZ) = \text{Exp}(sX)(\text{Exp}(sY)\text{Exp}(sZ)),$$

which by definition 2.19 and theorem 2.20 can be written as

$$\sum_{i=1}^{\infty} F_i \left(\sum_{j=1}^{\infty} F_j(sX, sY), sZ \right) = \sum_{i=1}^{\infty} F_i \left(sX, \sum_{j=1}^{\infty} F_j(sY, sZ) \right).$$

After expanding the previous expression one can see that the only terms which cannot be a Lie polynomial are $F_n(X, Y) + F_n(X + Y, Z)$ in the left side and $F_n(X, Y + Z) + F_n(Y, Z)$ on the right side. Thus, after rearranging the polynomials we get

$$F_n(X, Y) + F_n(X + Y, Z) \equiv_{Lie} F_n(X, Y + Z) + F_n(Y, Z). \quad (2.2)$$

We conclude from $\text{Exp}(aX)\text{Exp}(bX) = \text{Exp}((a + b)X)$ that $F(saX, sbX) = (a + b)sX$ so for any $k > 1$,

$$F_k(aX, bX) = 0. \quad (2.3)$$

In particular, if $a = 1$ and $b = 0$ (or vice versa) we have

$$F_k(X, 0) = F_k(0, X) = 0. \quad (2.4)$$

Moreover, since F_k is homogeneous of degree k , we get

$$F_k(aX, aY) = a^k F(X, Y). \quad (2.5)$$

First of all replace $Z = -Y$ in (2.2) and obtain

$$F_n(X, Y) + F_n(X + Y, -Y) \equiv_{Lie} F_n(X, Y - Y) + F_n(Y, -Y)$$

and by (2.3) and (2.4)

$$F_n(X, Y) \equiv_{Lie} -F_n(X + Y, -Y). \quad (2.6)$$

Now replacing $X = -Y$ in (2.2)

$$F_n(-Y, Y) + F_n(-Y + Y, Z) \equiv_{Lie} F_n(-Y, Y + Z) + F_n(Y, Z)$$

we get

$$0 \equiv_{Lie} F_n(-Y, Y + Z) + F_n(Y, Z).$$

And replacing in the previous expression $Y = X$ and $Z = Y$ we obtain

$$F_n(X, Y) \equiv_{Lie} -F_n(-X, X + Y). \quad (2.7)$$

The previous relations allow us to relate $F_n(X, Y)$ to $F_n(Y, X)$ as follows:

$$\begin{aligned} F_n(X, Y) &\equiv_{Lie} -F_n(-X, X + Y) && \text{by (2.7)} \\ &\equiv_{Lie} -(-F_n(Y, -X - Y)) && \text{by (2.6)} \\ &\equiv_{Lie} F_n(Y, -X - Y) \\ &\equiv_{Lie} -F_n(-Y, -X) && \text{by (2.7)} \\ &\equiv_{Lie} -(-1)^n F_n(Y, X) && \text{by (2.5)}. \end{aligned}$$

So the relation defined by

$$F_n(X, Y) \equiv_{Lie} -(-1)^n F_n(Y, X). \quad (2.8)$$

The next step is to replace $Z = -\frac{Y}{2}$ in (2.2) which gives us

$$\begin{aligned} F_n(X, Y) + F_n(X + Y, -Y/2) &\equiv_{Lie} F_n(X, Y/2) + F_n(Y, -Y/2) \\ &\equiv_{Lie} F_n(X, Y/2). \end{aligned} \quad (2.9)$$

Now we replace $X = -\frac{Y}{2}$ in (2.2), so

$$\begin{aligned} F_n(-Y/2, Y + Z) + F_n(Y, Z) &\equiv_{Lie} F_n(-Y/2, Y) + F_n(Y/2, Z) \\ &\equiv_{Lie} F_n(Y/2, Z). \end{aligned}$$

Thus, replacing $Y = X$ and $Z = Y$ we get

$$F_n(X, Y) + F_n(-X/2, X + Y) \equiv_{Lie} F_n(X/2, Y). \quad (2.10)$$

Relation (2.9) allows us to rewrite (2.10) as follows

$$\begin{aligned} F_n(X/2, Y) &\equiv_{Lie} F_n(X/2, Y/2) - F_n(X/2 + Y, -Y/2) && \text{by (2.9)} \\ &\equiv_{Lie} 2^{-n}F_n(X, Y) + F_n(X/2 + Y/2, Y/2) && \text{by (2.6)} \\ &\equiv_{Lie} 2^{-n}F_n(X, Y) + 2^{-n}F_n(X + Y, Y), && \text{by (2.5)}. \end{aligned}$$

$$\begin{aligned} F_n(-X/2, X + Y) & \\ &\equiv_{Lie} F_n(-X/2, X/2 + Y/2) - F_n(X/2 + Y, -X/2 - Y/2) && \text{by (2.9)} \\ &\equiv_{Lie} -F_n(X/2, Y/2) + F_n(Y/2, X/2 + Y/2) && \text{by (2.6) and (2.7)} \\ &\equiv_{Lie} -2^{-n}F_n(X, Y) + 2^{-n}F_n(Y, X + Y) && \text{by (2.5)}. \end{aligned}$$

So, (2.10) looks like

$$F_n(X, Y) \equiv_{Lie} 2^{1-n}F_n(X, Y) + 2^{-n}F_n(X + Y, Y) + 2^{-n}F_n(Y, X + Y).$$

And applying (2.8) we get

$$(1 - 2^{1-n})F_n(X, Y) \equiv_{Lie} 2^{-n}(1 + (-1)^n)F_n(X + Y, Y). \quad (2.11)$$

Notice, if n is odd, then $F_n(X, Y) \equiv_{Lie} 0$ as desired.

If n is even, we replace X by $X - Y$ in (2.11), so

$$(1 - 2^{1-n})F_n(X - Y, Y) \equiv_{Lie} 2^{1-n}F_n(X, Y). \quad (2.12)$$

The left side of the previous expression becomes

$$(1 - 2^{1-n})F_n(X - Y, Y) \equiv_{Lie} -(1 - 2^{1-n})F_n(X, -Y) \quad \text{by (2.6)}.$$

Replacing this last formula into (2.12), we obtain

$$-(1 - 2^{1-n})F_n(X, -Y) \equiv_{Lie} 2^{1-n}F_n(X, Y). \quad (2.13)$$

Finally, substituting $Y = -Y$ we get

$$\begin{aligned} -F_n(X, Y) &\equiv_{Lie} \frac{2^{1-n}}{1 - 2^{1-n}}F_n(X, -Y) \\ &\equiv_{Lie} -\left(\frac{2^{1-n}}{1 - 2^{1-n}}\right)^2 F_n(X, Y) \text{ by (2.13)} \end{aligned}$$

and since $\frac{2^{1-n}}{1-2^{1-n}} \neq 1$, hence $F_n(X, Y) \equiv_{Lie} 0$ as desired. \square

Theorem 2.25 (Cartan theorem). *Let G be a Lie group, and let H be a subgroup of G which is also a closed subset of the topological space G . Then there exists a unique differentiable structure on H such that H is a Lie subgroup of G .*

Proof. The uniqueness follows directly from corollary 2.18. We only need to show the existence of the differentiable structure. We will proceed by steps.

Step 1. We prove that \mathfrak{h} is a vector space. Let $\mathfrak{h} := \{X \in \mathfrak{g} : \text{Exp}(tX) \in H \text{ for all } t \in \mathbb{R}\}$. For any $X \in \mathfrak{h}$ and $s \in \mathbb{R}$, we get by Theorem 2.14 and definition of \mathfrak{h} that $\text{Exp}(t(sX)) = {}_{sX}(t) = \exp_X(st) = \text{Exp}((st)X) \in H$. Moreover, for any X and $Y \in \mathfrak{h}$ for any $t \in \mathbb{R}$ we get by (i) in corollary 2.21

$$\text{Exp}(t(X + Y)) = \lim_{n \rightarrow \infty} (\text{Exp}(t/n)X \text{Exp}(t/n)Y)^n \in H$$

since H is a closed subspace. Clearly, applying (ii) of the same corollary as before, we get that $[X, Y] \in \mathfrak{h}$.

Step 2. Let \mathfrak{m} be a subspace of \mathfrak{g} such that $\mathfrak{g} = \mathfrak{m} \oplus \mathfrak{h}$. Then there exists a neighborhood V of the $0 \in \mathfrak{m}$ such that if $0 \neq X \in V$ implies $\text{Exp}X \notin H$. By contradiction, suppose that every neighborhood V_n exists an element $0 \neq X_n \in V_n \subset \mathfrak{m}$ and $\text{Exp}X_n \in H$. By construction, $\lim_{n \rightarrow \infty} X_n = 0$. Since \mathfrak{g} is a finite dimensional vector space, we can consider its Euclidean norm $\|\cdot\|$ (note that this choice can be made since all norms are equivalent in finite dimensional vector spaces). Let $k := \{X \in \mathfrak{m} : 1 \leq \|X\| \leq 2\}$. Moreover, we will consider integer n_i such that $Y_i := n_i X_i \in k \subset \mathfrak{m}$. Since k is compact (is a bounded closed set in a Euclidean space) we have that $\lim_{i \rightarrow \infty} Y_i = Y \in k$ exists. It could be necessary to pass to a subsequence, but we will rename it to coincide with the original numeration.

Since $\lim_{i \rightarrow \infty} X_i = 0$ we see that $\lim_{i \rightarrow \infty} 1/n_i = 0$ and $\text{Exp}(Y_i/n_i) = \text{Exp}(X_i) \in H$. Now we will prove that $Y \in \mathfrak{h}$ which will lead to a contradiction since $\mathfrak{h} \cap \mathfrak{m} = \{0\}$ and $\|Y_j\| \geq 1$. Since $\text{Exp}(-Y_i/n_i) = \text{Exp}(Y_i/n_i)^{-1} \in H$, we can assume without lost of generality that $n_i > 0$. By definition of \mathfrak{h} and the previous assumption, $Y \in \mathfrak{h}$ if $\text{Exp}(tY) \in H$ for every real number $t > 0$. Let us define

$$k_i(t) := \lfloor tn_i \rfloor,$$

where $\lfloor \cdot \rfloor$ is the integer part function.

Since $tn_i - 1 < k_i(t) \leq tn_i$ and $\lim_{i \rightarrow \infty} 1/n_i = 0$ we get

$$\lim_{i \rightarrow \infty} k_i(t)/n_i = t.$$

Moreover, $\text{Exp}(k_i(t)Y_i/n_i) = \text{Exp}(Y_i/n_i)^{k_i(t)} \in H$ and $\lim_{i \rightarrow \infty} k_i(t)Y_i/n_i = tY$. However, since the exponential map is continuous, we get that

$$\text{Exp}(tY) = \lim_{i \rightarrow \infty} \text{Exp}(k_i(t)Y_i/n_i) \in \mathfrak{h}.$$

By definition of \mathfrak{h} we have $Y \in \mathfrak{h}$, which is the desired contradiction.

Step 3. We will show that there exists a neighborhood U of $e \in G$ such that

$$U \cap H = U \cap \text{Exp}(\mathfrak{h}) = \text{Exp}(V')$$

for some neighborhood V' of the zero in \mathfrak{h} . To show this, recall that Exp is a local diffeomorphism. So, we can find coordinates which allow us to choose neighborhoods $0 \in W \subset \mathfrak{h}$, $0 \in W' \subset \mathfrak{m}$ and U of the identity in G such that for $\widehat{W} = W \oplus W'$

$$\phi : \widehat{W} \rightarrow U, X + X' \mapsto \text{Exp}X\text{Exp}X'$$

is a diffeomorphism. Moreover if $x \in \phi(\widehat{W})$ we have $x = \text{Exp}(X)\text{Exp}(X')$ for some $X \in W \subset \mathfrak{h}$ and $X' \in W' \subset \mathfrak{m}$. Thus, if $x \in U \cap H$, then $\text{Exp}(X)\text{Exp}(X') \in H$ which implies $\text{Exp}(X') \in H$ since H is a group. We can shrink W' to fit V from *Step 2* and we see that $X' = 0$ which implies that $U \cap H = U \cap \text{Exp}(\mathfrak{h}) = \text{Exp}(V')$.

Step 4 We claim that H is a differentiable submanifold of G . Recall the useful characterization for submanifolds given by Proposition 2.11 of [SW73] which asserts

Proposition 2.26. *Let M be a submanifold of dimension m of a manifold N with dimension n . Then there exists a coordinate system z_1, \dots, z_n on a neighborhood V of $p \in M$ such that:*

- (i) $z_1(p) = \dots = z_n(p) = 0$.
- (ii) *The set $W := \{r \in V : z_{m+1}(r) = \dots = z_n(r) = 0\}$ together with the restrictions of z to W form a chart of M at p .*

Conversely, if a subset $M \subset N$ has a coordinate system at $p \in M$ satisfying (i) and (ii), then M is a submanifold on N .

Thus, for any $p \in H$ we must show that there is a neighborhood $U \ni p$ such that $U \cap H$ is a submanifold of U . Since we can translate any of these coordinate systems to the identity, it will suffice to find such coordinate functions which fulfill the conditions in the proposition on a neighborhood U of the identity.

But such a neighborhood was constructed in *Step 3* using the canonical coordinate system imposed by the exponential map. And, the required neighborhood W of the proposition to be $U \cap H$. Moreover, the topology in H makes the function $t \rightarrow \text{Exp}(tX)$ continuous for every $X \in \mathfrak{h}$. Finally, by Proposition 2.17, \mathfrak{h} is the corresponding Lie algebra of H . \square

Chapter 3

Gleason metrics and the construction of a Lie structure

3.1 Topological vector space

Hilbert's fifth problem concerns about the minimal hypothesis needed to place a differentiable structure (Lie structure) on a topological group. In order to achieve this goal we will start working on topological vector spaces.

Definition 3.1 (Topological vector space). A *topological vector space* is an \mathbb{R} -vector space V with a topology such that the vector space operations $+$: $V \times V \rightarrow V$ and \cdot : $\mathbb{R} \times V \rightarrow V$ are jointly continuous.

Here it is important to note that we require the operations to be *jointly* continuous, which means that we do not only require the sum and scalar product to be continuous in each component, but we need them to be continuous as functions of the product space $\mathbb{R} \times V$ with the product topology into V .

Example 3.2. Consider \mathbb{R} with the compact topology where a non-empty set is open if and only if its complement is compact. With this topology the space is not Hausdorff, the scalar product \cdot : $\mathbb{R} \times \mathbb{R} \rightarrow \mathbb{R}$ is jointly continuous. Addition is continuous in each component yet not jointly continuous since the set $+^{-1}([0, 1]^c) = \{(x, y) \in \mathbb{R} \times \mathbb{R} : x + y \notin [0, 1]\}$ cannot contain any open set in the product topology induced by the compact topology of \mathbb{R} .

We will be highly interested in those topological vector spaces whose topology is generated by a norm $\|\cdot\|$. In the finite dimensional case the following theorem will be highly helpful.

The proof of upcoming theorem is based on the proof given [[Cas12](#)].

Theorem 3.3. *Every finite-dimensional Hausdorff topological vector space has the topology induced by its norm.*

Proof. Let V be a finite-dimensional Hausdorff topological vector space, with topology \mathfrak{F} . We will denote \mathfrak{F}_n the topology induced by the norm. And we will identify $(V, \mathfrak{F}_n) \cong (\mathbb{R}^n, \mathfrak{F}_n)$. We need to show that $\mathfrak{F} = \mathfrak{F}_n$.

Since V is finite dimensional, we can choose a basis $\{v_1, v_1, \dots, v_n\}$ for V , and define the linear bijection

$$T : (V \cong \mathbb{R}^n, \mathfrak{F}_n) \rightarrow (V, \mathfrak{F})$$

as

$$T((x_1, x_2, \dots, x_n)) := \sum_{i=1}^n x_i v_i.$$

It will be enough to show that T is a homeomorphism. T is clearly continuous with respect to \mathfrak{F} since scalar multiplication and addition are continuous in (V, \mathfrak{F}) .

To prove that T is an open map, we need to take an open ball in \mathfrak{F}_n and see that its image under T is open in (V, \mathfrak{F}) .

It will suffice to verify the existence of an open bounded neighborhood of the origin in \mathfrak{F} . Since, it can be continuously translated and dilated to form a basis of \mathfrak{F}_n .

Since T is continuous, it sends compact sets in \mathfrak{F}_n onto compact sets of \mathfrak{F} . In particular for $S^{n-1} := \{x \in \mathbb{R}^n : \|x\| = 1\}$ we get $T(S^{n-1})$ is compact in V . Using the hypothesis of V being Hausdorff, we can take for all $x \in T(S^{n-1})$ disjoint open sets U_x and W_x such that $0 \in U_x$ and $x \in W_x$. Clearly $\{W_x\}_x$ is an open covering of $T(S^{n-1})$ and by its compactness there exist x_1, \dots, x_n such that $T(S^{n-1}) \subset \bigcup_{i=1}^n W_{x_i}$. Hence, by construction $0 \in \bigcap_{i=1}^n U_{x_i} =: U$ is an open neighborhood of the origin in V disjoint from $T(S^{n-1})$.

Such U could be unbounded. In order to show that U is bounded we proceed as follows. Using the jointly continuity of the scalar product we can find an open interval $(-\epsilon, \epsilon)$ in \mathbb{R} and an open neighborhood U' of the origin in V such that $U'' := (-\epsilon, \epsilon)U'$ is contained U . Note that U'' is disjoint from $T(S^{n-1})$ since U was disjoint from $T(S^{n-1})$ by construction. By construction U'' is the bounded open neighborhood of the origin which we were looking for. \square

Corollary 3.4. *In a Hausdorff topological vector space V , every finite-dimensional subspace W is closed.*

Proof. We only need to prove that $V \setminus W$ is open. Clearly for all $x \in V \setminus W$ the subspace $U := \text{span } \{W \cup \{x\}\}$ is finite-dimensional. After applying the previous theorem we can easily separate x from W since U has the usual norm topology. \square

Theorem 3.5. *Let V be a locally compact Hausdorff topological vector space. Then V is isomorphic to some \mathbb{R}^d as a topological vector space.*

Proof. Let V be a locally compact Hausdorff topological vector space. There exists a compact neighborhood K of the origin. Moreover, the set $\frac{1}{2}K$ is also a neighborhood of

the origin. Clearly the family $\{x + \frac{1}{2}\overset{\circ}{K}\}_{x \in K}$ is a covering of K and by its compactness there exist $\{x_1, \dots, x_s\} =: B$ such that

$$K \subset \bigcup_{i=1}^s \left(x_i + \frac{1}{2}\overset{\circ}{K} \right) \subset B + \frac{1}{2}K.$$

If we take $W := \text{span } \{B\}$ we have $K \subset W + \frac{1}{2}K$ and if we cover K again we obtain that

$$K \subset W + \frac{1}{2}K \subset W + \frac{1}{2} \left(W + \frac{1}{2}K \right) = W + \frac{1}{4}K.$$

Iterating this process n -times we obtain the following result:

$$K \subset W + 2^{-n}K.$$

If we fix any neighborhood U of the origin in V , then for all $x \in V$ there exists $n \in \mathbb{N}$ such that $2^{-n}x \in U$. Since U is a neighborhood of the origin, there exist an open set B such that $B \subset U$. Moreover, by the compactness of K , we know that K is bounded, so we can find an open set A such that $K \subset A$.

Hence, there exists an $N \in \mathbb{N}$ such that $2^{-N}A \subset B$. Taking the same M we have that

$$2^{-N}K \subset 2^{-N}A \subset B \subset U.$$

So we obtain that for any neighborhood of the origin U , there exists a sufficiently large N such that $2^{-N}K \subset U$, consequently $K \subset W + U$. Finally, since U is arbitrary, we have that

$$K \subset \overline{W}.$$

By corollary 3.4, since W is finite-dimensional, we have that $\overline{W} = W$; so $K \subset W$.

Finally, by construction we have $W \subset V$. On the other hand, for any $x \in V$ there exists an $N \in \mathbb{N}$ such that $2^{-N}x \in K \subset W$. Since W is a subspace and $x = 2^N 2^{-N}x \in W$, we have that $V \subset W$. This leads us to conclude that $V = W$. \square

3.2 Strong Gleason metric

Now we will introduce the metric structure that will allow us to construct a Lie structure in our given topological group. Here we will only see how to construct such differentiable structure based on a suitable metric that we will impose. Later on, we will see the necessary conditions under which we can impose such metric on a given topological group.

Before we start our discussion of the metric, which we will call a *strong Gleason Metric*, we are going to introduce an auxiliary norm-like function. It will help us to control the Gleason metric.

Definition 3.6 (Escape norm). Let G be a group. The *escape norm* associated to an open neighborhood U of the identity is a function $\|\cdot\|_U : G \rightarrow [0, 1]$ defined by

$$\|g\|_U := \inf_{n \in \mathbb{N}} \left\{ \frac{1}{n+1} : g, g^2, \dots, g^n \in U \right\}.$$

Note that if $g \notin U$ then $\|g\|_U = 1$, and $\|g\|_U = 0$ if and only if $\langle g \rangle \subset U$.

Definition 3.7 (strong Gleason metric). Let G be a topological group. A *strong Gleason metric* on G is a left-invariant metric $d : G \times G \rightarrow \mathbb{R}^+$ which generates the topology on G . Additionally we denote $\|g\| := d(g, e)$ where $e \in G$ is the identity element. Also we require d to fulfill the following properties for some fixed constant $C > 0$:

- (i) (Escape property) If $g \in G$ and $n \geq 1$ such that $n\|g\| \leq \frac{1}{C}$ then

$$\|g^n\| \geq \frac{1}{C}n\|g\|.$$

- (ii) (Commutator estimate) If $g, h \in G$ are such that $\|g\| \leq \frac{1}{C}$ and $\|h\| \leq \frac{1}{C}$ then

$$\|[g, h]\| \leq C\|g\|\|h\|.$$

where $[g, h] = g^{-1}h^{-1}gh$ is the commutator of two elements of the group.

- (iii) (Escape norm comparison) For a sufficiently small open neighborhood U of the identity we get that there exist positive constants K_1, K_2 such that

$$K_1\|g\|_U \leq \|g\| \leq K_2\|g\|_U.$$

It is important to note that $\|g\|$ is not a norm since G is not a vector space. Yet, the reader should think of $\|g\|$ as the *size of g* relative to the identity. In this sense $\|g\|$ somehow resembles an actual norm. Moreover, we will refer to $\|g\|$ as the norm of g .

Remark. Notice that the escape property in the previous definition gives us some local control over the norm of small elements. In other words, if an element $g \in G$ is small enough then the norm of the power of g is linearly related to the norm of g thanks to the triangle inequality presented in proposition 3.9

$$\frac{1}{C}n\|g\| \leq \|g^n\| \leq n\|g\|.$$

The following theorem gives a solution of the Hilbert's fifth problem under certain conditions on G .

Theorem 3.8. (*Lie structure from strong Gleason metrics*) Let G be a locally compact group with a strong Gleason metric. Then, G is isomorphic to a Lie group.

In order to prove this theorem we will need to do some hard work on the construction of an exponential map that will allow us to achieve the desired Lie structure.

The above theorem will be proven at the end of this chapter for which it is important to keep it in mind. Since most of the work presented here will be highly technical and one could lost sight of the ultimate goal of such work.

Proposition 3.9. *Let G be a locally compact group with a strong Gleason metric. Then we have the following results for any $0 < \epsilon < \frac{1}{C}$:*

- (i) $\|g^{-1}\| = \|g\|$,
- (ii) $d(g, h) = \|g^{-1}h\|$,
- (iii) $\|g_1 \cdots g_n\| \leq \sum_{i=1}^n \|g_i\|$,
- (iv) $\|ghg^{-1}\| \sim \|h\|$ whenever $\|g\|, \|h\| < \epsilon$,
- (v) $d(gk, hk) \sim d(g, h)$ whenever $\|g\|, \|h\|, \|k\| < \epsilon$, (approximate right invariance),
- (vi) $d(gh, hg) \ll \|g\|\|h\|$ whenever $\|g\|, \|h\| < \epsilon$.

Proof. (ii) follows immediately as $d(g, h) = \|h^{-1}g\| = \|(h^{-1}g)^{-1}\| = \|g^{-1}h\|$. Clearly (i) follows from (ii) taking $h = e$.

For (iii) a simple induction on n will give us the result, so we will only prove the basic case with $n = 2$. Consider $\|g_1g_2\| = d(g_1g_2, e) \leq d(g_1g_2, g_1) + d(g_1, e) = d(g_2, e) + \|g_1\| = \sum_{i=1}^2 \|g_i\|$.

To prove (iv) we will prove two inequalities, beginning with

$$\begin{aligned} \|ghg^{-1}\| &= \|ghg^{-1}h^{-1}h\| \leq \|[g, h]\| + \|h\| \\ &\leq C\|g\|\|h\| + \|h\| \leq (C\epsilon + 1)\|h\|. \end{aligned}$$

In the second step we used (iii) and (iv) and in the forth step we used the commutator estimate of definition 3.7.

On the other hand, applying the above inequality to g^{-1} instead of g and ghg^{-1} instead of h , we find

$$\|h\| = \|g^{-1}(ghg^{-1})g\| \leq (C\epsilon + 1)\|ghg^{-1}\|,$$

which proves (iv).

To prove the approximate right invariance (v) we only need to apply the previous result $d(gk, hk) = \|k^{-1}h^{-1}gk\| \sim \|h^{-1}g\| = d(g, h)$. Note that here it was necessary that $\|g\|, \|h\|, \|k\| < \epsilon$ to apply the result above.

Finally, in a similar way we can prove (vi) by calculating $d(gh, hg) = \|g^{-1}h^{-1}gh\| \leq C\|g^{-1}\|\|h^{-1}\| = C\|g\|\|h\|$ whenever $\|g\|, \|h\| < \epsilon$, so we get $d(gh, hg) \ll \|g\|\|h\|$. \square

Theorem 3.10. *Let G be a locally compact group with a strong Gleason metric. Then G is complete with this metric.*

Proof. Take a Cauchy sequence $\{g_n\}_n \subset G$. Then for all $\epsilon > 0$ there exists $N \in \mathbb{N}$ such that for all $n, m \geq N$ the distance $d(g_n, g_m) < \epsilon$. By the left invariance this can be restated as $d(g_m^{-1}g_n, e) = \|g_m^{-1}g_n\| < \epsilon$, hence

$$g_m^{-1}g_n \longrightarrow e$$

as $n, m \rightarrow \infty$.

By the local compactness of G we can find a compact neighborhood U of the identity and a natural number N such that whenever $n, m \geq N$, we have that $\{g_m^{-1}g_n\}_{n,m} \subset U$.

Let us take the sequence $\{g_N^{-1}g_n\}_n \subset \{g_m^{-1}g_n\}_{n,m} \subset U$. Since U is compact in a metric space, the sequence $\{g_N^{-1}g_n\}_n$ has a convergent subsequence $\{g_N^{-1}g_{n_k}\}_{n_k}$ such that

$$\lim_{n_k \rightarrow \infty} g_N^{-1}g_{n_k} = h \in U.$$

Since we fixed g_N^{-1} , the continuity of multiplication on G implies

$$\lim_{n_k \rightarrow \infty} g_{n_k} = g_N h.$$

So we found a convergent subsequence of $\{g_n\}_n$, and by the Cauchy assumption $\{g_n\}_n$ converges to $g_N h$. Therefore, G is complete under the strong Gleason metric d . \square

Definition 3.11. Let X be any set and $f, g : X \rightarrow \mathbb{R}$ functions. We say that

$$f \ll g$$

if there exists a positive constant K such that

$$f(x) \leq Kg(x)$$

for all $x \in X$.

Moreover, we will say that $f \sim g$ if $f \ll g \ll f$.

With this result in mind we will prove the following extremely useful lemma.

Lemma 3.12. *Let $0 < \epsilon < \frac{1}{C}$. If $n \geq 1$ and $\|g\|, \|h\| < \frac{\epsilon}{n}$ then*

$$(i) \quad d(g^n h^n, (gh)^n) \ll n^2 \|g\| \|h\|,$$

$$(ii) \quad d(g^n, h^n) \sim nd(g, h).$$

where the corresponding constants are independent of n, g and h .

Proof. We will begin to prove (i) by calculating

$$\begin{aligned} d(g^n h^n, (gh)^n) &= \|h^{-n} g^{-n} (gh)^n\| \\ &= \left\| \prod_{i=0}^{n-1} h^{i-n} g^{i-n} gh g^{n-i-1} h^{n-i-1} \right\| \\ &\leq \sum_{i=0}^{n-1} \|h^{i-n} g^{i-n} gh g^{n-i-1} h^{n-i-1}\|. \end{aligned}$$

This show us that if we have $\|h^{i-n} g^{i-n} gh g^{n-i-1} h^{n-i-1}\| \ll n\|g\|\|h\|$, then

$$d(g^n h^n, (gh)^n) \ll \sum_{i=0}^{n-1} n\|g\|\|h\| = n^2\|g\|\|h\|.$$

To prove the desired inequality let us do the following calculations using the results of proposition 3.9 and the left invariance of d

$$\begin{aligned} \|h^{i-n} g^{i-n} gh g^{n-i-1} h^{n-i-1}\| &= d(g^{n-i} h^{n-i}, gh g^{n-i-1} h^{n-i-1}) \\ &= d(g^{n-i-1} h^{n-i}, h g^{n-i-1} h^{n-i-1}) \\ &\sim d(g^{n-i-1} h, h g^{n-i-1}) \\ &\ll \|g^{n-i-1}\|\|h\| \ll (n-i-1)\|g\|\|h\| \\ &\ll n\|g\|\|h\|. \end{aligned}$$

This bound gives us the desired result.

To prove the second estimate (ii) notice that we are able to use the escape property since we can choose an ϵ small enough to get $2\epsilon \leq \frac{1}{C}$ so $\|k\| \leq \frac{2\epsilon}{n} \leq \frac{1}{nC}$. Call $k = h^{-1}g$. We obtain that $\|k\| = \|h^{-1}g\| = d(h, g) \leq \frac{2\epsilon}{n}$. Then we can use the triangle inequality to see that

$$\begin{aligned} d(g^n, h^n) &\leq d(g^n, h^n k^n) + d(h^n k^n, h^n) \ll d((hk)^n, h^n k^n) + \|k^n\| \\ &\ll n^2\|h\|\|k\| + n\|k\| = n\|k\|(n\|h\| + 1) \leq n\|k\|(\epsilon + 1) \end{aligned}$$

and

$$d(g^n, h^n) = d(h^n k^n, h^n) = \|k^n\| \geq \frac{1}{C} n\|k\|.$$

In conclusion we get that $d(g^n, h^n) \sim nd(g, h)$. □

3.3 Construction of a Lie structure from a strong Gleason metric

The construction of a Lie structure on a locally compact with strong Gleason metric group G will strongly rely on the definition of an *exponential map* given in definition 3.18. This

map will allow us to give G the differentiable structure that we are looking for. Now remember that in a Lie group we have a correspondence between the elements of its Lie algebra and the 1-parameter subgroups of G . This is the reason to start working in the space

$$L(G) := \{\varphi : \mathbb{R} \rightarrow G : \varphi \text{ is a continuous homomorphism}\}$$

of all 1-parameter subgroups of G .

Note that if G is a Lie group, then $L(G)$ must be isomorphic to an euclidean space since we could identify $L(G)$ with the lie algebra of G which is already an euclidean space. So the following part of this work will focus on the demonstration some important properties of $L(G)$ and the proof that it is a finite-dimensional topological vector space. By theorem 3.3 it is then isomorphic to some euclidean space.

Before we get into the detail of the construction of such a Lie structure, we will show a little useful result concerning basic analysis.

Theorem 3.13. *Let $f, g : X \rightarrow \mathbb{R}^+$ be real-valued bounded functions with bounds $C_1, C_2 > 0$ respectively and $f(x) \neq 0$. Suppose for all $x \in X$ and $n \in \mathbb{N}$ such that:*

$$f(x) < \frac{C_1}{n} \implies g(x) < \frac{C_2}{n}.$$

Then there exists a constant $K > 0$ such that for all $x \in X$

$$g(x) \leq Kf(x).$$

Proof. Suppose by contradiction that for all $n \in \mathbb{N}$ there exists an element $x_n \in X$ such that

$$f(x_n) < \frac{C_1}{n} \text{ and } g(x_n) < \frac{C_2}{n}$$

and $g(x_n) > nf(x_n)$.

Under the previous conditions we obtain

$$f(x_n) < \frac{g(x_n)}{n} \leq \frac{C_2}{n} = \frac{C_1}{\frac{nC_1}{C_2}} \leq \frac{C_1}{n \lfloor \frac{C_1}{C_2} \rfloor}$$

so

$$f(x_n) < \frac{C_1}{n \lfloor \frac{C_1}{C_2} \rfloor}.$$

By the hypothesis we get $g(x_n) < \frac{C_2}{n \lfloor \frac{C_1}{C_2} \rfloor}$. This inequality and $nf(x_n) < g(x_n)$ gives us

$$f(x_n) < \frac{C_2}{n^2 \lfloor \frac{C_1}{C_2} \rfloor} = \frac{C_1}{n^2 \lfloor \frac{C_1}{C_2} \rfloor \frac{C_1}{C_2}} \leq \frac{C_1}{n^2 \lfloor \frac{C_1}{C_2} \rfloor^2}.$$

We can repeat this process, which will give us as result

$$f(x_n) < \frac{C_1}{n^k \lfloor \frac{C_1}{c_2} \rfloor^k}$$

for every natural number k . Therefore, $f(x_n) = 0$ which is a contradiction. \square

Notice that in the previous theorem the constant K is independent of the parameters of the functions f and g . This fact will be fundamental for the future applications of this result.

Now we give $L(G)$ the **compact-open topology** which is generated by the balls

$$B_{r,K}(\varphi_0) = \{\varphi \in L(G) : \sup_{t \in K} \{d(\varphi(t), \varphi_0(t))\} < r\}$$

for $\varphi_0 \in L(G)$, $r > 0$ and a compact interval $K \subset \mathbb{R}$. Using the homomorphism property of 1-parameter subgroups, we can choose an arbitrary compact interval such as $[-1, 1]$.

Lemma 3.14. *$L(G)$ is locally compact.*

Proof. Since $L(G)$ has compact-open topology by theorem 46.8 and exercise 46.10 of [Mun00] we get that $L(G)$ is complete if (G, d) is complete. We already had this result in theorem 3.10.

Fix some $\varphi_0 \in L(G)$. By the continuity of φ_0 at 0 we have that for any $\epsilon > 0$ there exists an interval $I = [-T, T] \ni 0$ such that for all $t \in I$ we obtain that $\|\varphi_0(t)\| = \|\varphi_0(t)\varphi_0(0)^{-1}\| < \epsilon$.

Consider the open neighborhood $B := B_{\epsilon, I}(\varphi_0)$ of φ_0 . To prove that $L(G)$ is locally compact we will use the Arzelà -Ascoli theorem, which asserts that if X is a compact space, Y is a complete metric space. And $E \subset C(X, Y)$, then \overline{E} is compact in the compact-open topology if and only if E is equicontinuous and for all $x \in X$ the set $\{\overline{f(x)} : f \in E\}$ is compact in Y . So, we only need to show that \overline{B} is compact.

In this case we will take $X = I$, $Y = G$ and $E = B$. First of all, note that for each $\varphi \in B$ and $t \in I$ we have

$$\begin{aligned} \|\varphi(t)\| &= \|\varphi(t) - \varphi_0(t) + \varphi_0(t)\| \leq \|\varphi(t) - \varphi_0(t)\| + \|\varphi_0(t)\| \\ &\leq d(\varphi(t), \varphi_0(t)) + \epsilon \leq \sup_{t \in I} \{d(\varphi(t), \varphi_0(t))\} + \epsilon \leq 2\epsilon \end{aligned}$$

which means that the family is point-wise bounded. Since we get a globally uniform bound.

We will verify that B is equicontinuous. We will show that every $\varphi \in B$ is Lipschitz continuous with the same Lipschitz bound. We will go with an argument similar to the one presented in theorem 3.13. We will see if whenever $\frac{|t|}{T} \leq 1$ and $n \geq 1$ implies $\|\varphi(t/n)\| \leq \epsilon/n$, then there exists a positive constant M such that for every $t \in I$

$$\|\varphi(t)\| \leq M \frac{|t|}{T} \epsilon.$$

Suppose that for every natural number $N \in \mathbb{N}$ there exists some $t_N \in I$ such that $\frac{|t_N|}{T} \leq 1$ implies $\|\varphi(t_N/n)\| \leq \epsilon/n$. But $\|\varphi(t_N)\| > N \frac{|t_N|}{T} \epsilon$.

Since $t_N \in I$ we will always have $|t_N|/T \leq 1$ and by hypothesis $\|\varphi(t_N/n)\| \leq \epsilon/n$ so we get

$$N \frac{|t_N|}{T} \epsilon < \|\varphi(t_N)\| \leq N \|\varphi(t_N/n)\| \leq \epsilon$$

thus $N \frac{|t_N|}{T} \leq 1$. By hypothesis $\|\varphi(Nt_N/n)\| \leq \frac{\epsilon}{n}$. In particular $\|\varphi(t_N/N)\| \leq \frac{\epsilon}{N}$.

Moreover, by the contradiction hypothesis

$$N \frac{|t_N|}{T} \epsilon < \|\varphi(Nt_N)\| \leq n \|\varphi(Nt_N/n)\| \leq \epsilon.$$

This implies that $\frac{N^2|t_N|}{T} \leq 1$. Thus, by hypothesis $\|\varphi(N^2t_N/n)\| \leq \epsilon/n$. and we obtain $\|\varphi(t_N/N)\| \leq \epsilon/N^3$.

Iterating this argument, as in theorem 3.13, we get that

$$\frac{|t_N| N^k}{T} \leq 1$$

which leads us to

$$\|\varphi(N^k t_N/n)\| \leq \frac{\epsilon}{n}.$$

Furthermore,

$$\|\varphi(t_N/N)\| \leq \frac{\epsilon}{N^{k+1}},$$

which goes to zero as $k \rightarrow \infty$. This is a contradiction since $\frac{|t_N|}{N}$ is different from 0.

In conclusion we get that $\|\varphi(t)\| \leq M \frac{|t|}{T} \epsilon$. So, we can see that for all $t, t' \in I$ we conclude that

$$\begin{aligned} d(\varphi(t), \varphi(t')) &= d(\varphi(t')^{-1} \varphi(t), e) = \|\varphi(-t') \varphi(t)\| \\ &= \|\varphi(t - t')\| \leq |t - t'| \frac{\epsilon}{T}. \end{aligned}$$

Showing that every $\varphi \in B$ is Lipschitz with constant $\frac{\epsilon}{T}$. Then, B is equicontinuous and by Arzela-Ascoli \overline{B} is compact as desired. \square

Definition 3.15. On $L(G)$ we define the following operations

- (i) $(c\varphi)(t) := \varphi(ct)$,
- (ii) $(\varphi + \psi)(t) := \lim_{n \rightarrow \infty} (\varphi(t/n) \psi(t/n))^n$.

for any $\varphi, \psi \in L(G)$ and $c \in \mathbb{R}$.

Remark. To prove that a given sequence $\{a_n\}_n$ is Cauchy, fix an $\epsilon > 0$ and you want to see that there exists a natural number N such that

$$d(a_n, a_m) \leq \epsilon$$

for all $n, m \geq N$.

Note that it is enough to show that for all $\epsilon > 0$ there exists an $N > 0$ such that for all $n \geq N$ and $m \geq 1$ $d(a_n, a_{nm}) \leq \epsilon$, because

$$d(a_n, a_m) \leq d(a_n, a_{nm}) + d(a_{nm}, a_{n'}) < 2\epsilon$$

for every $n, m \geq N$.

Lemma 3.16. *If $\varphi, \psi \in L(G)$, then $\varphi + \psi$ is well defined and lies in $L(G)$.*

Proof. Well defined. Since $L(G)$ is complete it is sufficient to show that $\{(\varphi(t/n)\psi(t/n))^n\}_n$ is a Cauchy sequence. By previous remark it is enough to prove

$$d((\varphi(t/n)\psi(t/n))^n, (\varphi(t/nm)\psi(t/nm))^{nm}) \rightarrow 0$$

for n going to infinity independently of t . We are going to prove a slightly stronger claim:

$$d((\varphi(t/n)\psi(t/n))^{n'}, (\varphi(t/nm)\psi(t/nm))^{n'm}) \rightarrow 0$$

for $n, m \rightarrow \infty$ and $1 \leq n' \leq n$.

Applying lemma 3.12 we get that

$$\begin{aligned} & d(\varphi(t/n)\psi(t/n), (\varphi(t/(nm))\psi(t/(nm)))^m) \\ &= d(\varphi(t/(nm))^m \psi(t/(nm))^m, (\varphi(t/nm))\psi(t/(nm))^m) \\ &\ll m^2 \|\varphi(t/(nm))\| \|\psi(t/nm)\| \\ &\ll m^2 (\epsilon/(nm))^2 = (\epsilon/n)^2. \end{aligned}$$

where the last inequality comes from the Lipschitz bound given in theorem 3.14.

Applying lemma 3.12 again we get that

$$d((\varphi(t/n)\psi(t/n))^{n'}, (\varphi(t/(nm))\psi(t/(nm)))^{n'm}) \ll n'(\epsilon/n)^2 \leq \epsilon^2/n.$$

In this way we obtain the claim that gives us the result taking $n' = n$. Thus, the point-wise limit exists and since we are working in a space with the compact-open topology by corollary 46.4 and theorem 46.8 in [Mun00] we get that the limit is continuous as desired.

Homomorphism. By the density of Q in \mathbb{R} and the continuity of the elements on $L(G)$ to prove that $\varphi + \psi$ is a homomorphism it suffices to show that

$$(\varphi + \psi)((p + q)t) = (\varphi + \psi)(pt)(\varphi + \psi)(qt)$$

and

$$(\varphi + \psi)(-t) = (\varphi + \psi)(t)^{-1}$$

for any $p, q \in \mathbb{Q}$. Recall that p, q can be written without loss of generality as $p = a/c$ and $q = b/c$ with $a, b, c \in \mathbb{N}$, which will give us the following equality to prove:

$$(\varphi + \psi) \left(\left(\frac{a+b}{c} \right) t \right) = (\varphi + \psi) \left(\frac{a}{c} t \right) (\varphi + \psi) \left(\frac{b}{c} t \right).$$

By definition of the sum

$$(\varphi + \psi) \left(\frac{a}{c} t \right) = \lim_{n \rightarrow \infty} \left(\varphi \left(\frac{at}{nc} \right) \psi \left(\frac{at}{nc} \right) \right)^n.$$

Since $n \rightarrow \infty$, also $cn \rightarrow \infty$, so it will be enough to show that

$$(\varphi + \psi)((a+b)t) = (\varphi + \psi)(at)(\varphi + \psi)(bt)$$

for any integers a, b .

First of all using lemma 3.12 we can estimate the distance

$$d(\varphi(t/n)^a \psi(t/n)^a, (\varphi(t/n)\psi(t/n))^a) \ll a^2 \|\varphi(t/n)\| \|\psi(t/n)\| \ll \left(\frac{\epsilon}{n} \right)^2$$

and

$$d((\varphi(t/n)^a \psi(t/n)^a)^n, (\varphi(t/n)\psi(t/n))^{an}) \ll n \left(\frac{a\epsilon}{n} \right)^2 = \frac{a^2 \epsilon^2}{n} \rightarrow 0$$

as $n \rightarrow \infty$. Then, in the limit we have that

$$\begin{aligned} (\varphi + \psi)(at) &= \lim_{n \rightarrow \infty} (\varphi(at/n)\psi(at/n))^n = \lim_{n \rightarrow \infty} (\varphi(t/n)^a \psi(t/n)^a)^n \\ &= \lim_{n \rightarrow \infty} (\varphi(t/n)\psi(t/n))^{an}, \end{aligned}$$

analogously we have for $(\varphi + \psi)(bt) = \lim_{n \rightarrow \infty} (\varphi(t/n)\psi(t/n))^{bn}$ and $(\varphi + \psi)((a+b)t) = \lim_{n \rightarrow \infty} (\varphi(t/n)\psi(t/n))^{(a+b)n}$, hence

$$\begin{aligned} (\varphi + \psi)(at)(\varphi + \psi)(bt) &= \lim_{n \rightarrow \infty} (\varphi(t/n)\psi(t/n))^{an} \lim_{n \rightarrow \infty} (\varphi(t/n)\psi(t/n))^{bn} \\ &= \lim_{n \rightarrow \infty} (\varphi(t/n)\psi(t/n))^{(a+b)n} = (\varphi + \psi)((a+b)t) \end{aligned}$$

as desired.

To prove that $(\varphi + \psi)(-t) = (\varphi + \psi)(t)^{-1}$ it is enough to see that

$$\begin{aligned} (\varphi + \psi)(-t)^{-1} &= \lim_{n \rightarrow \infty} (\varphi(-t/n)\psi(-t/n))^{-n} = \lim_{n \rightarrow \infty} (\varphi(t/n)^{-1}\psi(t/n)^{-1})^{-n} \\ &= \lim_{n \rightarrow \infty} ((\psi(t/n)\varphi(t/n))^{-1})^{-n} = \lim_{n \rightarrow \infty} (\psi(t/n)\varphi(t/n))^n \\ &= \lim_{n \rightarrow \infty} (\psi(t/n)\varphi(t/n))^n \psi(t/n)\psi(t/n)^{-1} \\ &= \lim_{n \rightarrow \infty} \psi(t/n)(\varphi(t/n)\psi(t/n))^n \psi(t/n)^{-1} \\ &= \lim_{n \rightarrow \infty} (\varphi(t/n)\psi(t/n))^n = (\varphi + \psi)(t). \end{aligned}$$

Thus, $(\varphi + \psi)(-t)^{-1} = (\varphi + \psi)(t)$ so $e = (\varphi + \psi)(t)(\varphi + \psi)(-t)$. And by the uniqueness of the inverse we have $(\varphi + \psi)(-t) = (\varphi + \psi)(t)^{-1}$. \square

Lemma 3.17. $L(G)$ is a topological vector space.

Proof. Vector space axioms. We start showing that $L(G)$ is a vector space. It is clear that the $0 \in L(G)$ defined as $0(t) := e$ for every $t \in \mathbb{R}$ is the additive identity and $1 \in \mathbb{R}$ is the scalar multiplication identity. Since $a(b\varphi(t)) = a\varphi(bt) = \varphi(a(bt)) = \varphi((ab)t) = (ab)\varphi(t)$ scalar multiplication is compatible with field multiplication.

Distributivity of scalar multiplication over field addition is clear by the homomorphism property of φ ; since for all $t \in \mathbb{R}$

$$\begin{aligned} (a\varphi + b\varphi)(t) &= \lim_{n \rightarrow \infty} (a\varphi(t/n)b\varphi(t/n))^n = \lim_{n \rightarrow \infty} (\varphi(at/n)\varphi(bt/n))^n \\ &= \lim_{n \rightarrow \infty} (\varphi(at/n + bt/n))^n = \lim_{n \rightarrow \infty} (\varphi((a+b)t/n))^n \\ &= \lim_{n \rightarrow \infty} \varphi((a+b)nt/n) = \lim_{n \rightarrow \infty} \varphi((a+b)t) \\ &= \varphi((a+b)t) = (a+b)\varphi(t). \end{aligned}$$

Distributivity of scalar multiplication over vector addition is calculated as follows:

$$\begin{aligned} (a\varphi + a\psi)(t) &= \lim_{n \rightarrow \infty} (a\varphi(t/n)a\psi(t/n))^n = \lim_{n \rightarrow \infty} (\varphi(at/n)\psi(at/n))^n \\ &= \lim_{n \rightarrow \infty} (\varphi(t/n)\psi(t/n))^{an} = \left(\lim_{n \rightarrow \infty} (\varphi(t/n)\psi(t/n))^n \right)^a \\ &= ((\varphi + \psi)(t))^a = (\varphi + \psi)(at) = a(\varphi + \psi)(t) \end{aligned}$$

for all $t \in \mathbb{R}$.

Clearly the additive inverse of φ is $-\varphi$ because

$$\begin{aligned} (\varphi + (-\varphi))(t) &= \lim_{n \rightarrow \infty} (\varphi(t/n)(-\varphi(t/n)))^n = \lim_{n \rightarrow \infty} (\varphi(t/n)\varphi(-t/n))^n \\ &= \lim_{n \rightarrow \infty} (\varphi(t/n - t/n))^n = \lim_{n \rightarrow \infty} (\varphi(0))^n = \lim_{n \rightarrow \infty} e = e \end{aligned}$$

for all t .

The commutativity of addition is calculated as follows:

$$\begin{aligned} (\varphi + \psi)(t) &= \lim_{n \rightarrow \infty} (\varphi(t/n)\psi(t/n))^n = \lim_{n \rightarrow \infty} \varphi(t/n)(\psi(t/n)\varphi(t/n))^n \varphi(t/n)^{-1} \\ &= \lim_{n \rightarrow \infty} \varphi(t/n) \lim_{n \rightarrow \infty} (\psi(t/n)\varphi(t/n))^n \lim_{n \rightarrow \infty} \varphi(t/n)^{-1} \\ &= \varphi(0) \lim_{n \rightarrow \infty} (\psi(t/n)\varphi(t/n))^n \varphi(0)^{-1} = e(\psi + \varphi)(t)e^{-1} \\ &= (\psi + \varphi)(t). \end{aligned}$$

The only remaining axiom left to prove is the associativity of addition: if $\varphi, \psi, \phi \in L(G)$ then $((\varphi + \psi) + \phi)(t) = (\varphi + (\psi + \phi))(t)$ for all $t \in \mathbb{R}$. By the homomorphism property it will be enough to show this for a sufficiently small t . By definition of $(\varphi + \psi)(t) = \lim_{n \rightarrow \infty} (\varphi(t/n)\psi(t/n))^n$ we get that

$$\begin{aligned} nd((\varphi + \psi)(t/n), \varphi(t/n)\psi(t/n)) &\sim d((\varphi + \psi)(t/n)^n, (\varphi(t/n)\psi(t/n))^n) \\ &= d((\varphi + \psi)(t), (\varphi(t/n)\psi(t/n))^n) \\ &= d((\varphi(t/m)\psi(t/m))^m, (\varphi(t/n)\psi(t/n))^n) \\ &\sim d((\varphi(t/nm)\psi(t/nm))^{nm}, (\varphi(t/n)\psi(t/n))^n) \\ &\ll \frac{\epsilon^2}{n} \end{aligned}$$

so $d((\varphi + \psi)(t/n), \varphi(t/n)\psi(t/n)) \ll \frac{\epsilon^2}{n^2}$. In a similar way we obtain that $d(((\varphi + \psi) + \phi)(t/n), (\varphi + \psi)(t/n)\phi(t/n)) \ll \frac{\epsilon^2}{n^2}$ and $d((\varphi + (\psi + \phi))(t/n), \varphi(t/n)(\psi + \phi)(t/n)) \ll \frac{\epsilon^2}{n^2}$. If we apply lemma 3.12, we get that

$$\begin{aligned} d(((\varphi + \psi) + \phi)(t), (\varphi(t/n)\psi(t/n)\phi(t/n))^n) &\sim nd(((\varphi + \psi) + \phi)(t/n), \varphi(t/n)\psi(t/n)\phi(t/n)) \\ &\leq nd(((\varphi + \psi) + \phi)(t/n), (\varphi + \psi)(t/n)\phi(t/n)) \\ &\quad + nd((\varphi + \psi)(t/n)\phi(t/n), \varphi(t/n)\psi(t/n)\phi(t/n)) \\ &\ll n \frac{\epsilon^2}{n^2} + nd((\varphi + \psi)(t/n), \varphi(t/n)\psi(t/n)) \\ &\ll n \frac{\epsilon^2}{n^2} = \frac{\epsilon^2}{n}. \end{aligned}$$

Analogously

$$\begin{aligned} d((\varphi + (\psi + \phi))(t), (\varphi(t/n)\psi(t/n)\phi(t/n))^n) &\sim nd((\varphi + (\psi + \phi))(t/n), \varphi(t/n)\psi(t/n)\phi(t/n)) \\ &\leq nd((\varphi + (\psi + \phi))(t/n), \varphi(t/n)(\psi + \phi)(t/n)) \\ &\quad + nd(\varphi(t/n)(\psi + \phi)(t/n), \varphi(t/n)\psi(t/n)\phi(t/n)) \\ &\ll n \frac{\epsilon^2}{n^2} + nd((\psi + \psi)(t/n), \psi(t/n)\phi(t/n)) \\ &\ll n \frac{\epsilon^2}{n^2} = \frac{\epsilon^2}{n}. \end{aligned}$$

In conclusion

$$\begin{aligned}
& d(((\varphi + \psi) + \phi)(t), (\varphi + (\psi + \phi))(t)) \\
& \leq d(((\varphi + \psi) + \phi)(t), (\varphi(t/n)\psi(t/n)\phi(t/n))^n) \\
& \quad + d((\varphi(t/n)\psi(t/n)\phi(t/n))^n, (\varphi + (\psi + \phi))(t)) \\
& \ll \frac{\epsilon^2}{n} + \frac{\epsilon^2}{n} = 2\frac{\epsilon^2}{n} \rightarrow 0 \text{ as } n \rightarrow \infty.
\end{aligned}$$

Continuity of operations. The only thing left to prove is the continuity of the operations. To prove the joint continuity of the scalar multiplication, take $\epsilon > 0$, $c \in \mathbb{R}$ and $\phi_0 \in L(G)$. Define $I = [-c, c]$ and consider $B_{\frac{\epsilon}{2}, I}(\phi_0)$. Recall that by definition

$$B_{\frac{\epsilon}{2}, I} = \left\{ \phi \in L(G) : \sup_{t \in I} \{d(\phi(t), \phi_0(t))\} < \frac{\epsilon}{2} \right\}.$$

Since the family of functions $B_{\frac{\epsilon}{2}, I}(\phi_0)$ is uniformly Lipschitz continuous (lemma 3.14), there exists a constant K such that every homomorphism in $B_{\frac{\epsilon}{2}, I}(\phi_0)$ is Lipschitz continuous.

To show the continuity we will consider $(c - \frac{\epsilon}{4K}, c + \frac{\epsilon}{4K}) \subset \mathbb{R}$ and we want to see that

$$\left(c - \frac{\epsilon}{4K}, c + \frac{\epsilon}{4K}\right) B_{\frac{\epsilon}{2}, I}(\phi_0) \subset B_{\epsilon, I_0}(c\phi_0)$$

where $I_0 = [-1, 1]$. To achieve this we take $a \in (c - \frac{\epsilon}{4K}, c + \frac{\epsilon}{4K})$ and $\phi \in B_{\frac{\epsilon}{2}, I}(\phi_0)$ and we will verify

$$\begin{aligned}
a\phi \in B_{\epsilon, I_0}(c\phi_0) & \iff \sup_{t \in [-1, 1]} \{d(a\phi(t), c\phi_0(t))\} < \epsilon \\
& \iff \sup_{t \in [-1, 1]} \{d(\phi(at), \phi_0(ct))\} < \epsilon \\
& \iff \forall t \in [-1, 1] : d(\phi(at), \phi_0(ct)) < \epsilon.
\end{aligned}$$

Note that for all $t \in [-1, 1] = I_0$ we have

$$\begin{aligned}
d(\phi(at), \phi_0(ct)) & = d(\phi((c - c + a)t), \phi_0(ct)) \\
& = d(\phi(ct + (a - c)t), \phi_0(ct)) \\
& = d(\phi(ct)\phi((a - c)t), \phi_0(ct)).
\end{aligned}$$

Call $t' = ct$ then $t' \in [-c, c] = I$ and

$$\begin{aligned}
d(\phi(at), \phi_0(ct)) &= d(\phi(t')\phi(((a-c)/c)t'), \phi_0(t')) \\
&\leq d(\phi(t')\phi(((a-c)/c)t'), \phi(t')) + d(\phi(t'), \phi_0(t')) \\
&= d(\phi(((a-c)/c)t'), e) + d(\phi(t'), \phi_0(t')) \\
&\leq d(\phi(((a-c)/c)t'), e) + \sup_{t' \in I} \{d(\phi(t'), \phi_0(t'))\} \\
&\leq d(\phi(((a-c)/c)t'), e) + \frac{\epsilon}{2} \\
&= d(\phi((a-c)t), e) + \frac{\epsilon}{2} \\
&= d(\phi(at)\phi(-ct), e) + \frac{\epsilon}{2} \\
&= d(\phi(at)\phi(ct)^{-1}, e) + \frac{\epsilon}{2} \\
&= d(\phi(ct)^{-1}, \phi(at)^{-1}) + \frac{\epsilon}{2}.
\end{aligned}$$

Thus,

$$\begin{aligned}
d(\phi(at), \phi_0(ct)) &= d(\phi(-ct), \phi(-at)) + \frac{\epsilon}{2} \\
&\leq K\|at - ct\| + \frac{\epsilon}{2} \leq K\|a - c\| + \frac{\epsilon}{2} \\
&< K\frac{\epsilon}{2K} + \frac{\epsilon}{2} = \epsilon.
\end{aligned}$$

Now we are going to prove the continuity of the sum at the origin. Let $\epsilon > 0$ and take $\delta = \frac{\epsilon}{2T}$, whenever $\sup_{t \in I} \|\varphi(t)\| < \delta$ and $\sup_{t \in I} \|\phi(t)\| < \delta$. Recalling that for φ and ϕ we have the Lipschitz bound given by $\|\varphi(t)\| \ll \delta|t|$ and $\|\phi(t)\| \ll \delta|t|$, we are able to calculate

$$\begin{aligned}
\|(\phi + \varphi)(t)\| &= \left\| \lim_{n \rightarrow \infty} (\phi(t/n)\varphi(t/n))^n \right\| = \lim_{n \rightarrow \infty} \|(\phi(t/n)\varphi(t/n))^n\| \\
&\leq \lim_{n \rightarrow \infty} n\|\phi(t/n)\varphi(t/n)\| \leq \lim_{n \rightarrow \infty} n\|\phi(t/n)\| + n\|\varphi(t/n)\| \\
&\ll \lim_{n \rightarrow \infty} n\delta \left| \frac{t}{n} \right| + n\delta \left| \frac{t}{n} \right| = \lim_{n \rightarrow \infty} 2\delta|t| = 2\delta|t|
\end{aligned}$$

and taking supremum over $t \in I$ we get

$$\sup_{t \in I} \|(\phi + \varphi)(t)\| \ll 2\delta T = \epsilon$$

which gives us the continuity of the addition in $L(G)$. \square

3.4 Construction of an exponential map

Now we already have that $L(G)$ is a locally compact topological vector space. By exercise 46.10 and theorem 46.8 of [Mun00] there exists a metric which makes $L(G)$ complete as desired. By theorem 3.5 we get that $L(G)$ is isomorphic to a finite dimensional vector space \mathbb{R}^n with the usual topology. This important result will motivate the following definition, which will help us to construct a Lie structure in G at the end of this section.

Definition 3.18. We define the *exponential map* $\exp : L(G) \rightarrow G$ as $\exp(\phi) := \phi(1)$.

Note. It is continuous since in the compact-open topology every evaluation is continuous.

We will introduce an auxiliary notion which will allow us to prove a slightly general version of the Arzela-Ascoli theorem.

Definition 3.19 (Asymptotically equi-continuous). Let X, Y be metric spaces. A family of functions $\{\phi_n : X \rightarrow Y\}_{n \in \mathbb{N}}$ is called *asymptotically equi-continuous* if for any $\epsilon > 0$ there exist $\delta > 0$ and $N \in \mathbb{N}$ such that $d(x, y) < \delta$ implies $d(\phi_n(x), \phi_n(y)) < \epsilon$ for every $n \geq N$.

Theorem 3.20 (Arzela-Ascoli-like theorem). *Let $I \subset \mathbb{R}$ be a compact interval and Y be a locally compact complete metric space. Then a subsequence of continuous functions $\{\phi_n : I \rightarrow Y\}_{n \in \mathbb{N}}$ converges uniformly if $\{\phi_n\}_n$ is point wise bounded and asymptotically equi-continuous.*

Proof. Fix an enumeration $\{x_i\}_{i \in \mathbb{N}}$ of the rational numbers contained in I . Since $\{\phi_n\}_n$ is point wise bounded, then $\{\phi_n(x_1)\}_n$ is a bounded sequence in Y . Henceforth, by the local compactness on Y we can find a convergent subsequence $\{\phi_{n_{k_1}}(x_1)\}_{k_1}$.

Now we take the sequence $\{\phi_{n_{k_1}}(x_2)\}_{k_1}$ which is also bounded and analogously have a convergent subsequence $\{\phi_{n_{k_2}}(x_2)\}_{k_2}$. By induction on this process, we get a chain of subsequences

$$\{\phi_{k_1}\} \supset \{\phi_{k_2}\} \cdots \supset \{\phi_{k_n}\}$$

such that for each k_i the sequence $\{\phi_{n_{k_i}}\}_{k_i}$ converges for x_1, \dots, x_i . Taking the diagonal subsequence $\varphi_m := \phi_{m,m}$ where $\phi_{m,m}$ is the m -th term in the family $\{\phi_{n_{k_m}}\}_{k_m}$.

By construction, the sequence $\{\varphi_m\}_m$ converges for all rational number x_i in the interval I . Therefore, given any $\epsilon > 0$ and rational x_i there exists an integer N_{x_i} such that for every $n, m \geq N_{x_i}$

$$d(\varphi_n(x_i), \varphi_m(x_i)) < \epsilon/3.$$

Since the family $\{\varphi_m\}_m$ is asymptotically equi-continuous, for any $\epsilon > 0$ there exist an $N \in \mathbb{R}$ and $\delta > 0$ such that for all $s, t \in (x - \delta, x + \delta) =: U_x$ and $x \in I$

$$d(\varphi_n(s), \varphi_n(t)) < \epsilon/3$$

for all $n \geq N$.

Clearly the collection $\{U_x\}_x$ forms an open cover of I . Since I is compact, this covering admits a finite subcover U_1, \dots, U_l . Consequently for any $t \in I$ we can find a $j \in \{1, \dots, l\}$ and $x_k \in \mathbb{Q} \cap I$ such that $t, x_k \in U_j$. As a result we get

$$\begin{aligned} d(\varphi_n(t), \varphi_m(t)) &\leq d(\varphi_n(t), \varphi_n(x_k)) + d(\varphi_n(x_k), \varphi_m(x_k)) + d(\varphi_m(x_k), \varphi_m(t)) \\ &\leq \frac{\epsilon}{3} + \frac{\epsilon}{3} + \frac{\epsilon}{3} = \epsilon \end{aligned}$$

whenever $n, m \geq \max\{N, N_{x_k}\}$. For this reason, the sequence $\{\varphi_m\}_m$ is uniformly Cauchy, and therefore converges to a continuous function ϕ . \square

Proposition 3.21. *For any neighborhood K of the origin in $L(G)$, $\exp(K)$ is a neighborhood of the identity on G .*

Proof. By the continuity of \exp and since $L(G)$ is isomorphic as a topological vector space to some \mathbb{R}^n , we can choose a compact star-shaped neighborhood of the origin in $L(G)$ such that $\exp(K)$ is contained in the $B_\epsilon(0)$ where ϵ is small enough to fulfill that $\epsilon < \frac{1}{C}$. Clearly $\exp(K)$ is compact by the continuity of \exp .

Suppose by contradiction that $\exp(K)$ is not a neighborhood of the identity in G . This means that any open set containing the identity cannot be a subset of $\exp(K)$. Moreover, since G is a metric space we can restate the previous affirmation as, for any $n \in \mathbb{R}$ the open ball $B_{1/n}(e) \not\subset \exp(K)$. Thus for every $n \in \mathbb{N}$ we can choose a point $g_n \in B_{1/n}(e)$ such that $g_n \notin \exp(K)$ and clearly by construction $g_n \rightarrow e$ as: $n \rightarrow \infty$, in particular $\|g_n\| \rightarrow 0$.

By the compactness of $\exp(K)$ we can find for each n an element $h_n \in \exp(K)$ such that it minimizes the distance $d(g_n, h_n)$. Then, if we define $k_n := h_n^{-1}g_n$ we obtain

$$\|k_n\| = \|(h_n)^{-1}g_n\| = d(g_n, h_n) \leq d(g_n, e) = \|g_n\| \text{ and } \|h_n\| \leq d(g_n, h_n) + \|g_n\| \leq 2\|g_n\|$$

hence $\|k_n\|, \|h_n\| \rightarrow 0$ as $n \rightarrow \infty$.

Let $N_n := \lfloor \frac{\epsilon}{\|k_n\|} \rfloor$. Clearly, $N_n \leq \frac{\epsilon}{\|k_n\|}$ and $N_n \rightarrow \infty$ as $n \rightarrow \infty$. Moreover,

$$\|k_n^{N_n}\| = d(k_n^{N_n}, e^{N_n}) \sim N_n d(k_n, e) = N_n \|k_n\| \leq \epsilon$$

for all $n \in \mathbb{N}$.

We define the approximate one-parameter subgroups $\phi_n : [-1, 1] \rightarrow G$

$$\phi_n(t) := k_n^{\lfloor tN_n \rfloor}$$

and as a consequence we get $\|\phi_n(t)\| = \|k_n^{\lfloor tN_n \rfloor}\| \sim \lfloor tN_n \rfloor \|k_n\| \leq (|t|N_n + 1) \frac{\epsilon}{N_n}$. In order to get an approximate homomorphism property recall the following properties of the integer part function

$$\begin{aligned} \lfloor tN_n \rfloor &\leq tN_n, \\ \lfloor sN_n \rfloor &\leq sN_n. \end{aligned}$$

Adding the previous equations and by the properties of the integer part

$$\begin{aligned} 1 + \lfloor (t+s)N_n \rfloor &\geq tN_n + sN_n \\ -1 - \lfloor (t+s)N_n \rfloor &\leq -tN_n - sN_n \leq -\lfloor tN_n \rfloor - \lfloor sN_n \rfloor. \end{aligned}$$

Finally we get

$$\lfloor tN_n \rfloor + \lfloor sN_n \rfloor - \lfloor (t+s)N_n \rfloor \leq 1. \quad (3.1)$$

Now we can calculate the following approximate homomorphism property:

$$\begin{aligned} d(\phi_n(t+s), \phi_n(t)\phi_n(s)) &= d(k_n^{\lfloor (t+s)N_n \rfloor}, k_n^{\lfloor tN_n \rfloor + \lfloor sN_n \rfloor}) \\ &= \|k_n^{\lfloor tN_n \rfloor + \lfloor sN_n \rfloor - \lfloor (t+s)N_n \rfloor}\| \\ &\sim (\lfloor tN_n \rfloor + \lfloor sN_n \rfloor - \lfloor (t+s)N_n \rfloor) \|k_n\| \\ &\leq \|k_n\| \rightarrow 0 \text{ as } n \rightarrow \infty. \end{aligned}$$

This approximate homomorphism property will allow us to prove that the family $\{\phi_n\}_n$ is asymptotically equi-continuous. Fix an $\eta > 0$. By the previous result there exists N_1 such that $d(\phi_n(t+s), \phi_n(t)\phi_n(s)) \leq \eta/2$ for $n \geq N$. We also can consider N_2 such that $\epsilon/N_2 \leq \eta/4$ and finally take some $\delta > 0$ such that $\delta < \frac{\eta}{4\epsilon}$. If $N = \max\{N_1, N_2\}$ and $s \in \mathbb{R}$ such that $\|s\| < \delta$ we get for every $n \geq N$

$$\begin{aligned} d(\phi_n(t+s), \phi(t)) &\leq d(\phi_n(t+s), \phi_n(t)\phi_n(s)) + d(\phi_n(t)\phi_n(s), \phi_n(t)) \\ &= d(\phi_n(t+s), \phi_n(t)\phi_n(s)) + d(\phi_n(s), e) \\ &= d(\phi_n(t+s), \phi_n(t)\phi_n(s)) + \|\phi_n(s)\| < \frac{\eta}{2} + \epsilon \left(|s| + \frac{1}{N_n} \right) \\ &< \frac{\eta}{2} + \frac{\eta}{4\delta}\delta + \frac{\eta}{4} = \eta. \end{aligned}$$

Since it is independent of n and of t , we see that the family of ϕ_n is asymptotically equi-continuous. Moreover, the family $\{\phi_n\}_n$ is point-wise bounded since for any $t \in [-1, 1]$ we get

$$\|\phi_n(t)\| = \|k_n^{\lfloor tN_n \rfloor}\| \sim \lfloor tN_n \rfloor \|k_n\| \leq (|t|N_n) \frac{\epsilon}{N_n} = |t|\epsilon$$

which is independent of n . Now, by theorem 3.20, a subsequence of $\{\phi_n\}_n$ converges uniformly to a continuous homomorphism $\hat{\phi} : [-1, 1] \rightarrow G$ which can be extended to a 1-parameter subgroup $\phi \in L(G)$. Moreover, recall that $\|\phi_n(t)\| \sim |t|\epsilon$; in particular $\|\phi_n(1)\| \sim \epsilon$ for every natural number n . By the continuity of the norm we get $\|\phi(1)\| \sim \epsilon$.

Since $h_n \in \exp(K)$, we can find a $\psi_n \in K$ such that $\exp(\psi_n) = \psi_n(1) = h_n$.

We are interested in finding a bound for $\|\psi_n(t)\|$ in terms of h_n . By the continuity of ψ_n in the interval $I = [-1, 1]$, there exists some $\delta > 0$ such that $\|\psi_n(t)\| \leq \delta$ for all $t \in I$, in particular for $t = 1$. Furthermore, by the triangle inequality $\|\psi_n(1/m)^k\| \leq \delta$ whenever $1 \leq k \leq m$.

Thus calculating the escape norm with respect to the open $B_\epsilon(e)$ we get

$$\|\psi_n(1/m)\|_{B_\delta(e)} \leq \frac{1}{m+1} \leq \frac{1}{m}.$$

By the escape norm comparison property we get that

$$\|\psi_n(1/m)\| \ll \|\psi_n(1/m)\|_{B_\delta(e)} \leq \frac{1}{m}.$$

This last inequality allows us to apply the escape property if we choose an m large enough such that $\frac{1}{m} \leq \frac{1}{C}$. As a consequence,

$$\|\psi_n(1/m)^m\| \geq \frac{m\|\psi_n(1/m)\|}{C}.$$

Consequently,

$$\|\psi_n(1)\| \geq \frac{m\|\psi_n(1/m)\|}{C} \geq \frac{k\|\psi_n(1/m)\|}{C} \geq \frac{\|\psi_n(k/m)\|}{C}$$

so $\|\psi_n(k/m)\| \leq C\|\psi_n(1)\|$ for all $1 \leq k \leq m$ and m large enough.

Let k/m be a such that for any $t \in I$

$$\left| t - \frac{k}{m} \right| \leq \frac{1}{L}\|\psi_n(1)\|$$

where L is the Lipschitz constant for ψ_n which exists since $\psi \in L(G)$. Notice the we can choose $m = \max\{m_1, m_2\}$ where m_1 and m_2 are such that $\frac{1}{m_1} \leq \frac{1}{C}$ and the other one fulfills the condition $\left| t - \frac{k}{m_2} \right| \leq \frac{1}{L}\|\psi_n(1)\|$.

Hence, we get

$$\begin{aligned} \|\psi_n(t)\| &\leq d(\psi_n(t), \psi_n(k/m)) + \|\psi_n(k/m)\| \\ &\leq L \left| t - \frac{k}{m} \right| + C\|\psi_n(1)\| \\ &\leq L \left(\frac{1}{L}\|\psi_n(1)\| \right) + C\|h_n\| \\ &\leq (1+C)\|h_n\| \ll \|h_n\|. \end{aligned}$$

We claim now that $\exp(\psi_n + \frac{1}{N_n}\phi)$ is close to g_n . So we calculate, using the triangle inequality,

$$\begin{aligned} &d(\exp(\psi_n + \frac{1}{N_n}\phi), g_n) \\ &\leq d(\exp(\psi_n + \frac{1}{N_n}\phi), \exp(\psi_n)\exp(\frac{1}{N_n}\phi)) + d(\exp(\psi_n)\exp(\frac{1}{N_n}\phi), g_n). \end{aligned}$$

The first term in the previous expression can be estimated as follows:

$$d(\exp(\psi_n + \frac{1}{N_n}\phi), \exp(\psi_n) \exp(\frac{1}{N_n}\phi)) = d((\psi_n + \frac{1}{N_n}\phi)(1), (\psi_n)(1)(\frac{1}{N_n}\phi)(1)).$$

We will proceed with the following argument. For any $\gamma > 0$ there exists an M such that for every $m \geq M$, in particular for $\gamma = \frac{\epsilon \|h_n\|}{N_n}$,

$$\begin{aligned} & d((\psi_n + \frac{1}{N_n}\phi)(1), (\psi_n)(1)(\frac{1}{N_n}\phi)(1)) \\ & \leq d((\psi_n + \frac{1}{N_n}\phi)(1), (\psi_n(1/m) \frac{1}{N_n}\phi(1/m))^m) \\ & \quad + d((\psi_n(1/m) \frac{1}{N_n}\phi(1/m))^m, (\psi_n)(1)(\frac{1}{N_n}\phi)(1)) \\ & \leq \gamma + d((\psi_n(1/m) \frac{1}{N_n}\phi(1/m))^m, (\psi_n)(1/m)^m (\frac{1}{N_n}\phi)(1/m)^m) \\ & \ll \frac{\epsilon \|h_n\|}{N_n} + m^2 \|\psi_n(1/m)\| \frac{1}{N_n} \|\phi(1/m)\| \\ & \ll \frac{\epsilon \|h_n\|}{N_n} + m^2 \left| \frac{1}{m} \right| \|h_n\| \frac{1}{N_n} \frac{1}{m} \|\phi(1)\| = \|h_n\| \frac{1}{N_n} (2\epsilon) \ll \|h_n\| \frac{1}{N_n}. \end{aligned}$$

In the previous calculation we use that $\|\psi_m(1)\| = \|\psi_n(t/m)^m\| \sim m \|\psi_n(t/m)\|$ and the analogous result for ϕ .

To analyze the second term, we will carry out the following calculation:

$$\begin{aligned} d(\exp(\psi_n) \exp(\frac{1}{N_n}\phi), g_n) &= d((\psi_n)(1)(\frac{1}{N_n}\phi)(1), h_n k_n) \\ &= d((\frac{1}{N_n}\phi)(1), k_n) \\ &\sim \frac{1}{N_n} d(\phi(1), k_n^{N_n}). \end{aligned}$$

In total we obtain

$$d(\exp(\psi_n + \frac{1}{N_n}\phi), g_n) \ll \frac{\|h_n\|}{N_n} + \frac{d(\phi(1), k_n^{N_n})}{N_n}.$$

Here it is important to notice two important facts which arise from the construction of h_n and ϕ . First of all by construction $\|h_n\| \rightarrow 0$ as $n \rightarrow \infty$. In second place, recall $\phi_n(t) = k_n^{\lfloor tN_n \rfloor}$, in particular $\exp(\phi_n) = \phi_n(1) = k_n^{N_n}$. And, since ϕ is the uniform limit of ϕ_n , we have $k_n^{N_n} = \phi_n(1) \rightarrow \phi(1)$. Moreover, $d(\phi(1), k_n^{N_n}) \rightarrow 0$ as $n \rightarrow \infty$.

These two previous annotations are resumed in $\|h_n\| + d(\phi(1), k_n^{N_n}) \rightarrow 0$ as $n \rightarrow \infty$. We will call this term $M_n := \|h_n\| + d(\phi(1), k_n^{N_n})$ which clearly converges to 0 as n grows bigger.

Additionally, recall that by construction $N_n \rightarrow \infty$ as $n \rightarrow \infty$ and $\psi_n \rightarrow 0$. This implies $\psi_n + \frac{1}{N_n}\phi$ converges point-wise to the function 0 as $n \rightarrow \infty$. In particular $(\psi_n + \frac{1}{N_n}\phi)(1) \rightarrow e$ so $\exp(\psi_n + \frac{1}{N_n}\phi) \rightarrow e$ for $n \rightarrow \infty$.

Now, by hypothesis K is an open neighborhood of the identity in G . By the previous result, there must exist an $N \in \mathbb{N}$ such that for all $n \geq N$ $\exp(\psi_n + \frac{1}{N_n}\phi) \in K$. By construction h_n was the element in K which minimizes the distance $d(g_n, k)$ for all $h \in K$. Thus,

$$\|k_n\| = \|h_n^{-1}g_n\| = d(h_n, g_n) \leq d(g_n, k)$$

for any $k \in K$. In particular

$$\|k_n\| \leq d(g_n, \exp(\psi_n + 1/N_n\phi)) \ll \frac{M_n}{N_n} = \frac{M_n}{\lfloor \frac{\epsilon}{\|k_n\|} \rfloor} \leq \frac{M_n}{\frac{\epsilon}{\|k_n\|} - 1} = \frac{\|k_n\| M_n}{\epsilon - \|k_n\|}.$$

Thus,

$$\epsilon - \|k_n\| \leq M_n \Rightarrow \epsilon \leq M_n + \|k_n\|.$$

But, this is a contradiction since M_n and $\|k_n\|$ converge to 0 as n goes to infinity. So, their sum can not be greater than a strictly positive number. \square

The previous proposition together with theorem 5.8 shows that any group with a strong Gleason metric is locally euclidean as proven in corollary 5.9. This result is necessary to apply the following results and get that G is in fact a Lie group.

To get that the group G is a Lie group we must prove that the transition functions are C^∞ . Furthermore, we will use the following theorem.

Theorem 3.22 (Criterion for Lie structure). *Let G be a topological group. G is a Lie group if and only if there is a neighborhood of the identity in G which is isomorphic to a $C^{1,1}$ group.*

In our case the neighborhood of the identity which will be isomorphic to a $C^{1,1}$ group is $\exp(K)$ constructed in theorem 3.21. The reason why $\exp(K)$ is a $C^{1,1}$ group is the fact that the Baker-Campbell-Hausdorff formula gives us control over group operations near the origin, which is the required condition for a $C^{1,1}$ group.

In complete detail, if G is a Lie group, the proof is trivial. In the other direction the proof is presented in lemma 1.2.34 of [Tao12].

Another prove of this result is given by Kuranichi in lemma 6 of [Kur50].

Chapter 4

Gleason-Yamabe theorem for compact groups

In this chapter we will develop some technical tools which will allow us to prove the *Gleason-Yamabe theorem for compact groups*. This result will be needed as an intermediate result to prove the Gleason-Yamabe theorem for locally compact groups. Moreover, the tools that we are going to formulate will let us construct Gleason metrics on topological groups with certain properties, which we will study later.

4.1 Haar Measure

One of the most important results for topological groups is the existence of a unique left Haar measure whenever the group is locally compact. From now on we will always work in Hausdorff spaces to avoid technical difficulties, otherwise we can take an adequate quotient as indicated in corollary 2.3 of [Fol94]. We will begin with some basic definitions.

Definition 4.1 (σ -algebra). Let X be a set. A σ -algebra \mathcal{A} over X is a subset of its power set $\mathcal{P}(X)$ such that

- (i) $\emptyset \in \mathcal{A}$.
- (ii) If $A \in \mathcal{A}$ then $A^c \in \mathcal{A}$.
- (iii) If $\{A_n\}_{n \in \mathbb{N}} \subset \mathcal{A}$ then $\bigcup_{n \in \mathbb{N}} A_n \in \mathcal{A}$.

Definition 4.2. If X is a topological space, we define the *Borel algebra* over X as the smallest σ -algebra containing all the open sets of X . It will be denoted by $\mathcal{B}(X)$.

The elements of the σ -algebra are called measurable sets. Given a set X and a σ -algebra \mathcal{A} over X , a pair (X, \mathcal{A}) is called a measurable space.

Definition 4.3. Let (X, \mathcal{A}) be a measurable space. A function $\mu : \mathcal{A} \rightarrow \mathbb{R}^+$ is called a *measure* if and only if it satisfies:

- (i) $\mu(\emptyset) = 0$.
- (ii) For any $A \in \mathcal{A}$ we have $\mu(A) \geq 0$.
- (iii) (σ -additivity) For any countable collection $\{A_n\}_{n \in \mathbb{N}}$ of disjoint sets we have

$$\mu\left(\bigcup_n A_n\right) = \sum_n \mu(A_n).$$

A triplet (X, \mathcal{A}, μ) consisting of a space X , σ -algebra \mathcal{A} and measure μ is called a *measure space*.

Definition 4.4. Let $(X, \mathcal{B}(X), \mu)$ be a measure space. We call μ a *regular measure* if for every $A \in \mathcal{B}(X)$

$$\mu(A) = \inf\{\mu(U) : A \subset U \text{ and } U \text{ is open}\}$$

and

$$\mu(A) = \sup\{\mu(K) : K \subset A \text{ and } K \text{ is compact}\}.$$

Definition 4.5. Let $(X, \mathcal{B}(X), \mu)$ be a measure space. μ is *locally finite* if for every $x \in X$ there exists an open neighbourhood U_x of x such that $\mu(U_x) < \infty$

Definition 4.6. Let G be a topological group. And $(G, \mathcal{B}(G), \mu)$ be a measure space. We call μ *left-invariant* if for all $E \in \mathcal{B}(G)$ and all $g \in G$

$$\mu(gE) = \mu(E).$$

We can define a *right-invariant* measure on $(G, \mathcal{B}(G))$ considering Eg instead of gE . Finally we are able to define a *left Haar measure* as follows:

Definition 4.7 (Left Haar measure). Let G be a locally compact topological group. A *left Haar measure* μ is a measure on $(G, \mathcal{B}(G))$ which is left invariant, regular and locally finite.

To show the existence of a left Haar measure in locally compact groups, we will use the Riesz representation theorem which we will enunciate without proof, since this is a well known result in functional analysis. For a proof see [Wei80].

Theorem 4.8 (Riesz representation theorem). *Let X be a locally compact Hausdorff space, and let I be a positive linear functional on $C_c(X)$ (functions with compact support). Then there exists a unique measure μ on $\mathcal{B}(X)$ such that*

$$I(f) = \int_X f d\mu \text{ for every } f \in C_c(X)$$

which is locally finite and regular.

Definition 4.9. We say that $f \in C(G)$ is *left (right) uniformly continuous* if $\|L_y f - f\|_\infty \rightarrow 0$ ($\|R_y f - f\|_\infty \rightarrow 0$) as $y \rightarrow e$.

Proposition 4.10. *If $f \in C_c(G)$, then f is left and right uniformly continuous.*

Proof. We will present the proof for left uniform continuity. Given $f \in C_c(G)$ and $\epsilon > 0$, call $K = \text{supp}(f)$. By the continuity of f for all $x \in K$ there exists a neighborhood U_x of the identity in G such that $\|f(y^{-1}x) - f(x)\| < \epsilon/2$. We can choose a symmetric neighborhood V_x of the identity such that $e \in V_x V_x \subset U_x$.

Clearly the collection of sets $\{V_x\}_{x \in K}$ is an open covering of K and using the compactness of K we can find x_1, \dots, x_n such that $K \subset \bigcup_{i=1}^n V_{x_i} x_i$. Let $V := \bigcap_{i=1}^n V_{x_i}$, we claim that $\|L_y f - f\|_{\text{sup}} < \epsilon$ for every $y \in V$.

If neither x nor $y^{-1}x$ are in K then $f(y^{-1}x) = f(x) = 0$ so we finally get the desired result.

If $x \in K$ then there exists an i such that $x \in V_{x_i} x_i$, in other words $xx_i^{-1} \in V_{x_i}$. Thus, if we take $y \in V \subset V_{x_i}$ we get $y^{-1}x = y^{-1}(xx_i^{-1})x_i \in U_{x_i} x_i$ by the symmetry of the neighborhood V_{x_i} . But we see

$$\|f(y^{-1}x) - f(x)\| \leq \|f(y^{-1}x) - f(x_i)\| + \|f(x_i) - f(x)\| < \frac{1}{2}\epsilon + \frac{1}{2}\epsilon = \epsilon$$

as desired.

On the other hand if $y^{-1}x \in K$ then we can find a j such that $y^{-1}xx_j^{-1} \in V_{x_j}$. Moreover since $y \in V \subset V_{x_j}$ we can write $x = y(y^{-1}xx_j^{-1})x_j \in U_{x_j} x_j$ and again we get

$$\|f(y^{-1}x) - f(x)\| \leq \|f(y^{-1}x) - f(x_j)\| + \|f(x_j) - f(x)\| < \frac{1}{2}\epsilon + \frac{1}{2}\epsilon = \epsilon.$$

□

Now we will get into the proof of the existence of a left Haar measure in locally compact groups.

Theorem 4.11. *Every locally compact Hausdorff group G possesses a left Haar measure.*

Proof. If $f, \phi \in C_c^+(G)$ are positive continuous functions with compact support, we define

$$(f : \phi) := \inf \left\{ \sum_{j=1}^n c_j : f \leq \sum_{j=1}^n c_j L_{x_j} \phi \right\}$$

where $L_{x_j} \phi(g) := \phi(x_j^{-1}g)$ is the left translate of the function ϕ by x_j and $c_j > 0$. Note that this definition makes sense by the compactness of the support of f because it can be covered by a finite number of left translates of the set $\{g \in G | \phi(g) > \frac{1}{2}\|\phi\|_\infty\}$.

Note that the quantity $(f : \phi)$ has the following properties:

- (i) $(f : \phi) = (L_y f : \phi)$ for any $y \in G$,
- (ii) $(f_1 + f_2 : \phi) \leq (f_1 : \phi) + (f_2 : \phi)$,
- (iii) $(cf : \phi) = c(f : \phi)$ for any $c > 0$,
- (iv) $(f_1 : \phi) \leq (f_2 : \phi)$ whenever $f_1 \leq f_2$,
- (v) $(f : \phi) \geq \|f\|_\infty / \|\phi\|_\infty$ and
- (vi) $(f : \phi) \leq (f : \psi)(\psi : \phi)$ for any $\psi \in C_c^+(G)$

All of these properties are obvious by the definition except for (v) and (vi). To see (v), it is enough to note that since $f \leq \sum_{j=1}^n c_j L_{x_j} \phi$ we get

$$\begin{aligned} \|f\|_\infty &\leq \left\| \sum_{j=1}^n c_j L_{x_j} \phi \right\|_\infty \leq \sum_{j=1}^n \|c_j L_{x_j} \phi\|_\infty \\ &= \sum_{j=1}^n c_j \|L_{x_j} \phi\|_\infty = \sum_{j=1}^n c_j \|\phi\|_\infty = \|\phi\|_\infty \sum_{j=1}^n c_j. \end{aligned}$$

As a consequence we get that $\|f\|_\infty / \|\phi\|_\infty \leq \sum_{j=1}^n c_j$ and taking the infimum $\|f\|_\infty / \|\phi\|_\infty \leq (f : \phi)$.

Finally (vi) can be easily seen after noting that if $f \leq \sum_i c_i L_{x_i} \psi$ and $\psi \leq \sum_j b_j L_{y_j} \phi$ then $f \leq \sum_{i,j} c_i b_j L_{x_i y_j} \phi$.

Now we will fix an $f_0 \in C_c^+(G)$ and we will define

$$I_\phi(f) := \frac{(f : \phi)}{(f_0 : \phi)}$$

for any $f, \phi \in C_c^+(G)$. Clearly by (vi), (ii), (iii), and (iv) the functional I_ϕ is left invariant, sub-additive, homogeneous of degree 1 and monotone. Furthermore, we can use (vi) to see $(f : \phi) \leq (f : f_0)(f_0 : \phi)$ and $(f_0 : \phi) \leq (f_0 : f)(f : \phi)$; these inequalities will lead to

$$(f_0 : f)^{-1} \leq I_\phi(f) \leq (f : f_0). \quad (4.1)$$

To get the result of the theorem, after applying theorem 4.8, will be enough to get that I_ϕ is additive rather than sub-additive. Thus, I_ϕ will be a restriction to $C_c^+(G)$ of a functional defined on $C_c(G)$. The following technical lemma will give as an approximation for the desired missing inequality.

Lemma 4.12. *If $f_1, f_2 \in C_c^+(G)$ and $\epsilon > 0$, then there is a neighborhood V of $e \in G$ such that if $\text{supp}(\phi) \subset V$ then $I_\phi(f_1) + I_\phi(f_2) \leq I_\phi(f_1 + f_2) + \epsilon$.*

Proof. Take $g \in C_c^+(G)$ such that $g = 1$ on $\text{supp}(\phi)$ and let $\delta > 0$. Let us define $h := f_1 + f_2 + \delta g$ and $h_i := \frac{f_i}{h}$ for $i = 1, 2$. Note that $h_i = 0$ whenever $f_i = 0$, so $h_i \in C_c^+(G)$ and by proposition 4.10 there exist a neighborhood U of the identity $e \in G$ such that $\|h_i(x) - h_i(y)\| < \delta$ for $i = 1, 2$ and $y^{-1}x \in U$.

Suppose $\phi \in C_c^+(G)$ and $\text{supp}(\phi) \subset V$. If $h \leq \sum_{j=1}^n c_j L_{x_j} \phi$, then

$$\begin{aligned} f_i(x) &= h(x)h_i(x) \leq \left(\sum_{j=1}^n c_j L_{x_j} \phi(x) \right) h_i(x) \\ &= \sum_{j=1}^n \left(c_j \phi(x_j^{-1}x) h_i(x) \right) \leq \sum_{j=1}^n \left(c_j \phi(x_j^{-1}x) (h_i(x_j) + \delta) \right) \end{aligned}$$

because if $x_j^{-1}x \in V$ then $h_i(x) - h_i(x_j) \leq \|h_i(x) - h_i(x_j)\| < \delta$. Moreover, by construction, $h_1 + h_2 \leq 1$ which gives us

$$\begin{aligned} (f_1 : \phi) + (f_2 : \phi) &\leq \sum_{j=1}^n c_j (h_1(x_j) + \delta) + \sum_{j=1}^n c_j (h_2(x_j) + \delta) \\ &= \sum_{j=1}^n c_j (1 + 2\delta). \end{aligned}$$

Taking the infimum over the sums that define $(h : \phi)$, we get $(f_1 : \phi) + (f_2 : \phi) \leq (h : \phi)(1 + 2\delta)$ and after dividing by the normalization factor $(f_0 : \phi)$ we see that

$$\begin{aligned} I_\phi(f_1) + I_\phi(f_2) &\leq I_\phi(h)(1 + 2\delta) = I_\phi(f_1 + f_2 + \delta g)(1 + 2\delta) \\ &\leq (I_\phi(f_1 + f_2) + \delta I_\phi(g))(1 + 2\delta) \\ &= I_\phi(f_1 + f_2) + [2\delta I_\phi(f_1 + f_2) + \delta(1 + 2\delta)I_\phi(g)]. \end{aligned}$$

Note that by equation 4.1 the result will follow if we take δ such that $2\delta(f_1 + f_2 : f_0) + \delta(1 + 2\delta)(g : f_0) < \epsilon$. \square

Now we will proceed with the final part of the proof of theorem 4.11. For each $f \in C_c^+(G)$ we define the intervals $X_f := [(f_0 : f)^{-1}, (f : f_0)]$ and consider the space $X := \prod_f X_f$ which by Tychonoff's theorem is compact and Hausdorff. This product consists of all functionals from $C_c^+(G)$ to $(0, \infty)$ whose values lie in the interval X_f . By construction and equation (4.1) for every $\phi \in C_c^+(G)$ we get that $I_\phi \in X$.

Let us define for each neighborhood V of the identity the set

$$K(V) := \overline{\{I_\phi \mid \text{supp}(\phi) \subset V\}} \subset X.$$

The family $\{K(V)\}$ has the *finite intersection property* since for any finite sub-collection we have $K(\bigcap_{i=1}^n V_i) \subset \bigcap_{i=1}^n K(V_i)$. This can be easily verified by taking $F_\phi \in K(\bigcap_{i=1}^n V_i)$.

Then $\text{supp}(\phi) \in \bigcap_{i=1}^n V_i$, so $F_\phi \in K(V_i)$ for every $i = 1, \dots, n$ and finally $F_\phi \in \bigcap_{i=1}^n K(V_i)$. So every finite intersection is non-empty and by the compactness of X there exists some

$$I \in \bigcap_{e \in V} K(V).$$

In other words, for any neighborhood V of $e \in G$, $\epsilon > 0$ and functions $f_1, \dots, f_n \in C_c^+(G)$, there exists $\phi \in C_c^+(G)$ such that $\text{supp}(\phi) \subset V$ and $\|I(f_j) - I_\phi(f_j)\| < \epsilon$ for any $j \in \{1, \dots, n\}$. Now we need to verify that I commutes with left translations, addition and multiplication by scalars.

First of all recall that for any $\phi \in C_c^+(G)$ the functional I_ϕ is left invariant and homogeneous of degree 1. Take any $y \in G$ and constant $c \geq 0$. Consider $\epsilon > 0$, any neighborhood V of the identity and the finite collection of functions $A := \{f, L_y f, cf\} \subset C_c^+(G)$. By construction of I , there exists $\phi \in C_c^+(G)$ such that $\text{supp}(\phi) \subset V$ and $\|I(f) - I_\phi(f)\| < \frac{\epsilon}{3}$ for any $f \in A$. Thus

$$\begin{aligned} \|I(f) - I(L_y f)\| &\leq \|I(f) - I_\phi(f)\| + \|I_\phi(f) - I_\phi(L_y f)\| \\ &\quad + \|I_\phi(L_y f) - I(L_y f)\| < \frac{\epsilon}{3} + 0 + \frac{\epsilon}{3} < \epsilon \end{aligned}$$

and

$$\begin{aligned} \|I(cf) - cI(f)\| &\leq \|I(cf) - I_\phi(cf)\| + \|I_\phi(cf) - cI_\phi(f)\| \\ &\quad + \|cI_\phi(f) - cI(f)\| < \frac{\epsilon}{3} + 0 + c\frac{\epsilon}{3} < \epsilon(1+c). \end{aligned}$$

Since ϵ is arbitrary small, the claim follows. Finally we need to prove the additivity of I . For any $\epsilon > 0$, neighborhood $e \in V$ and functions $f_1, f_2, f_1 + f_2 \in C_c^+(G)$, there exists $\phi \in C_c^+(G)$ such that $\text{supp}(\phi) \subset V$ and $\|I(f) - I_\phi(f)\| < \frac{\epsilon}{4}$ by definition of I . And, by lemma 4.12 $I_\phi(f_1) + I_\phi(f_2) \leq I_\phi(f_1 + f_2) + \frac{\epsilon}{4}$, so

$$\begin{aligned} \|I(f_1) + I(f_2) - I(f_1 + f_2)\| &\leq \|I(f_1) - I_\phi(f_1)\| + \|I(f_2) - I_\phi(f_2)\| \\ &\quad + \|I_\phi(f_1) + I_\phi(f_2) - I_\phi(f_1 + f_2)\| \\ &\quad + \|I_\phi(f_1 + f_2) - I(f_1 + f_2)\| \\ &< \frac{\epsilon}{4} + \frac{\epsilon}{4} + \frac{\epsilon}{4} + \frac{\epsilon}{4} = \epsilon. \end{aligned}$$

These calculations show us that I is a left invariant linear functional on $C_c^+(G)$ which can be easily extended to $C_c(G)$ by the decomposition of any $f \in C_c(G)$ in its positive and negative components $f = g - h$ with $g, h \in C_c^+(G)$ and the definition $I(f) = I(g) - I(h)$. If we consider a different decomposition $f = g' - h'$, then $g - h = g' - h'$. Hence $g + h' = g' + h$ and $I(g) + I(h') = I(g') + I(h)$ which give us a well defined extension of I to $C_c(G)$. Finally by the Riesz representation theorem 4.8 there exists a regular, locally finite measure μ on $\mathcal{B}(G)$.

The only thing to verify is the left invariance of μ . Take any $A \in \mathcal{B}(G)$ and $g \in G$. Then

$$\mu(A) = \int_G \chi_A d\mu = I(\chi_A) = I(L_g \chi_A) = I(\chi_{gA}) = \int_G \chi_{gA} d\mu = \mu(gA) \quad \square$$

4.2 Gleason-Yamabe theorem for compact groups

In this section we will use the spectral theorem for compact self-adjoint operators to prove a baby version of the Peter-Weyl theorem which will help us to prove the Gleason-Yamabe theorem for compact groups.

Theorem 4.13 (Spectral theorem for compact self-adjoint operators). *Let H be a Hilbert space and $T : H \rightarrow H$ a compact self-adjoint operator. Then there exists an orthonormal system of eigenvectors with corresponding eigenvalues $\{\lambda_n\}_{n \in \mathbb{N} \cup \{\infty\}}$ which can be chosen to be such that $|\lambda_1| \geq |\lambda_2| \geq \dots > 0$ and the only point of accumulation of the sequence $\{\lambda_n\}_{n \in \mathbb{N} \cup \{\infty\}}$ is 0. Moreover, we can decompose H in the following way*

$$H = \ker(T) \oplus \bigoplus_n V_{\lambda_n}$$

where V_{λ_n} are the eigenspaces of T with eigenvalues λ_n which are finite dimensional.

Note that in the preceding theorem $\ker(T)$ is not required to be finite dimensional; this fact will be important in the following result.

Definition 4.14. Let G be a locally compact group and μ its Haar measure. For $1 \leq p < \infty$ let us define

$$L^p(G) := \{f : G \rightarrow \mathbb{C} : \|f\|_p < \infty\}$$

where

$$\|f\|_p := \left(\int_G |f|^p d\mu \right)^{1/p}.$$

Remark. It is easy to verify that $L^2(G)$ is a Hilbert space, as noted in in chapter 5 of [Len90], with the inner product defined as follows for any $f, g \in L^2(G)$

$$\langle f, g \rangle := \int_G f(x) \overline{g(x)} d\mu(x).$$

Definition 4.15. Let (X, \mathcal{A}) be a measurable space. Let $A_1, \dots, A_n \subset X$ be measurable sets and a_1, \dots, a_n be real numbers. A *simple function* is a function $f : X \rightarrow \mathbb{R}$ of the form

$$f(x) := \sum_{i=1}^n a_i \chi_{A_i}(x)$$

Lemma 4.16. Fix $f \in L^2(G)$. Given $g \in G$ and a sequence $\{g_n\} \subset G$ such that $g_n \rightarrow g$ as $n \rightarrow \infty$. Then

$$\lim_{n \rightarrow \infty} \|L_{g_n} f - L_g f\|_2 = 0.$$

Proof. We will strongly use the fact that the functions with compact support are dense in $L^2(G)$ [Coh94]. Let $K \subset G$ be a compact set and $f = \chi_K$ its indicator function. Clearly the function $L_g f$ is the indicator function of the compact set $g^{-1}K$ for any $g \in G$. By definition, we get that $L_{g_n} f \rightarrow L_g f$ converges in measure, which means that $\mu(g_n^{-1}K \Delta g^{-1}K) \rightarrow 0$ as $n \rightarrow \infty$.

Now it is easy to see that

$$\begin{aligned} \|L_{g_n} f - L_g f\|_2 &= \left(\int_G \chi_{g_n^{-1}K}(x) - \chi_{g^{-1}K}(x) d\mu(x) \right)^{1/2} \\ &= \left(\int_{g_n^{-1}K \Delta g^{-1}K} 1 d\mu(x) \right)^{1/2} = \mu(g_n^{-1}K \Delta g^{-1}K)^{1/2} \rightarrow 0 \end{aligned}$$

as $n \rightarrow \infty$.

Now let $f \in L^2(G)$ be an arbitrary function and $\epsilon > 0$. In chapter 3 csection 4 of [Coh94] we see that simple functions are dense in the set of all compactly supported functions and these last ones are dense in $L^2(G)$, we can find a compactly supported function f_0 such that $\|f - f_0\|_2 < \epsilon/5$ and we can find a simple function f_1 such that $\|f_0 - f_1\|_2 < \epsilon/5$. By an extension of the previous argument to simple functions there exists an $N \in \mathbb{N}$ such that $\|L_{g_n} f_1 - L_g f_1\|_2 < \epsilon/5$ for all $n \geq N$. It follows that

$$\begin{aligned} \|L_{g_n} f - L_g f\|_2 &\leq \|L_{g_n} f - L_{g_n} f_0\|_2 + \|L_{g_n} f_0 - L_g f_0\|_2 + \|L_g f_0 - L_g f\|_2 \\ &= 2\|f - f_0\|_2 + \|L_{g_n} f_0 - L_g f_0\|_2 < \frac{2\epsilon}{5} + \|L_{g_n} f_0 - L_g f_0\|_2 \\ &\leq \frac{2\epsilon}{5} + \|L_{g_n} f_0 - L_{g_n} f_1\|_2 + \|L_{g_n} f_1 - L_g f_1\|_2 + \|L_g f_1 - L_g f_0\|_2 \\ &\leq \frac{2\epsilon}{5} + 2\|f_0 - f_1\|_2 + \|L_{g_n} f_1 - L_g f_1\|_2 < \frac{4\epsilon}{5} + \|L_{g_n} f_1 - L_g f_1\|_2 \\ &< \frac{4\epsilon}{5} + \frac{\epsilon}{5} = \epsilon \end{aligned}$$

which is the desired result. □

Definition 4.17 (Convolution). For any $f, g \in L^2(G)$ we define the convolution

$$(f * g)(x) := \int_G f(y)g(y^{-1}x)d\mu(y)$$

Proposition 4.18. If $f, g \in L^2(G)$ then $f * g$ is well defined, continuous and lies in $L^2(G)$.

Proof. To see that the convolution is well defined, we only need to verify that the integral is bounded since a function is integrable if and only if its absolute value is integrable as well as show in proposition 2.3.7 of [Coh94]. But, if we apply the Hölder's inequality we get that

$$|(f * g)(x)| \leq \int_G |f(y)g(y^{-1}x)| d\mu(y) \leq \|f\|_2 \|g\|_2 < \infty.$$

To see that $f * g \in L^2(G)$ is a consequence of Fubini's theorem, which we can use since G is locally compact and the Haar measure is locally finite, we will calculate:

$$\begin{aligned} \int_G |(f * g)(x)|^2 d\mu(x) &= \int_G \left| \int_G f(y)g(y^{-1}x) d\mu(y) \right|^2 d\mu(x) \\ &\leq \int_G \int_G |f(y)|^2 |g(y^{-1}x)|^2 d\mu(y) d\mu(x) \\ &\leq \int_G |f(y)|^2 \left(\int_G |g(y^{-1}x)|^2 d\mu(x) \right) d\mu(y) \\ &\leq \int_G |f(y)|^2 (\|g\|_2) d\mu(y) = \|g\|_2 \int_G |f(y)|^2 d\mu(y) \\ &\leq \|g\|_2 \|f\|_2 < \infty \end{aligned}$$

which shows as that $\|f * g\|_2 < \infty$.

The only thing left to prove is the continuity of the convolution. Fix $x \in G$ and a convergent sequence $x_n \rightarrow x$. We have

$$\begin{aligned} |(f * g)(x_n) - (f * g)(x)| &= \left| \int_G f(y)g(y^{-1}x_n) d\mu(y) - \int_G f(y)g(y^{-1}x) d\mu(y) \right| \\ &\leq \int_G |f(y)| |g(y^{-1}x_n) - g(y^{-1}x)| d\mu(y) \\ &\leq \|f\|_2 \|L_{x_n}g - L_xg\|_2 \rightarrow 0 \end{aligned}$$

by lemma 4.16 which give us the continuity of the convolution. □

Definition 4.19. For any $g \in L^2(G)$ we can define $T_g : L^2(G) \rightarrow L^2(G)$ as $T_g(f) := f * g$. Which by the previous proposition is a bounded linear operator.

Remark. If g satisfies $g(x^{-1}) = \overline{g(x)}$ for all $x \in G$, then T_g is self-adjoint since for all

$f, h \in L^2(G)$

$$\begin{aligned} \langle T_g f, h \rangle &= \int_G T_g f(x) \overline{h(x)} dx = \int_G \int_G f(y) g(y^{-1}x) \overline{h(x)} dy dx \\ &= \int_G \int_G f(y) \overline{h(x) g(y^{-1}x)} dy dx = \int_G f(y) \overline{\left(\int_G h(x) g(y^{-1}x) dx \right)} dy \\ &= \int_G f(y) \overline{\left(\int_G h(x) g(x^{-1}y) dx \right)} dy = \int_G f(y) \overline{T_g(h)} dy = \langle f, T_g h \rangle \end{aligned}$$

in the worst case scenario g could be taken as the identity function.

Moreover, we easily get for every $g \in L^2(G)$ and $h \in G$

$$\begin{aligned} T_g(L_h f)(x) &= \int_G f(h^{-1}y) g(y^{-1}x) dy = \int_G f(h^{-1}hz) g(z^{-1}h^{-1}x) d(hz) \\ &= \int_G f(z) g(z^{-1}h^{-1}x) d(z) = (f * g)(h^{-1}x) = L_h T_g(f)(x), \end{aligned}$$

so $T_g(L_h f) = L_h T_g(f)$.

Now we will need to show that T is a compact operator. We will present a slightly strong theorem from [Bum98] which will give us that T is compact.

Theorem 4.20 (Hilbert-Schmidt). *Let X be a locally compact space with a positive Borel measure and assume that $L^2(X)$ is a separable Hilbert space. Let $K \in L^2(X \times X)$. Then the operator*

$$(Tf)(x) := \int_X K(x, y) f(y) dy$$

is a compact operator.

The proof of the previous theorem can be found [Bum98]. To apply this theorem to our case, we will only need to take $K(x, y) = g(y^{-1}x) \in L^2(G)$ by definition 4.17.

Definition 4.21. A linear subspace V of $L^2(G)$ is called *invariant* if for all $y \in G$ we have $L_y V \subset V$

Theorem 4.22 (Baby Peter-Weyl theorem). *Let G be a compact Hausdorff group with Haar measure μ and let $y \in G \setminus \{e\}$. Then there exists a finite-dimensional invariant subspace of $L^2(G)$ and $L_y|_V \neq id$.*

Proof. Suppose by contradiction that for every finite dimensional invariant subspace of $L^2(G)$ there exists a $y \in G$ such that L_y is the identity. Thus, $L_y - id$ annihilates every finite dimensional subspace. By theorems 4.20 and 4.13 if we take a $g \in L^2(G)$ as in

the previous remark, T_g will be a compact self-adjoint operator. This will allow us to decompose

$$L^2(G) = \ker(T_g) \oplus \bigoplus_n V_{\lambda_n}$$

where each V_{λ_n} is finite dimensional.

By hypothesis $V_{\lambda_n} \subset \ker(L_y - id)$ which also means that $\text{range}(L_y - id) \subset \ker(T_g)$. Thus, $T_g(L_y - id)(f) = 0$ for any $f \in L^2(G)$, in other words

$$T_g(L_y f) = T_g f$$

which by the previous remark can be written as $L_y(f * g) = f * g$.

To find a contradiction it will be enough to construct functions $f, g \in L^2(G)$ which contradict this last one equality. Consider an open symmetric neighborhood U of the identity in G , which exist by proposition 2.1 in [Fol94], such that $y \notin U^2$ and take $f = g = \chi_U$. First of all note that since U was chosen symmetric, we have that $g(x^{-1}) = 1 = \overline{g(x)}$ for every $g \in G$. Consequently T_g is a self-adjoint compact operator.

Finally note that $\int_G \chi_U(z) \chi_U(z^{-1}x) dz \neq 0$ if $z \in U$ and $x \in zU$, which means that $x \in U^2$ so $\text{supp}(f * g) \subset U^2$. Since $y \notin U^2$ we conclude $\text{supp}(L_y(f * g)) \not\subset U^2$. This means that $\text{supp}(f * g) \neq \text{supp}(L_y(f * g))$ which makes impossible that $L_y(f * g) = f * g$. This gives us the desired contradiction and proves the theorem. \square

Finally we are able to prove the Gleason-Yamabe theorem for compact groups.

Theorem 4.23 (Gleason-Yamabe theorem for compact groups). *Let G be a compact Hausdorff group and let U be a neighborhood of the identity. Then there exists a compact normal subgroup H contained in U such that G/H is isomorphic to a linear group.*

Proof. Note that if $U = G$ then we could take $H = G$ and the result will be trivial. Assume $U \subsetneq G$ and let $g \in G \setminus U$. By theorem 4.22 we can find a finite dimensional invariant subspace $V_g \subset L^2(G)$ on which L_g is different from the identity. Let us define the continuous homomorphism

$$\hat{\rho} : G \rightarrow L(V_g), \hat{\rho}(g) = L_g.$$

If we identify $V_g \cong \mathbb{C}^{n_g}$ for some finite n_g , the space $L(V_g)$ can be easily identified with $GL_{n_g}(\mathbb{C})$. Under this identification we can define a continuous homomorphism

$$\rho_g : G \rightarrow GL_{n_g}(\mathbb{C})$$

such that $\rho_g(g)$ is non trivial. Note that by the continuity of ρ_g we can find an open neighborhood U_g of g , where ρ_g is non trivial.

Since G is compact, so is $G \setminus U$, and we can find a finite family $\{g_1, \dots, g_k\}$ such that $\{U_{g_i}\}_{i=1}^k$ covers $G \setminus U$ and for each $g \in G$ at least one of $\{\rho_{g_1}, \dots, \rho_{g_k}\}$ is non trivial. Therefore we can construct the following function

$$\varrho := \bigoplus_{i=1}^k \rho_{g_i} : G \rightarrow \bigoplus_{i=1}^k GL_{n_i}(\mathbb{C})$$

where $GL_{n_i}(\mathbb{C})$ is an abbreviation of $GL_{g_{n_i}}(\mathbb{C})$. Consider $H = \ker(\varrho)$ which is always a normal subgroup of G by the first isomorphism theorem and compact since is closed ($H = \varrho^{-1}(\{0\})$) in a compact space, and in this case is contained in U . And finally, applying the first homomorphism theorem we get that $G/H \cong \varrho(G)$. By the continuity of ϱ we get that $\varrho(G)$ is compact and hence a closed subgroup of $GL_{n_1+\dots+n_k}(G)$. Furthermore G/H is isomorphic to a linear group (hence a Lie group). \square

Chapter 5

Building metrics on groups

In this chapter we will finally prove the Gleason-Yamabe theorem. We will use the tools developed up to this point, namely the criterion for a Lie structure whenever we have Gleason metric, the existence of a left Haar measure, and the compact case of the Gleason-Yamabe theorem.

5.1 Construction of metrics from functions

Definition 5.1. Let G be a topological group, and let $\psi : G \rightarrow \mathbb{R}^+$ be a bounded non-negative function. Then we define the pseudo-metric $d_\psi : G \times G \rightarrow \mathbb{R}^+$ as

$$d_\psi(g, h) := \sup_{x \in G} |L_g \psi(x) - L_h \psi(x)|$$

and the semi-norm

$$\|g\|_\psi = d_\psi(g, e).$$

Proposition 5.2. *Under the conditions of the previous definition we have the following properties for any $g, h, k \in G$:*

- (i) $d_\psi(g, h) \geq 0$ and if $g = h$ then $d_\psi(g, h) = 0$,
- (ii) $d_\psi(g, h) = d_\psi(h, g)$,
- (iii) $d_\psi(g, h) \leq d_\psi(g, k) + d_\psi(k, h)$,
- (iv) If $\psi \in C_c(G)$, then d_ψ is continuous,
- (v) $d_\psi(g, h) \leq \|\psi\|_\infty$ with equality holds if and only if $g^{-1}h \notin KK^{-1}$, where $K = \text{supp}(\psi)$,
- (vi) $d_\psi(kg, kh) = d_\psi(g, h)$.

Proof. Clearly (i), (ii) and (iii) follow from the properties of the absolute value and supremum in \mathbb{R} .

To see (iv) it is enough to notice that translations and norm are continuous functions, thus $|L_g\psi(x) - L_h\psi(x)|$ is continuous. Consequently, if we write down

$$d_\psi(g, h) = \|L_g\psi - L_h\psi\|_\infty$$

we see that the continuity of d_ψ came from the continuity of the supremum in the space of continuous functions.

The property (v) is a consequence of the positivity of ψ . Notice that in $\|L_g\psi - L_h\psi\|_\infty$ we are just subtracting translates of a positive compactly supported function. In the worst case scenario we could get that $\text{supp}(L_g\psi) \cap \text{supp}(L_h\psi) = \emptyset$. Hence, we get $\|L_g\psi - L_h\psi\|_\infty = \|\psi\|_\infty$ which only occurs whenever $hK \cap gK = \emptyset$ so $g^{-1}h \notin KK^{-1}$.

The left invariance (vi) follows from a straight forward calculation

$$\begin{aligned} d_\psi(kg, kh) &= \sup_{x \in G} |L_{kg}\psi(x) - L_{kh}\psi(x)| \\ &= \sup_{x \in G} |L_g\psi(k^{-1}x) - L_h\psi(k^{-1}x)| \\ &= \sup_{k^{-1}x \in G} |L_g\psi(k^{-1}x) - L_h\psi(k^{-1}x)| \\ &= \sup_{y \in G} |L_g\psi(y) - L_h\psi(y)| \\ &= d_\psi(g, h). \end{aligned} \quad \square$$

Note. We define the following notation $\partial_g\psi := \psi(g^{-1}x) - \psi(x)$. Taking this into account we can write down $\|g\|_\psi = \sup_{x \in G} |\partial_g\psi(x)|$.

Definition 5.3 (Weak strong Gleason metric). Let G be a topological group. A *weak strong Gleason metric* is a left-invariant metric which generates the topology on G . Additionally there exists some constant $C > 0$ which obeys the following properties:

- (i) For all $n \geq 1$ such that $n\|g\| \leq \frac{1}{C}$ then $\|g^n\| \geq \frac{1}{C}n\|g\|$
- (ii) For a sufficiently small open neighborhood U of the identity there exist that the exists positive constants K_1, K_1 such that

$$K_1\|g\|_U \leq \|g\| \leq K_2\|g\|_U$$

Remark. Observe that a strong Gleason metric is a weak strong Gleason which satisfies additionally the commutator estimate

$$\|[g, h]\| \leq C\|g\|\|h\|$$

whenever $\|g\|, \|h\| \leq \frac{1}{C}$.

Theorem 5.4. *Every weak strong Gleason metric is a strong Gleason metric.*

Proof. Let $\epsilon > 0$, and let $\psi \in C_c(G)$ be a non-negative function supported on the ball $B_\epsilon(e)$. For instance we could use the following function

$$\psi(x) := \left(1 - \frac{\|x\|}{\epsilon}\right)_+.$$

This function is clearly Lipschitz since

$$\|\psi(x) - \psi(y)\| = \left|1 - \frac{\|x\|}{\epsilon} - 1 + \frac{\|y\|}{\epsilon}\right| = \frac{|\|y\| - \|x\||}{\epsilon} \leq \frac{d(x, y)}{\epsilon}$$

Moreover, we can see that $\|\partial_g \psi\|_\infty \ll \|g\|$ by the following calculation for every $x \in \text{supp}(\psi)$

$$\|\partial_g \psi(x)\| = \|\psi(g^{-1}x) - \psi(x)\| \leq \frac{d(g^{-1}x, x)}{\epsilon} = \frac{\|x^{-1}gx\|}{\epsilon}. \quad (5.1)$$

To obtain the desired result it will be enough to show that $\|x^{-1}gx\| \ll \|g\|$ for $x \in B_\epsilon(e)$ and $g \in G$.

We can prove it by the following argument. Let $g \in G$ and $n \in \mathbb{N}$ such that $n\|g\| \leq \epsilon$. The triangle inequality gives us $\|g^n\| \leq n\|g\| \leq \epsilon$ hence if we define $g^x = x^{-1}gx$, we obtain $\|(g^x)^n\| = \|x^{-1}g^n x\| \leq 2\|x\| + \|g^n\| \leq 3\epsilon$. And, by property (ii) in definition 5.3 we get

$$\|g^x\| \leq \|g^x\|_{B_{5\epsilon}(e)} \leq \frac{1}{n+1}.$$

Since whenever $\|g\| \leq \frac{\epsilon}{n}$ implies that $\|x^{-1}gx\| \leq \frac{1}{n+1}$ then we get that $\|x^{-1}gx\| \ll \|g\|$ by theorem 3.13. Which finally gives us that

$$\|\partial_g \psi(h)\|_\infty \ll \|g\|. \quad (5.2)$$

Let μ be the left-invariant Haar measure on G . So we can define ϕ as the convolution $\phi := \psi * \psi$. In order to obtain that the Gleason weak metric on G is in fact a Gleason metric, we need to prove the commutator estimate. We will begin with the following inequality

$$\|\partial_g \partial_h \phi\|_\infty \ll \|g\| \|h\| \quad (5.3)$$

whenever $h, g \in B_\epsilon(e)$.

Thanks to the left-invariance of the Haar measure we can write down

$$\begin{aligned}
\partial_h \phi(x) &= \partial_h((\psi * \psi)(x)) = (\psi * \psi)(h^{-1}x) - (\psi * \psi)(x) \\
&= \int_G \psi(y)\psi(y^{-1}h^{-1}x)dy - \int_G \psi(y)\psi(y^{-1}x) dy \\
&= \int_G \psi(h^{-1}z)\psi(z^{-1}hh^{-1}x) d(h^{-1}z) - \int_G \psi(y)\psi(y^{-1}x) dy \\
&= \int_G \psi(h^{-1}z)\psi(z^{-1}x) dz - \int_G \psi(y)\psi(y^{-1}x) dy \\
&= \int_G \psi(h^{-1}y)\psi(y^{-1}x) dy - \int_G \psi(y)\psi(y^{-1}x) dy \\
&= \int_G (\psi(h^{-1}y) - \psi(y))\psi(y^{-1}x) dy \\
&= \int_G (\partial_h \psi)(y)\psi(y^{-1}x) dy.
\end{aligned}$$

We can do a similar calculation to prove the following result

$$\begin{aligned}
\partial_g \partial_h \phi(x) &= \partial_h \psi(g^{-1}x) - \partial_h \psi(x) \\
&= \int_G (\partial_h \psi)(y)\psi(y^{-1}g^{-1}x) dy - \int_G (\partial_h \psi)(y)\psi(y^{-1}x) dy \\
&= \int_G (\partial_h \psi)(y)(\psi(y^{-1}g^{-1}x) - \psi(y^{-1}x)) dy \\
&= \int_G (\partial_h \psi)(y)(\psi(y^{-1}g^{-1}yy^{-1}x) - \psi(y^{-1}x)) dy \\
&= \int_G (\partial_h \psi)(y)(\psi(y^{-1}g^{-1}yy^{-1}x) - \psi(y^{-1}x)) dy \\
&= \int_G (\partial_h \psi)(y)(\psi((g^y)^{-1}y^{-1}x) - \psi(y^{-1}x)) dy \\
&= \int_G (\partial_h \psi)(y)(\partial_{g^y} \psi)(y^{-1}x) dy
\end{aligned}$$

where $g^y := y^{-1}gy$. If $h \in B_\epsilon(e)$, then the integrand in the previous expression is different from zero whenever $y \in B_{2\epsilon}(e)$ because the support of ψ is contained in $B_\epsilon(e)$.

By the previous result and (5.1) we see that

$$\begin{aligned}
|\partial_g \partial_h \phi(x)| &\leq \int_G |(\partial_h \psi)(y)| |(\partial_{g^y} \psi)(y^{-1}x)| dy \\
&\ll \int_G \|h\| |(\partial_{g^y} \psi)(y^{-1}x)| dy \text{ by (5.2)} \\
&\ll \|h\| \int_G \sup_{y \in G} \|g^y\| dy \\
&\ll \|h\| \sup_{y \in B_{2\epsilon}(e)} \|g^y\|.
\end{aligned}$$

Until here we have proven that $\|\partial_g \partial_h \phi\|_\infty \ll \|g\| \sup_{y \in B_{2\epsilon}(e)} \|g^y\|$. So, the only remaining claim to prove (5.3) is that if $g \in B_\epsilon(e)$ and $y \in B_{2\epsilon}(e)$ then

$$\|g^y\| \ll \|g\|.$$

We can achieve this by the following argument. For all natural numbers n such that $n\|g\| \leq \epsilon$ then $\|g^n\| \leq n\|g\| \leq \epsilon$ which means that $\|(g^y)^n\| = \|y^{-1}g^n y\| \leq 2\|y\| + \|g^n\| \leq 5\epsilon$. And, by property (ii) in definition 5.3 we get

$$\|g^y\| \leq \|g^y\|_{B_{5\epsilon}(e)} \leq \frac{1}{n+1}.$$

Since, whenever $\|g\| \leq \frac{\epsilon}{n}$ implies that $\|g^y\| \leq \frac{1}{n+1}$, we get that $\|g^y\| \ll \|g\|$. This completes the proof of the inequality (5.3).

Now we need to prove that the norms $\|\cdot\|$ and $\|\cdot\|_\phi$ are comparable if we are close enough to the origin.

Take and $g \in B_\epsilon(e)$ and observe that

$$\begin{aligned}
\|g\|_\phi &= \sup_{x \in G} |\partial_g \phi(x)| \leq \sup_{x \in G} \int_G |\partial_g \psi(y)| |\psi(y^{-1}x)| dy \\
&\ll \sup_{x \in G} \int_G \|g\| |\psi(y^{-1}x)| dy = \|g\| \sup_{x \in G} \int_G |\psi(y^{-1}x)| dy \\
&\leq \|g\| \sup_{x \in G} \int_{B_\epsilon(e)} dy \ll \|g\|.
\end{aligned}$$

Note that here the compactness of the support of ψ is fundamental in order to assure $\int_{B_\epsilon(e)} dy < \infty$.

On the other hand let n be a natural number such that $n\|g\|_\phi < \|\phi\|_\infty$. Thus, by the triangle inequality $\|g^n\|_\phi < \|\phi\|_\infty$, which means that the functions ϕ and $L_{g^n} \phi$ have overlapping supports. This can be asserted by (v) in proposition 5.2. Hence, $g^n \in B_{4\epsilon}(e)$ and by the property (ii) in definition 5.3 we get that

$$\|g\| \leq \|g\|_{B_{4\epsilon}(e)} \leq \frac{1}{n+1}.$$

Then we get that whenever $\|g\|_\phi \leq \frac{\|\phi\|_\infty}{n}$ implies $\|g\| \leq \frac{1}{n+1}$. Thus, $\|g\| \ll \|g\|_\phi$ as desired.

From equation (5.3) and the previous comparison results we get that $\|\partial_g \partial_h \phi\|_\infty \ll \|g\|_\phi \|h\|_\phi$ whenever g, h are small enough. Now we carry out the following calculation

$$\begin{aligned}
\|[g, h]\|_\phi &= \|L_{[g, h]}\phi - \phi\|_\infty = \sup_{x \in G} |L_{[g, h]}\phi(x) - \phi(x)| \\
&= \sup_{x \in G} |\phi(g^{-1}h^{-1}ghx) - \phi(x)| \\
&= \sup_{y \in G} |\phi(g^{-1}h^{-1}y) - \phi(h^{-1}g^{-1}y)| \\
&= \|L_g L_h \phi - L_h L_g \phi\|_\infty \\
&= \|L_g L_h \phi - L_g \phi - L_h \phi + \phi - L_h L_g \phi + L_h \phi + L_g \phi - \phi\|_\infty \\
&= \|\partial_g(L_h \phi - \phi) - \partial_h(L_g \phi - \phi)\|_\infty \\
&= \|\partial_g(\partial_h \phi) - \partial_h(\partial_g \phi)\|_\infty \\
&\leq \|\partial_g(\partial_h \phi)\|_\infty + \|\partial_h(\partial_g \phi)\|_\infty \ll \|g\| \|h\|.
\end{aligned}$$

And, since for g, h are sufficiently small, we get $\|[g, h]\| \ll \|[g, h]\|_\infty \ll \|g\| \|h\|$ which completes the prove that every weak Gleason metric is indeed a Gleason metric. \square

5.2 Groups with NSS property

The only remaining thing is to find the necessary condition we should impose on a topological group in order to get a metrizable group with a Gleason metric.

Definition 5.5 (No small subgroup property). A topological group G has the *no small subgroup* property (abbreviated NSS) if there exists an open neighborhood U of the identity which does not contain any subgroup of G different from the trivial subgroup.

We say that a group G is NSS if it has the NSS property as defined before.

Proposition 5.6. *Any Lie group is NSS.*

Proof. Recall that the exponential map $\exp : \mathfrak{g} \rightarrow G$ is a local diffeomorphism. Thus, there exists an open neighborhood U of the zero which is diffeomorphic to some open neighborhood V of the identity.

If G is not NSS, then every open neighborhood in the identity has a non-trivial subgroup. Since G is locally compact, the identity has an open precompact neighborhood V' . Consider $W = V \cap V'$ which is an open neighborhood of the identity, so it contains a non-trivial subgroup H . Moreover, $\overline{H} \subset \overline{W}$, since \overline{H} is closed subgroup of a Lie group. Thus it is a Lie subgroup and by the diffeomorphism we get $\exp^{-1}(\overline{H})$ is a Lie algebra. Furthermore, $\exp^{-1}(\overline{H})$ is a subspace of \mathfrak{g} contained in \overline{U} .

This last affirmation is a contradiction, because an open bounded neighborhood of the identity cannot contain any subspace of \mathfrak{g} . Thus, G is NSS. \square

Now we will prove a theorem which will help us to establish the local structure of G through the \exp map and $L(G)$. The following two theorems are needed to complete the proof of theorem 3.22.

The proof of the next theorem was originally presented by Kuranishi and Gleason. But the proof we give was given by [Yam53].

Theorem 5.7. *Let G be a locally compact group with NSS, then there exists a neighborhood V of $e \in G$ such that if $x, y \in V$, with $x^2 = y^2$, then $x = y$.*

Proof. Let \hat{V}_0 be the neighborhood of the identity in G given by the NSS property. By proposition 2.1 in [Fol94] we can choose a compact symmetric neighborhood V_0 if $e \in G$ which is contained in \hat{V}_0 , thus V_0 also fulfills NSS.

Let W be neighborhood of $e \in G$ with $W^2 \subset V_0$ and by the compactness of V_0 choose a compact symmetric neighborhood V of e such that $V(g^{-1}Vg) \subset W$ for any $g \in V_0$.

Suppose $x, y \in V$, such that $x^2 = y^2 \in V$ and $x \neq y$, define $a = x^{-1}y$. Then, $a = x^{-1}eye \in V(e^{-1}Ve) \subset W$ since $V(g^{-1}Vg) \subset W$ for any $g \in V_0$. Moreover, notice that

$$axa = x^{-1}yxx^{-1}y = x^{-1}y^2 = x^{-1}x^2 = x,$$

so we have that $a = x^{-1}a^{-1}x \in W$. Notice that this implies that $a^m = x^{-1}a^{-m}x$.

We are going to prove by induction that $a, \dots, a^n \in V_0$ for every positive integer n .

Base cases: We have already proven that $a \in W \subset V_0$, furthermore, $a^2 \in W^2 \subset V_0$.

Induction hypothesis: Let us assume that $a^k \in V_0$ for every $k < n$.

Inductive step: If n is even, then in particular for $m := \frac{n}{2}$ we have $a^m \in V_0$, so

$$a^n = a^{2m} = x^{-1}a^{-m}xa^m \in V(c^{-1}Vc) \subset W \subset V_0$$

taking $c = a^m \in V_0$.

On the other hand, if n is odd, define $m = \frac{n-1}{2}$, Clearly $a^m \in V_0$, we have already seen that $a^{2m} \in V(c^{-1}Vc) \subset W$ for $c = a^m$, thus $a^n = a^{2m+1} = a^{2m}a \in W^2 \subset V_0$. The, we get that $a^n \in V_0$. Concluding the induction.

Hence $a, \dots, a^n \in V_0$ for every positive integer n , since V_0 is symmetric then $a, \dots, a^n \in V_0$ for every integer n . But V_0 was by construction a neighborhood of the identity which fulfills NSS, this implies that $a = e$ hence $x = y$ as desired. \square

An slightly different proof can be found in [Kur50].

Theorem 5.8. *Let G be a locally compact group with NSS then there exists a neighborhood K of $0 \in L(G)$ such that the map $\exp|_K : K \rightarrow \exp(K)$ is injective.*

Proof. For any neighborhood U of $0 \in L(G)$ we have by theorem 3.21 that $\exp(U)$ is a neighborhood of $e \in G$. By theorem 5.7 we know that there exists a neighborhood V of the identity in G such that if $x^2 = y^2$ then $x = y$ for all $x, y \in V$. Consider $W = \exp(U) \cap V$.

Suppose that there exists 1-parameter subgroups $\phi, \psi \exp^{-1}(W)$ such that $\exp(\phi) = \exp(\psi)$ thus $\phi(1) = \psi(1)$. By the homomorphism property of ϕ and ψ we have that $\phi(\frac{1}{2})^2 = \psi(\frac{1}{2})^2$, which implies that $\phi(\frac{1}{2}) = \psi(\frac{1}{2})$ by theorem 5.7.

Inductively we get that $\phi(\frac{1}{2^k}) = \psi(\frac{1}{2^k})$ for any $k \in \mathbb{N}$, hence $\phi(\frac{1}{2^k})^n = \psi(\frac{1}{2^k})^n$ for any $n, k \in \mathbb{N}$ which by the homomorphism property of ϕ and ψ implies that $\phi(\frac{n}{2^k}) = \psi(\frac{n}{2^k})$ for any $n, k \in \mathbb{N}$.

In other words, ϕ and ψ agree on every dyadic rational which means that ϕ and ψ agree on every real number since the dyadic rationals are dense in \mathbb{R} . This means that $\phi = \psi$ which implies that \exp is an injective map restricted to the neighborhood $K = \exp^{-1}(W)$. \square

The previous theorem is the missing piece for the following result which allow us to prove that G is locally euclidean and as in theorem 3.22 that in fact G is a Lie group.

Corollary 5.9. *Let G be a locally compact group with NSS, then G is locally euclidean.*

Proof. Theorem 5.8 asserts that there exists a neighborhood of $0 \in L(G)$ such that \exp map is injective restricted to such neighborhood. Recall that \exp is continuous by definition 3.18, and theorem 3.21 says that for every neighborhood of the $0 \in L(G)$ its image under \exp is a neighborhood of the identity in G . So there exist a neighborhood of $0 \in L(G)$ which is homeomorphic to a neighborhood of $e \in G$. This last neighborhood can be continuously translated to any point in G so the \exp map give us a local homeomorphism between $L(G)$ and G . Which is the desired result since $L(G)$ is isomorphic to some \mathbb{R}^d thanks to lemma 3.17 and theorem 3.5. \square

Proposition 5.10. *Let G be a locally compact NSS group. For any neighborhood V of the identity, there exists a positive integer N such that if $g, g^2, \dots, g^N \in U$, then $g \in V$. Where U is the open neighborhood which comes from the NSS property of G .*

Proof. This affirmation can be restated as, for any sufficiently small (which fulfills that NSS property and has compact closure) neighborhood U of the identity and for any other neighborhood of the identity V , there exists an integer $N \in \mathbb{N}$ such that for all $g \in G$ such that $g, \dots, g^N \in U$ then $g \in V$.

By contradiction suppose that there exists a neighborhood U sufficiently small, as explained before, and a neighborhood V of the identity such that for every $n \in \mathbb{N}$ there exists an element $g_n \in G$ such that $g_n, \dots, g_n^n \in U$ and $g_n \notin V$.

Note that for the previous construction we get that

$$\begin{aligned} g_1 &\in U \\ g_2, g_2^2 &\in U \\ g_3, g_3^2, g_3^3 &\in U \\ &\vdots \\ g_n, \dots, g_n^n &\in U \\ &\vdots \end{aligned}$$

by hypothesis the closure of U is compact, so we get that the sequence $\{g_n\}_n$ has a convergent subsequence $g_{n_k} \rightarrow h \in \overline{U}$ for $k \rightarrow \infty$. Note that since $\{g_n\}_n \not\subset V$ we get $\{g_{n_k}\}_k \not\subset V$. Furthermore, $\lim_{k \rightarrow \infty} g_{n_k} = h \notin V$ which means that h cannot be the identity.

By the continuity of the group operations we get that $g_{n_k}^2 \rightarrow h^2$. In addition, since $\{g_n^2\}_{n \geq 2} \subset U$ any subsequence satisfies $\{g_{n_k}^2\}_{n_k \geq 2} \subset U$ and by the uniqueness of the limit we get that $h^2 \in \overline{U}$. This argument can be made for every $n \in \mathbb{N}$, so $h^n \in \overline{U}$, and by construction $h \neq e$ which contradicts the NSS property of G . \square

Lemma 5.11. *Let G be a locally compact NSS group. For any two sufficiently small neighborhoods U, V of the identity, we have*

$$\|g\|_U \ll \|g\|_V \ll \|g\|_U$$

for all $g \in G$.

Proof. By the symmetry of the result it will be enough to show the second inequality, which is equivalent to prove that for every $n \in \mathbb{N}$ such that $g, \dots, g^n \in U$ implies that $g, \dots, g^n \in V$. But this result follows from an analogous argument to the one presented in Theorem 5.10 modifying it: for any sufficiently small neighborhood U of the identity, for any other neighborhood of the identity V and for all $k \in \mathbb{N}$, there exists an integer $N \in \mathbb{N}$ such that for all $g \in G$ with $g, \dots, g^{kN} \in U$ implies $g, \dots, g^k \in V$.

A demonstration of the previous affirmation can be achieved by repeating the argument in theorem 5.10 for every k . \square

Proposition 5.12. *(Approximate triangle inequality) Let G be a locally compact NSS group, and U_0 be a sufficiently small open neighborhood of the identity. Then, for any n and any $g_1, \dots, g_n \in G$ we get*

$$\|g_1 \cdots g_n\|_{U_0} \ll \sum_{i=1}^n \|g_i\|_{U_0}.$$

Proof. Since U_0 is a neighborhood of the identity, we can find a symmetric neighborhood contained in U_0 which will be small enough too. So we can suppose that U_0 is symmetric itself.

Claim. *We claim that if somehow we have a positive constant M independent of g_1, \dots, g_n such that*

$$\|g_1 \cdots g_n\|_{U_0} \leq M \sum_{i=1}^n \|g_i\|_{U_0}. \quad (5.4)$$

From this M we can build up a better constant which will be independent of M . This independence will be a crucial one which later on will be strongly used in the proof of the approximate triangle inequality proposition.

Proof. Claim proof. Let us define a *modified escape norm* function $\|\cdot\|_{*,U_0} : G \rightarrow \mathbb{R}^+$ as follows

$$\|g\|_{*,U_0} := \inf \left\{ \sum_{i=1}^n \|g_i\|_{U_0} : g = g_1 \cdots g_n \right\} \quad (5.5)$$

where the infimum ranges over all the possible finite decomposition of g in products of elements of G .

By (5.4) for any fixed $g \in G$ we see $\frac{1}{M} \|g\|_{U_0} \leq \sum_{i=1}^n \|g_i\|_{U_0}$ and after taking the infimum we get that $\frac{1}{M} \|g\|_{U_0} \leq \|g\|_{*,U_0}$. On the other hand since g is itself a decomposition g we get $\|g\|_{*,U_0} \leq \|g\|_{U_0}$. In conclusion

$$\frac{1}{M} \|g\|_{U_0} \leq \|g\|_{*,U_0} \leq \|g\|_{U_0}. \quad (5.6)$$

Clearly by equation (5.5), for all $g, h \in G$ we have a triangular inequality for the modified escape norm

$$\|gh\|_{*,U_0} \leq \|g\|_{*,U_0} + \|h\|_{*,U_0} \quad (5.7)$$

since every decomposition of g and every decomposition on h can be multiplied to obtain a decomposition of gh .

By the symmetry of U_0 we get that $\|g\|_{*,U_0} = \|g^{-1}\|_{*,U_0}$. Now we are able to define a left invariant pseudo-metric d_* on G as

$$d_*(g, h) := \|g^{-1}h\|_{*,U_0}.$$

Recall that the distance from a point g to a set $A \subset G$ with respect to d_* is defined by $d_*(g, A) := \inf_{a \in A} d_*(g, a)$. Furthermore, we define the function $\psi : G \rightarrow \mathbb{R}^+$ by

$$\psi(x) := (1 - Md_*(x, U_0))_+.$$

Here it is crucially important to notice that the support of ψ is contained in U_0^2 . We want to show that

$$\text{supp}(\psi) = \left\{ x : d_*(x, U_0) \leq \frac{1}{M} \right\} \subset U_0^2.$$

If you take $x \in \text{supp}(\psi)$, then $\inf_{y \in U_0} \|x^{-1}y\|_{*,U_0} \leq \frac{1}{M}$. By inequality (5.6) we get that

$$\frac{1}{M} \inf_{y \in U_0} \|x^{-1}y\|_{U_0} \leq \inf_{y \in U_0} \|x^{-1}y\|_{x,U_0} \leq \frac{1}{M}.$$

Thus, $\inf_{y \in U_0} \|x^{-1}y\|_{U_0} \leq 1$ and by definition of infimum we can find $y \in U_0$ such that $\|x^{-1}y\|_{U_0} \leq 1$. In particular $x^{-1}y \in U_0$ and by symmetry $y^{-1}x \in U_0$ so $x = yy^{-1}x \in U_0^2$ as desired.

Now as in the equation (5.1) we want to prove a bound of the form

$$|\partial_g \psi(x)| \ll M \|g\|_{U_0}$$

for any $g \in G$ and $x \in U_0^2$. Here, since we can describe the support in terms independent of M we will be able to show that the constant implied in this inequality is indeed independent from M . This independence will be fundamental in the later development of the present argument.

By expanding the partial symbol, we get

$$\begin{aligned} |\partial_g \psi(x)| &= |\psi(g^{-1}x) - \psi(x)| = |1 - Md_*(g^{-1}x, U_0) - 1 + Md_*(x, U_0)| \\ &= M |d_*(x, U_0) - d(g^{-1}x, U_0)|. \end{aligned}$$

Recall that by the triangle inequality we get for any $x, y \in G$ and $H \subset G$

$$d_*(x, H) \leq d_*(x, y) + d_*(y, H).$$

So we get that

$$|\partial_g \psi(x)| = M |d_*(x, U_0) - d(g^{-1}x, U_0)| \leq Md_*(x, g^{-1}x) = M \|x^{-1}gx\|_{*,U_0}.$$

To complete this inequality we only need to proof that $\|x^{-1}gx\|_{*,U_0} \ll \|g\|_{U_0}$.

Thanks to (5.6) it suffices to see that $\|x^{-1}gx\|_{U_0} \ll \|g\|_{U_0}$ because

$$\|x^{-1}gx\|_{*,U_0} \leq \|x^{-1}gx\|_{U_0} \ll \|g\|_{U_0}.$$

Moreover, for every $n \in \mathbb{N}$ such that $\|g\|_{U_0} \leq \frac{1}{n+1}$ we conclude $g, \dots, g^n \in U_0$ and since $x \in U_0^2$ we get $x^{-1}gx, \dots, x^{-1}g^n x \in U_0^5$. And, calculating its escape norm we see $\|x^{-1}gx\|_{U_0^5} \leq \frac{1}{n+1}$. So, whenever $\|g\|_{U_0} \leq \frac{1}{n+1}$, we get that $\|x^{-1}gx\|_{U_0^5} \leq \frac{1}{n+1}$ which implies that $\|x^{-1}gx\|_{U_0^5} \ll \|g\|_{U_0}$.

Moreover, since the escape norm functions are comparable for small neighborhoods of the identity, we get that

$$\|x^{-1}gx\|_{U_0} \ll \|x^{-1}gx\|_{U_0^5} \ll \|g\|_{U_0}$$

as desired. In this last expression it is very important to notice that the implied constants do not depend on M in any way. They only depend on the open neighborhood U_0 and its powers.

To emulate the proof of the preceding theorem we will need to convolve ψ with another function. In this case we will use the following function $\eta : G \rightarrow [0, 1]$ defined as

$$\eta(x) := \begin{cases} \sup_{j=0, \dots, L} \left\{ 1 - \frac{j}{L} : x \in U_1^j U_0 \right\} & x \in U_1^L U_0 \\ 0 & x \notin U_1^L U_0 \end{cases}$$

for any natural number L independent of M and an open neighborhood of the identity $U_1 \subset U_0$. Notice that we can choose U_1 sufficiently small depending on L and U_0 such that $U_1^L U_0 \subset U_0^2$ and indeed $\text{supp}(\eta) \subset U_0^2$.

Note that by definition η is a simple function, which means that it has finite range. And, clearly for any $x \in G$ and $g \in U_1$ we get the bound

$$|\partial_g \eta(x)| \leq \frac{1}{L}$$

if $x \notin U_1^L U_0$ then in the worst case $g^{-1}x \in U_1^L U_0 \setminus U_1^{L-1} U_0$ and then we get the result. On the other hand, if $x \in U_1^L U_0$, then there exist some $j = 0, \dots, L$ such that $x \in U_1^j U_0 \setminus U_1^{j-1} U_0$. But in this case $g^{-1}x \in U_1^{j-1} U_0$ or $g^{-1}x \in U_1^{j+1} U_0$; in either case we get that $|\psi(g^{-1}x) - \psi(x)| \leq \frac{1}{L}$. Finally if $x \in U_0$ then in the worst case scenario $g^{-1}x \in U_1 U_0 \setminus U_0$ but then the result is obvious.

Since the functions ψ and η are compactly supported, bounded and Borel measurable (ψ is continuous and η is simple), we can form their convolution $\phi := \psi * \eta$. Indeed, $\text{supp}(\phi) \subset U_0^4$. It is also important to notice that we have a lower bound for $\|\phi\|_\infty$. Note that

$$\|\phi\|_\infty \geq \phi(e) = \int_G \psi(y)\eta(y^{-1})dy \geq \mu(\text{supp}(\psi)) \geq \mu(U_0) \gg 1.$$

The last inequality can be achieved by choosing a suitable multiple of the measure μ . So, the implied constant will depend only on μ and U_0 .

Moreover as in the proof of the previous theorem, for every n such that $n\|g\|_\phi < \|\phi\|_\infty$ implies $\|g^n\|_\phi < \|\phi\|_\infty$. This last inequality implies that ϕ and $L_g \phi$ have overlapping supports (otherwise we will have the equality $\|g^n\|_\phi = \|\phi\|_\infty$). Hence, $g^n \in U_0^8$ which suggest $\|g\|_{U_0^8} \leq \frac{1}{n+1}$.

Since, whenever $\|g\|_\phi < \|\phi\|_\infty/n$ implies $\|g\|_{U_0^8} \leq 1/n + 1$, thus

$$\|g\|_{U_0^8} \ll \|g\|_\phi$$

and by the comparison of the functions $\|\cdot\|_{U_0^8}$ and $\|\cdot\|_{U_0}$ we get

$$\|g\|_{U_0} \ll \|g\|_{\phi}.$$

The converse inequality can be seen as

$$\begin{aligned} \|g\|_{\phi} &= \|\partial_g \phi\|_{\infty} = \|\partial_g(\psi * \eta)\|_{\infty} = \|(\partial_g \psi) * \eta\|_{\infty} \\ &\leq \sup_{x \in G} \int_G |\partial_g \psi(y)| |\eta(g^{-1}x)| dy \\ &\ll M \|g\|_{U_0} \sup_{x \in G} \int_G |\eta(g^{-1}x)| dy \\ &\leq \mu(\text{supp}(\eta)) M \|g\|_{U_0} \\ &\leq \mu(U_0^2) M \|g\|_{U_0}. \end{aligned}$$

Since the $\mu(U_0^2)$ does not depend on M we get

$$\|g\|_{\phi} \ll M \|g\|_{U_0}.$$

Finally we see

$$\|g\|_{U_0} \ll \|g\|_{\phi} \ll M \|g\|_{U_0}. \quad (5.8)$$

Recall that for any $g, h \in G$ we have

$$\partial_g \partial_h \phi(x) = \int_G \partial_h \psi(y) \partial_{g^y} \eta(y^{-1}x) dy.$$

Notice that if $h \in U_0$ then the term $\partial_h \psi(y)$ has support in the set U_0^3 , so the integral vanishes unless $y \in U_0^3$.

By the continuity of the conjugation function $(\cdot)^y : G \rightarrow G$, we can find a small neighborhood of the identity $U_2 \subset U_1$ such that $g^y \in U_1$ if $g \in U_2$ and $y \in U_0^3$. Moreover, for $h \in U_0$ and $g \in U_2$ we get

$$\begin{aligned} |\partial_g \partial_h \phi(x)| &\leq \int_G |\partial_h \psi(y)| |\partial_{g^y} \eta(y^{-1}x)| dy \\ &= \int_{U_0^3} |\partial_h \psi(y)| |\partial_{g^y} \eta(y^{-1}x)| dy \\ &\ll M \|h\|_{U_0} \int_{U_0^3} |\partial_{g^y} \eta(y^{-1}x)| dy \\ &\leq M \|h\|_{U_0} \frac{1}{L} \int_{U_0^3} dy = \frac{M}{L} \|h\|_{U_0} \mu(U_0^3). \end{aligned}$$

Since U_0^3 is independent of M , we get that $|\partial_g \partial_h \phi(x)| \ll \frac{M}{L} \|h\|_{U_0}$.

Now the following calculation

$$\begin{aligned}
\sum_{i=0}^n \partial_g \partial_{g^i} \phi(x) &= \sum_{i=0}^n \partial_g (\phi(g^{-i}x) - \phi(x)) \\
&= \sum_{i=0}^n \phi(g^{-i-1}x) - \phi(g^{-1}x) - \phi(g^{-i}x) + \phi(x) \\
&= \sum_{i=0}^n \phi(g^{-i-1}x) - \phi(g^{-i}x) - \partial_g \phi(x) \\
&= -\phi(x) + \phi(g^{-n}x) - n\partial_g \phi(x) \\
&= \partial_{g^n} \phi(x) - n\partial_g \phi(x)
\end{aligned}$$

shows us that $\partial_{g^n} \phi = n\partial_g \phi + \sum_{i=0}^n \partial_g \partial_{g^i} \phi$. Taking norm and supremum we get

$$\begin{aligned}
n\|g\|_\phi &= \|g^n\|_\phi + \sup_{x \in G} \left| \sum_{i=0}^n \partial_g \partial_{g^i} \phi(x) \right| \\
&\leq \|g^n\|_\phi + \sum_{i=0}^n \sup_{x \in G} |\partial_g \partial_{g^i} \phi(x)| \\
&\ll \|g^n\|_\phi + \sum_{i=0}^n \sup_{x \in G} \frac{M}{L} \|g\|_{u_0} \\
&= \|g^n\|_\phi + n \frac{M}{L} \|g\|_{u_0} \\
&= \|\phi\|_\infty + n \frac{M}{L} \|g\|_{u_0}.
\end{aligned}$$

Thus, we get

$$\|g\|_\phi \ll \frac{1}{n} + \frac{M}{L} \|g\|_{u_0}.$$

This is not yet enough. We need to construct a better constant. We want to prove that there exists a positive constant K for all M, L such that

$$\|g\|_\phi \leq K \left(\frac{M}{L} + 1 \right) \|g\|_{U_0}.$$

Note that by the choosing of the quantifiers we require K to be independent of M .

We will prove that for all $n \in \mathbb{N}$ such that $\|g\|_{U_0} \leq \frac{1}{n}$ implies $\|g\|_\phi \leq \frac{1}{n} + \frac{M}{L} \|g\|_{u_0}$. Then, $\|g\|_\phi \ll \left(\frac{M}{L} + 1 \right) \|g\|_{U_0}$

Proceeding by contradiction we will get for every $K > 0$ there exists some M_K, L_K and g_K such that

$$\|g_K\|_\phi \geq K \left(\frac{M_K}{L_K} + 1 \right) \|g_K\|_{U_0}.$$

Since $\|g_K\|_\phi \leq \|\phi\|_\infty$, we see that there exists a positive constant R such that $1 \ll \|\phi\|_\infty \leq R$ and $\|g\|_\phi \leq R$. Moreover, we obtain

$$\|g_K\|_{U_0} \leq \frac{R}{K \left(\frac{M_K}{L_K} + 1 \right)} = \frac{1}{\frac{K}{R} \left(\frac{M_K}{L_K} + 1 \right)} \leq \frac{1}{\lfloor \frac{K}{R} \rfloor \lfloor \left(\frac{M_K}{L_K} + 1 \right) \rfloor}.$$

By hypothesis we obtain

$$\begin{aligned} \|g_K\|_\phi &\leq \frac{1}{\lfloor \frac{K}{R} \rfloor \lfloor \left(\frac{M_K}{L_K} + 1 \right) \rfloor} + \frac{M_K}{L_K} \|g_K\|_{U_0} \\ &\leq \frac{1}{\lfloor \frac{K}{R} \rfloor \lfloor \left(\frac{M_K}{L_K} + 1 \right) \rfloor} + \frac{M_K}{L_K} \frac{1}{\lfloor \frac{K}{R} \rfloor \lfloor \left(\frac{M_K}{L_K} + 1 \right) \rfloor} \end{aligned}$$

so

$$\|g_K\|_\phi \leq \frac{1 + \frac{M_K}{L_K}}{\lfloor \frac{K}{R} \rfloor \lfloor \left(\frac{M_K}{L_K} + 1 \right) \rfloor} \leq \frac{R \left(1 + \frac{M_K}{L_K} \right)}{\lfloor \frac{K}{R} \rfloor \lfloor \left(\frac{M_K}{L_K} + 1 \right) \rfloor}.$$

Thus,

$$\frac{R \left(1 + \frac{M_K}{L_K} \right)}{\lfloor \frac{K}{R} \rfloor \lfloor \left(\frac{M_K}{L_K} + 1 \right) \rfloor} \geq \|g_K\|_\phi \geq K \left(\frac{M_K}{L_K} + 1 \right) \|g_K\|_{U_0}$$

and clearly

$$\|g_K\|_{U_0} \leq \frac{\frac{R}{K}}{\lfloor \frac{K}{R} \rfloor \lfloor \left(\frac{M_K}{L_K} + 1 \right) \rfloor} \leq \frac{1}{\frac{K}{R} \lfloor \frac{K}{R} \rfloor \lfloor \left(\frac{M_K}{L_K} + 1 \right) \rfloor} \leq \frac{1}{\lfloor \frac{K}{R} \rfloor^2 \lfloor \left(\frac{M_K}{L_K} + 1 \right) \rfloor}.$$

We can repeat this procedure infinitely many times and get

$$\|g_K\|_{U_0} \leq \frac{1}{\lfloor \frac{K}{R} \rfloor^m \lfloor \left(\frac{M_K}{L_K} + 1 \right) \rfloor}.$$

Since K is arbitrary, we can choose $K > R$. In this case we get that $\|g_K\|_{U_0} \rightarrow 0$ as $m \rightarrow \infty$. But this is a contradiction since the $g_K \neq e$ and G is NSS so the only group element with escape norm equal to zero is the identity.

This previous result allows us to write $\|g\|_\phi \ll \left(\frac{M}{L} + 1\right) \|g\|_{U_0}$ where the implied constant does not depend on M . Note that since $d_\phi(\cdot, \cdot)$ is a pseudo-metric, we have the following triangular inequality

$$\|g_1 \cdots g_n\|_\phi \leq \sum_{i=1}^n \|g_i\|_\phi$$

an combining this with the previous result and (5.8) we get

$$\|g_1 \cdots g_n\|_{U_0} \leq \|g_1 \cdots g_n\|_\phi \leq \sum_{i=1}^n \|g_i\|_\phi \ll \left(\frac{M}{L} + 1\right) \sum_{i=1}^n \|g_i\|_{U_0}.$$

Here note that we can consider any arbitrarily large L , and $\frac{M}{L} \rightarrow 0$ as $L \rightarrow \infty$ since the M was fixed at the beginning of the proof. Moreover, the constants implied in the \ll symbol were always constructed independent of M which gives us the claim

$$\|g_1 \cdots g_n\|_{U_0} \ll \sum_{i=1}^n \|g_i\|_{U_0}. \quad \square$$

Proof of proposition 5.12 The previous construction depends on the existence of a positive constant M . But, we are not sure of the existence of such constant. So we will consider a slight modification of the function $\|\cdot\|_{U_0}$.

Recall that during the above argument we used only some properties of $\|\cdot\|_{U_0}$ namely: that they are comparable for small enough open sets, the symmetry of the function for a symmetric open U_0 and that it is, in fact, a bounded function. Fix any $\epsilon > 0$ and consider the function

$$\begin{aligned} \|\cdot\|^* : G &\rightarrow \mathbb{R}^+ \\ g &\mapsto \|g\|_{U_0} + \epsilon. \end{aligned}$$

Notice that $\epsilon\|g_1 \cdots g_n\|^*$ is bounded since $\|g_1 \cdots g_n\|_{U_0} \leq 1$. Analogously we get that $\sum_{i=1}^n (\|g_i\| + \epsilon) > \epsilon$ so it is bounded below and we get the following result:

$$\frac{\epsilon(\|g_1 \cdots g_n\|_{U_0} + \epsilon)}{\sum_{i=1}^n (\|g_i\|_{U_0} + \epsilon)} \leq S$$

for some constant S . Thus,

$$\|g_1 \cdots g_n\|^* \leq \frac{S}{\epsilon} \sum_{i=1}^n \|g_i\|^*.$$

Since the function $\|\cdot\|^*$ is also comparable for small enough open sets, symmetric for a symmetric open U_0 and bounded. We can apply the previous result taking $M = \frac{S}{\epsilon}$ and we get that

$$\|g_1 \cdots g_n\|^* \leq A \left(\frac{M}{L} + 1\right) \sum_{i=1}^n \|g_i\|^*$$

where the constant A is independent of M . Moreover, we can again take $L \rightarrow \infty$ and get

$$\|g_1 \cdots g_n\|^* \leq A \sum_{i=1}^n \|g_i\|^*$$

which can be written as

$$\|g_1 \cdots g_n\|_{U_0} + \epsilon \leq A \sum_{i=1}^n (\|g_i\|_{U_0} + \epsilon).$$

Here it is crucially important (as we remark before) that the constant A is independent of $M = \frac{S}{\epsilon}$ which in particular makes A independent of ϵ . Thus, we can take $\epsilon \rightarrow 0$ and get the desired result

$$\|g_1 \cdots g_n\|_{U_0} \leq A \sum_{i=1}^n \|g_i\|_{U_0}. \quad \square$$

Theorem 5.13. *Every locally compact NSS group admits a weak Gleason metric.*

Proof. Let U_0 be a sufficiently small open neighborhood of the identity such that U_0 is pre-compact and fulfills the NSS property. By (5.6) the escape norm and the modified escape norms are comparable.

In addition to this we have already seen that d_* is a left invariant pseudo-metric. Moreover, since G is NSS then $\|g^{-1}h\|_{*,U_0} = 0$ if $d_*(g, h) = 0$ which implies that

$$\begin{aligned} 0 &= \inf \left\{ \sum_{i=1}^n \|g_i\|_{U_0} : g^{-1}h = g_1 \cdots g_n \right\} \\ &\gg \inf \{ \|g_1 \cdots g_n\|_{U_0} : g^{-1}h = g_1 \cdots g_n \}. \end{aligned}$$

Thus, one can find g_1, \dots, g_n such that $\|g_1 \cdots g_n\|_{U_0} \leq 0$ and $g^{-1}h = g_1 \cdots g_n$. But since G is NSS the only element with escape norm equal to zero is the identity, so $g^{-1}h = g_1 \cdots g_n = e$ and clearly $h = g$. This implies that d_* is a genuine metric.

Now we claim that d_* generates the topology of G . Thank to the left invariance of d_* it is enough to prove that any open neighborhood of the identity contains a ball around the identity and conversely.

Let U be a neighborhood of the identity and consider $U' \subset U$ a smaller neighborhood of the identity. From (5.6) and theorem 5.6 follows that $\|\cdot\|_{U'}$ is comparable to $\|\cdot\|_{*,U_0}$. Furthermore, the ball $B_{1/n}^*(e) \subset U'$; if $x \in B_{1/n}^*(e)$ then $\|x\|_{*,U_0} = d_*(x, e) < \frac{1}{n}$. Thus,

$$\|x\|_{U'} \ll \|x\|_{U_0} \ll \|x\|_{*,U_0} < \frac{1}{n},$$

so we get that there is a constant A such that

$$\|x\|_{U'} \leq \frac{A}{n} \leq \frac{1}{\lfloor n/A \rfloor}$$

which implies that $g, \dots, g^{\lfloor n/A \rfloor} \in U'$ as desired.

In the other hand, if we take the ball $B_r^*(e)$ then for any positive integer m we can find an open neighborhood U_m such that $U_m^m \subset U_0$. This implies that for any $g \in U_m$ we get $\|g\|_{U_0} \leq \frac{1}{m+1}$ since $g^m \in U_m^m \subset U_0$. Thus for a sufficiently large m we get that $U_m \subset B_r(e)$, since

$$d_*(g, e) = \|g\|_{*,U_0} \ll \|g\|_{U_0} \leq \frac{1}{m+1} < r.$$

Finally we need to verify that d_* fulfills the conditions in Definition 5.3. First of all we need to verify that for all $g \in G$ and $n \geq 1$ such that $n\|g\|_{*,U_0} < \epsilon$, we obtain $\|g^n\|_{*,U_0} \gg n\|g\|_{*,U_0}$. If $g = e$ the claim is trivial. Assume that $g \neq e$. We get that there must exist some natural number m such that $g^m \notin U_0$ and $g, \dots, g^{m-1} \in U_0$ so,

$$\|g\|_{*,U_0} \gg \|g\|_{U_0} \gg \frac{1}{m}.$$

Take now U_1 a symmetric neighborhood of the identity small enough such that $U_1^2 \subset U_0$ and choose $\epsilon > 0$ such that $B_\epsilon^*(e) \subset U_1$.

We have that $\|g^i\|_{*,U_0} \leq i\|g\|_{*,U_0} \leq n\|g\|_{*,U_0} < \epsilon$ for all $1 \leq i \leq n$. Moreover, $g^i \in U_1$ because $g^i \in B_\epsilon^*(e) \subset U_1$. Furthermore, this implies that $n < m$ because otherwise it contradicts $g^m \notin U_0$.

Now, let $m+j$ be the first multiple of n which is larger than m . Thus we can write $m+j = kn$ for some natural number k . Also j can be made such that $1 \leq j \leq n$. Notice that $g^{m+j} = g^{kn} = (g^n)^k \notin U_1$, because if $g^{m+j} \in U_1$ then $g^m = g^{m+j}g^{-j} \in U_1^2 \subset U_0$ which is a contradiction.

Then $\|g^n\|_{U_1} \geq \frac{1}{k} \geq \frac{n}{m+j}$ since $g^{m+j} \notin U_1$. Notice that since $j \leq n$ and $n \leq m$, thus

$$\frac{n}{m+j} \geq \frac{n}{2m}$$

which means that

$$\frac{n}{m+j} \gg \frac{n}{m}$$

which lead us to

$$\|g^n\|_{*,U_0} \geq \|g^n\|_{U_0} \ll \|g^n\|_{*,U_0} \leq \frac{n}{m+j} \gg \frac{n}{m} = n\|g\|_{U_0} \geq \frac{n}{M}\|g\|_{*,U_0}.$$

Finally notice that the property (ii) in the Definition 5.3 comes directly from the inequality (5.6). \square

Theorem 5.14. *Every NSS locally compact group G is isomorphic to a Lie group.*

Proof. Since every locally compact group admits a weak Gleason metric by Theorem 5.13, G admits a Gleason metric. Moreover, G is isomorphic to a Lie group by the results in Chapter 3. \square

Corollary 5.15. *A locally compact group is NSS if and only if it is isomorphic to a Lie group.*

Proof. The proof follows immediately from Theorems 5.6 and Theorem 5.13. \square

Bibliography

- [Bum98] Daniel Bump. *Automorphic Forms and Representations*. Cambridge Studies in Advanced Mathematics. Cambridge University Press, 1998. ISBN: 9780521658188. URL: <http://books.google.com.co/books?id=QQ1cr7B6XqQC>.
- [Cas12] B. Casselman. “Introduction to topological vector spaces”. Online essay. 2012. URL: <http://www.math.ubc.ca/~cass/research/pdf/TVS.pdf>.
- [Coh94] D.L. Cohn. *Measure theory*. Birkhäuser Boston, 1994. ISBN: 97808 17630034. URL: <http://books.google.com.co/books?id=uY2wQgAACAAJ>.
- [Fol94] G.B. Folland. *A Course in Abstract Harmonic Analysis*. Studies in Advanced Mathematics. Taylor & Francis, 1994. ISBN: 9780849384905. URL: <http://books.google.com.co/books?id=0VwYZI1DypUC>.
- [Hil02] D. Hilbert. “Mathematical problems”. In: *Bulletin of the American Mathematical Society* 8.10 (1902), pp. 437–479.
- [Kur50] M. Kuranishi. “On euclidean local groups satisfying certain conditions”. In: *Proceedings of the American Mathematical Society* 1.3 (1950), pp. 372–380.
- [Len90] R. Lenz. *Lecture Notes in Computer Science*. Ed. by G. Goos and J. Hartmanis. Vol. 413. Springer-Verlag, 1990. ISBN: 3-540-52290-5. URL: <http://webstaff.itn.liu.se/~reile/lncs.html>.
- [Mun00] J.R. Munkres. *Topology*. Prentice Hall, Incorporated, 2000. ISBN: 9780131816299. URL: <http://books.google.com.co/books?id=XjoZAQAIAAJ>.
- [Sti08] J. Stillwell. *Naive Lie theory*. Undergraduate Texts in Mathematics. Springer, 2008. ISBN: 9780387782157. URL: <http://books.google.com.co/books?id=SuR50AgxyDIC>.
- [SW73] A.A. Sagle and R. Walde. *Introduction to Lie groups and Lie algebras*. Pure and Applied Mathematics. Elsevier Science, 1973. ISBN: 9780080873664. URL: <http://books.google.com.co/books?id=LA37coRrnaMC>.
- [Tao12] T. Tao. “Hilbert’s fifth problem and related topics”. Department of Mathematics, UCLA, Los Angeles, CA. 2012.

- [Var84] V.S. Varadarajan. *Lie Groups, Lie Algebras, and Their Representation*. Graduate Texts in Mathematics. Springer, 1984. ISBN: 9780387909691.
- [War71] F.W. Warner. *Foundations of differentiable manifolds and Lie groups*. Graduate Texts in Mathematics. Springer, 1971. ISBN: 9780387908946. URL: <http://books.google.com.co/books?id=iaeUqc2yQVQC>.
- [Wei80] J. Weidmann. *Linear operators in Hilbert spaces*. Graduate Texts in Mathematics. Springer-Verlag, 1980. ISBN: 9780387904276. URL: http://books.google.com.co/books?id=_SDvAAAAMAAJ.
- [Yam51] H. Yamabe. “Note on locally compact groups”. In: *Osaka Mathematical Journal* 3.1 (1951), pp. 77–82.
- [Yam53] H. Yamabe. “A Generalization of A Theorem of Gleason”. In: *Annals of Mathematics* 58.2 (1953), pp. 351–365.