Equivariant algebraic topology applied to some problems in topological combinatorics

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Title in English

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Abstract: In this thesis we present several results on geometric combinatorics whose solution can be achieved by means of results and tools from algebraic topology. The combinatorial problems are related to known problems as the topological Borsuk problem, equilateral sets in metric spaces, Tverberg type problems, evasiveness of graph properties, etc. Among the results and tools used in the solutions of these problems we find theorems of Borsuk-Ulam and Dold, theorems of Smith and Oliver, group cohomology, cohomology theories, numerical and cohomological index theories.

Keywords: Equivariant topology. Borsuk-Ulam theorem. Dold's theorem. G-spaces. Borsuk' number. Fadell-Husseini index. Tverberg's theorem. Morava K-theory. Graph Properties. Evasiveness. Graph automorphisms. Euler characteristic.

Título en español

Topología algebraica equivariante aplicada a algunos problemas de combinatoria topológica

Resumen: En esta tesis se muestran varios resultados de problemas de combinatoria geométrica cuya solución se logra mediante el uso de resultados y herramientas provenientes de la topología algebraica. Los problemas que resolvemos están relacionados con problemas conocidos como el problema de Borsuk topológico, conjuntos equiláteros en espacios métricos, problemas tipo Tverberg, evasividad de propiedades de grafos, entre otros. Entre las herramientas usadas para resolver estos problemas encontramos teoremas de topología algebraica como los teoremas de Borsuk-Ulam y de Dold, teoremas de Smith y Oliver, resultados de cohomología de grupos, teorías de cohomolgía, índices numérico y cohomológico de G-espacios.

Palabras clave: Topología equivariante. Teorema de Borsuk-Ulam. Teorema de Dold. G-espacios. Número de Borsuk. Índice de Fadell-Husseini. Teorema de Tverberg. K-teoría de Morava. Propiedades de grafos. Evasividad. Automorfismos de grafos. Caracteística de Euler.

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Contents

Introduction

We say a problem is solved by *topological methods* if the method of solution uses topological information coming from a topological space associated to the problem. Many combinatorial problems (for example, the the Ham Sandwich theorem, the Kneser conjecture and evasiveness of graph properties), can be rephrased and put into a "topological setting" which is suitable for applying results and tools from algebraic topology. In many instances, we can identify a pattern in the solutions: the *configuration space/ test map (CS/TM) scheme*.

In the CS/TM scheme we associate to the combinatorial problem a topological space X. This topological space X, called the *configuration space*, codifies all *potential* solutions to the combinatorial problem. The next step in the CS/TM scheme is to define a map f, called the *test map*, from X into a Euclidean space $\mathbb{R}^m$ and we identify a *special* subspace Δ of $\mathbb{R}^m$. The map f codifies the required conditions for a potential solution to be an actual solution to the problem and the subspace Δ tests if a potential solution $x \in X$ is a true solution of the problem, in the sense that x solves the combinatorial problem if and only if $f(x) \in \Delta$. Thus, our combinatorial problem results to be equivalent to the affirmation that $im\ f \cap \Delta \neq \varnothing$. Then we assume, on the contrary, that the image of f does not intersect Δ, so that we get a map $X \to \mathbb{R}^m \setminus \Delta$. On the other hand, the combinatorial problem comes naturally with certain *group of symmetries* G. All of the spaces involved in the CS/TM scheme become G-spaces and the test map f becomes G-equivariant. Hence, one faces the problem of proving that there are no G-equivariant maps $X \to \mathbb{R}^m \setminus \Delta$ (a standard problem in equivariant algebraic topology).

In some cases, we just study topological properties of the configuration space X to solve the problem, but in general we have to study the configuration space as well as the target of the test map.

One of the first places where these topological methods appeared is in Lovász's solution of the Kneser conjecture ([32], 1978). The Kneser conjecture is the assertion that the *chromatic number* of the *Kneser graph* $KG_{n,k}$ is $n - 2k + 2$, where $k > 0$ and $n \geq 2k - 1$. In Lovász's solution of the Kneser conjecture, it is used a variation of the Borsuk-Ulam theorem, namely the Lusternik-Schnirelman theorem. We recall the statement of the Borsuk-Ulam theorem: for any continuous map $f : S^d \to \mathbb{R}^d$, there exists $x \in S^d$ such that $f(x) = f(-x)$. The Lusternik-Schnirelman theorem says that if a collection of $d+1$ subsets $A_1, \ldots, A_{d+1}$ of S^d cover S^d, where each A_i is either open or closed, then there exists $x \in S^d$ an some A_i such that $x, -x \in A_i$, that is, some A_i must contain a pair of antipodal points.

One important problem in geometrical and topological combinatorics is the topological Tverberg problem. We start with Radon's theorem ([44], 1921): any set of $d+2$ points in $\mathbb{R}^d$ can be partitioned into two subsets A_1 and A_2 in such a way that their convex hulls intersect, that is, $conv\ A_1 \cap conv\ A_2 \neq \varnothing$. One nice application of Radon's theorem is in the proof of Helly's theorem (discovered by E. Helly in 1913), which says that a collection of n convex subsets of $\mathbb{R}^d$ with $n > d$ have nonempty intersection if subfamilies of size $d+1$ have nonempty intersections. The first published proof of Helly's theorem was by Radon in 1921 although Helly presented his own proof in 1923.

H. Tverberg shows the following generalization of Radon's theorem ([58], 1966): for $q \geq 2$ and $d \geq 1$, any collection of $(q-1)(d+1)+1$ points in $\mathbb{R}^d$ can be partitioned into q subsets $A_1, \ldots, A_q$ in such a way that the convex hulls $\{conv\ A_j\}$ have nonempty intersection. This result is known as the Tverberg theorem. Later, I. Bárány, S. Shlosman and A. Szücs ([5], 1981) propose the following problem: show that if $f : \Delta^N \to \mathbb{R}^d$ is any continuous function, where $N = (q-1)(d+1)$ and Δ^N is the N-dimensional simplex, then there are q disjoint faces of Δ^N, $F_1, \ldots, F_q$ such that the images $f(F_j)$ have a non-empty intersection. This is the *topological Tverberg problem*. In [5], Bárány *et al* prove that this is true when q is a prime number (this is apparently the first time G-equivariant maps and the CS/TM scheme appear explicitly in the solution of a combinatorial problem).

M. Özaydin, in an unpublished preprint [42], 1987, extended the result of [5] to power primes $q = p^k$. Several authors have given proofs of Özaydin achievement: A. Volovikov [59]; K. Sarkaria [53], based on characteristic classes; R. Živaljević, [63], whose proof is based on the Fadell-Husseini index. Recently, F. Frick has given counterexamples to the Tverberg problem in the non-prime-power case [18], 2015.

P. Conner and F. Floyd introduced the notion of numerical index and co-index of G-spaces [12], 1960. The numerical index is a useful tool for solving combinatorial problems. A result of fundamental importance in this context is the following theorem of A. Dold [13]: for an n-connected G-space X and a free cell G-complex Y of dimension at most n, there are no G-equivariant maps $X \to Y$. Dold's theorem generalizes the Borsuk-Ulam theorem.

The work of K. Sarkaria, [50] (1990), [51] (1991), [52] (1991), [53] (2000), also Bárany *et al* ([5]), shows how naturally topological constructions as *join* of spaces and simplicial complexes, *deleted joins* and *deleted products*, appear in many combinatorial problems as configuration spaces and how their topology helps to solve the combinatorial problem.

R. Živaljević noticed that there was a common pattern in the solutions all those combinatorial problems. Živaljević synthesizes this pattern into the CS/TM scheme (see chapter 14 of [14], [62] (1996), see also [7]). A very nice exposition of the CS/TM scheme is found in J. Matoušek's book [35].

We mention that there are also several variations of the indexes and there are other tools from algebraic topology that have also many applications; for instance, obstruction theory and characteristic classes have proved to be successful in several combinatorial problems (see [8, 29, 31, 63]).

E. Fadell and S. Husseini introduced the ideal valued cohomological index using equivariant cohomology [15, 16]. They used it to study zero sets of equivariant maps, Borsuk-Ulam and Bourgin-Yang type theorems and critical point theory.

For a G-space X, its index $Ind_{G,R}X$ is an ideal in the ring $H^*(BG;R)$, where R is a commutative ring with unit. The important property of the ideal-valued index is that it gives a necessary condition to the existence of a G-equivariant map $X \to Y$ between G-spaces X and Y. In fact, if there is a G-equivariant map $X \to Y$, then $Ind_{G,R}Y \subseteq Ind_{G,R}X$. This has proved to be very useful for proving non-existence of equivariant maps between G-spaces when the action on the target is not free (where Dold's theorem does not apply). See [9, 10, 11, 63] for nice applications to combinatorial problems in which the fundamental tool used in the solutions is the Fadell-Husseini index.

One combinatorial problem of interest is the evasiveness conjecture for graph properties. A graph property $\mathcal{P}$ is a collection of graphs which is closed under isomorphism of graphs. The *complexity* $c(\mathcal{P})$ of a graph property $\mathcal{P}$ is the number of questions of the form "is $\{i, j\}$ an edge of G?," one is obligated to ask in the worst case, in order to determine whether G belongs to $\mathcal{P}$; $\mathcal{P}$ is called evasive if $c(\mathcal{P}) = n(n-1)/2$ (the largest possible), otherwise $\mathcal{P}$ is non-evasive. Examples of non-evasive graph properties are very difficult to find. One of the most famous examples of a non-evasive graph property is the property of being a *skorpion graph* (see [6, 30]).

Monotone graph properties are those that are closed under removal of edges, and nontrivial graph properties are those that are both nonempty and not the collection of all graphs. The *evasiveness conjecture* (or Karp's conjecture) asserts that every nontrivial monotone graph property is evasive.

In [49], it was conjectured that there is a constant $\epsilon > 0$ such that any nontrivial monotone digraph property on n vertices has complexity at least ϵn^2. This statement, known as the Aandreraa-Rosenberg conjecture, was proven in [48]. Karp's conjecture is a very strong version of this conjecture (see [49]).

Topological methods are used in the evasiveness conjecture by J. Kahn, M. Saks, and D. Sturtevant in [27], 1984. They associate to each monotone graph property a simplicial complex (this is the configuration space of the problem) and prove that the complex associated to monotone non-evasive graph properties is *collapsible*. This is the central result of the *topological approach to evasiveness*. Then, by combining this topological approach with the theory of fixed points of actions of groups on *acyclic* complexes, Kahn *et al* give a proof of Karp's conjecture in the prime power case. In particular, they use results of P. Smith [54] and R. Oliver [41]. They also prove the six vertices case by using this topological approach.

Karp's conjecture is proven in [61] by A. C.-C. Yao for *bipartite graph properties*. Yao's proof is based on this topological approach, too.

There are plenty of families of nontrivial monotone graphs properties which are known to be evasive (see for example [24]), the techniques are based mainly on the topological approach and in many cases discrete Morse theory is a useful tool (see [17], [24]).

For background on the CS/TM scheme and how the topological methods have been applied in many combinatorial problems see [7], also chapter 14 by R. Živaljević in [14] and J. Matoušek's book [35]. Applications of the numerical G-index to combinatorial problems can be found in [35, 62]. Some applications of the Fadell-Husseini index to combinatorial problems are found in [8, 9, 10, 63]. For background on graph properties, the evasiveness conjecture and the topological approach of Kahn, Saks and Sturtevant, see [6, 27, 30, 38].

Our work is divided into 3 chapters. Chapter 1 deals with graph properties and the evasiveness conjecture. Chapters 2 and 3 treat cohomological indexes and applications to some combinatorial problems. Chapter 1 can be taken independently.

In chapter 1 we show the topological approach to evasiveness and the proof of the evasiveness conjecture for the prime power case given in [27]. Then we study the sizes of automorphism groups of graphs on $2p$ vertices (p prime) and use this to estimate the Euler characteristic of the simplicial complex associated to a nontrivial monotone graph property $\mathcal{P}$. If we add the condition of being non-evasive, then the Euler characteristic of the associated simplicial complex is 1. Our estimates for $\chi(\mathcal{P})$ yield that $\mathcal{P}$ has to contain some classes of *special* graphs. We apply our estimates to reproduce a proof of the evasiveness conjecture in the 6 vertices case. Although our estimates reduces the problem, the use of *Oliver groups* is essential to deal with the different cases that appear. We also apply our estimates to the 10 vertices case (not solved yet) and make a reduction of the problem. Unfortunately, we do not complete the proof of Karp's conjecture in the 10 vertices case. We end with a description of potential counterexamples to the evasiveness conjecture in the 10 vertices case.

By using the action of some Oliver groups on the (contractible) simplicial complex associated to a nontrivial monotone and non-evasive property of graphs, we obtain lower bounds for the dimension of such simplicial complexes. In particular we prove that if $\mathcal{P}$ is a nontrivial monotone and non-evasive graph property on $2p$ vertices, where $p > 3$ is a prime number, then the dimension of the associated simplicial complex is at least $4p - 1$. This improves, in the particular case of $2p$ vertices, a general lower bound of Bjorner for the dimension of vertex-homogeneous simplicial complexes with Euler characteristic equal to 1 (see [33]).

In chapter 2 we treat ideal-valued index theories. They can be defined for cohomology theories in a similar way to the Fadell-Husseini index.

We compile some basic properties of the Fadell-Husseini index as well as some tools of calculation such as the Serre spectral sequence of a fibration and formulas for indexes of spheres, $E_N G$-spaces and the configuration spaces $F(\mathbb{R}^d, p)$.

We also show a sketch of Živaljević's proof for the topological Tverberg theorem. Then we change from equivariant cohomology to Morava K-theories to have an ideal-valued cohomological index based on a different cohomology theory. We reproduce the proof of the topological Tverberg theorem using this *new* ideal-valued index. Our calculations of the new index are based totally on the willing to reproduce a proof of the topological Tverberg theorem.

In chapter 3, we introduce the concept of *r-regular m-gons.* We say that m (distinct) points $x_0, x_1, \ldots, x_{m-1}$ in a metric space (X, ρ) form a regular m-gon with respect to ρ if $\rho(x_0, x_1) = \rho(x_1, x_2) = \cdots = \rho(x_{m-1}, x_0)$; it is possible that an m-gon is regular with respect to various metrics on M simultaneously, and so is that we introduce r-regular m-gons.

Our main result is that if we have r metrics in $\mathbb{R}^d$ that induce the Euclidean topology of $\mathbb{R}^d$, p is a prime number and $r < d$, then there exists an r-regular p-gon in $\mathbb{R}^d$ with respect to the r given metrics. We appeal to the CS/TM scheme and Fadell-Husseini index in our proof. Our result in the case $d = 2$, $r = 1$, $p = 3$, reduces to Soibelman's proof that the Borsuk number of $\mathbb{R}^2$ is 3 ([55]).

In the paper of P. Blagojević and G. Ziegler [10], "Tetrahedra on deformed spheres and integral group cohomology," it is proved, by means of the CS/TM scheme and Fadell-Husseini index, the following theorem: if $f : S^2 \to \mathbb{R}^3$ is continuous and injective, and ρ is a metric in $\mathbb{R}^3$ that induces the Euclidean topology of $\mathbb{R}^3$, then there are four distinct points y_0, y_1, y_2, y_3 in the image of f such that $\rho(y_0, y_1) = \rho(y_1, y_2) = \rho(y_2, y_3) = \rho(y_3, y_0)$ and $\rho(y_0, y_2) = \rho(y_1, y_3)$. Based on this result, we consider the problem of finding *r-regular m-gons on deformed spheres.* Our main result here can be stated as follows: given a continuous injective map $f : S^d \to \mathbb{R}^{d+1}$ and $(p-1)/2$ metrics in $\mathbb{R}^d$, each metric inducing the Euclidean topology of $\mathbb{R}^d$, $p \geq 3$ prime, and if $(p-1)^2/2$ is not of the form $jd+1$ for any $j = 1, 2, \ldots, p-1$, then there are p distinct points $y_0, y_1, \ldots, y_{p-1}$ in the image of f such that

$$\rho_j(y_0, y_j) = \rho_j(y_j, y_{2j}) = \rho_j(y_{2j}, y_{3j}) = \cdots = \rho_j(y_{(p-1)j}, y_0),$$

for all $j = 1, 2, \ldots, (p-1)/2$. The subscripts are considered as the elements of the finite field $\mathbf{F}_p$.

CHAPTER 1

Evasiveness of Graph Properties

1.1 Graph properties and the evasiveness conjecture

We consider only simple graphs on the fixed set of n vertices $V = \{1, 2, \ldots, n\}$. A graph $G = (V, E)$ is thus determined by its edge set $E \subseteq \binom{V}{2}$, which allows us to identify G with E. Two graphs $G = (V, E)$ and $G' = (V, E')$ are *isomorphic* if there is a permutation σ of V (i.e., $\sigma \in S_n$), such that $\{i, j\} \in E$ if and only if $\{\sigma(i), \sigma(j)\} \in E'$. A *graph property* $\mathcal{P}$ is a collection of graphs, or a family of subsets of $\binom{V}{2}$, which is closed under isomorphism of graphs, namely a graph G is in $\mathcal{P}$ if and only if any graph G' isomorphic to G is also in $\mathcal{P}$.

Consider the following game in which there are two players X and Y, a graph property $\mathcal{P}$ that both players X and Y know and a graph G that only player Y knows. The goal of player X is to determine whether the graph G is in $\mathcal{P}$; player X is allowed to ask Y questions of the form "*is the edge* $\{i, j\}$ *in* G*?*". Player Y

answers *yes* or *no* to these questions. The game ends when player X has determined if G is in $\mathcal{P}$ or not.

A *strategy* for the player X is an algorithm that, depending on the answer player Y gives at each stage of the game, assigns an edge for asking the next question, or if possible, gives one of the answers, "G is in $\mathcal{P}$" or "G is not in $\mathcal{P}$". The minimal number k for which there is a strategy for player X such that regardless of the graph G and the answers of player Y, player X can always end the game by asking at most k questions, is the *complexity* $c(\mathcal{P})$ of the graph property $\mathcal{P}$; therefore, there is a strategy for X such that X can always reach his goal by asking at most $c(\mathcal{P})$ questions, but there is some *extreme* case in which exactly $c(\mathcal{P})$ questions are required. We have that $c(\mathcal{P}) \leq \binom{n}{2}$. In the extreme case that $c(\mathcal{P}) = \binom{n}{2}$, we say that the property $\mathcal{P}$ is *evasive*, otherwise we say $\mathcal{P}$ is *non-evasive*.

For example, let us fix $V = \{1, 2, 3\}$. There are three edges: $e_1 = \{1, 2\}, e_2 = \{1, 3\}$ and $e_3 = \{2, 3\}$. Let $\mathcal{P}$ be the property of "having exactly one edge". This is a graph property consisting of exactly three graphs (one isomorphism class).

The first question in our strategy can be "is e_1 in G?", and suppose the answer to this question is yes. At this point we know that the (unknown) graph G has at least one edge, but it is impossible for us to know if any other edge is in G. We are forced to ask, say, "is e_2 in G?". If the answer to this question is yes, then the game ends. But let us think of the worst case, that is, the answer to that question is no. Since we do not know if e_3 is in G, we are forced to ask the third edge. In fact, for any strategy, there is some "worst case" in which all 3 questions have to be asked in order to finish the game. Thus, this property $\mathcal{P}$ is evasive.

Few graph properties are known to be non-evasive. For example, the property $\mathcal{P}$ consisting of all graphs on six vertices that are isomorphic to one of the graphs shown in figure 1.1 is non-evasive (see [30] for more examples). A famous example of a non-evasive graph property is the property of being a *scorpion graph* (see [6],[30]), defined for $n \geq 5$ and that has complexity $\leq 6n - 13$. For $n \geq 11$, the

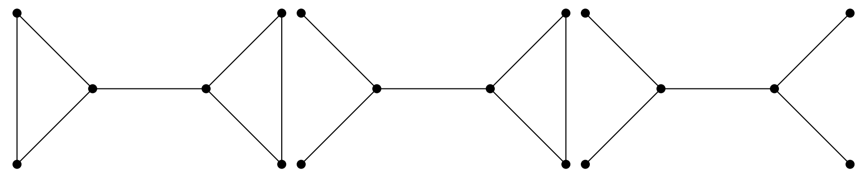


FIGURE 1.1

property of scorpion graphs is non-evasive.

We say a graph property $\mathcal{P}$ is *monotone* if it is closed under removal of edges. This means that if $G \in \mathcal{P}$ and $H \subseteq G$, then $H \in \mathcal{P}$. A property $\mathcal{P}$ is called *trivial* if it is either empty or is the family of all subsets of $\binom{V}{2}$, otherwise $\mathcal{P}$ is called *nontrivial.*

In the earlies seventies R. Karp proposed the following conjecture.

The evasiveness conjecture: every nontrivial monotone graph property is evasive.

In the following two sections we review the topological approach to evasiveness of Kahn, Saks and Sturtevant. The last 5 sections of the chapter treat estimations of Euler characteristic, graphs on p and $2p$ vertices, Oliver groups and lower bounds for simplicial complexes associated to non-evasive monotone graph properties and the application of our work to the cases of 6 and 10 vertices.

1.2 Simplicial Complexes

Let V be a finite set. An *(abstract) simplicial complex* on V is a collection K of subsets of V such that (i) $\{v\} \in K$ for all $v \in V$ and (ii) $A \in K$ and $B \subseteq A$ implies $B \in K$. If $A \in K$, A is a *face* or a *simplex* of K. The *dimension* of A is $|A| - 1$ and the dimension of K, $dimK$, is the maximum dimension of its faces. If the whole set of vertices V is a face of K we say that K is a *simplex*, that is, K consists of all subsets of V. If K has f_i faces of dimension i, the Euler characteristic of K, $\chi(K)$, is defined by

$$\chi(K) = \sum_{i \geq 0} (-1)^i f_i. \tag{1.2.1}$$

The *geometric realization* of K, $|K|$ is constructed as follows. If $V = \{v_1, v_2, \ldots, v_n\}$, we identify v_i with the standard basis vector $e_i \in \mathbb{R}^n$. Then, $|K|$ is the subspace of $\mathbb{R}^n$ obtained as the union of all convex hulls $\langle A \rangle = conv\{e_i : \ v_i \in A\}$ for $A \in K$.

The *automorphism group* of K, $Aut(K)$, is the collection of all permutations of V which leave K invariant. If Γ is a subgroup of $Aut(K)$, then Γ acts on $|K|$ by extending linearly the action on vertices, and we write $|K|^\Gamma$ for the fixed points of this action.

The space $|K|^\Gamma$ can be described in an abstract way as follows. Define K^Γ to be the simplicial complex such that (i) the vertices of K^Γ are the orbits of the action of Γ on V that are also faces of K and (ii) if $A_1, A_2, \ldots, A_r$ are vertices of K^Γ, then $\{A_1, A_2, \ldots, A_r\}$ is a face of K^Γ if $A_1 \cup A_2 \cup \cdots \cup A_r$ is a face of K.

If we identify each vertex A_i of K^Γ with the barycenter of $|A_i|$ in $|K|$, then the geometric realization of K^Γ is just $|K|^\Gamma$.

A *free face* of K is a nonempty face A of K such that A is not maximal under inclusion in K, but it is contained in exactly one inclusion-maximal face B of K, where we require that $dimB = dimA + 1$. An *elementary collapse* of K consists of the removal from K of a free face along with the maximal face containing it. We say that K *collapses* to a complex K', and denote this by $K \searrow K'$, if K' can be obtained from K by a sequence of elementary collapses and say that K is *collapsible* if it collapses to a complex consisting of a single vertex (see figure 1.2).

For a vertex v, the *link* of v, denoted $lk_K(v)$, is the simplicial complex on vertices $V \setminus \{v\}$ given by $lk_K(v) = \{A \subseteq V \setminus \{v\} : \ A \cup \{v\} \in K\}$. The *deletion* of v, $del_K(v)$,

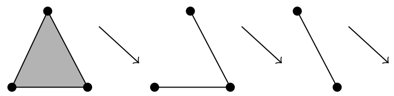


FIGURE 1.2. Collapsing a simplicial complex to a vertex.

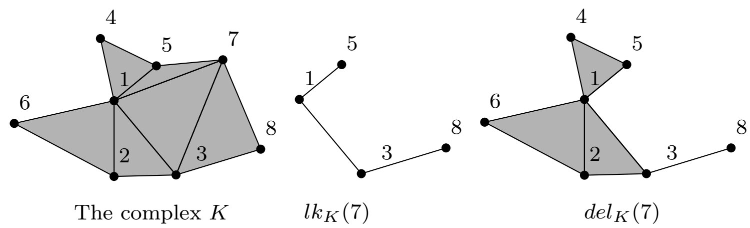


FIGURE 1.3

is the simplicial complex on $V \setminus \{v\}$ given by $del_K(v) = \{A \subseteq V \setminus \{v\} : \; A \in K\}$. The *(Alexander) dual* of K, K^*, is the simplicial complex on V given by $K^* = \{A \subseteq V : \; V \setminus A \notin K\}$.

1.3 The Topological Approach

Evasiveness can be defined for simplicial complexes as follows. There is a game, two players X and Y, a simplicial complex K which is known to both X and Y, and a subset A of V known only to Y. Player X has to determine if A is in K by asking questions of the form "*is the vertex v in A?*" As before, we can define the complexity of K and call K evasive if its complexity equals the size of V, otherwise, call K non-evasive.

The dual of a non-evasive complex is again non-evasive, for determining whether A is in K is equivalent to determine whether $V \setminus A$ is not in K^*.

We can regard a nonempty monotone graph property $\mathcal{P}$ on n vertices as an abstract simplicial complex $\Delta\mathcal{P}$ as follows. The set of vertices of $\Delta\mathcal{P}$ is the set of two-element subsets of $V = \{1, 2, \ldots, n\}$ (that is, the set of all possible edges $\{i, j\}$, $1 \leq i < j \leq n$), and the simplices of $\Delta\mathcal{P}$ are the collections of such edges that correspond to graphs belonging to $\mathcal{P}$. Thus, simplices of $\Delta\mathcal{P}$ of dimension k

correspond to graphs in $\mathcal{P}$ having $k+1$ edges. The two concepts of evasiveness defined for properties of graphs $\mathcal{P}$ coincide.

The result that relates the complexity problem with topology is the following.

Theorem 1.3.1. ***(Kahn-Saks-Sturtevant, [27])*** *A non-evasive complex is collapsible.*

Proof. Let K be a non-evasive complex on a set of vertices V. Then, there is some vertex v_0 in K such that for any $A \subseteq V$, asking "*is* $v_0 \in A$*?*" as the first question will always lead to determine whether A is in K without making the total of $n = |V|$ questions. Determining whether A is in K is equivalent to determining whether one of the following two holds: $A \setminus \{v_0\}$ is in $lk_K(v_0)$ or $A \setminus \{v_0\}$ is in $del_K(v_0)$. Thus, we have that either of these two queries on $A \setminus \{v_0\}$ can be determined by asking less than $n-1$ questions, that is, both $lk_K(v_0)$ and $del_K(v_0)$ are non-evasive complexes. An induction argument applies. Assuming that the theorem is true for all non-evasive complexes on $n-1$ vertices, then $lk_K(v_0)$ and $del_K(v_0)$ are collapsible. If a sequence of free faces $A_1, A_2, \dots, A_m$ permits collapsing $lk_K(v_0)$ to a vertex x, then the sequence $A_1 \cup \{v_0\}, A_2 \cup \{v_0\}, \dots, A_m \cup \{v_0\}, \{v_0\}$ permits collapsing K to $del_K(v_0)$ and since $del_K(v_0)$ is collapsible, so is K. □

Remark 1.3.2. In the proof of theorem 1.3.1 we see that if K is non-evasive, then there exists some vertex $v_0 \in V$ such that both $lk_K(v_0)$ and $del_K(v_0)$ are non-evasive complexes. This can be used to give an equivalent inductive definition of non-evasive complex. Define a simplicial complex to be K non-evasive if it consists of a single vertex or if there exists some vertex $v_0 \in V$ such that both $lk_K(v_0)$ and $del_K(v_0)$ are non-evasive. Theorem 1.3.4 is true if we adopt this inductive definition of non-evasive complex.

Corollary 1.3.3. *A non-evasive complex is* $\mathbb{Z}$*-acyclic.*

The next ingredient for giving the proof of Karp's conjecture in the prime power case is the action of the group $Aut(K)$. We use the following result of R. Oliver (see [41]):

Theorem 1.3.4. ***(Oliver)*** *Let K be a simplicial complex, Γ be a finite subgroup of $Aut(K)$ and p be a fixed prime. Assume that*
i) $|K|$ is $\mathbb{Z}/p$-acyclic and
ii) Γ has a normal p-subgroup Γ_1 such that the quotient Γ/Γ_1 is cyclic.
Then $\chi(|K|^\Gamma) = 1$. In particular, $|K|^\Gamma$ is nonempty.

We call a group Γ satisfying condition ii) in theorem 1.3.4 an *Oliver group.* For example, all finite p-groups are Oliver groups.

Lemma 1.3.5. *Let K be a nonempty $\mathbb{Z}$-acyclic complex on V and Γ be a vertex-transitive subgroup of $Aut(K)$. If Γ is an Oliver group, then K is a simplex.*

Proof. We claim that there is a face A of K which is invariant under Γ. A face A is invariant under Γ if and only if, regarding Γ acting on $|K|$, there is some point x in the relative interior of $|A|$ fixed by Γ. Since Γ is an Oliver group, by theorem 1.3.4 there is a point x in $|K|$ fixed by Γ. Take A to be the unique face of K such that x is in the relative interior of $|A|$. Since Γ acts transitively on V, $A = V$ and so K is a simplex. □

Theorem 1.3.6. ***(Kahn-Saks-Sturtevant, [27])*** *The evasiveness conjecture is true if $|V|$ is a power of a prime.*

Proof. Let $|V| = p^r$ be a prime power and $\mathcal{P}$ be a nonempty monotone and non-evasive graph property on vertices V. We will see that $\mathcal{P}$ is trivial. Identify V with the finite field $GF(p^r)$ and consider the group Γ of all affine transformations $\phi_{a,b} : GF(p^r) \to GF(p^r),\ x \mapsto ax + b$; for $a, b\ GF(p^r), a \neq 0$. This group Γ acts doubly transitively on V, that is, Γ acts transitively on the set of ordered pairs of elements of V and so Γ acts transitively on the set of unordered pairs. We are just saying that Γ is a vertex-transitive subgroup of $Aut(\Delta\mathcal{P})$. The group Γ has a normal p-subgroup $\Gamma_1 = \{\phi_{1,b} :\ b \in GF(p^r)\}$ with quotient $\Gamma/\Gamma_1 \cong GF(p^r)^*$ (the nonzero elements of $GF(p^r)$), which is known to be cyclic. Thus, Γ is an Oliver group. By Lemma 1.3.5, $\Delta\mathcal{P}$ is a simplex. This means that the whole set of vertices $\binom{V}{2}$ belongs to $\Delta\mathcal{P}$, in other words, the complete graph on V is in $\mathcal{P}$ and $\mathcal{P}$ is trivial. □

1.4 Euler Characteristic and Automorphisms of Graphs

In what follows, we are going to use the notation for graphs of Harary [20].

For a graph G, $V(G)$ is its vertex set and $E(G)$ is its edge set. The complete graph on n vertices is denoted by K_n. A graph G is *bipartite* if its vertex set can be partitioned into two disjoint subsets V_1 and V_2 so that each edge of G has the form $\{v_1, v_2\}$, where $v_1 \in V_1$ and $v_2 \in V_2$. A *complete bipartite graph* $K_{n,m}$ has $|V_1| = n$, $|V_2| = m$ and edge set $\{\{v_1, v_2\} : v_1 \in V_1, v_2 \in V_2\}$. An *n-cycle graph* is a graph G with $V = \{v_1, v_2, v_3, \ldots, v_{n-1}, v_n\}$ and $E = \{\{v_1, v_2\}, \{v_2, v_3\}, \ldots, \{v_{n-1}, v_n\}, \{v_n, v_1\}\}$. An n-cycle graph is denoted by C_n.

For a graph G, its complement $\overline{G}$ has $V(\overline{G}) = V(G)$ and $E(\overline{G}) = \binom{V}{2} \setminus E(G)$. For graphs G_1 and G_2 with $V(G_1) \cap V(G_1) = \varnothing$, the *union* graph $G_1 \sqcup G_2$ has $V(G_1 \sqcup G_2) = V(G_1) \cup V(G_2)$ and $E(G_1 \sqcup G_2) = E(G_1) \cup E(G_2)$. If $V(G_1) \cap V(G_2) = \varnothing$, the *join* graph $G_1 + G_2$ has $V(G_1 + G_2) = V(G_1) \cup V(G_2)$ and $E(G_1 + G_2) = E(G_1) \cup E(G_2) \cup \{\{v_1, v_2\} : v_1 \in V(G_1), v_2 \in V(G_2)\}$. For instance, $\overline{K}_n$ is the graph on n vertices without edges and $K_{n,m} = \overline{K}_m + \overline{K}_n$.

If the graph G is the union of k connected components isomorphic to a graph H, then G is denoted by kH.

The *automorphism group* of a graph G on n vertices, denoted $Aut(G)$, is the group of permutations of the set $V(G)$ that preserve the set of edges $E(G)$. For instance, $Aut(K_n) \cong S_n$, $Aut(C_n) \cong D_n$, $Aut(G) = Aut(\overline{G})$. If G is a connected graph, then $Aut(kG) \cong Aut(G) \wr S_k$ (*wreath product*). If G_1 and G_2 are disjoint and connected non-isomorphic graphs, then $Aut(G_1 \sqcup G_2) \cong Aut(G_1) \times Aut(G_2)$. We have that $Aut(G_1 + G_2) \cong Aut(G_1) \times Aut(G_2)$ if no component of $\overline{G}_1$ is isomorphic to a component of $\overline{G}_2$.

If $\mathcal{P}$ is a nonempty monotone and non-evasive graph property, then $\Delta\mathcal{P}$ is collapsible and this implies that $\chi(\Delta\mathcal{P}) = 1$. Therefore, if $\chi(\Delta\mathcal{P}) \neq 1$, then $\mathcal{P}$, being monotone and nontrivial, is evasive. This is why we estimate the Euler characteristic of $\Delta\mathcal{P}$ where $\mathcal{P}$ is a monotone graph property.

For a graph G on n vertices, let $[G]$ denote its isomorphism class and for any pair of graphs G, H on n vertices, let us write $[G] \leq [H]$ if and only if G is isomorphic to some subgraph of H. We can write the Euler characteristic $\chi(\Delta\mathcal{P})$ as

$$\chi(\Delta\mathcal{P}) = \sum_{[G]\subseteq\mathcal{P}} (-1)^{m_G-1}|[G]|, \qquad (1.4.1)$$

where $\mathcal{P}$ is supposed to be nonempty and monotone, m_G represents the number of edges of G (so that G corresponds to a face of dimension $m_G - 1$ of $\Delta\mathcal{P}$), $|[G]|$ is the size of the isomorphism class $[G]$ and the sum is taken over all isomorphism classes of graphs contained in $\mathcal{P}$.

In many instances, there is a common divisor $d > 1$ of all the sizes $|[G]|$ for $G \in \mathcal{P}$. As a result, d divides $\chi(\Delta\mathcal{P})$ so that $\chi(\Delta\mathcal{P}) \neq 1$ and we can conclude that $\mathcal{P}$ is evasive. Therefore, $\mathcal{P}$ is nontrivial monotone and non-evasive, then $\mathcal{P}$ must contain some graphs G for which d is not a divisor of $|[G]|$.

In order to study the divisors of $|[G]|$, we observe that $|[G]| = \frac{n!}{|Aut(G)|}$, so we investigate the divisors of $|Aut(G)|$. For the following basic result see [20], chapter 14.

Lemma 1.4.1. *Let G be a graph on n vertices, then the group $Aut(G)$ decomposes as*

$$Aut(G) \cong (Aut(G_1) \wr S_{n_1}) \times \cdots \times (Aut(G_s) \wr S_{n_s}), \qquad (1.4.2)$$

where the G_i's are the distinct connected components of G and n_i is the number of components of G isomorphic to G_i. If m_i is the number of vertices of G_i, then $n = n_1m_1 + n_2m_2 + \cdots + n_sm_s$, $Aut(G_i)$ is isomorphic to a subgroup of S_{m_i} and $|Aut(G)|$ divides $\prod_i n_i! \cdot (m_i!)^{n_i}$.

Consider $2C_3 = C_3 \sqcup C_3$. In 1.4.2 we have $s = 1, m_1 = 3, n_1 = 2$, then $Aut(2C_3) \cong Aut(C_3) \wr S_2 \cong S_3 \wr S_2$. For the graph $3K_2$ we have that $Aut(3K_2) \cong Aut(K_2) \wr S_3 \cong S_2 \wr S_3$, in particular $|Aut(3K_2)| = 3! \cdot 2^3$.

Lemma 1.4.2. *Suppose that a graph G on n vertices has exactly k vertices of a fixed degree r, where $0 \leq k \leq n$. Then $Aut(G)$ is isomorphic to a subgroup of $S_k \times S_{n-k}$. In particular, $\binom{n}{k}$ divides $|[G]|$.*

In fact, every element of $Aut(G)$ preserves the set of vertices of degree r and also preserves the set of vertices that are not of degree r. Then, $|Aut(G)|$ divides $k!(n-k)!$ and $\frac{n!}{k!(n-k)!}$ divides $|[G]|$.

A graph G is called *regular* if all of its vertices have the same degree. If such degree is r, we say that G is r-regular. When studying the divisors of the size of the automorphism group of a regular graph, the following result of N. Wormald is very useful.

Theorem 1.4.3. ***(Wormald, [60])*** *Let G be a connected r-regular graph on n vertices, where $r > 0$. Then $|Aut(G)|$ divides*

$$rn \prod_p p^\beta \tag{1.4.3}$$

where the product is taken over all prime numbers $p \leq r-1$ and β is given by

$$\sum_{p^\alpha \leq r-1} \left\lfloor \frac{n-2}{p^\alpha} \right\rfloor \tag{1.4.4}$$

Corollary 1.4.4. *In the hypothesis of theorem 1.4.3, if $r < 3$, then $|Aut(G)|$ divides rn.*

1.5 Graphs on p and $2p$ vertices

In this section we describe all graphs G on p vertices and also on $2p$ vertices for which p is not a divisor of $|[G]|$. We assume that p is an odd prime.

First, we deal with graphs on p vertices. In the hypothesis of lemma 1.4.2 we have that $\binom{p}{k}$ divides $|[G]|$. Since p is prime, p divides $\binom{p}{k}$ for $0 < k < p$. Then, the only cases in which p may not be a divisor of $|[G]|$ are $k = p$ (G is a regular graph) and $k = 0$ (there are no vertices of degree r in G). When we vary the degree r, we conclude that if p does not divide $|[G]|$, then G is regular.

Lemma 1.5.1. *If G is a graph on p vertices and p does not divide $|[G]|$, then G is a regular graph.*

Since $|[G]| = \frac{p!}{|Aut(G)|}$, we have that p does not divide $|[G]|$ if and only if p divides $|Aut(G)|$, which means that $Aut(G)$ contains an element of order p. The elements of order p in the symmetric group S_p are precisely the p-cycles. We assume, without lost of generality, that the p-cycle $\sigma = (01 \cdots p-1)$ is an automorphism of G, where the vertices $0, 1, \ldots, p-1$ are regarded as the elements of the finite field $\mathbb{F}_p$.

Let us suppose that G is not the graph $\overline{K}_p$. If the edge $\{r, s\}$ is in G where $r < s$, reorder the p-cycle σ if necessary to assume that $r = 0$. Thus, we can assume that the edge $\{0, s\}$ is in G. The $(s-1)$-th power of the p-cycle σ, $\sigma^{s-1} = (0\ s\ 2s \cdots (p-1)s)$, is again a p-cycle and an automorphism of G. We find that G contains the edges $\{0, s\}, \{s, 2s\}, \{2s, 3s\}, \ldots, \{(p-1)s, 0\}$. These set of edges form a cycle graph of length p that is a subgraph of G and that we denote by $C(s)$. With this notation we have $C(s) = C((p-1)s)$. Remember that the vertices $0, 1, \ldots, p-1$ are the elements of $\mathbb{F}_p$ and since $(p-1)s = -s = p-s$, we get $C(s) = C(p-s)$. In figure 1.4 we illustrate the example for $p = 7$, $s = 2$, $\sigma = (0123456)$, $C(2) = C(5)$.

Every element $s \in \{1, 2, \ldots, (p-1)/2\}$ such that $\{0, s\} \in G$ determines a cycle graph $C(s)$ that is a subgraph of G. Suppose that $\{0, s\}, \{0, r\}$ are in G,

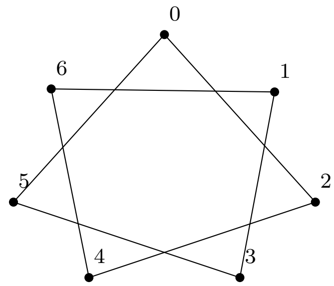


FIGURE 1.4. $C(2)$, $p = 7$, $\sigma = (0123456)$.

where $s, r \in \{1, 2, \ldots, (p-1)/2\}$ and $r \neq s$. The cycles $C(s)$ and $C(r)$ have sets of edges $\{0, s\}, \{s, 2s\}, \ldots, \{(p-1)s, 0\}$ and $\{0, r\}, \{r, 2r\}, \ldots, \{(p-1)r, 0\}$ respectively. We claim that $C(s)$ and $C(r)$ have no common edges. In fact, if we have $\{is, (i+1)s\} = \{jr, (j+1)r\}$, then there are two options: (i) $is = jr$, $is + s = jr + r$ which implies that $r = s$ and (ii) $is = jr + r$, $is + s = jr$, which implies $s = -r = k - r$ and this contradicts that $s \in \{1, 2, \ldots, (p-1)/2\}$.

Therefore, the graph G can be decomposed as a union of disjoint cycle graphs of length p,

$$G = C(s_1) \cup C(s_2) \cup \cdots \cup C(s_l) \tag{1.5.1}$$

where $\{0, s_i\} \in G$ and $s_i \in \{1, 2, \ldots, (p-1)/2\}$. As a consequence, G is an l-regular graph with lp edges.

For a nonempty subset $S = \{s_1, s_2, \ldots, s_l\} \subseteq \{1, 2, \ldots, (p-1)/2\}$, $C(S)$ or $C(s_1, s_2 \ldots, s_l)$ will denote the graph $C(s_1) \cup C(s_2) \cup \cdots \cup C(s_l)$. The notation $C(\varnothing)$ will denote the graph $\overline{K_p}$. For the complement graph of a graph of the form $C(S)$ we have $\overline{C(S)} = C(\overline{S})$.

All the cycle graphs $C(s)$ are isomorphic, in fact, given $s \in \{1, 2, \ldots, (p-1)/2\}$, the map $\mathbb{F}_p \to \mathbb{F}_p$ given by $x \mapsto sx$ gives an isomorphism between $C(1)$ and $C(s)$.

Each cycle graph $C(s_i)$ in 1.5.1 is fixed under the action of the p-cycle $\sigma = (01 \cdots p-1)$ and as a consequence G remains fixed under the action of σ, that is, $\sigma \in Aut(G)$. We have proved the following.

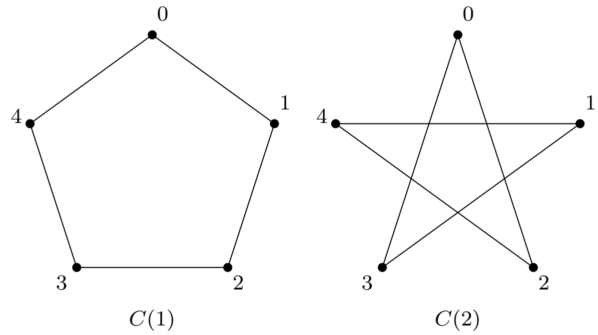


FIGURE 1.5

Lemma 1.5.2. *If G is a graph on p vertices, then p does not divide $|[G]|$ if and only if G is isomorphic to a graph of the form $C(S)$, where $S \subseteq \{1, 2, \ldots, (p-1)/2\}$.*

Example 1.5.3. Let us show how combining lemma 1.5.2 and Euler characteristic of monotone graph properties, we can give a proof of Karp's conjecture in the case of 5 vertices. The vertices of our graphs are going to be the elements of the field $\mathbb{F}_5 = \{0, 1, 2, 3, 4\}$. By lemma 1.5.2, the only graphs G on 5 vertices such that 5 does not divide $|Aut(G)|$ are $\overline{K}_5, C(1) \cong C(2) \cong C_5$ and $C(1, 2) \cong K_5$ (see $C(1)$ and $C(2)$ in figure 1.5). For any graph G not isomorphic to one of these four graphs, we have that 5 divides $|[G]|$.

Let $\mathcal{P}$ be a nontrivial monotone and non-evasive graph property on 5 vertices. Then, $\chi(\Delta\mathcal{P}) = 1$ and $\mathcal{P}$ must contain some of the 4 graphs for which 5 does not divide the size of their automorphism group. Since the graph $\overline{K}_5$ represents the empty simplex, $\overline{K}_5$ does not contribute to $\chi(\Delta\mathcal{P}) = 1$ and since $\mathcal{P}$ is nontrivial, $\mathcal{P}$ does not contain the complete graph K_5. Then $\mathcal{P}$ must contain $[C_5]$. We know that $Aut(C_5) \cong D_5$ and $|[C_5]| = 5!/10 = 12$. The cycle C_5 represents a face of dimension 4, so $\chi(\Delta\mathcal{P})$ has the form $5m + 12$, but $5m + 12$ can never be equal to 1. This proves that such a property $\mathcal{P}$ cannot exist.

Remark 1.5.4. We have proved indeed that if $\mathcal{P}$ is any nontrivial monotone graph property on 5 vertices that contains the 5-cycle C_5, then $\chi(\Delta\mathcal{P})$ has the form $5m + 2$. In particular, $\Delta\mathcal{P}$ is not $\mathbb{Z}$-acyclic. This is also true for any other nontrivial monotone graph property $\mathcal{P}$ on 5 vertices, because in those cases, $\chi(\Delta\mathcal{P})$ is divisible

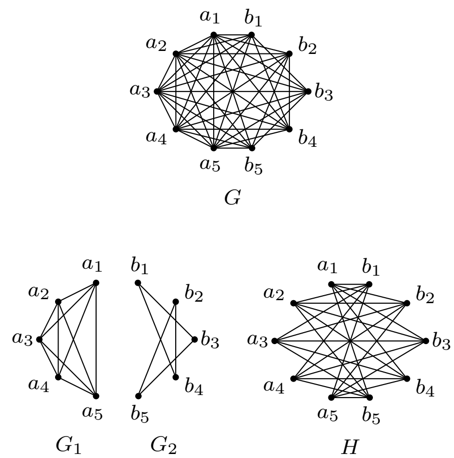


FIGURE 1.6

by 5 and in particular $\chi(\Delta\mathcal{P}) \neq 1$. As a result, the simplicial complex associated to any nontrivial monotone graph property on 5 vertices is not $\mathbb{Z}$-acyclic.

Now we go to graphs on $2p$ vertices. If G is a graph on $2p$ vertices (p an odd prime) and G has exactly k vertices of a fixed degree r, then $\binom{2p}{k}$ divides $|[G]|$ by lemma 1.4.2. The only way p does not divide $\binom{2p}{k} = \frac{(2p)!}{k!(2p-k)!}$ is that p^2 divides $k!(2p-k)!$, which occurs if and only if $k = p$ or $k = 2p$. Thus, we assume that p does divide $|[G]|$ and consider the following two cases.

Case 1. $k = p$. In this case there are exactly p vertices of degree r, say they are $a_1, \ldots, a_p$. The other p vertices, say $b_1, \ldots, b_p$, must have the same degree s, where $s \neq r$. Each automorphism of G preserves the $a_i's$ and also preserves the $b_j's$. The group $Aut(G)$ is isomorphic to a subgroup of $S_{\{a_i\}} \times S_{\{b_j\}} \cong S_p \times S_p$.

Let G_1 be the subgraph of G which is the subgraph *induced* by the vertices $a_1, \ldots, a_p$, G_2 be the subgraph of G induced by the vertices b_j's and let H be the subgraph of G whose edges are all edges of G having the form $\{a_i, b_j\}$ (see figure 1.6).

If p^2 divides the size of $Aut(G)$, then $Aut(G)$ contains a subgroup of order p^2. The subgroups of order p^2 in $S_p \times S_p$ have the form $\langle\sigma\rangle \times \langle\tau\rangle$ where both σ and τ

have order p. Thus, $Aut(G) \subseteq S_{\{a_i\}} \times S_{\{b_j\}}$ contains a subgroup of the form $\langle\alpha\rangle \times \langle\beta\rangle$ where both elements α and β have order p (α and β are p-cycles).

Let t be the number of $b_j's$ that are adjacent to a_1. Then, for each $i = 1, 2, \ldots, p$, the number of $b_j's$ adjacent to a_i is also t because automorphisms of G preserve the adjacency and $\alpha \times id$ acts transitively on the $a_i's$ while $\alpha \times id$ fixes the b_j's. Thus, the number of edges of H is $m_H = pt$. Similarly, if u is the number of $a_i's$ adjacent to b_1, then $m_H = pu$. Hence, $u = t$.

If $t = 0$, then H has no edges. It results that $G = G_1 \sqcup G_2$ and since no component of G_1 is isomorphic to a component of G_2, we have that $Aut(G) \cong Aut(G_1) \times Aut(G_2)$. Therefore, p does not divide $|[G]|$ if and only if p divides both $|Aut(G_1)|$ and $|Aut(G_2)|$. When we require that p divides $|Aut(G_1)|$ and $|Aut(G_2)|$, the conclusion is that G_1 and G_2 are isomorphic to graphs of the form $C(S)$ as in lemma 1.5.2. In particular, G_1 and G_2 are regular graphs of different degree.

If $t > 0$, the permutation $id \times \beta$ permutes transitively the b_j's and fixes a_1; this implies that $t = p$, $H = K_{p,p}$ and $G = G_1 + G_2$. Therefore, $\overline{G} = \overline{G_1} \cup \overline{G_2}$ lies in the case $t = 0$ above, so $\overline{G}_1$ and $\overline{G}_2$ and also G_1 and G_2 are isomorphic to graphs of the form $C(S)$ as in lemma 1.5.2.

Case 2. $k = 2p$. In this case G is a regular graph of degree r. When $r = 0$, $G = \overline{K}_{2p}$. We assume $r > 0$. By lemma 1.4.1, the group $Aut(G)$ decomposes as

$$Aut(G) \cong (Aut(G_1) \wr S_{n_1}) \times \cdots \times (Aut(G_s) \wr S_{n_s}),$$

where the G_i's are the distinct connected components of G and n_i is the number of components of G isomorphic to G_i. If m_i is the number of vertices of G_i, then $2p = n_1m_1 + n_2m_2 + \cdots + n_sm_s$, $Aut(G_i)$ is isomorphic to a subgroup of S_{m_i} and

$|Aut(G)|$ divides $\prod_i n_i! \cdot (m_i!)^{n_i}$. Since there are no isolated vertices in G, we have that $m_i \geq 2$ for all i.

We want to determine conditions under which p^2 divides $\prod_i n_i! \cdot (m_i!)^{n_i}$.

If some $n_i \geq p$, then $2p \geq n_i m_i \geq 2n_i \geq 2p$ which implies $s = 1$, $n_1 = p$ and $m_1 = 2$. It follows that $G \cong pK_2$; besides, $|Aut(G)| = p!2^p$ is not divisible by p^2.

If $n_i < p$ for all i, then the only cases in which p^2 divides $\prod_i n_i! \cdot (m_i!)^{n_i}$ are the following:
(i) $m_i = 2p$ for some i. In this case G is a regular connected graph of degree $r > 0$.
(ii) $m_i = p$ and $m_j = p$ for $i \neq j$. In this case G is the union of two non-isomorphic regular connected graphs on p vertices, both of degree r.
(iii) $m_i = p$ and $n_i = 2$ for some i. G is the union of two copies of a regular connected graph of degree r on p vertices.

In the case that G is regular and connected of degree $r > 0$ (case (i)), consider $\overline{G}$ which is also regular of degree $2p - r - 1$ and $|Aut(G)| = |Aut(\overline{G})|$. If $\overline{G}$ is also connected, take the one between G and $\overline{G}$ that has fewer degree, so we can assume that G is connected of degree $r \leq \frac{2p-1}{2}$. This condition on r is equivalent to $r < p$. Then, we apply Wormald's theorem (theorem 1.4.3) to obtain that $|Aut(G)|$ divides $r(2p) \prod_{q \leq r-1} q^\beta$ and we see from this that p^2 is not a divisor of $|Aut(G)|$ (although p could be a divisor of $|Aut(G)|$).

On the other hand, if $\overline{G}$ is not connected and p^2 divides $|Aut(G)| = |Aut(\overline{G})|$, then $\overline{G}$ lies in case (ii) or case (iii). Thus, $\overline{G}$ is the union of two regular graphs on p vertices of the same degree, $\overline{G} = H_1 \sqcup H_2$ and $G = \overline{H}_1 + \overline{H}_2$.

We conclude that the only way that p^2 divides $|Aut(G)|$, where G is a regular graph of degree r, is that G is of the form $G_1 \sqcup G_2$ or $G_1 + G_2$, where G_1 and G_2 have the form $C(S)$ as in lemma 1.5.2.

We resume our classification in the following lemma:

Lemma 1.5.5. *Let G be a graph on $2p$ vertices such that p^2 divides $|Aut(G)|$, or equivalently that p does not divide $|[G]|$. Then, G is isomorphic to a graph of the* *form $G_1 \sqcup G_2$ or to a graph of the form $G_1 + G_2$, where G_1 and G_2 are graphs on p vertices isomorphic to graphs $C(S)$ as in lemma 1.5.2. The graphs G_1 and G_2 are allowed to be isomorphic.*

Remark 1.5.6. To determine all graphs G on $2p$ vertices such that p does not divide $|[G]|$, it suffices to determine all graphs H on p vertices such that p does not divide $|[H]|$ (lemma 1.5.2 contains the precise description of such graphs H) and then take unions $G_1 \sqcup G_2$ and joins $G_1 + G_2$ of such graphs. Graphs of the form $G_1 + G_2$ are obtained from those of the form $G_1 \sqcup G_2$ by the relation $\overline{G_1 \sqcup G_2} = \overline{G_1} + \overline{G_2}$.

1.6 Oliver groups and lower bounds for the dimension of non-evasive monotone graph properties

In this section we make use of some Oliver groups acting on the simplicial complex associated to a nontrivial monotone and non-evasive graph property $\mathcal{P}$ to prove that $\mathcal{P}$ must contain some special graphs. Such special graphs are the orbits of the Oliver group acting on the two-element subsets of $\{1, 2, \dots, n\}$ (vertices of $\Delta\mathcal{P}$). This will permit us to give lower bounds for the dimension of $\Delta\mathcal{P}$.

Let $\mathcal{P}$ be a nonempty monotone and non-evasive graph property. If Γ is an Oliver group acting on $|\Delta\mathcal{P}|$, then $|\Delta\mathcal{P}|^\Gamma \neq \varnothing$ (theorem 1.3.4). In the abstract version of $|\Delta\mathcal{P}|^\Gamma$ we obtain that $\mathcal{P}^\Gamma \neq \varnothing$ and this means that there are some graphs in $\mathcal{P}$ that are also (union of) orbits of the action of Γ on the set of two-element subsets of $\{1, 2, \dots, n\}$.

We will consider Oliver groups that are subgroups of S_n because they automatically act on $\mathcal{P}$ giving automorphisms of $\Delta\mathcal{P}$. We will conclude that at least one of the orbits of Γ acting on the 2-element subsets of $\{1, 2, \dots, n\}$ must be a graph belonging to $\mathcal{P}$.

For the first example, suppose $n = p^r$ is a prime power and consider $GF(p^r)$, the finite field of p^r elements. For $a, b\ GF(p^r), a \neq 0$ let $\phi_{a,b} : GF(p^r) \to GF(p^r)$ be the affine linear map defined by $\phi_{a,b}(x) = ax + b$. If $\Gamma_{p^r} = \{\phi_{a,b} : \ a, b \ GF(p^r), a \neq 0\}$, then Γ_{p^r} is an Oliver group. This group Γ_{p^r} is the one used in the proof of theorem 1.3.6. The normal p-subgroup with cyclic quotient is $\Gamma_1 = \{\phi_{1,b} : b \in GF(p^r)\}$. The group Γ_{p^r} acts doubly transitively on $GF(p^r)$ which implies that there is just one orbit of Γ_{p^r} acting on the two-element subsets of $\{1, 2, \ldots, p^r\}$. This orbit is the complete graph K_{p^r}.

Now suppose that $n = p^r + t$ and consider the group $\Gamma_{p^r} \times \mathbb{Z}/t$ where Γ_{p^r} acts on the vertices $1, 2, \ldots, p^r$ as above, while fixes the vertices $p^r + 1, \ldots, n$. The factor $\mathbb{Z}/t$ acts on $1, 2, \ldots, p^r$ trivially and acts on the remaining t vertices $p^r + 1, \ldots, n$ by permuting them cyclically.

We know that Γ_{p^r} has the normal p-subgroup Γ_1 with cyclic quotient. Then, Γ_1 is also a normal p-subgroup of $\Gamma \times \mathbb{Z}/t$ with quotient isomorphic to $(\Gamma/\Gamma_1) \times \mathbb{Z}/t$ which is cyclic when $p^r - 1$ and t are coprimes. Thus, the group $\Gamma \times \mathbb{Z}/t$ is an Oliver group if $p^r - 1$ and t are coprimes.

The orbits of the action of $\Gamma_{p^r} \times \mathbb{Z}/t$ on the two element subsets of $\{1, 2, \ldots, n\}$ are the complete graph on the vertices $1, 2, \ldots, p^r$ with the remaining t vertices isolated, that is, $K_{p^r} \sqcup \overline{K}_t$; the complete bipartite graph $K_{p^r,t}$ and the orbits of the form $\overline{K}_{p^r} \sqcup G$, where G is one of the graphs on t vertices $p^r + 1, \ldots, n$ that are fixed under the action of $\mathbb{Z}/t$.

Proposition 1.6.1. *Any nonempty monotone and non-evasive graph property on $n = p^r + t$ vertices, where $(p^r - 1, t) = 1$, has to contain some of the following graphs: $K_{p^r} \sqcup \overline{K}_t, K_{p^r,t}$ or one of the graphs of the form $\overline{K}_{p^r} \sqcup G$, where G is one of the graphs on t vertices $p^r + 1, \ldots, n$ that are fixed under the action of $\mathbb{Z}/t$.*

Corollary 1.6.2. *Any nonempty monotone and non-evasive property $\mathcal{P}$ of graphs on $n = p^r + 1$ vertices, where $n > 3$, has to contain one of the two graphs $K_{p^r} \sqcup K_1$ or $K_{p^t,1}$. In particular, dim $\Delta\mathcal{P} \geq n-2$. Moreover, if $\mathcal{P}$ is nontrivial then $\mathcal{P}$ cannot contain both $K_{p^r} \sqcup K_1$ and $K_{p^t,1}$.*

Proof. Applying proposition 1.6.1, $\mathcal{P}$ has to contain $K_{p^r} \sqcup K_1$ or $K_{p^t,1}$. These two graphs have $p^r(p^r-1)/2 = (n-1)(n-2)/2$ and $p^r = n-1$ edges respectively. In any case $dim\ \mathcal{P} \geq n-2$ if $n > 3$. The two graphs $A = K_{p^r} \sqcup K_1$ and $B = K_{p^t,1}$ are the only two orbits of the Oliver group $\Gamma_{p^r} \times \{*\}$ acting on the 2-element subsets of $1, 2, \ldots, n$. Thus, A and B represent the only two (potential) vertices of the simplicial complex $\mathcal{P}^{\Gamma_{p^r} \times \{*\}}$. If both A and B belong to $\mathcal{P}$, then A and B are vertices of $\mathcal{P}^{\Gamma_{p^r} \times \{*\}}$. By theorem 1.3.4, $\chi(|\mathcal{P}^{\Gamma_{p^r} \times \{*\}}|) = 1$, which obligates $\mathcal{P}^{\Gamma_{p^r} \times \{*\}}$ to contain the simplex $\{A, B\}$. This means that $K_n = A \cup B$ belongs to $\mathcal{P}$ which implies that $\mathcal{P}$ is trivial. □

Another particular case of proposition 1.6.1 we can analyze is the case $n = 2^r + 2$. Let $\mathcal{P}$ be any nonempty monotone and non-evasive property of graphs on $2^r + 2$ vertices. Proposition 1.6.1 gives us the three graphs $A = \overline{K}_{2^r} \sqcup K_2$, $B = K_{2^r} \sqcup \overline{K_2}$ and $C = K_{2^r,2}$. The graph A already belongs to $\mathcal{P}$ because A is one of the vertices of $\Delta\mathcal{P}$. We have that A is one vertex of $\Delta\mathcal{P}$ and A is a fixed point of $\Gamma := \Gamma_{2^r} \times \mathbb{Z}/2$.

The transitivity of the group $Aut(\Delta\mathcal{P})$ permits to assume that, in the inductive definition of non-evasive complexes, $lk_{\Delta\mathcal{P}}A$ and $del_{\Delta\mathcal{P}}A$ are non-evasive. Since A is a fixed point of Γ, Γ acts on $lk_{\Delta\mathcal{P}}A$ and $del_{\Delta\mathcal{P}}A$ and theorem 1.3.4 can be applied. In particular, we have that $(lk_{\Delta\mathcal{P}}A)^\Gamma \neq \varnothing$.

By the definition of the link we have that $(lk_{\Delta\mathcal{P}}A)^\Gamma = lk_{\Delta\mathcal{P}^\Gamma}A$. Then, at least one of the two B or C must be in $lk_{\Delta\mathcal{P}^\Gamma}A$. Thus, $\mathcal{P}$ contains B or C. If $\mathcal{P}$ contains both B and C, then B and C belong to $lk_{\Delta\mathcal{P}^\Gamma}A$, but the equation $\chi(lk_{\Delta\mathcal{P}^\Gamma}A) = 1$ obligates $lk_{\Delta\mathcal{P}^\Gamma}A$ to contain $B \cup C = K_{2^r} + \overline{K}_2$, this is the complete graph on $2^r + 2$ vertices without one edge. Thus, we have that $\Delta\mathcal{P}$ is isomorphic to the boundary of the $\left(\binom{n}{2} - 1\right)$-dimensional simplex: a sphere of dimension $\binom{n}{2} - 1$. Therefore, $\Delta\mathcal{P}$ is not a collapsible simplicial complex. This contradiction proves that $\mathcal{P}$ cannot contain both B and C.

If we assume that $B \in \mathcal{P}$, then the equation $\chi(\Delta\mathcal{P}^\Gamma) = 1$ implies that $A \cup B \in \mathcal{P}$ and $dim\Delta\mathcal{P} \geq dim(A \cup B) = 2^{r-1}(2^r - 1)$. Similarly, if $C \in \mathcal{P}$, then $A \cup C \in \mathcal{P}$

and $dim\Delta\mathcal{P} = dim(A \cup C) = 2^{r+1}$.

The following two results are generalizations of the corresponding results for 6 vertices in [27]. In the proof of each result an Oliver group is used in combination with theorem 1.3.4. Both results concern monotone non-evasive properties of graphs on $2p$ vertices.

Proposition 1.6.3. *Let $\mathcal{P}$ be a nontrivial monotone and non-evasive graph property on $2p$ vertices, where p is prime. Then, all perfect matchings belong to $\mathcal{P}$.*

Proof. Let Γ be the group generated by the permutations $(1\ p+1), (2\ p+2), \ldots, (p\ 2p)$ and $\alpha = (1\ 2 \cdots\ p)(p{+}1\ p{+}2 \cdots 2p)$. The subgroup H of Γ generated by $(1\ p+1), (2\ p+2), \ldots, (p\ 2p)$ is a normal 2-subgroup with quotient isomorphic to the subgroup of Γ generated by α, which is cyclic. Then, Γ is an Oliver group and $\mathcal{P}^\Gamma \neq \varnothing$.

We claim that every orbit of Γ acting on the two-element subsets of $\{1, 2, \ldots, 2p\}$ contains a perfect matching. One of such orbits is the set of edges $\{1, p\}, \{2, p+1\}, \ldots, \{p, 2p\}$ (a perfect matching!).

Let G be any other orbit and $\{r, s\} \in G$. We can assume that $r, s \in \{1, 2, \ldots, p\}$, for if $r \leq p$ and $s > p$ then $s - p \neq r$ and the permutation $(s - p\ s)$ sends $\{r, s\}$ to $\{r, s - p\}$, and $s - p \in \{1, 2, \ldots, p\}$. If $r, s > p$, then $(r - p\ r)(s - p\ s)$ sends $\{r, s\}$ to $\{r-p, s-p\}$ and both $r-p, s-p \in \{1, 2, \ldots, p\}$. The orbit of $\{r, s\}$ under α is a cycle graph of length p with vertices $\{1, 2, \ldots, p\}$ which is a subgraph of G. Since α^{1-r} sends r to 1, G contains an edge of the form $\{1, t\}$ with $1 < t \leq p$. Now, $(1\ p)$ sends $\{1, t\}$ to $\{p, t\}$ and so $\{p, t\} \in G$, then by considering the action of α on $\{p, t\}$, we conclude that $\{p, t\}, \{p + 1, t + 1\}, \{p + 2, t + 2\}, \ldots, \{2p, t + p\} \in G$ (if necessary, when $t + i > p$ we subtract p to obtain a value between 1 and p). This set of edges is a perfect matching and this ends the proof of our claim.

Since $\mathcal{P}^\Gamma \neq \varnothing$, at least one of the orbits of Γ belongs to $\mathcal{P}$ and since $\mathcal{P}$ is monotone, $\mathcal{P}$ contains a perfect matching. Then, $\mathcal{P}$ contains all perfect matchings, for $\mathcal{P}$ is closed under isomorphism of graphs. □

Proposition 1.6.4. *Let $\mathcal{P}$ be a nontrivial monotone and non-evasive graph property on $2p$ vertices, where p is and odd prime. Then at least one of $2C_p$, $K_{p,p}$ belongs to $\mathcal{P}$.*

Proof. The set of $2p$ vertices is going to be the union of two disjoint copies of the finite field $\mathbb{F}_p = \{0, 1, \ldots, p-1\}$, the second copy of $\mathbb{F}_p$ will be labeled $\mathbb{F}'_p = \{0', 1', \ldots, (p-1)'\}$. Let Γ be the group generated by the permutations $\alpha = (0\ 0')(1\ 1')\cdots(p-1\ (p-1)'), \beta = (01\cdots p-1), \gamma = (0'1'\cdots(p-1)')$. The subgroup of Γ generated by β and γ is a normal p-subgroup of Γ whose quotient is cyclic isomorphic to $\langle\alpha\rangle$. Thus, Γ is an Oliver group and $\mathcal{P}^\Gamma \neq \varnothing$.

We investigate the orbits of Γ acting on the two-element subsets of $\mathbb{F}_p \cup \mathbb{F}'_p$. Note that if G is one of such orbits, then for $r, s \in \mathbb{F}_p$, $\{r, s\} \in G$ if and only if $\{r', s'\} \in G$ (because α sends $\{r, s\}$ to $\{r', s'\}$).

Let G_0 be the orbit $\{0, 0'\}$. The orbit of $\{0, 0'\}$ under β gives us all edges of the form $\{x, 0'\}$ with $x \in \mathbb{F}_p$. Then, fixing $x \in \mathbb{F}_p$, the orbit of $\{x, 0'\}$ under γ gives all edges of the form $\{x, y'\}$ with $y' \in \mathbb{F}'_p$. We conclude that all edges $\{x, y'\}$, $x \in \mathbb{F}_p$, $y' \in \mathbb{F}'_p$ are in G_0. The group Γ preserves the set of edges $\{\{x, y'\} : x \in \mathbb{F}_p, y' \in \mathbb{F}'_p\}$, so $G_0 = \{\{x, y'\} : x \in \mathbb{F}_p, y' \in \mathbb{F}'_p\}$ (G_0 is isomorphic to the complete bipartite graph $K_{p,p}$).

Let G be any orbit of Γ different from G_0. Then, G does not contain any edge of the form $\{x, y'\}$ with $x \in \mathbb{F}_p$, $y' \in \mathbb{F}'_p$. Let $\{r, s\} \in G$, then $r, s \in \mathbb{F}_p$ or $r, s \in \mathbb{F}'_p$. We can assume that $r, s \in \mathbb{F}_p$, since $\{r, s\} \in G$ if and only if $\{r', s'\} \in G$. In order to determine G, it suffices to determine the orbit of β acting on the two-element subsets of $\mathbb{F}_p$. If K represents the orbit of $\{r, s\}$ under β, then $G = K \cup \alpha(K)$. Now, the permutation β^{p-r} sends $\{r, s\}$ to $\{0, s-r\}$. Let $t = s - r \neq 0$. The orbit of $\{0, t\}$ under β consists of the p edges $\{0, t\}, \{1, t+1\}, \{2, t+2\}, \ldots, \{p-1, t+p-1\}$, this set of edges is precisely K. We want to describe K in a more convenient way. The permutation β^t is also a generator of $\langle\beta\rangle$ and the orbit of $\{0, t\}$ under β^t is described as the set of edges $\{0, t\}, \{t, 2t\}, \{2t, 3t\}, \ldots, \{(p-2)t, (p-1)t\}, \{(p-1)t, 0\}$. This is K. In the notation of lemma 1.5.2, $K = C(t)$. Thus, we have that $G = C(t) \cup \alpha(C(t)) = C(t) \cup C(t')$ (since $\alpha(C(t)) = C(t')$).

There are $(p+1)/2$ orbits of Γ, $G_0 \cong K_{p,p}$ and $C(t) \cup C(t') \cong 2C_p$ for $t = 1, 2, \ldots, (p-1)/2$.
Since $\mathcal{P}^{\Gamma} \neq \varnothing$, at least one the orbits of Γ belongs to $\mathcal{P}$ and this ends the proof. □

Corollary 1.6.5. *Let $\mathcal{P}$ be a nontrivial monotone and non-evasive graph property on $2p$ vertices, where $p > 3$ is prime. Then, $dim \Delta\mathcal{P} \geq 4p - 1$.*

Proof. By proposition 1.6.4, at least one of $2C_p$, $K_{p,p}$ belongs to $\mathcal{P}$. If $K_{p,p}$ is in $\mathcal{P}$, then $\Delta\mathcal{P}$ contains a face of dimension $p^2 - 1$. If $\mathcal{P}$ does not contain $K_{p,p}$, then $\mathcal{P}$ contains $2C_p$. Therefore, $\mathcal{P}$ contains all the graphs $C(t) \cup C(t')$ in the proof of lemma 1.6.4. Since $(p-1)/2 \geq 2$, there are at least two orbits of the form $C(t) \cup C(t')$. By theorem 1.3.4, $\chi(|\Delta\mathcal{P}^{\Gamma}| = 1$, then $\mathcal{P}$ must contain a graph which is the union of two of the orbits $C(t) \cup C(t')$. This union has $4p$ edges and is a face of $\Delta\mathcal{P}$ of dimension $4p - 1$. Since $p^2 - 1 \geq 4p - 1$, we find that, in any case, $dim \Delta\mathcal{P} \geq 4p - 1$. □

Remark 1.6.6. A result of Bjorner establishes that that for a vertex homogeneous simplicial complex K on a finite set of cardinality m with $\chi(K) = 1$, the dimension of K satisfies $dim K \geq M - 1$, where M is the maximum prime power dividing m (see [33]). Corollary 1.6.5 says that for a nontrivial monotone and non-evasive property $\mathcal{P}$ of graphs on $2p$ vertices, $dim \Delta\mathcal{P} \geq 4p - 1$; $\Delta\mathcal{P}$ is a simplicial complex on a vertex set of $p(2p-1)$ elements, thus Bjorner's bound for the dimension gives $2p - 2$ in the best case. The bound $4p - 1$ is a better lower bound.

1.7 Evasiveness of Graph Properties on Six Vertices

In this section we show in detail a proof of the evasiveness conjecture for properties of graphs on six vertices. The idea of estimating the Euler characteristic of the simplicial complex associated to a graph property yields that a nontrivial monotone and non-evasive graph property on 6 vertices contains exactly one of the two graphs $2K_3, K_{3,3}$ (this also follows from proposition 1.6.4).

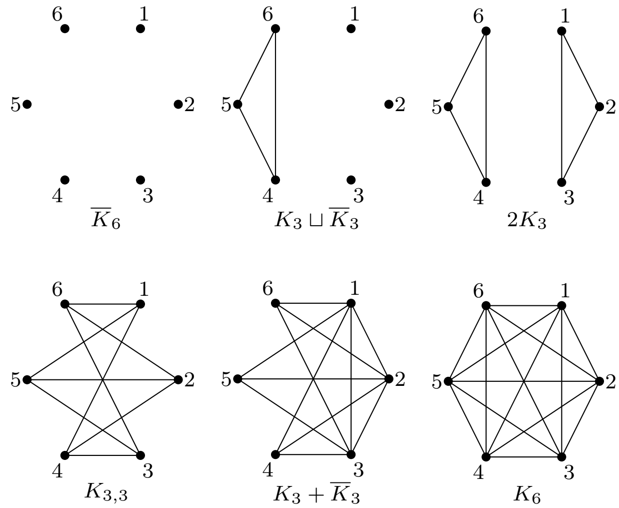


FIGURE 1.7

The first step in the proof is to find all graphs G on 6 vertices for which 3 is not a divisor of $|[G]|$. Then, as for the 5 vertices case (example 1.5.3), $\mathcal{P}$ has to contain some of these graphs in order to satisfy $\chi(\Delta\mathcal{P}) = 1$ and this yields various cases for $\mathcal{P}$, depending on which of those graphs $\mathcal{P}$ contains. By considering the action of several Oliver groups, we show that none of the different cases that result can happen, reaching a contradiction and finishing the proof.

The classification of graphs G on 6 vertices for which 3 does not divide $|[G]|$ is the content of the following lemma.

Lemma 1.7.1. *Let G be a graph on 6 vertices and suppose that 3 is not a divisor of $|[G]|$. Then, G is isomorphic to one of the following 6 graphs: $\overline{K}_6, K_3 \sqcup \overline{K}_3, 2K_3, K_{3,3}, K_3 + \overline{K}_3, K_6$ (See figure 1.7).*

Proof. Apply lemma 1.5.2 to $p = 3$ to find that the only graphs on 3 vertices for which 3 is not a divisor of the size of the automorphism group are $\overline{K}_3$ and $K_3 = C_3 = C(1)$. Then, apply lemma 1.5.5 to conclude that the graphs on 6 vertices for which 3 is not a divisor of the size of their automorphism group are $\overline{K}_6 = \overline{K}_3 \sqcup \overline{K}_3, K_3 \sqcup \overline{K}_3, 2K_3, K_{3,3} = \overline{2K_3}, K_3 + \overline{K}_3 = \overline{K_3 \sqcup \overline{K}_3}, K_6 = \overline{\overline{K}_6}$. □

Let $\mathcal{P}$ be a nontrivial monotone and non-evasive graph property on 6 vertices. Then, $\mathcal{P}$ does not contain K_6, and the graph $\overline{K}_6$ represents the empty simplex which does not contribute to $\chi(\mathcal{P})$. If none of the graphs $K_3 \sqcup \overline{K}_3, 2K_3, K_{3,3}, K_3 + \overline{K}_3$ belongs to $\mathcal{P}$, then for each $G \in \mathcal{P}$, 3 is a divisor of $|[G]|$ and $\mathcal{P}$ cannot be non-evasive. We must have that $\mathcal{P}$ contains some of the 4 graphs $K_3 \sqcup \overline{K}_3, 2K_3, K_{3,3}, K_3 + \overline{K}_3$.

The following table shows the automorphism group, the size of the isomorphism class of these 4 graphs and the dimension of the faces that they represent in $\Delta\mathcal{P}$:

G	$Aut(G)$	$\lvert[G]\rvert$	$dimG$
$K_3 \sqcup \overline{K}_3$	$S_3 \times S_3$	20	2
$2K_3$	$S_3 \wr S_2$	10	5
$K_{3,3}$	$S_3 \wr S_2$	10	8
$K_3 + \overline{K}_3$	$S_3 \times S_3$	20	11

Lemma 1.7.2. *If $\mathcal{P}$ is a nontrivial monotone and nonevasive graph property on 6 vertices, then $\mathcal{P}$ satisfies exactly one of the following:*

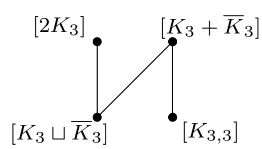
1) From the 4 graphs $K_3 \sqcup \overline{K}_3, 2K_3, K_{3,3}, K_3 + \overline{K}_3$, $\mathcal{P}$ just contains $K_{3,3}$.

2) From the 4 graphs $K_3 \sqcup \overline{K}_3, 2K_3, K_{3,3}, K_3 + \overline{K}_3$, $\mathcal{P}$ just contains $K_3 \sqcup \overline{K}_3$ and $2K_3$.

3) From the 4 graphs $K_3 \sqcup \overline{K}_3, 2K_3, K_{3,3}, K_3 + \overline{K}_3$, $\mathcal{P}$ just contains $K_3 \sqcup \overline{K}_3, K_{3,3}$ and $K_3 + \overline{K}_3$.

We will say that $\mathcal{P}$ is of *type* 1, 2 or 3 if $\mathcal{P}$ satisfies 1), 2) or 3) in lemma 1.7.2, respectively. Note that if $\mathcal{P}$ is of type i, then $\mathcal{P}^* = \{\overline{G} : G \notin \mathcal{P}\}$ is of type $3 - i$.

Proof. We have the following relations: $[K_3 \sqcup \overline{K}_3] \leq [2K_3], [K_3 \sqcup \overline{K}_3] \leq [K_3 + \overline{K}_3], [K_{3,3}] \leq [K_3 + \overline{K}_3]$ that can be represented by the following poset:



$\mathcal{P}$ could just contain any of the 15 nonempty subsets of the set $\{K_3 \sqcup \overline{K}_3, 2K_3, K_{3,3}, K_3 + \overline{K}_3\}$.

Note that $\mathcal{P}$ cannot just contain $2K_3$, for if $2K_3 \in \mathcal{P}$, then $K_3 \sqcup \overline{K}_3 \in \mathcal{P}$ (since $\mathcal{P}$ is monotone and $[K_3 \sqcup \overline{K}_3] \leq [2K_3]$).

On the other hand $\mathcal{P}$ could just contain $K_3 \sqcup \overline{K}_3$, for example. If this is the case, we estimate $\chi(\Delta\mathcal{P})$. For all graphs G in $\mathcal{P}$ that are not isomorphic to $K_3 \sqcup \overline{K}_3$, 3 is a divisor of $|[G]|$, then $\chi(\Delta\mathcal{P})$ has the form $3m + 20$ ($K_3 \sqcup \overline{K}_3$ is a 2-dimensional face of $\mathcal{P}$, and $|[K_3 \sqcup \overline{K}_3]| = 20$).

The following table shows all the possibilities of which nonempty subsets of $\{K_3 \sqcup \overline{K}_3, 2K_3, K_{3,3}, K_3 + \overline{K}_3\}$ can be contained in $\mathcal{P}$ and the form that $\chi(\Delta\mathcal{P})$ takes in each case:

$\mathcal{P}$ contains just	$\chi(\Delta\mathcal{P})$
$K_3 \sqcup \overline{K}_3$	$3m + 20$
$K_{3,3}$	$3m + 10$
$K_3 \sqcup \overline{K}_3, 2K_3$	$3m + 10$
$K_3 \sqcup \overline{K}_3, K_{3,3}$	$3m + 30$
$K_3 \sqcup \overline{K}_3, 2K_3, K_{3,3}$	$3m + 20$
$K_3 \sqcup \overline{K}_3, K_{3,3}, K_3 + \overline{K}_3$	$3m + 10$
$K_3 \sqcup \overline{K}_3, 2K_3, K_{3,3}, K_3 + \overline{K}_3$	$3m$

The only cases in which $\chi(\Delta\mathcal{P}) = 1$ can happen are the cases in which $\chi(\Delta\mathcal{P})$ has the form $3m + 10$. This ends the proof. □

Up to this point, we have reduced the possibilities for $\mathcal{P}$ to be of type 1, 2 or 3. This is as far as our estimations of Euler characteristic can help. To end the proof of evasiveness in the six vertices case, we need to use Oliver groups as in [27]. We will show that none of the three types can really happen. The ideas we are going to show are already found in [27], but there are some results we can obtain directly from lemma 1.7.2. For instance, notice that in any case, exactly one of the two graphs $2K_3, K_{3,3}$ belongs to $\mathcal{P}$.

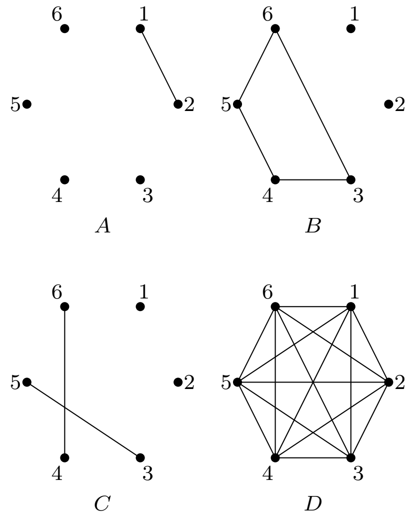


FIGURE 1.8. Orbits under the action of $\langle(12),(3456),(35)\rangle$

First, we want to show that $\mathcal{P}$ cannot be of type 1 nor 3. Assume that $\mathcal{P}$ is of type 1. Then, $\mathcal{P}$ contains $K_{3,3}$, but it does not contain $K_3 \sqcup \overline{K}_3$. Following [27], consider the group $\Gamma = \langle(12),(3456),(35)\rangle$ which is a 2-group (all its elements have order equal to a power of 2) and therefore Γ is an Oliver group. This group Γ acts on the vertices of the simplicial complex $\Delta\mathcal{P}$, and the orbits A, B, C, D of this action of Γ are shown in figure 1.8. Since all of the graphs A, B, C,$A \cup B$ and $A \cup C$ are isomorphic to subgraphs of $K_{3,3}$, they belong to $\mathcal{P}$. We can check that the graph $A \cup D, B \cup D, C \cup D$ and $B \cup C$ contain a copy of $K_3 \sqcup \overline{K}_3$ as a subgraph, so they cannot be in $\mathcal{P}$ (as $\mathcal{P}$ is monotone and does not contain $K_3 \sqcup \overline{K}_3$). If $D \in \mathcal{P}$, then $\mathcal{P}^\Gamma$ is a simplicial complex whose faces are $\varnothing, A, B, C, D, \{A, B\}, \{A, C\}$, but this implies that $\chi(\mathcal{P}^\Gamma) = 2$ which contradicts theorem 1.3.4. Hence, $\mathcal{P}^\Gamma$ is a simplicial complex with faces $\varnothing, A, B, C, \{A, B\}, \{A, C\}$.

As $\Delta\mathcal{P}$ is a non-evasive complex, there is some vertex X of $\Delta\mathcal{P}$ such that $lk_{\Delta\mathcal{P}}(X)$ and $del_{\Delta\mathcal{P}}(X)$ are nonevasive. The transitivity of $Aut(\Delta\mathcal{P})$ (remember that being a property of graphs implies that the group S_n, acting in the natural way on the vertices of the simplicial complex $\Delta\mathcal{P}$, is a subgroup of $Aut(\Delta\mathcal{P})$) permits us to take $X = A$. The vertex A of $\Delta\mathcal{P}$ is fixed by Γ, so Γ acts on $lk_{\Delta\mathcal{P}}(A)$ and theorem 1.3.4 can be applied to conclude that $\chi((lk_{\Delta\mathcal{P}}(A))^\Gamma) = 1$.

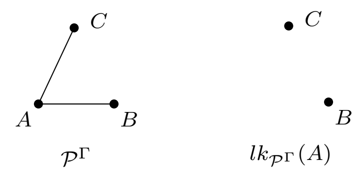


FIGURE 1.9

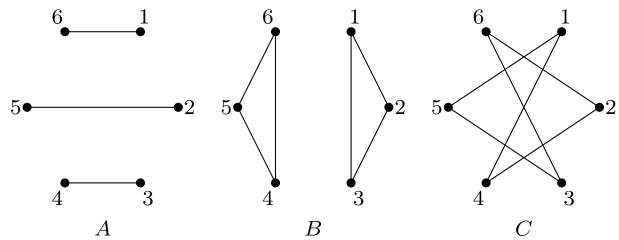


FIGURE 1.10. Orbits under the action of $\langle(153624)\rangle$

Now, $(lk_{\Delta\mathcal{P}}(A))^\Gamma = lk_{\Delta\mathcal{P}^\Gamma}(A)$ is a complex with just two vertices and no edges (see figure 1.9). Its Euler characteristic is 2, which contradicts $\chi((lk_{\Delta\mathcal{P}}(A))^\Gamma) = 1$.

This ends the proof that $\mathcal{P}$ cannot be of type 1.

If $\mathcal{P}$ is of type 3, then $\mathcal{P}^*$ is of type 1. The argument above applied to $\mathcal{P}^*$ gives that $\mathcal{P}^*$ cannot be of type 1 and as consequence, $\mathcal{P}$ cannot be of type 3.

Now, we want to prove that $\mathcal{P}$ cannot be of type 2. In fact, suppose that $\mathcal{P}$ is of type 2. In this case, $\mathcal{P}^*$ is also of type 2. Both $\mathcal{P}$ and $\mathcal{P}^*$ contain $K_3 \sqcup \overline{K}_3$ and $2K_3$, but they do not contain $K_{3,3}$ nor $K_3 + \overline{K}_3$.

Consider the group $\Gamma = \langle(153624)\rangle$. The orbits A, B, C of Γ are in figure 1.10. The group Γ is an Oliver group (Γ is cyclic of order 6, so Γ_1 can be taken as any subgroup of order 3). By proposition 1.6.3, $A \in \mathcal{P}$. As $\mathcal{P}$ is of type 2, $B \in \mathcal{P}$ too.

We claim that C is also in $\mathcal{P}$. On the contrary, $\mathcal{P}^\Gamma$ has just two vertices A, B and theorem 1.3.4 implies that $A \cup B \in \mathcal{P}$, but C is isomorphic to a subgraph of $A \cup B$, so C is in $\mathcal{P}$.

This argument also applies to $\mathcal{P}^*$ (as $\mathcal{P}^*$ is also of type 2), so we have that $A, B, C \in \mathcal{P}^*$. By the definition of $\mathcal{P}^*$, $\overline{A} = B \cup C, \overline{B} = A \cup C, \overline{C} = A \cup B$ are not in $\mathcal{P}$, so $\mathcal{P}^\Gamma$ consists only of three vertices and this contradicts that $\chi(\mathcal{P}^\Gamma) = 1$.

We have finished the proof that every nontrivial monotone graph property on 6 vertices has to be evasive.

Remark 1.7.3. The estimation of the Euler characteristic by means of divisors of the sizes of isomorphism classes of graphs on n vertices is not enough to give a proof of evasiveness. The use of another tool is necessary for completing the proof (theorem 1.3.4, for instance). In fact, there are nontrivial monotone graphs properties $\mathcal{P}$ satisfying $\chi(\mathcal{P}) = 1$ (see [24], pp. 124, also see [33]). In [24], there is an example of a monotone graph property on 6 vertices which is $\mathbb{Q}$-acyclic and therefore its Euler characteristic is 1. It is not $\mathbb{Z}$-acyclic.

1.8 Evasiveness and graphs on Ten Vertices

In this section we consider graph nontrivial monotone graph properties $\mathcal{P}$ on 10 vertices and make a treatment similar to the case of 6 vertices. We will label the ten vertices for our graphs as $0, 1, 2, 3, 4, 0', 1', 2', 3', 4'$.

First, we apply lemma 1.5.2 to $p = 5$ to find all graphs G on the 5 vertices 0,1,2,3,4, such that 5 does not divide $|[G]|$ (see example 1.5.3). They are $\overline{K}_5, C_5 \cong C(1) \cong C(2)$ and $K_5 = C(1,2)$ (note that $C(2) = \overline{C(1)}$). Correspondingly, for the 5 vertices $0', 1', 2', 3', 4'$, we have the graphs $C(1') \cong C(2')$ and $C(1', 2')$.

Suppose that, for a graph G on 10 vertices, 5 does not divide $|[G]|$ and apply lemma 1.5.5. Then, G is isomorphic to $\overline{K}_{10}$, K_{10}, or one of the 10 graphs $G_i, \overline{G}_i$, $i = 1, 2, 3, 4, 5$ shown in figure 1.11.

If $\mathcal{P}$ does not contain any of the 10 graphs in figure 1.11, then 5 divides $\chi(\mathcal{P})$ and so $\mathcal{P}$ is evasive.

Let $\mathcal{P}$ be a nontrivial monotone and non-evasive graph property on ten vertices. Then, $\mathcal{P}$ contains some of the graphs in figure 1.11. The following table contains

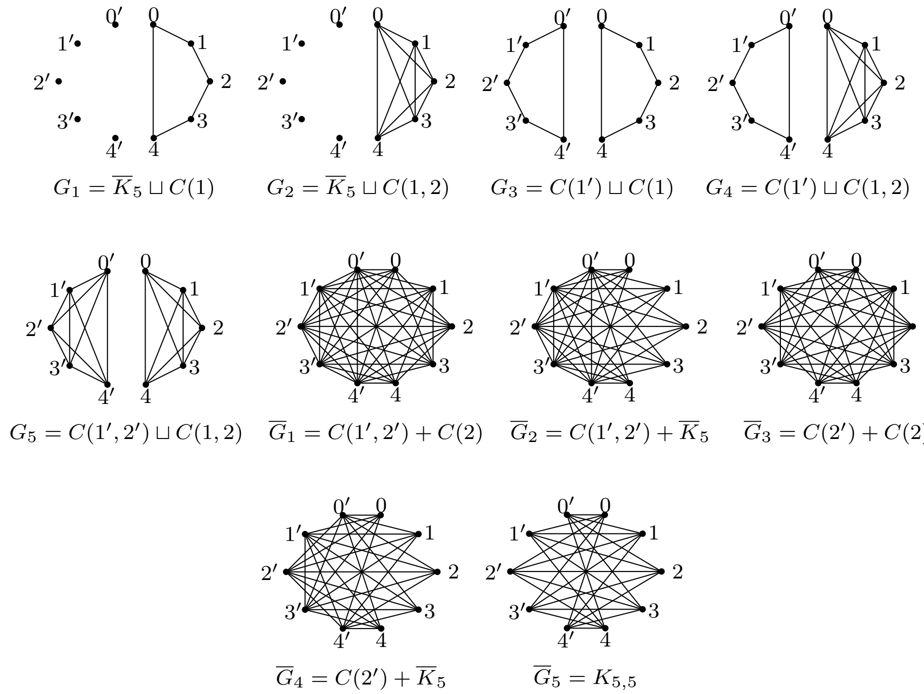
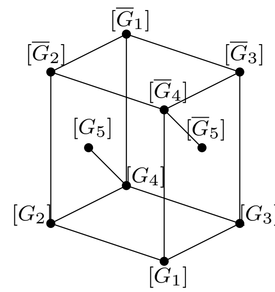


FIGURE 1.11

the automorphism groups of G_i for $i = 1,2,3,4,5$, also the sizes $|[G_i]|$ and the contribution modulo 5 to $\chi(\Delta\mathcal{P})$ when $G_i \in \mathcal{P}$, that is $(-1)^{m_{G_i}-1}|[G_i]|(mod\ 5)$:

G_i	$Aut(G_i)$	$\|[G_i]\|$	$(-1)^{m_{G_i}-1}\|[G_i]\|(mod\ 5)$
G_1	$D_{10} \times S_5$	$2^4 \cdot 3^3 \cdot 7$	+4
G_2	$S_5 \times S_5$	$2^2 \cdot 3^2 \cdot 7$	-2
G_3	$D_5 \wr S_2$	$2^5 \cdot 3^4 \cdot 7$	-4
G_4	$S_5 \times D_{10}$	$2^4 \cdot 3^3 \cdot 7$	+4
G_5	$S_5 \wr S_2$	$2 \cdot 3^2 \cdot 7$	-1

Note that if G is a graph on 10 vertices having m_G edges, then $\overline{G}$ has $m_{\overline{G}} = 45 - m_G$ edges. The Hasse diagram of the isomorphism classes $[G_i]$'s and $[\overline{G_j}]$'s is the following:



If $\mathcal{P}$ is non-evasive, then $\mathcal{P}$ contains some of the isomorphism classes $[G_i], [\overline{G}_j]$. The set of such isomorphism classes contained in $\mathcal{P}$ becomes an *order ideal* of the poset above. Therefore, $\chi(\Delta\mathcal{P})$ is congruent modulo 5 to the sum of the numbers $(-1)^{m_G-1}|[G]|$ where $[G]$ belongs to that order ideal. We need to determine all the order ideals of the the poset above and also determine which of them can make $\chi(\Delta\mathcal{P})$ congruent to 1 *mod* 5. There are exactly 9 of these order ideals:

$$I_1 = \{[\overline{G_5}]\}$$
$$I_2 = \{[G_1], [\overline{G_4}], [\overline{G_5}]\}$$
$$I_3 = \{[G_1], [G_3], [\overline{G_5}]\}$$
$$I_4 = \{[G_1], [G_2], [\overline{G_2}], [\overline{G_4}], [\overline{G_5}]\}$$
$$I_5 = \{[G_1], [G_2], [G_3], [G_4], [G_5]\}$$
$$I_6 = \{[G_1], [G_3], [\overline{G_3}], [\overline{G_4}], [\overline{G_5}]\}$$
$$I_7 = \{[G_1], [G_2], [G_3], [G_4], [\overline{G_2}], [\overline{G_4}], [\overline{G_5}]\}$$
$$I_8 = \{[G_1], [G_2], [G_3], [\overline{G_2}], [\overline{G_3}], [\overline{G_4}], [\overline{G_5}]\}$$
$$I_9 = \{[G_1], [G_2], [G_3], [G_4], [\overline{G_1}], [\overline{G_2}], [\overline{G_3}], [\overline{G_4}], [\overline{G_5}]\}$$

In a similar way to lemma 1.7.2, we get that from the 10 isomorphism classes $[G_i], [\overline{G}_i]$, $i = 1, 2, 3, 4, 5$, $\mathcal{P}$ contains exactly those isomorphism classes of graphs belonging to a fixed set I_k, $k = 1, \ldots, 9$. We say that $\mathcal{P}$ is of *type* k if $\mathcal{P}$ contains exactly I_k. $\mathcal{P}$ is of type k if and only if $\mathcal{P}^*$ is of type $10 - k$.

We will consider some Oliver groups to show that types 1, 3, 7 and 9 cannot happen. The first result is the following lemma.

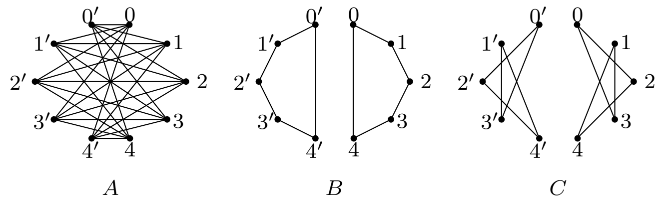


FIGURE 1.12. Orbits of $\langle(00')(11')(22')(33')(44'),(01234),(0'1'2'3'4')\rangle$

First we show that $\mathcal{P}$ cannot be of type 3 nor 7. Proposition 1.6.4 implies that $\mathcal{P}$ contains one of the 2, $K_{5,5} \cong \overline{G}_5$ or $2C_5 \cong G_3$. The corresponding Oliver group is $\Gamma = \langle(00')(11')(22')(33')(44'),(01234),(0'1'2'3'4')\rangle$. The orbits of Γ, $A \cong \overline{G}_5, B \cong G_3, C \cong G_3$ are shown in figure 1.12. By theorem 1.3.4, $\chi(\mathcal{P}^\Gamma) = 1$.

If $\mathcal{P}$ is of type 3 or 7, then $G_3, \overline{G}_5 \in \mathcal{P}$. Thus, we have that $A, B, C \in \mathcal{P}$ and since $\chi(\mathcal{P}^\Gamma) = 1$, the graphs $A \cup B \cong A \cup C$ are in $\mathcal{P}$. We have that $A \cup B \cong \overline{G}_3$, but $\overline{G}_3$ does not belong to $\mathcal{P}$ (see the order ideals I_3 and I_7 above).

To show that $\mathcal{P}$ cannot be of type 1 nor 9, we need the following result of P. A. Smith [54].

Theorem 1.8.1. *If Γ is a p-group acting on a $\mathbb{Z}/p$-acyclic complex K, then $|K|^\Gamma$ is also $\mathbb{Z}/p$-acyclic.*

Suppose that $\mathcal{P}$ is of type 1 and let $\Gamma = \langle(00'),(12341'2'3'4')\rangle$. Then, Γ is a 2-group. The (potential) vertices of $\mathcal{P}^\Gamma$ are the orbits of Γ acting on the vertices of the simplicial complex $\mathcal{P}$. These orbits are shown in figure 1.13.

Since $\mathcal{P}$ is of type 1, $\mathcal{P}$ contains $\overline{G}_5$ but does not contain G_1. The graphs in the following list are in $\mathcal{P}$ because each of them is isomorphic to a subgraph of $\overline{G}_5$:

$$A, B, C, D, E, A \cup B, A \cup C, A \cup D, A \cup E, B \cup D. \tag{1.8.1}$$

The graphs in the following list are not in $\mathcal{P}$ because each of them contains a subgraph isomorphic to G_1:

$$B \cup C, B \cup E, B \cup F, C \cup D, C \cup F, D \cup E, D \cup F, E \cup F. \tag{1.8.2}$$

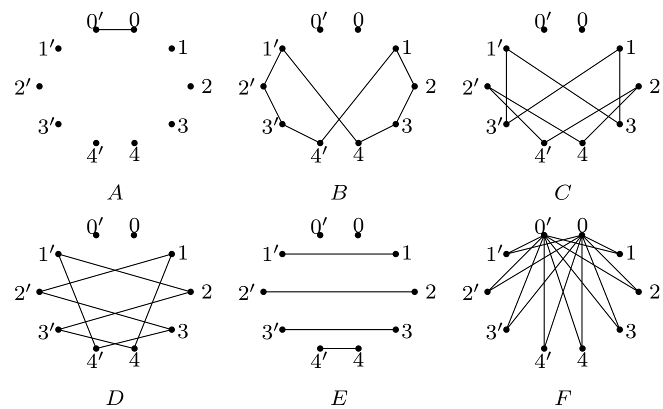


FIGURE 1.13. Orbits of $\langle (00'), (12341'2'3'4') \rangle$

Observe that the graphs $F, A \cup F$ and $C \cup E$ are not in these lists.

Now, A is one of the vertices of the simplicial complex $\Delta\mathcal{P}$ and A is also one of the fixed points of Γ. Therefore, Γ acts on $lk_{\Delta\mathcal{P}}(A)$. Moreover $lk_{\Delta\mathcal{P}}(A)$ is a non-evasive complex. The fixed point set of the action of Γ on $lk_{\Delta\mathcal{P}}(A)$ is given by $lk_{\Delta\mathcal{P}}(A)^\Gamma = lk_{\Delta\mathcal{P}^\Gamma}(A)$.

From 1.8.1 we see that B, C, D, E are vertices of $lk_{\Delta\mathcal{P}}(A)^\Gamma$. Since $lk_{\Delta\mathcal{P}}(A)$ is $\mathbb{Z}/2$-acyclic, $lk_{\Delta\mathcal{P}}(A)^\Gamma$ is $\mathbb{Z}/2$-acyclic by theorem 1.8.1. Then, $lk_{\Delta\mathcal{P}}(A)^\Gamma$ is connected.

The graph F cannot be a vertex of the simplicial complex $lk_{\Delta\mathcal{P}}(A)^\Gamma$ because, on the contrary, F would be an isolated vertex of $lk_{\Delta\mathcal{P}}(A)^\Gamma$. Thus, $lk_{\Delta\mathcal{P}}(A)^\Gamma$ has precisely the vertices B, C, D, E. The only other faces of $lk_{\Delta\mathcal{P}}(A)^\Gamma$ can be $\{B, D\}$ and $\{C, E\}$. In any case $lk_{\Delta\mathcal{P}}(A)^\Gamma$ results to be non-connected. This contradiction proves that $\mathcal{P}$ cannot be of type 1.

If $\mathcal{P}$ is of type 9, then $\mathcal{P}^*$ is of type 1 and the argument above applies to $\mathcal{P}^*$ to conclude that $\mathcal{P}^*$ cannot be of type 1. Thus, $\mathcal{P}$ cannot be of type 9.

Remark 1.8.2. We have not found the appropriate Oliver groups to use in order to discard the 5 remaining possible types for $\mathcal{P}$. If a nontrivial monotone and non-evasive graph property on 10 vertices $\mathcal{P}$ exists, then it must contain one of the order ideals I_k for $k = 2, 4, 5, 6, 8$. The first thing we can do is to test the property *generated* by I_k in order to know whether the associated simplicial complex is $\mathbb{Z}$-

acyclic. This could be done by using a computer program.

Another option is to study a commod divisor of the sizes of automorphism groups of graphs, other than 5, to get more conditions on the graphs that belong to $\mathcal{P}$.

CHAPTER 2

Ideal-valued cohomological index

2.1 Ideal-Valued Cohomological Index Theories

Cohomology theories

A*cohomology theory* h^* (on the appropriate category $\mathcal{T}$ of topological pairs (X, A) and maps of pairs) is a sequence of contravariant functors $h^q : \mathcal{T} \to \mathcal{A}$, where $\mathcal{A}$ is the category of abelian groups and homomorphisms, along with natural transformations $\delta^q : h^q \circ R \to h^{q+1}$, $q \in \mathbb{Z}$ (R is the functor that sends pair (X, A) to the pair $(A, \varnothing)$ and a map $f : (X, A) \to (Y, B)$ to $f|A$). These natural transformations are called the *connecting homomorphisms*. It is customary to write f^* instead of $h^q(f)$ for a map $f : (X, A) \to (Y, B)$. The following axioms are required:

Homotopy: If $f_0 \simeq f_1 : (X, A) \to (Y, B)$, then $f_0^* = f_1^* : h^q(Y, B) \to h^q(X, A)$ for all $q \in \mathbb{Z}$.

Excision: For every pair (X, A) and a subset $U \subseteq A$ satisfying $\overline{U} \subseteq Int\ A$, the inclusion $j : (X \setminus U, A \setminus U) \to (X, A)$ induces an isomorphism $h^q(X, A) \cong h^q(X \setminus U, A \setminus U)$, for all $q \in \mathbb{Z}$.

Exactness: For every pair (X, A), there is a long exact sequence

$$\cdots \xrightarrow{\delta^{q-1}} h^q(X,A) \xrightarrow{i^*} h^q(X) \xrightarrow{j^*} h^q(A) \xrightarrow{\delta^q} h^{q+1}(X,A) \longrightarrow \cdots$$

where $i : (X, \emptyset) \hookrightarrow (X, A)$ and $j : (A, \emptyset) \hookrightarrow (X, \emptyset)$ are the inclusions, and $h^q(X)$ means $h^q(X, \emptyset)$.

The cohomology theory h^* is *multiplicative* if it has products

$$h^p(X,A) \otimes h^q(X,B) \to h^{p+q}(X, A \cup B)$$

for any pairs (X, A) and (X, B) with $\{A, B\}$ excisive and any $p, q \in \mathbb{Z}$.

When we want to compute $h^*(X)$ for (a CW-complex) X we have the *Atiyah-Hirzebruch spectral sequence* (AHSS): a (cohomological type) spectral sequence $\{E_r^{*,*}, d_r\}$ that (under suitable conditions) converges to $h^*(X)$ and that has E_2-page given by

$$E_2^{s,t} = H^s(X; h^t(pt)), \tag{2.1.1}$$

the singular cohomology of X with coefficients in $h^*(pt)$ (pt represents the one-point space), see [57].

The Borel construction and equivariant cohomology

For each (finite) group G there exists the associated *classifying space* BG and the universal G-bundle $EG \to BG$ (see [39] for a construction of the universal G-bundle). The space EG is characterized as a contractible cell G-complex with a free action of G and BG is the associated quotient EG/G. The "universality" means that for any free cell G-complex X and the associated G-bundle $X \to X/G$, there is a homotopically unique map $\alpha_X : X/G \to BG$ such that the bundle $X \to X/G$ is isomorphic to the pull back of $EG \to BG$ along α_X.

If X is a free cell G-complex, one good way to define its equivariant cohomology $H^*_G(X;R)$ is as the singular cohomology of the quotient X/G. This "does not work" when X is not free. In that case, the equivariant cohomology $H^*_G(X;R)$ is defined as the singular cohomology of the *homotopy quotient*, or *Borel construction*, X_G. For any G-space X one defines the homotopy quotient $X_G := EG \times_G X$, where G acts on $EG \times X$ diagonally. When X is free, X_G is (weakly) homotopy equivalent to X/G. The equivariant cohomology of X is defined by $H^*_G(X;R) := H^*(X_G;R)$.

The equivariant cohomology of a point pt, $H^*_G(pt;R)$, is the singular cohomology of BG, $H^*(BG;R)$. For example, $H^*_{\mathbb{Z}/2}(pt;\mathbb{F}_2) = H^*(B\mathbb{Z}/2;\mathbb{F}_2) = H^*(\mathbb{R}P^\infty;\mathbb{F}_2) = \mathbb{F}_2[t]$, where t has degree 1. If p is an odd prime number, then $H^*_{\mathbb{Z}/p}(pt;\mathbb{F}_p) = H^*(B\mathbb{Z}/p;\mathbb{F}_p) = \mathbb{F}_p[a,b]/(a^2)$, where a has degree 1 and b has degree 2.

Ideal-valued cohomological index for cohomology theories

Based on the definition of the Fadell-Husseini index, we can define ideal-valued cohomological index using a cohomology theory h^* ([15], [16], [63]).

For a G-space X there is a (unique) map $X \to *$. Then we have a map $EG\times X \to EG$ and by passing to quotients, we get $q_X : X_G \to BG$. For a cohomology theory h^*, we denote $h^*_G(X) := h^*(X_G)$; for a pair (X,A), $h^*_G(X,A) := h^*(X_G,A_G)$, where A is an invariant subspace of X.

We have a map $q^*_X : h^*_G(pt) = h^*(BG) \to h^*_G(X)$ induced by $q_X : X_G \to BG$. From this, $h^*_G(X)$ becomes an $h^*_G(pt)$-module by defining $\lambda \cdot x := q^*_X(\lambda)x$, for $\lambda \in h^*_G(pt)$ and $x \in h^*_G(X)$.

We define the *index* of X with respect to h^*, denoted $Ind^{h^*}_G X$, as the kernel of $q^*_X : h^*_G(pt) \to h^*_G(X)$: $Ind^{h^*}_G X := ker\ q^*_X$. The index $Ind^{h^*}_G X$ depends on the action of G on X. The following are called the *axioms for an index-valued cohomological index theory* (see [15, 16, 28, 63]):

Monotonicity: If there is a G-equivariant map $f : X \to Y$ between G-spaces X and Y then

$$Ind_G Y \subseteq Ind_G X.$$

Additivity: If $\{X = X_1 \cup X_2, X_1, X_2\}$ is excisive, where X_1 and X_2 are G-invariant subspaces of X, then

$$Ind_G\ X_1 \cdot Ind_G\ X_2 \subseteq Ind\ X.$$

Continuity: If A is a closed G-invariant subspace of X, then there is a G-invariant neighborhood U of A in X such that

$$Ind_G\ A = Ind_G\ U.$$

Index theorem: If $f : X \to Y$ is G-equivariant, $B \subseteq Y$ a closed G-invariant subspace and $A := f^{-1}(B)$, then

$$Ind_G\ A \cdot Ind_G\ (Y \setminus B) \subseteq Ind_G\ X.$$

Remark 2.1.1. The monotonicity axiom gives a necessary condition for the existence equivariant maps between G-spaces. In fact, if there is some equivariant map $f : X \to Y$, the commutative diagram

$$\begin{array}{ccc} X & \xrightarrow{f} & Y \\ & \searrow \quad \swarrow & \\ & * & \end{array}$$

induces the following commutative diagram in cohomomolgy

$$\begin{array}{ccc} h^*_G(X) & \xleftarrow{f^*} & h^*_G(Y) \\ {\scriptstyle q^*_X} \nwarrow & & \nearrow {\scriptstyle q^*_Y} \\ & h^*(BG) & \end{array}$$

from which we obtain $Ind_G\, Y \subseteq Ind_G\, X$.
The axiom of additivity is satisfied if h^* is multiplicative with unit. Monotonicity and additivity are the two axioms we are going to need in our work.

The continuity axiom is satisfied if h^* is assumed to be *continuous* (for any pair (X, A) with A closed in X, $h^*(A) \cong \varinjlim h^*(U)$ where the direct limit is taken over all neighborhoods U of A in X and the isomorphism is induced by the inclusions). The index theorem is a direct consequence of the other 3 axioms.

The cohomology theories we are interested in satisfy to be multiplicative and continuous, namely singular cohomology with coefficients in a field and Morava K-theories although continuity is not that important in our work.

For references, see [16, 28].

2.2 The Fadell-Husseini Index

The ideal-valued cohomological index we obtain by using equivariant cohomology with coefficients in R (a commutative ring with unit) is the well known Fadell-Husseini index (see for example [15, 16, 63]), which we denote $Ind_{G,R}X$ to include the coefficient ring R. The Fadell-Husseini index satisfies the axioms for an ideal-valued cohomological index theory (under suitable conditions, see [15, 16]).

In the CS/TM scheme one wants to prove that there is no equivariant map $X \to Y$ between two G-spaces X and Y arising from a combinatorial problem. The space Y is frequently the unit sphere $S(W)$ of a G-representation W and its ideal-valued index is a principal ideal $\langle u \rangle \subseteq H^*(BG, R)$. If there is some equivariant map $X \to Y$, the monotonicity axiom of the index theory implies that u lives in the ideal-valued index of X. One tries to prove on the other hand that the element u really does not belong to the index of X. That is why we need formulas for computing indexes of spheres and tools to determine that a given element in $H^*(BG, R)$ does not belong to $Ind_{G,R}X$.

We want to list some of the properties of the Fadell-Husseini index and some tools of calculations.

The Serre spectral sequence and Fadell-Husseini index

For the following theorem see [34], also see [21] and [57].

Theorem 2.2.1. *Let R be a commutative ring with unit. For a fibration $F \to E \to B$ with B path-connected and F connected, there is a first quadrant spectral sequence $\{E_r^{*,*}, d_r\}$, converging to $H^*(E;R)$ (as an algebra), with*

$$E_2^{p,q} \cong H^p(B; \mathcal{H}^q(F;R)),$$

the cohomology of the base space B with local coefficients in the cohomology of the fiber F.

The map $q_X : X_G \to BG$ is a fibration with fiber X. There is a spectral sequence $\{E_r^{*,*}, d_r\}$ with $E_2^{p,q} = H^p(BG; \mathcal{H}^q(X;R))$ where $\mathcal{H}^q(X;R)$ is a system of local coefficients, this local coefficient system is determined by the action of $G = \pi_1(BG)$ on $H^*(X;R)$. Therefore, the E_2-page of the spectral sequence is interpreted as the cohomology of the group G with coefficients in the G-module $H^*(X,R)$ (see for example [1, 9]),

$$E_2^{p,q} = H^p(G; H^q(X;R)). \tag{2.2.1}$$

The map $q_X^* : H^*(BG,R) \to H^*(X_G,R)$ can be represented as the composition (known as the *edge homomorphism*)

$$H^*(BG;R) \to E_2^{*,0} \to E_3^{*,0} \to E_4^{*,0} \to \cdots \to E_\infty^{*,0} \subseteq H^*(X_G;R). \tag{2.2.2}$$

Remark 2.2.2. We will use the Serre spectral sequence of the fibration $X \to X_G \to BG$ to show that a given element $u \in E_2^{*,0} = H^*(BG;R)$ does not belong to $Ind_{G,R}X$. What we will do is to check that u survives forever in the spectral sequence.

Products

If X is a G-space and Y an H-space, then $X \times Y$ is a $G \times H$-space. The map $q_{X\times Y} : (X\times Y)_{G\times H} \to B(G\times H)$ can be identified with the map $q_X \times q_Y : X_G \times Y_H \to BG \times BH$ and the map $q^*_{X\times Y}$ can be identified with

$$q^*_X \otimes q^*_Y : H^*(BG;\mathbb{K}) \otimes H^*(BH;\mathbb{K}) \to H^*_G(X;\mathbb{K}) \otimes H^*_H(Y;\mathbb{K}),$$

where $\mathbb{K}$ is a field. The kernel of this map can be expressed as follows.

Lemma 2.2.3. $\ker q^*_X \otimes q^*_Y = \ker q^*_X \otimes H^*(BH;\mathbb{K}) + H^*(BG;\mathbb{K}) \otimes \ker q^*_Y$, *that is* $Ind_{G\times H,\mathbb{K}} X \times Y = Ind_{G,\mathbb{K}} \otimes H^*(BH;\mathbb{K}) + H^*(BG;\mathbb{K}) \otimes Ind_{H,\mathbb{K}} Y$.

Corollary 2.2.4. *If $Y = pt$ in lemma 2.2.5, then*

$$Ind_{G\times H,\mathbb{K}} X = Ind_{G,\mathbb{K}} X \otimes H^*(BH;\mathbb{K})$$

Corollary 2.2.5. *If $H^*(BG;\mathbb{K}) \cong \mathbb{K}[x_1,\ldots,x_n], H^*(BH;\mathbb{K}) \cong \mathbb{K}[y_1,\ldots,y_m]$, $Ind_{G,\mathbb{K}} X = \langle f_1,\ldots,f_r\rangle$ and $Ind_{H,\mathbb{K}} Y = \langle g_1,\ldots,g_s\rangle$, then*

$$Ind_{G\times H,\mathbb{K}} X \times Y = \langle f_1,\ldots,f_r,g_1,\ldots,g_s\rangle \subseteq \mathbb{K}[x_1,\ldots,x_n,y_1,\ldots,y_m].$$

The above results on the index of products can be consulted in [15].

Remark 2.2.6. The central idea of the proof of these results is the *Künneth isomorphism.* When we wanted to have similar formulas for the index of a product, using other cohomology theories, we found that (in some sense) the singular cohomology with coefficients in a field and the Morava K-theories are essentially the unique cohomology theories that have Künneth isomorphisms (see [45]). That is one of the reasons we are going to treat ideal-valued index for Morava K-theories (section 2.3 below).

Spheres, E_nG-spaces and the configuration space $F(\mathbb{R}^d, p)$

A free G-space X is an E_nG*-space* if it is an $(n-1)$-connected, n-dimensional CW-complex equipped with a G-invariant CW-structure. E_nG-spaces exist; in fact, the join $G^{*(n+1)}$ of $n+1$ copies of the group G, where G is regarded as a 0-dimensional simplicial complex, is an E_nG-space.

The following properties of the Fadell-Husseini index can be found in [63].

Proposition 2.2.7. *i) For the antipodal $\mathbb{Z}/2$-action on a sphere S^n, we have that*

$$Ind_{\mathbb{Z}/2,\mathbb{F}_2} S^n = \langle t^{n+1} \rangle \subseteq \mathbb{F}_2[t].$$

In general, if X is an $E_n\mathbb{Z}/2$-space, then

$$Ind_{\mathbb{Z}/2,\mathbb{F}_2} X = \langle t^{n+1} \rangle \subseteq H^*(B\mathbb{Z}/2; \mathbb{F}_2) = \mathbb{F}_2[t].$$

ii) For an odd prime p, we have $H^*(B\mathbb{Z}/p, \mathbb{F}_p) = \mathbb{F}_p[a,b]/\langle a^2 \rangle$, where $\deg(a) = 1$ and $\deg(b) = 2$. The unit sphere $S^{2n-1} \subseteq \mathbb{C}^n$ is a $\mathbb{Z}/p$-space where $\mathbb{Z}/p$ is seen as the subgroup of S^1 of p-th roots of 1. S^1 acts on S^{2n-1} by complex multiplication and so does $\mathbb{Z}/p$. Then

$$Ind_{\mathbb{Z}/p,\mathbb{F}_p} S^{2n-1} = \langle b^n \rangle \subseteq \mathbb{F}_p[a,b]/\langle a^2 \rangle.$$

If X is an $E_{2n-1}\mathbb{Z}/p$, then

$$Ind_{\mathbb{Z}/p,\mathbb{F}_p} X = \langle b^n \rangle \subseteq \mathbb{F}_p[a,b]/\langle a^2 \rangle.$$

iii) For an $E_{2n}\mathbb{Z}/p$-space X, we have that

$$Ind_{\mathbb{Z}/p,\mathbb{F}_p} X = \langle ab^n \rangle \subseteq \mathbb{F}_p[a,b]/\langle a^2 \rangle.$$

iv) For a finite group G and an E_nG-space X, we have

$$Ind_{G,\mathbb{K}}X \subseteq \bigoplus_{d>n} H^d(BG;\mathbb{K}),$$

that is, for every (homogeneous) element $x \in Ind_G X \subseteq H^*(BG;\mathbb{K})$, $deg(x) > n$.

v) Let U, V be two G-representations and $S(U), S(V)$ be the associated unit G-spheres. Assume that the vector bundles $U \to U_G \to BG$ and $V \to V_G \to BG$ are orientable over $\mathbb{K}$. If $Ind_{G,\mathbb{K}}S(U) = \langle f \rangle \subseteq H^*(BG;\mathbb{K})$ and $Ind_{G,\mathbb{K}}S(V) = \langle g \rangle \subseteq H^*(BG;\mathbb{K})$, then

$$Ind_{G,\mathbb{K}}S(U \oplus V) = \langle f \cdot g \rangle \subseteq H^*(BG;\mathbb{K}).$$

vi) Let $V = \mathbb{C}$ be the 1-dimensional complex $(\mathbb{Z}/p)^k$-representation associated with the vector $(\alpha_1, \ldots, \alpha_n) \in \mathbb{F}_p^n$ (If ω_i is the generator of the i^{th} copy of $\mathbb{Z}/p$ in $(\mathbb{Z}/p)^n$, then the vector α is characterized by the equality $\omega_i \cdot v = \zeta^{\alpha_i} v$ where $\zeta = e^{2\pi i/p}$, $v \in V$ and $i = 1, \ldots, n$). Then

$$Ind_{(\mathbb{Z}/p)^n,\mathbb{F}_p}S(V) = \langle \alpha_1 b_1 + \cdots + \alpha_n b_n \rangle \subseteq \mathbb{F}_p[b_1, \ldots, b_n] \subseteq H^*(B(\mathbb{Z}/p)^n;\mathbb{F}_p).$$

For the following result see [11].

Proposition 2.2.8. *Let G be a finite group and V an n-dimensional complex G-representation. Then*

$$Ind_{G,\mathbb{Z}}S(V) = \langle c_n(V_G) \rangle \subseteq H^*(BG;\mathbb{Z}),$$

where $c_n(V_G)$ is the n-th Chern class of the bundle $V \to EG \times_G V \to BG$.

If $G = \mathbb{Z}/n$, the cyclic group of orden n, and $U = \mathbb{C}^2$ is the $\mathbb{Z}/n$-representation in which the generator of $\mathbb{Z}/n$ acts by multiplication by $e^{2\pi i/n}$, then $H^*(B\mathbb{Z}/n;\mathbb{Z}) =$

$\mathbb{Z}[u]/\langle nu\rangle$, where $u = c_1(U_{\mathbb{Z}/n})$ is the first Chern class of the vector bundle $U \to U_{\mathbb{Z}/n} \to B\mathbb{Z}/n$ (see [3]). The cohomology class u is a generator of $H^2(\mathbb{Z}/n;\mathbb{Z})$. We have that

$$Ind_{\mathbb{Z}/n,\mathbb{Z}}S(U) = \langle u\rangle \subseteq \mathbb{Z}[u]/\langle nu\rangle. \tag{2.2.3}$$

The *configuration space* of n distinct points in the topological space X is the space

$$F(X,n) = \{(x_1, x_2, \dots, x_n) \in X^n : x_i \neq x_j \text{ for all} i \neq j\}. \tag{2.2.4}$$

The symmetric group S_n acts on $F(X,n)$ by permuting coordinates; the cyclic group $\mathbb{Z}/n$ acts on $F(X,n)$ by restricting the action of S_n. The following result is found in [9] (Theorem 6.1):

Proposition 2.2.9. *Let p be a prime and $d > 1$. Then*

$$Ind_{\mathbb{Z}/p,\mathbb{F}_p}F(\mathbb{R}^d,p) = \begin{cases} \langle t^{(d-1)(p-1)+1}\rangle \subseteq \mathbb{F}_2[t], & \textit{for } p=2,\\ \langle ab^{\frac{(d-1)(p-1)}{2}}, b^{\frac{(d-1)(p-1)}{2}+1}\rangle \subseteq \mathbb{F}_p[a,b]/\langle a^2\rangle, & \textit{for } p \textit{ odd.}\end{cases}$$

The topological Tverberg theorem

We are going to sketch the proof of the topological Tverberg theorem given by Živaljević in [63], where Fadell-Husseini index is used.

Theorem 2.2.10. *(Topological Tverberg theorem) If $q = p^k$ is a prime power and $d \geq 1$, then for any continuous map $f : \Delta^{(q-1)(d+1)} \to \mathbb{R}^d$, there exist q disjoint faces of $\Delta^{(q-1)(d+1)}$, $F_1, F_2, \dots, F_q$ such that*

$$f(F_1) \cap f(F_2) \cap \dots \cap f(F_q) \neq \varnothing.$$

The symmetric group S_q acts on $\mathbb{C}^q$ by permuting the coordinates; this S_q-representation is the sum of a trivial S_q-representation, $\{(z,\dots,z) : z \in \mathbb{C}\}$, and the representation $W_{\mathbb{C}}(1,q) = \{(z_1,\dots,z_q) : z_1 + \dots + z_q = 0\}$. The S_q-

representation $W_{\mathbb{C}}(1,q)$ is called the *standard* (complex) S_q-representation. Let $W_{\mathbb{C}}(m,q) := W_{\mathbb{C}}(1,q)^{\oplus(m)}$. The standard real S_q-representation is defined similarly.

Let $N = (q-1)(d+1)$, X an $E_N S_q$-space and $W_{\mathbb{R}}(d+1,q)$ the corresponding standard real S_q-representation. The CS/TM scheme yields that the topological Tveberg theorem is true for $q \geq 2$ and $d \geq 1$, if there is no S_q-equivariant map $X \to S(W_{\mathbb{R}}(d+1,q))$.

It suffices to consider the case in which $d+1$ is even. If this is the case, then $W_{\mathbb{R}}(d+1,q)$ and $W_{\mathbb{C}}((d+1)/2,q)$ are isomorphic S_q-representations (see [31, 53]).

When $q = p^k$, the restriction of the action of S_q on $\mathbb{C}^q$ to the subgroup $(\mathbb{Z}/p)^k$, gives a $(\mathbb{Z}/p)^k$-representation that is isomorphic to the *regular* representation of $(\mathbb{Z}/p)^k$, $\mathbb{C}[(\mathbb{Z}/p)^k]$, which decomposes as (see [19], [63]):

$$\mathbb{C}^q \cong \mathbb{C}[(\mathbb{Z}/p)^k] \cong \bigoplus_{\alpha \in \mathbb{F}_p^k} V_\alpha. \tag{2.2.5}$$

Therefore, $W_{\mathbb{C}}(1,q)$ decomposes (as $(\mathbb{Z}/p)^k$-representation) as follows:

$$W_{\mathbb{C}}(1,q) \cong \bigoplus_{\alpha \in \mathbb{F}_p^k \setminus \{0\}} V_\alpha. \tag{2.2.6}$$

The idea is to prove that there is no $(\mathbb{Z}/p)^k$-equivariant map $X \to S(W_{\mathbb{C}}((d+1)/2,q))$. To show this, Fadell-Husseini index with coefficients in $\mathbb{F}_p$ is used (we assume for simplicity that p is odd). If there exist such an equivariant map, we must have

$$Ind_{(\mathbb{Z}/p)^k,\mathbb{F}_p} S(W_{\mathbb{C}}((d+1)/2,q)) \subseteq Ind_{(\mathbb{Z}/p)^k,\mathbb{F}_p} X. \tag{2.2.7}$$

Since X is an $E_N(\mathbb{Z}/p)^k$-space, if a cohomology class belongs to $Ind_{(\mathbb{Z}/p)^k,\mathbb{F}_p} X$, then that class must have degree greater than N. On the other hand, by 2.2.6 and proposition 2.2.7 v) and vi), $Ind_{(\mathbb{Z}/p)^k,\mathbb{F}_p} S(W_{\mathbb{C}}(1,q))$ is the principal ideal in $\mathbb{F}_p[b_1,\ldots,b_k]$ generated by $\prod_{\alpha \in \mathbb{F}_p^k \setminus \{0\}} (\alpha_1 b_1 + \cdots + \alpha_k b_k)$.

By proposition 2.2.7 v), $Ind_{(\mathbb{Z}/p)^k,\mathbb{F}_p} S(W_{\mathbb{C}}(\frac{d+1}{2},q))$ is the principal ideal in $\mathbb{F}_p[b_1,\ldots,b_k]$ generated by the polynomial $\prod_{\alpha \in \mathbb{F}_p^k \setminus \{0\}} (\alpha_1 b_1 + \cdots + \alpha_k b_k)^{(d+1)/2}$, which is a cohomology class of degree $(q-1)(d+1)/2 = N$. By 2.2.7, this cohomology class must belong to $Ind_{(\mathbb{Z}/p)^k,\mathbb{F}_p}X$. This contradiction ends the proof.

Remark 2.2.11. Our calculations of the ideal-valued cohomological index for Morava K-theories are going to be based on the willing of reproducing this proof of the topological Tverberg theorem. In particular, we will need to make use of results on the Morava K-theories of some classifying spaces, the Künneth isomorphisms and the Atiyah-Hirzebruch spectral sequence that for Morava K-theories converges to the desire object.

What we need essentially to give a proof of the topological Tverberg theorem by means of ideal-valued index using a cohomology theory h^*

Once the ideal-valued index $Ind_G^{h^*}X$ is defined, it satisfies (with suitable conditions on h^* as being multiplicative and continuous) the axioms of monotonicity, additivity, continuity and the index theorem of an ideal-valued cohomological index theory. In the proof of the topological Tverberg theorem (using Fadell-Husseini index) we only used the axioms of monotonicity and additivity. We are going to show a list of properties that h^* and $Ind_G^{h^*}X$ satisfy that are sufficient for proving the topological Tverberg theorem. This is done by following Živaljević's proof.

P1. Monotonicity axiom of the index. The monotonicity property of the ideal-valued index $Ind_G^{h^*}X$ is the basic property we have in order to prove the there are no $(\mathbb{Z}/p)^k$-equivariant maps

$$X \to S(W_{\mathbb{C}}((d+1)/2, q)),$$

where X is an $E_N(\mathbb{Z}/p)^k$-space.

P2. Additivity axiom of the index. We use it to prove that for G-representations U and V,

$$Ind_G^{h^*}S(U) \cdot Ind_G^{h^*}S(V) \subseteq Ind_G^{h^*}S(U \oplus V).$$

P3. Index of $E_N(B\mathbb{Z}/p)^k$-spaces. If X is an $E_N(B\mathbb{Z}/p)^k$-space and $z \in Ind_G^{h^*}X$, we need that $deg(z) \neq N$.

P4. Künneth isomorphisms for h^*. With Künneth isomorphisms we can prove the analog to lemma 2.2.3:

$$Ind_{G\times H}^{h^*}X \times Y = Ind_G^{h^*}X \otimes h^*(BH) + h^*(BG) \otimes Ind_H^{h^*}Y,$$

and the analog to corollary 2.2.4:

$$Ind_{G\times H}^{h^*}X = Ind_G^{h^*}X \otimes h^*(BH).$$

P5. Index of $S(V_\alpha)$. We need to know the index of S^1 with the action of $\mathbb{Z}/p$ as p-th roots of 1. Then, with the use of the formula $Ind_{G\times H}^{h^*}X = Ind_G^{h^*}X \otimes h^*(BH)$ as above, the index of the unit sphere of the 1-dimensional complex $(\mathbb{Z}/p)^k$-representation associated to $(1, 0, \ldots, 0)$ can be computed. Then, with the appropriate automorphism of $(\mathbb{Z}/p)^k$ we can compute the index of the unit sphere $S(V_\alpha)$. Here, we hope that the index $Ind_{(\mathbb{Z}(p)^k}^{h^*}S(V_\alpha)$ contains an element u of degree 2.

The argument continues as follows: by combining P5 with P2 we get that $Ind_G^{h^*}S(U) \cdot Ind_G^{h^*}S(V) \subseteq Ind_G^{h^*}S(U \oplus V)$, so we can compute the index of the unit sphere $S(W_\mathbb{C}(1,q))$ of the standard complex representation, and the index of $S(W_\mathbb{C}(1,q)^{\oplus(d+1)/2})$. We hope to end with an element in $Ind_{(\mathbb{Z}/p)^k}^{h^*}S(W_\mathbb{C}(1,q)^{\oplus(d+1)/2})$ of degree $N = (q-1)(d+1)$. Once this is done, the assumption of the existence of a $(\mathbb{Z}/p)^k$-equivariant map $X \to S(W_\mathbb{C}((d+1)/2, q))$ yields that $Ind_G^{h^*}X$ contains an element of degree N, the contradiction that ends the proof of the topological Tverberg theorem by using ideal-valued cohomological index for h^*.

In section 2.3 we are going to verify that properties $P1 - P5$ are satisfied for Morava K-theories, so that we will have a proof of the topological Tverberg theorem using the ideal-valued index in Morava K-theories. We will have a proof only for odd-prime powers.

2.3 Morava K-theories and ideal-valued index

We present Morava K-theories in an axiomatic way by listing relevant properties that they satisfy (see [22, 23, 25, 45, 46]).

Given a prime number p, there exist a sequence of multiplicative cohomology theories called *Morava K-theories* and denoted $K(n)^*$, $n \geq 0$. The following properties are satisfied:

1. $K(0)^*(X) = H^*(X, \mathbb{Q})$.

2. **(Coefficients).** For $n \geq 1$, the coefficient ring of $K(n)^*$ is given by

$$K(n)^*(pt) = \mathbb{F}_p[v_n, v_n^{-1}], \tag{2.3.1}$$

   where the degree of v_n is $2(p^n - 1)$.

3. Every graded module over $K(n)^*(pt)$ is free (that is, $K(n)^*(pt)$ is a *graded field*). Also, $K(n)^*(X)$ is a module over $K(n)^*(pt)$.

4. **(Periodicity).** For $n \geq 1$, $K(n)$ is $2(p^n - 1)$-periodic. Multiplication by v_n induces isomorphisms

$$K(n)^i(X) \cong K(n)^{i+2(p^n-1)}(X), \tag{2.3.2}$$

   for any space X and $i \in \mathbb{Z}$.

5. $K(1)^*(X)$ is one of the $p - 1$ (isomorphic) summands of the mod. p complex K-theory of X.

6. **(Kunneth isomorphism).** For spaces X and Y,

$$K(n)^*(X \times Y) \cong K(n)^*(X) \otimes_{K(n)^*(pt)} K(n)^*(Y). \tag{2.3.3}$$

7. **(Complex oriented).** There exists an element $x \in K(n)^2(\mathbb{C}P^\infty)$ such that x restricts to a generator of $K(n)^2(\mathbb{C}P^1)$ under the inclusion $\mathbb{C}P^1 \subset \mathbb{C}P^\infty$. For the space $\mathbb{C}P^\infty$,

$$K(n)^*(\mathbb{C}P^\infty) = K(n)^*(pt)[[x]]. \tag{2.3.4}$$

8. For the classifying space $B\mathbb{Z}/p^i$ (p odd),

$$K(n)^*(B\mathbb{Z}_{p^i}) = K(n)^*(pt)[x]/(x^{p^{ni}}). \tag{2.3.5}$$

9. **(AHSS).** For a space X, the Atiyah-Hirzebruch spectral sequence of $K(n)^*(X)$, that is, the spectral sequence having E_2-page given by

$$E_2^{s,t} = H^s(X; K(n)^t(pt)), \tag{2.3.6}$$

converges to $K(n)^*(X)$ (see [47]).

Let $F_s^{s+t} = ker\{K(n)^{s+t}(X) \to K(n)^{s+t}(X^{s-1})\}$ (assuming X is a CW-complex with skeletons X^s). Then, for the E_∞-page we have $E_\infty^{s,t} \cong F_s^{s+t}/F_{s+1}^{s+t}$. The only nonzero rows of this spectral sequence occur when t is an integer multiple of $2(p^n-1)$. In particular, if $0 < s+t < 2(p^n-1)$, then $E_\infty^{0,s+t} = E_\infty^{1,s+t-1} = \cdots = E_\infty^{s+t-1,1} = 0$ and

$$K(n)^{s+t}(X) = F_0^{s+t} = F_1^{s+t} = \cdots = F_{s+t}^{s+t}. \tag{2.3.7}$$

The ideal-valued index for Morava K-theories

Recall that for a G-space X we have the corresponding map $q_X : X_G \to BG$, where X_G is the homotopy quotient $X_G = EG \times_G X$. We assume $n \geq 1$. There is an induced map $K(n)^*(BG) \to K(n)^*(X_G)$ whose kernel is defined to be the index of the G-space X and that we will denote by $\mathcal{I}nd^n_G X$:

$$\mathcal{I}nd^n_G X := ker\left(q_X^* : K(n)^*(BG) \to K(n)^*(X_G)\right). \tag{2.3.8}$$

The index $\mathcal{I}nd^n_G X$ satisfy the axioms of monotonicity, additivity, continuity and the index theorem for ideal-valued index theories. In particular properties P1, P2 and P4 are satisfied automatically by Morava K-theories and the corresponding ideal-valued index theories.

In order to reproduce a proof of the topological Tverberg theorem, we study the index of some particular type of G-spaces: $E_N(\mathbb{Z}/p)^k$-spaces and $(\mathbb{Z}/p)^k$-representation spheres, where p is an odd prime.

$E_N(\mathbb{Z}/p)^k$-spaces

We verify property P3. Assume that p is an odd prime number and $n \geq 1$. If X is an $E_N G$-space, then X can be completed to a space Y of type EG by adding only free G-cells of dimension at least $N+1$. With this model for EG, the quotient Y/G is a model for BG whose N-skeleton is precisely X/G. Since X is free, there is a homotopy equivalence $X_G \simeq X/G$ and the map $q_X^* : K(n)^*(BG) \to K(n)^*(X/G)$ can be interpreted as the induced by the inclusion $X/G \hookrightarrow Y/G$. Thus, $\mathcal{I}nd^n_G X$ is the kernel of the map $K(n)^*(BG) \to K(n)^*(X/G)$ induced by the inclusion of the N-skeleton X/G of Y/G.

Let L^N be the N-skeleton of the classifying space $B\mathbb{Z}/p$ (if N is odd, L^N is a *lens space*). Let $F_s^{s+t} = ker\{K(n)^{s+t}(B\mathbb{Z}/p) \to K(n)^{s+t}(L^{s-1})\}$. The Atiyah-Hirzebruch spectral sequence of $K(n)^*(B\mathbb{Z}/p)$ has E_2-page

$$E_2^{s,t} = H^s(B\mathbb{Z}/p; K(n)^t(pt)), \tag{2.3.9}$$

converges to $K^*(B\mathbb{Z}/p)$ and on the E_∞-page, $E_\infty^{s,t} \cong F_s^{s+t}/F_{s+1}^{s+t}$.

The mod. p cohomology of $B\mathbb{Z}/p$ is given by $H^*(B\mathbb{Z}/p; \mathbb{F}_p) = \mathbb{F}_p[a,b]/\langle a^2 \rangle$, where $deg(a) = 1$ and $deg(b) = 2$. The Atiyah-Hirzebruch spectral sequence of $K(n)^*(B\mathbb{Z}/p)$ has just one non-trivial differential, namely d_{2p^n-1}, acting on a and b by $d_{2p^n-1}(a) = b^{p^n}$ and $d_{2p^n-1}(b) = 0$ (see [23], [45]). We have

$$K(n)^*(B\mathbb{Z}_p) = K(n)^*(pt)[b]/\langle b^{p^n} \rangle. \tag{2.3.10}$$

For $0 < s+t < 2(p^n - 1)$, we have $E_\infty^{0,s+t} = E_\infty^{1,s+t-1} = \cdots = E_\infty^{s+t-1,1} = 0$; thus, $K(n)^{s+t}(B\mathbb{Z}/p) = F_0^{s+t} = F_1^{s+t} = \cdots = F_{s+t}^{s+t}$.

If $s+t$ is odd, then $K(n)^{s+t}(B\mathbb{Z}/p) = 0$. We also have $E_\infty^{s+t,0} = 0$ and we conclude that $F_{s+t+1}^{s+t} = F_{s+t}^{s+t} = 0$, that is, $ker\{K(n)^{s+t}(B\mathbb{Z}/p) \to K(n)^{s+t}(L^{s+t})\} = 0$.

If $s+t$ is even, then $F_{s+t}^{s+t} = K(n)^{s+t}(B\mathbb{Z}/p) = b^{(s+t)/2}\mathbb{F}_p$. We have $F_{s+t}^{s+t}/F_{s+t+1}^{s+t} \cong E_\infty^{s+t,0} = b^{(s+t)/2}\mathbb{F}_p$. Thus, $F_{s+t+1}^{s+t} = 0$, that is, $ker\{K(n)^{s+t}(B\mathbb{Z}/p) \to K(n)^{s+t}(L^{s+t})\} = 0$.

We have proved that if $0 < N < 2(p^n - 1)$, then

$$ker\{K(n)^N(B\mathbb{Z}/p) \to K(n)^N(L^N)\} = 0. \tag{2.3.11}$$

The Künneth isomorphism implies that

$$K(n)^*(B(\mathbb{Z}/p)^k) \cong K(n)^*(pt)[b_1, \ldots, b_k]/\langle b_1^{p^n}, \ldots, b_k^{p^n} \rangle, \tag{2.3.12}$$

where each b_i has degree 2. For the AHSS of $K(n)^*((B\mathbb{Z}/p)^k)$ we must have that $E_\infty^{s+t,0} = 0$ and $K(n)^{s+t}((B\mathbb{Z}/p)^k) = 0$ when $0 < s+t < 2(p^n-1)$ and $s+t$ is odd; and $E_\infty^{s+t,0} = K(n)^{s+t}((B\mathbb{Z}/p)^k) = \{\text{elements of degree } s+t\}$, when $0 < s+t < 2(p^n-1)$ and $s+t$ is even. If L^N denotes the N-skeleton of $(B\mathbb{Z}/p)^k$ we obtain, with a similar argument as above that if $0 < N < 2(p^n - 1)$, then

$$ker\{K(n)^N((B\mathbb{Z}/p)^k) \to K(n)^N(L^N)\} = 0. \tag{2.3.13}$$

This result translated in terms of the ideal-valued index of an $E_N(B\mathbb{Z})^k$-space gives us the following:

Proposition 2.3.1. *Let X be an $E_N(B\mathbb{Z}/p)^k$. If z is in $\mathcal{I}nd^n_{(B\mathbb{Z}/p)^k}X$, then $deg(z) \neq N$.*

Representation spheres

It only remains to treat property P5. The group $\mathbb{Z}/p$ seen as the group of p-th roots of unity, acts on $S^1 \subseteq \mathbb{C}$ by complex multiplication. This action is free and $S^1/\mathbb{Z}/p$ is identified with S^1. The mod p cohomology of S^1 is given by

$$H^i(S^1, \mathbb{F}_p) = \begin{cases} \mathbb{F}_p, & i = 0, 1, \\ 0, & \text{otherwise.} \end{cases} \tag{2.3.14}$$

The AHSS for $K(n)^*(S^1)$ collapses at the second page and we find that $K(n)^*(S^1)$ has no elements in degree 2. Since $K(n)^*(B\mathbb{Z}/p) = K(n)^*(pt)[b]/(b^{p^n})$ where the degree of b is 2, $K(n)^*(B\mathbb{Z}_p)$ is concentrated in even degrees. Therefore, the map $K(n)^*(B\mathbb{Z}/p) \to K(n)^*(S^1/\mathbb{Z}/p)$ maps x to zero.

Lemma 2.3.2. *$\mathcal{I}nd^n_{\mathbb{Z}/p}S^1 = \langle b\rangle/\langle b^{p^n}\rangle \subseteq K(n)^*(pt)[b]/\langle b^{p^n}\rangle$.*

Let $V = \mathbb{C}$ be the 1-dimensional complex representation of the abelian group $(\mathbb{Z}/p)^k$ associated to the vector $(\alpha_1, \ldots, \alpha_k) \in \mathbb{F}_p^k$.

First, for the action corresponding to the vector $(1, 0, \ldots, 0) \in \mathbb{F}_p^k$, we can identify the map $E(\mathbb{Z}/p)^k \times_{(\mathbb{Z}/p)^k} S(V) \to B(\mathbb{Z}/p)^k$ with

$$B(\mathbb{Z}/p)^{k-1} \times (E\mathbb{Z}/p \times_{\mathbb{Z}/p} S(V)) \to B(\mathbb{Z}/p)^{k-1} \times B\mathbb{Z}/p. \tag{2.3.15}$$

From this, the kernel of the map

$$K(n)^*(B(\mathbb{Z}/p)^k) \to K(n)^*(E(\mathbb{Z}/p)^k \times_{(\mathbb{Z}/p)^k} S(V)) \tag{2.3.16}$$

is the ideal in $K(n)^*(B(\mathbb{Z}/p)^k) \cong K(n)^*[b_1, \ldots, b_k]/\langle b_1^{p^n}, \ldots, b_k^{p^n}\rangle$ generated by b_1 (see lemma 2.2.3 and corollary 2.2.4).

Lemma 2.3.3. $\mathcal{I}nd^n_{(\mathbb{Z}/p)^k} S(V) = \langle b_1\rangle/\langle b_1^{p^n}, \ldots, b_k^{p^n}\rangle$.

For the action associated to the vector $(\alpha_1, \ldots, \alpha_k)$, assume without loss of generality that $\alpha_1 \neq 0$ and consider the automorphism $T : (\mathbb{Z}/p)^k \to (\mathbb{Z}/p)^k$ that is represented by the standard matrix

$$\begin{bmatrix} \alpha_1 & \alpha_2 & \alpha_3 & \cdots & \alpha_k \\ 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 1 \end{bmatrix} \tag{2.3.17}$$

regarding the domain of T with the action corresponding to $(\alpha_1, \ldots, \alpha_k)$ and its target with the action corresponding to $(1, 0, \ldots, 0)$.

We examine the induced map $T^* : K(n)^*(B(\mathbb{Z}/p)^k) \to K(n)^*(B(\mathbb{Z}/p)^k)$. If $\iota_i : \mathbb{Z}/p \to (\mathbb{Z}/p)^k$ is the inclusion of the i-th copy and $\pi_j : (\mathbb{Z}/p)^k \to \mathbb{Z}/p$ the projection on the j-th copy, then the map

$$\mathbb{Z}/p \xrightarrow{\iota_i} (\mathbb{Z}/p)^k \xrightarrow{T} (\mathbb{Z}/p)^k \xrightarrow{\pi_j} \mathbb{Z}/p \tag{2.3.18}$$

is just multiplication by T_{ji}, the ji-entry of the matrix representing T, and this induces a map $K(n)^*(B\mathbb{Z}/p) \to K(n)^*(B\mathbb{Z}/p)$.

We look at the E_2-page of the AHSS for $K(n)^*(B\mathbb{Z})$ and find that the induced map $E_2^{r,s} \to E_2^{r,s}$ is multiplication by T_{ji} for $r = 1, 2$, multiplication by T_{ji}^2 for $r = 3, 4$ and so on; all these maps commute with differentials in the spectral sequence. When we pass to the 3-page, the resulting induced map $E_3^{r,s} \to E_3^{r,s}$ is multiplication by T_{ji} for $r = 1, 2$, by T_{ji}^2 for $r = 3, 4$ and so on. Continuing this process, for the E_∞-page the induced map $E_\infty^{r,s} \to E_\infty^{r,s}$ is the identity for $r = 0$, is multiplication by T_{ji} for $r = 1, 2$, by T_{ji}^2 for $r = 3, 4$ and so on.

We find that the generator b_j of $K(n)^*(B\mathbb{Z}/p) \cong K(n)^*[b_j]/(b_j^{p^n})$ goes to $T_{ji}b_i \in K(n)^*(B\mathbb{Z}/p) \cong K(n)^*[b_i]/(b_i^{p^n})$. Hence, for the induced map $T^* : K(n)^*(B(\mathbb{Z}/p)^k) \to K(n)^*(B(\mathbb{Z}/p)^k)$, considering $K(n)^*(B(\mathbb{Z}/p)^k) \cong K(n)^*[b_1, \dots, b_k]/(b_1^{p^n}, \dots, b_k^{p^n})$, we have that

$$T^*(b_1) = \sum_i (\iota_i \circ T \circ \pi_1)^*(b_1) = \sum_i T_{1i}b_i = \sum_i \alpha_i b_i. \tag{2.3.19}$$

Lemma 2.3.4. *For the representation V of $\mathbb{Z}/p^k$ associated to $(\alpha_1, \dots, \alpha_k)$, the index of the sphere $S(V)$ is given by*

$$Ind^n_{(\mathbb{Z}/p)^k} S(V) = \langle \alpha_1 b_1 + \cdots + \alpha_k b_k \rangle / \langle b_1^{p^n}, \dots, b_k^{p^n} \rangle.$$

For a complex representation $V = \oplus_j V_j$ of $(\mathbb{Z}/p)^k$, where V_j is 1-dimensional corresponding to the vector $(\alpha_1^j, \dots, \alpha_k^j) \in \mathbb{F}_p^k$, we have that the element $\prod_j(\alpha_1^j b_1 + \cdots + \alpha_k^j b_k)$ modulo $\langle b_1^{p^n}, \dots, b_k^{p^n} \rangle$ belongs to $\mathcal{I}nd_{(\mathbb{Z}/p)^k} S(V)$.

Since the standard complex representation of $(\mathbb{Z}/p)^k$, $W_\mathbb{C}(1, p^k) = \{(z_1, \dots, z_{p^k}) : z_1 + \cdots z_{p^k} = 0\}$ decomposes as $W_\mathbb{C}(1, p^k) = \bigoplus_{\alpha \in \mathbb{F}_p^k \setminus \{0\}} V_\alpha$, the element $\prod_{\alpha \neq 0}(\alpha_1 b_1 + \cdots + \alpha_k b_k)$ modulo $\langle b_1^{p^n}, \dots, b_k^{p^n} \rangle$ belongs to $\mathcal{I}nd^n_{(\mathbb{Z}/p)^k} S(W_\mathbb{C}(1, p^k))$. For a sum $W_\mathbb{C}(1, p^k)^{\oplus(m)}$ we have that the element $\prod_{\alpha \neq 0}(\alpha_1 b_1 + \cdots + \alpha_k b_k)^m$ modulo $\langle b_1^{p^n}, \dots, b_k^{p^n} \rangle$ lives in $\mathcal{I}nd^n_{(\mathbb{Z}/p)^k} S(W_\mathbb{C}(1, p^k)^{\oplus(m)})$.

Topological Tverberg theorem for q and odd-prime power using Morava K-theory indexes

We have all we need on order to reproduce the proof of the topological Tverberg theorem in the case of odd-prime powers. Recall that for proving the topological Tverberg theorem for $q = p^k$ (p odd prime) and $d \geq 1$ it suffices to prove that for d odd, there is no $(\mathbb{Z}/p)^k$-equivariant map $X \to S(W_\mathbb{C}(1, q)^{\oplus(d+1)/2})$, where X in an $E_N(\mathbb{Z}/p)^k$-space, $N = (q-1)(d+1)$.

We assume that there exists such an equivariant map $X \to S(W_{\mathbb{C}}(1,q)^{\oplus(d+1)/2})$, then the monotonicity axiom of the ideal-valued index theory implies

$$\mathcal{I}nd^{n}_{(\mathbb{Z}/p)^k} S(W_{\mathbb{C}}(1,q)^{\oplus(d+1)/2}) \subseteq \mathcal{I}nd^{n}_{(\mathbb{Z}/p)^k} X. \tag{2.3.20}$$

First, the element $\prod_{\alpha\neq 0}(\alpha_1 b_1 + \cdots + \alpha_k b_k)^{(d+1)/2}$ modulo $\langle b_1^{p^n}, \ldots, b_k^{p^n}\rangle$ represents a class of degree $N = (p-1)(d+1)$ living in $\mathcal{I}nd^{n}_{(\mathbb{Z}/p)^k} S(W_{\mathbb{C}}(1,q)^{\oplus(d+1)/2})$, so by 2.3.20 we have that $\prod_{\alpha\neq 0}(\alpha_1 b_1 + \cdots + \alpha_k b_k)^{(d+1)/2}$ modulo $\langle b_1^{p^n}, \ldots, b_k^{p^n}\rangle$ belongs to $\mathcal{I}nd^{n}_{(\mathbb{Z}/p)^k} X$.

On the other hand, proposition 2.3.1 says that if a class z belongs to $\mathcal{I}nd^{n}_{(\mathbb{Z}/p)^k} X$, then its degree is not N. We have not reached a contradiction yet because the class $\prod_{\alpha\neq 0}(\alpha_1 b_1 + \cdots + \alpha_k b_k)^{(d+1)/2}$ modulo $\langle b_1^{p^n}, \ldots, b_k^{p^n}\rangle$ could be zero. This problem is avoided by choosing n big enough so that $(p^k - 1)(d+1)/2 < p^n$. Now we have a contradiction, which ends the proof.

CHAPTER 3

Regular m-gons

3.1 Regular m-gons, equilateral sets, and the topological Borsuk problem

In this section we introduce the concept of r-regular m-gons. Our ideas are based mainly in the paper of Y. Soibelman [55], in which the topological Borsuk number of $\mathbb{R}^2$ is calculated by means of topological methods and the paper of P. Blagojević and G. Ziegler [10] where Fadell-Husseini index is used in a problem related to the Borsuk problem for $\mathbb{R}^3$.

First of all, we review the topological Borsuk problem (see [55]). Let (X, ρ) be a metric space. For a compact subset K of X, $b_{(X,\rho)}(K)$ denotes the *Borsuk number* of K, that is, the minimal number of parts of K of smaller diameter necessary to partition K. The *Borsuk number* of (X, ρ) is defined to be $B(X, \rho) = \max_K b_{X,\rho}(K)$. If $\Omega(\rho)$ denotes the set of metrics on X equivalent to ρ, then the *topological Borsuk number* of (X, ρ) is defined by $B(X) = \min_{\tau \in \Omega(\rho)} B(X, \tau)$ The *topological Borsuk problem* consists of estimating $B(\mathbb{R}^n)$ for the Euclidean metric (see [55]). In particular, one wants to know if $B(\mathbb{R}^n) \geq n+1$. Soibelman proved in [55] that $B(\mathbb{R}^2) = 3$, but it is not known if $B(\mathbb{R}^3) \geq 4$.

A nonempty subset S of (X, ρ) is called *equilateral with respect to* ρ or ρ-*equilateral*, or just *equilateral*, if there is some positive real number c such that $\rho(x, y) = c$ for all $x, y \in S$, $x \neq y$. In order to prove that $B(X) \geq m$, it suffices to show that for any metric $\tau \in \Omega(\rho)$ there exists some τ-equilateral subset $S \subseteq X$ of size m (see [55]). The following result can be found in [43].

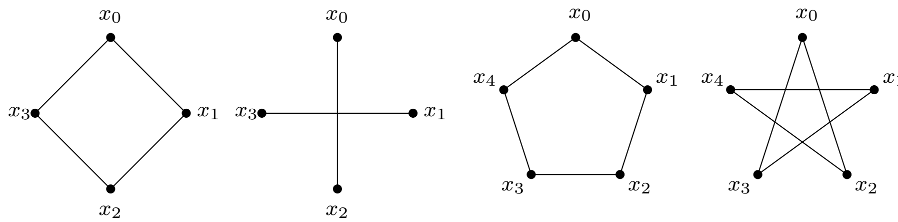
Theorem 3.1.1. ***(Petty)*** *Every equilateral set of size 3 in a finite dimensional normed space of dimension at least 3, can be extended to an equilateral set of size 4.*

Theorem 3.1.1 is another instance of a problem solved by topological methods (see [43]).

Theorem 3.1.1 implies that for any metric ρ on $\mathbb{R}^3$ induced by a norm we have $B(\mathbb{R}^2, \rho) \geq 4$. In fact, restrict ρ to a 2-plane X in $\mathbb{R}^3$ and apply the result of Soibelman that the Borsuk number of $\mathbb{R}^2$ is 3 to obtain a ρ-equilateral set in X of size 3. This equilateral set in X is also an equilateral set of size 3 in $(\mathbb{R}^3, \rho)$, so apply theorem 3.1.1 to extend it to a ρ-equilateral set of size 4. Thus, $B(\mathbb{R}^4, \rho) \geq 4$. This gives us a partial answer to the Borsuk problem for $\mathbb{R}^3$.

An *m-gon* in X is an m-cycle graph G whose vertices are m distinct elements in X. If G has vertices $x_0, x_1, \ldots, x_{m-1}$ and edges $\{x_i, x_{i+1}\}$ for $i = 0, 1, \ldots, m-1$ (where $x_m = x_0$), we say that G is a *regular m-gon* in (X, ρ), or a *regular m-gon with respect to* ρ, if there is some constant $c > 0$ such that $\rho(x_i, x_{i+1}) = c$ for all $i = 0, 1, \ldots, m-1$. Given r metrics on X, $\rho_1, \ldots, \rho_r$, we say that G is an *r-regular m-gon* (with respect to $\rho_1, \ldots, \rho_r$) if it is a regular m-gon for each metric ρ_i.

Let $\mathbb{Z}/m = \langle \omega | \ \omega^m = 1 \rangle$. Given an m-gon G with vertices $x_0, x_1, \ldots, x_{m-1}$ and edges $\{x_i, x_{i+1}\}$ for $i = 0, 1, \ldots, m-1$ (where $x_m = x_0$), $\mathbb{Z}/m$ acts naturally on the set of vertices of G by $\omega \cdot x_i = x_{i+1}$, and this induces an action of $\mathbb{Z}/m$ on the set of edges of G: $\omega \cdot \{x_i, x_{i+1}\} = \{\omega \cdot x_i, \omega \cdot x_{i+1}\}$.
If K_m denotes the complete graph on vertices $x_0, \ldots, x_{m-1}$, then the action of



(a) Orbits of $\mathbb{Z}/4$ on K_4 (b) Orbits of $\mathbb{Z}/5$ on K_5

FIGURE 3.1. Orbits of $\mathbb{Z}/m$ acting on K_m

$\mathbb{Z}/m$ on $x_0, x_1, \ldots, x_{m-1}$ induces an action on the edges of K_m. The set of edges of K_m decomposes as a union of disjoint orbits, some of these orbits represents m-gons, some other n-gons for some $n < m$ and some other orbits are not even n-gons for any n (see figure 3.1 (a)). But, if $m = p$ is an odd prime number, then each orbit of $\mathbb{Z}/p$ acting on the set of edges of K_p is actually a p-gon, and there are $(p-1)/2$ of them (see figure 3.1 (b)). If the subscripts of the x_i's are thought as the elements of the finite field $\mathbb{F}_p$, then these $(p-2)/2$ p-gons, that we will denote by $C_1, C_2, \ldots, C_{(p-1)/2}$, can be described as follows: C_t has edges $\{x_0, x_t\}, \{x_t, x_{2t}\}, \ldots, \{x_{(p-1)t}, x_{0t}\}$.

3.2 r-regular p-gons in $\mathbb{R}^d$

Now we make precise the problem we want to solve that involves r-regular p-gons. We restrict to p and odd prime because the orbits of the action of $\mathbb{Z}/p$ on the edges of the complete graph K_p (as above) gives us actual p-gons. We will also have free actions of $\mathbb{Z}/p$ on the spaces that appear in the CS/TM scheme. Although we obtain that there exist r-regular p-gons in $\mathbb{R}^d$ whenever $r < d$, our main result is more general and technical.

The problem

We want conditions on p (odd prime number), $r \geq 1$ and $d \geq 1$ such that given r metrics $\rho_1, \rho_2, \ldots, \rho_r$ on $\mathbb{R}^d$ (where each metric is equivalent to the Euclidean metric of $\mathbb{R}^d$), and a sequence $\{t_1, t_2, \ldots, t_r\}$, $t_i \in \{1, 2, \ldots, (p-1)/2\}$, there exist p distinct points $x_0, x_1 \ldots, x_{p-1}$ in $\mathbb{R}^d$ such that the p-gon C_{t_r} (as described above) is ρ_i-regular, $i = 1, 2, \ldots, r$.

The problem in the CS/TM scheme

The configuration space for our problem is the configuration space of p distinct points in $\mathbb{R}^d$, $F(\mathbb{R}^d, p) = \{(x_0, x_1, \ldots, x_{p-1}) \in (\mathbb{R}^d)^p : x_i \neq x_j, i \neq j\}$. Consider $\mathbb{Z}/p = \langle \omega | \ \omega^p = 1 \rangle$ acting on $(\mathbb{R}^d)^p$ by

$$\omega \cdot (x_0, x_1, \ldots, x_{p-1}) = (x_1, x_2, \ldots, x_{p-1}, x_0). \tag{3.2.1}$$

The group $\mathbb{Z}/p$ acts freely on $F(\mathbb{R}^d, p)$.

We define $V = \mathbb{R}^p \times \cdots \times \mathbb{R}^p$, the product of r copies of $\mathbb{R}^d$, and define our test map $f = (f_1, f_2, \ldots, f_r) : F(\mathbb{R}^d, p) \to V$ where

$$f_i(x_0, \ldots, x_{p-1}) = (\rho_i(x_0, x_{t_i}), \rho_i(x_{t_i}, x_{2t_i}), \ldots, \rho_i(x_{(p-1)t_i}, x_0)). \tag{3.2.2}$$

The group $\mathbb{Z}/p$ acts on $\mathbb{R}^p$ by $\omega \cdot (y_0, y_1, \ldots, y_{p-1}) = (y_1, y_2, \ldots, y_{p-1}, y_0)$ and $\mathbb{Z}/p$ acts on V diagonally. The map $f : F(\mathbb{R}^d, p) \to V$ becomes $\mathbb{Z}/p$-equivariant.

The existence of p distinct points $x_0, x_1 \ldots, x_{p-1}$ in $\mathbb{R}^d$ such that C_{t_i} is ρ_i-regular for any i is equivalent to the existence of an element $(x_0, \ldots, x_{p-1}) \in F(\mathbb{R}^d, p)$ such that for each $i = 1, \ldots, r$, $f_i(x_0, \ldots, x_{p-1}) \in \Delta$, where $\Delta = \{(t, t, \ldots, t) \in \mathbb{R}^d : t \in \mathbb{R}\}$. This is equivalent to say that $im\ f \cap \Delta^r \neq \varnothing$.

We assume that $im\ f \cap \Delta^r = \varnothing$; thus, we get a $\mathbb{Z}/p$-equivariant map $f : F(\mathbb{R}^d, p) \to V \setminus \Delta^r$. Note that the action of $\mathbb{Z}/p$ on $\mathbb{R}^p \setminus \Delta$ is free and so is the action of $\mathbb{Z}/p$ on $V \setminus \Delta^r$.

The results

Lemma 3.2.1. *Let p be an odd prime number, $d \geq 1$ and $r \geq 1$. If $d > r$, then there is no $\mathbb{Z}/p$-equivariant map $F(\mathbb{R}^d, p) \to V \setminus \Delta^r$.*

Proof. We have an $\mathbb{Z}/p$-equivariant map $g : V \setminus \Delta^r \to S((\Delta^r)^\perp)$ (projecting and normalizing). Assume that there is a $\mathbb{Z}/p$-equivariant map $h : F(\mathbb{R}^d, p) \to V \setminus \Delta^r$. The composition $g \circ h$ gives a $\mathbb{Z}/p$-equivariant map $F(\mathbb{R}^d, p) \to S((\Delta^r)^\perp)$. The actions of $\mathbb{Z}/p$ on $F(\mathbb{R}^d, p)$ and $S((\Delta^r)^\perp)$ are free.

Since Δ^r is an r-dimensional subspace of $V = (\mathbb{R}^p)^r$, we have that $(\Delta^r)^\perp$ is an $r(p-1)$-dimensional subspace of V. Thus, $S((\Delta^r)^\perp)$ is a sphere of dimension $r(p-1)-1$.

We calculate Fadell-Husseini indexes. By proposition 2.2.7, ii),

$$Ind_{\mathbb{Z}/p,\mathbb{F}_p} S((\Delta^r)^\perp) = \langle b^{r(p-1)/2} \rangle \subseteq \mathbb{F}_p[a,b]/\langle a^2 \rangle$$

and by proposition 2.2.9,

$$Ind_{\mathbb{Z}/p,\mathbb{F}_p} F(\mathbb{R}^d, p) = \langle ab^{(d-1)(p-1)/2}, b^{(d-1)(p-1)/2+1} \rangle \subseteq \mathbb{F}_p[a,b]/\langle a^2 \rangle.$$

The existence of a $\mathbb{Z}/p$-equivariant map $F(\mathbb{R}^d, p) \to S((\Delta^r)^\perp)$ yields the inclusion $Ind_{\mathbb{Z}/p,\mathbb{F}_p} S((\Delta^r)^\perp) \subseteq Ind_{\mathbb{Z}/p,\mathbb{F}_p} F(\mathbb{R}^d, p)$. From this we deduce that $r(p-1) > (d-1)(p-1)$, that is, $r > d-1$. This ends the proof. □

By lemma 3.2.1, the map $f : F(\mathbb{R}^d, p) \to V \setminus \Delta^r$ above must satisfy $im\, f \cap \Delta^r \neq \varnothing$. Hence, we have proved our main result:

Theorem 3.2.2. *Let p be an odd prime number, $\rho_1, \rho_2, \ldots, \rho_r$ be r metrics on $\mathbb{R}^d$ that are equivalent to the Euclidean metric of $\mathbb{R}^d$ and fix a sequence $\{t_1, t_2, \ldots, t_r\}$, where*

$t_i \in \{1, 2, \ldots, (p-1)/2\}$. If $d > r$, then there are p distinct points $x_0, x_1 \ldots, x_{p-1}$ in $\mathbb{R}^d$ such each p-gon C_{t_i} is ρ_i-regular, $i = 1, 2, \ldots, r$.

Remark 3.2.3. The continuity of the map $f = (f_1, f_2, \ldots, f_r)$ follows from the fact that a metric $\rho : \mathbb{R}^d \times \mathbb{R}^d \to \mathbb{R}$ that induces the usual topology of $\mathbb{R}^d$ is a symmetric and continuous function. Nothing else about the r metrics is used in the proof of theorem 3.2.2. So, in theorem 3.2.2 we can take r symmetric continuous maps $\rho_i : \mathbb{R}^d \times \mathbb{R}^d \to \mathbb{R}$; with this generality, the conclusion of theorem 3.2.2 is that for each $i \in \{1, 2, \ldots, r\}$ there is some $c_i \in \mathbb{R}$ (not necessarily positive) such that

$$\rho_i(x_0, x_{t_i}) = \rho_i(x_{t_i}, x_{2t_i}) = \ldots = \rho_i(x_{(p-1)t_i}, x_0) = c_i. \tag{3.2.3}$$

Corollary 3.2.4. *Let p be an odd prime number and $\rho_1, \rho_2, \ldots, \rho_{(p-1)/2}$ be $(p-1)/2$ metrics on $\mathbb{R}^d$ that are equivalent to the Euclidean metric of $\mathbb{R}^d$. If $d > (p-1)/2$, then there are p distinct points $x_0, x_1, \ldots, x_{p-1}$ in $\mathbb{R}^d$ such that for each $t \in \{1, 2, \ldots, (p-1)/2\}$, the p-gon C_t is ρ_t-regular.*

Proof. In theorem 3.2.2 put $r = (p-1)/2$ and take the sequence $\{1, 2, \ldots, (p-1)/2\}$. □

Corollary 3.2.5. *Let p be an odd prime number and ρ be a metric on $\mathbb{R}^d$ that is equivalent to the Euclidean metric of $\mathbb{R}^d$. If $d > (p-1)/2$, then there are p distinct points $x_0, x_1, \ldots, x_{p-1}$ in $\mathbb{R}^d$ such that for each $t = 1, 2, \ldots, (p-1)/2$, the p-gon C_t is regular with respect to ρ.*

Proof. In theorem 3.2.2 put $r = (p-1)/2$, $\rho_1 = \rho_2 = \cdots = \rho_{(p-1)/2}$ and take the sequence $\{1, 2, \ldots, (p-1)/2\}$. □

Corollary 3.2.6. *Let p be an odd prime number and $\rho_1, \rho_2, \ldots, \rho_r$ be r metrics on $\mathbb{R}^d$ that are equivalent to the Euclidean metric of $\mathbb{R}^d$. If $d > r$, then there exists an r-regular p-gon with respect to $\rho_1, \rho_2, \ldots, \rho_r$.*

Proof. In theorem 3.2.2 take the constant sequence $\{1, 1, \ldots, 1\}$. □

Corollary 3.2.7. *The Borsuk number of $\mathbb{R}^2$ is 3.*

Proof. In theorem 3.2.2 take $d = 2$, $r = 1$ and $p = 3$. Note that a regular 3-gon is the same as an equilateral set of size 3. □

Remark 3.2.8. Soibelman proves in [55] that the Borsuk number of $\mathbb{R}^2$ is 3 by the same method (the case $d = 2, r = 1, p = 3$). He remarks that for the topological Borsuk problem in $\mathbb{R}^n$, $n > 2$, there is an S_{n+1}-equivariant map $(\mathbb{R}^n)^{n+1} \to \mathbb{R}^{n(n+1)/2}$. As in the CS/TM scheme shown above, we get an equivariant map between spheres, but the actions of various subgroups of S_{n+1} are not free on the target, so Dold's theorem cannot be applied in the same way. Our results are based on the idea of having actual free actions of some groups (i. e. $\mathbb{Z}/p$) on the unit sphere of some convenient spaces on the target. We do not get equilateral sets, but r-regular p-gons.

Example 3.2.9. *Bi-equilateral triangles do not always exist in* $\mathbb{R}^2$. Theorem 3.2.2 (more accurately, corollary 3.2.6) ensures that there exist *bi-equilateral triangles* (that is, 2-regular triangles) for any pair of given metrics ρ_1 and ρ_2 in $\mathbb{R}^d$ whenever $d > 2$. We will exhibit two metrics in $\mathbb{R}^2$ for which there are no bi-equilateral triangles.

Consider the Euclidean norm $\|(x, y)\|_2 = \sqrt{x^2 + y^2}$ and the taxicab norm $\|(x, y)\|_1 = |x|+|y|$. By regarding $\mathbb{R}^2$ as the field of complex numbers $\mathbb{C}$, for the Euclidean norm, we can assume, without lost of generality, that all equilateral triangles have their vertices living in the unit circle $\{z \in \mathbb{C} : |z| = 1\}$. Let z_0, z_1, z_2 be the cubic roots of 1: $z_0 = 1, z_2 = \cos(2\pi/3) + i\sin(2\pi/3), z_2 = \cos(4\pi/3) + i\sin(4\pi/3)$. Now we can describe all Euclidean equilateral triangles with their vertices in the unit circle. In fact, for $0 \le \theta \ \le 2\pi$, the 3 complex numbers

$$\begin{cases} z_0(\theta) = \cos\theta + i\sin\theta, \\ z_1(\theta) = \cos(\theta + 2\pi/3) + i\sin(\theta + 2\pi/3), \\ z_2(\theta) = \cos(\theta + 4\pi/3) + i\sin(\theta + 4\pi/3) \end{cases}$$

are the vertices of an equilateral triangle. It suffices to consider $0 \le \theta \le 2\pi/3$ for obtaining all equilateral triangles with their vertices in the unit circle.

Now we calculate the length of the edges of such triangles using the taxicab norm:

$$\begin{cases} d_{01}(\theta) & = \|z_1(\theta) - z_0(\theta)\|_1 = |\cos(\theta + 2\pi/3) - \cos\theta| + |\sin(\theta + 2\pi/3) - \sin\theta|, \\ d_{02}(\theta) & = \|z_2(\theta) - z_0(\theta)\|_1 = |\cos(\theta + 4\pi/3) - \cos\theta| + |\sin(\theta + 4\pi/3) - \sin\theta|, \\ d_{12}(\theta) & = \|z_2(\theta) - z_1(\theta)\|_1 \\ & = |\cos(\theta + 4\pi/3) - \cos(\theta + 2\pi/3)| + |\sin(\theta + 4\pi/3) - \sin(\theta + 2\pi/3)|, \end{cases}$$

these expressions reduce to

$$\begin{cases} d_{01}(\theta) = \sqrt{3}|\sin(\theta + \pi/3)| + \sqrt{3}|\cos(\theta + \pi/3)|, \\ d_{02}(\theta) = \sqrt{3}|\sin(\theta + 2\pi/3)| + \sqrt{3}|\cos(\theta + 2\pi/3)|, \\ d_{12}(\theta) = \sqrt{3}|\sin(\theta + \pi)| + \sqrt{3}|\cos(\theta + \pi)|. \end{cases}$$

There is no θ between 0 and $2\pi/3$ such that $d_{01}(\theta) = d_{02}(\theta) = d_{12}(\theta)$. In fact, for $0 \leq \theta \leq 2\pi/3$, the equality $d_{01}(\theta) = d_{02}(\theta)$ holds only for $\theta = 0, \pi/4, \pi/2$. The common values $d_{01}(\theta) = d_{02}(\theta)$ are $\frac{\sqrt{3}}{2}(1 + \sqrt{3}), 3\sqrt{2}/2, \frac{\sqrt{3}}{2}(1 + \sqrt{3})$ respectively; but, the respective values of $d_{12}(\theta)$ are $\sqrt{3}, \sqrt{6}, \sqrt{3}$. This shows that there are no equilateral triangle with respect to the Euclidean metric that is also equilateral with respect to the taxicab metric in $\mathbb{R}^2$. Note that the three triangles found are indeed *isosceles* triangles with respect to the taxicab norm. In fact, a gereral result (see [36]) shows that given a circle with two distance functions on it, d_1 and d_2, where d_1 is the restricted distance coming from a smooth embedding of S^1 into a Riemannian manifold, and d_2 is symmetric and continuous, then there are three points in S^1 forming a d_1-equilateral triangle and, at the same time, a "d_2-isosceles" triangle.

3.3 Regular p-gons on deformed spheres

Theorem 3.2.2 shows the existence of regular p-gons in $(\mathbb{R}^d, \rho)$ for p prime if $p < d$; it does not apply to m-gons when m is not prime, in particular it does not apply to *regular quadrilaterals*. For proving the existence of regular quadrilaterals, Fadell-

Husseini index with integer coefficients is needed. In fact, we have the following result of Blagojević and Ziegler (see [10]):

Theorem 3.3.1. *For every injective continuous map $f : S^2 \to \mathbb{R}^3$, there are four distinct points y_0, y_1, y_2, y_3 in the image of f such that*

$$d(y_0, y_1) = d(y_1, y_2) = d(y_2, y_3) = d(y_3, y_0)$$

and

$$d(y_0, y_2) = d(y_1, y_3).$$

Here, d is the Euclidean metric of $\mathbb{R}^3$. The result is still true if we change d by any metric ρ equivalent to d.

From this theorem, the existence of regular quadrilaterals follows for any metric ρ in $\mathbb{R}^3$ (take f as the inclusion $S^2 \hookrightarrow \mathbb{R}^3$). The proof of this theorem relies on the use of Fadell-Husseini index with integer coefficients (It is shown in [10] that the use of mod. 2 coefficients is not enough). This theorem is related Borsuk problem for $\mathbb{R}^3$ and also to the *square peg problem* (see [10, 37]).

It is interesting to look for the existence of certain symmetries in the image of continuous maps $f : S^d \to \mathbb{R}^{d+1}$. That is the reason why we are going to analyze r-regular p-gons in the image of continuous injective maps $f : S^d \to \mathbb{R}^{d+1}$.

Admissible pairs (d, p)

Let p be an odd prime and $d \geq 1$. We call the pair (d, p) *admissible* if given any continuous injective map $f : S^d \to \mathbb{R}^{d+1}$ and $(p-1)/2$ metrics $\rho_1, \ldots, \rho_{(p-1)/2}$ on $\mathbb{R}^{d+1}$, each metric equivalent to the Euclidean metric of $\mathbb{R}^{d+1}$, there can be found p distinct points in the image of f, $f(x_0), \ldots, f(x_{p-1})$, such that

$$\rho_j(f(x_0), f(x_j)) = \rho_j(f(x_j), f(x_{2j})) = \cdots = \rho_j(f(x_{(p-1)j}), f(x_0)), \tag{3.3.1}$$

for all $j = 1, 2, \ldots, (p-1)/2$.

We want conditions on d and p such that the pair (d, p) is admissible. We will prove the following:

Theorem 3.3.2. *If $p \leq 2d+1$ and $(p-1)^2/2$ is not of the form $jd+1$ for any $j = 1, 2, \ldots, p-1$, then the pair (d, p) is admissible.*

A direct consequence of this theorem is:

Corollary 3.3.3. *If $p \leq 2d+1$ and d is even, then (d, p) is admissible.*

Example 3.3.4. Corollary 3.3.3 implies that $(2, 5)$ is an admissible pair. In particular, taking the Euclidean metric d of $\mathbb{R}^3$ an a continuous injective map $f : S^2 \to \mathbb{R}^3$, there are 5 distinct points y_0, y_1, y_2, y_3, y_4 in the image of f such that

$$d(y_0, y_1) = d(y_1, y_2) = d(y_2, y_3) = d(y_3, y_4) = d(y_4, y_0)$$

and

$$d(y_0, y_2) = d(y_2, y_4) = d(y_4, y_1) = d(y_1, y_3) = d(y_3, y_0)$$

(compare with theorem 3.3.1).

Example 3.3.5. The triplet $(3, 5)$ is admissible. In fact we have that $(p-1)^2/2 = 8$ and for $j = 1, 2, 3, 4$, $jd+1$ gives the values $4, 7, 10, 13$.
Thus, for any continuous and injective map $f : S^3 \to \mathbb{R}^4$, two metrics ρ_1 and ρ_2 that induce the usual topology of $\mathbb{R}^4$, there are 5 distinct points in the image of f, $y_0 = f(x_0), y_1 = f(x_1), y_2 = f(x_2), y_3 = f(x_3), y_4 = f(x_4)$, such that

$$\rho_1(y_0, y_1) = \rho_1(y_1, y_2) = \rho_1(y_2, y_3) = \rho_1(y_3, y_4) = \rho_1(y_4, y_0),$$

$$\rho_2(y_0, y_2) = \rho_2(y_2, y_4) = \rho_2(y_4, y_1) = \rho_2(y_1, y_3) = \rho_2(y_3, y_1).$$

Admissible pairs in the CS/TM scheme

Consider the map $F = (f_1, f_2, \dots, f_r) : (S^d)^p \to (\mathbb{R}^p)^{(p-1)/2}$ where $f_j(x_0, x_1, \dots, x_{p-1})$ equals

$$(\rho_j(f(x_0), f(x_1)), \rho_j(f(x_j), f(x_{2j})), \dots, \rho_j(f(x_{(p-1)j}), f(x_0)), \tag{3.3.2}$$

for $(x_0, x_1, \dots, x_{p-1}) \in (S^d)^p$, $j = 1, 2, \dots, (p-1)/2$. Let $L = \{(x, \dots, x) \in (S^d)^p : x \in S^d\}$ and $\Delta = \{(t, \dots, t) \in \mathbb{R}^p : t \in \mathbb{R}\}$. Let us restrict F to $X_{d,p} = (S^d)^p \setminus L$. We assume that $im\ F \cap \Delta^{(p-1)/2} = \varnothing$ so that we can consider F as a map $F : X_{d,p} \to Y_p := (\mathbb{R}^p)^{(p-1)/2} \setminus \Delta^{(p-1)/2}$. The group $\mathbb{Z}/p$ acts on these spaces by permuting coordinates and making the map F equivariant. Since projecting and normalizing are equivariant, we get an equivariant map $Y_p \to S(\Delta^{(p-1)/2})^{\perp} \cong S^{(p-1)^2/1-1}$.

Proposition 3.3.6. *The pair (d, p) is admissible if there is no $\mathbb{Z}/p$-equivariant map $X_{d,p} \to S(\Delta^{(p-1)/2})^{\perp}$.*

Proof. If there is no $\mathbb{Z}/p$-equivariant map $X_{d,p} \to S(\Delta^{(p-1)/2})^{\perp}$, then for the map $F : X_{d,p} = (S^d)^p \setminus L \to (\mathbb{R}^p)^{(p-1)/2}$ we have $im\ F \cap \Delta^{(p-1)/2} \neq \varnothing$, that is, there exists $(x_0, x_1, \dots, x_{p-1}) \in X_{d,p}$ such that for every $j = 1, 2, \dots, (p-1)/2$,

$$\rho_j(f(x_0), f(x_j)) = \rho_j(f(x_j), f(x_{2j})) = \cdots = \rho_j(f(x_{(p-1)j}), f(x_0)).$$

It only remains to show that $x_0, x_1, \dots, x_{p-1}$ are distinct. In fact, if $x_r = x_s$ for some r, s, then there are $j \in \{1, 2, \dots, (p-1)/2\}$ and $t \in \mathbb{F}_p$ such that $r = tj$ and $s = (t+1)j$. Then, $\rho_j(f(x_{tj}), f(x_{(t+1)j})) = 0$ and

$$\rho_j(f(x_0), f(x_j)) = \rho_j(f(x_j), f(x_{2j})) = \ldots = \rho_j(f(x_{(p-1)j}), f(x_0)) = 0,$$

which implies that $f(x_0) = f(x_1) = \cdots = f(x_{p-1})$, and since f is injective, $x_0 = x_1 = \cdots = x_{p-1}$, which contradicts that $(x_0, x_1, \dots, x_{p-1}) \in X_{d,p}$. Thus, the points $x_0, x_1, \dots, x_{p-1} \in S^d$ are distinct. □

Theorem 3.3.2 follows from proposition 3.3.6 once we have proved that there is no $\mathbb{Z}/p$-equivariant map $X_{d,p} \to S(\Delta^{(p-1)/2})^{\perp}$. Thus, we are interested in finding conditions on d and p such that there is no equivariant map $X_{d,p} \to S(\Delta^{(p-1)/2})^{\perp}$. For doing this, we are going to compare the Fadell-Husseini index with coefficients in $\mathbb{F}_p$ of the $\mathbb{Z}/p$-spaces $X_{d,p}$ and $S(\Delta^{(p-1)/2})^{\perp}$.

We are going to consider the following problem, which is more general: to find conditions on a triplet (d, r, p) such that there is no $\mathbb{Z}/p$-equivariant map $X_{d,p} \to S(\Delta^r)^{\perp}$.

The index of the sphere $S(\Delta^r)^{\perp}$

The unit sphere $S(\Delta^r)^{\perp}$ is a sphere of dimension $r(p-1)-1$ with a free action of $\mathbb{Z}/p$, in fact it is an $E_{2n-1}\mathbb{Z}/p$-space with $n = r(p-1)/2)$. Then we have

$$Ind_{\mathbb{Z}/p,\mathbb{F}_p} S(\Delta^r)^{\perp} = \langle b^{r(p-1)/2} \rangle \subseteq \mathbb{F}_p[a,b]/\langle a^2 \rangle = H^*(B\mathbb{Z}/p;\mathbb{F}_p). \tag{3.3.3}$$

We are not going to compute explicitly the index of $X_{d,p}$, but rather we are going to give conditions on d, r, p such that the element $b^{r(p-1)/2} \in H^*(B\mathbb{Z}/p;\mathbb{F}_p)$ does not belong to the index of $X_{d,p}$. We do this by using the Serre spectral sequence of the fibration $X_{d,p} \to (X_{d,p})_{\mathbb{Z}/p} \to B\mathbb{Z}/p$ to show that the element $b^{r(p-1)/2} \in H^*(B\mathbb{Z}/p;\mathbb{F}_p) = E_2^{*,0}$ survives forever. This means that $b^{r(p-1)/2}$ is not in $Ind_{\mathbb{Z}/p,\mathbb{F}_p} X_{d,p}$.

The E_2-page of the Serre spectral sequence of the fibration $X_{d,p} \to (X_{d,p})_{\mathbb{Z}/p} \to B\mathbb{Z}/p$, $E_2^{s,t} = H^s(B\mathbb{Z}/p; H^t(X_{d,p};\mathbb{F}_p))$ has to be understood as the cohomology of the group $\mathbb{Z}/p$ with coefficients in the $\mathbb{Z}/p$-module $H^*(X_{d,p};\mathbb{F}_p)$.

The cohomology $H^*(X_{d,p};\mathbb{Z}/p)$ as a $\mathbb{Z}/p$-module

We will use Lefschetz duality (see [40], theorem 70.2, page 415):

Theorem 3.3.7. *If (X, A) is a compact triangulated relative homology n-manifold which is orientable, then there are isomorphisms*

$$H^k(X \setminus A; G) \cong H_{n-k}(X, A; G),$$

for all G.

We apply theorem 3.3.7 to $((S^d)^p, L)$ and $G = \mathbb{Z}/p$ to have

$$H^*(X_{d,p}; \mathbb{Z}/p) = H^*((S^d)^p \setminus L; \mathbb{Z}/p) = H_{pd-*}((S^d)^p, L; \mathbb{Z}/p). \tag{3.3.4}$$

To compute the homology $H_*((S^d)^p, L; \mathbb{Z}/p)$ we are going to use the long exact sequence of the pair $((S^d)^p, L)$. Since $L \cong S^d$, $H_i(L; \mathbb{Z}/p) = \mathbb{Z}/p$ if $i = 0, d$ and $H_i(L; \mathbb{Z}/p) = 0$ otherwise.

An induction argument using the the Künneth isomorphism yields that $H_i((S^d)^p; \mathbb{Z}/p) \neq 0$ only for $i = 0, d, 2d, \dots, pd$, and that $H_{jd}((S^d)^p; \mathbb{Z}/p)$ is the direct sum of all terms of the form $A_1 \otimes A_2 \otimes \cdots \otimes A_p$, where j of the A_l's are equal to $H_d(S^d; \mathbb{Z}/p)$ and the remaining A_l's are equal to $H_0(S^d; \mathbb{Z}/p)$. If $N := (\mathbb{Z}/p)^{\oplus(p)}$ with the action of $\mathbb{Z}/p$ given by permuting the p copies of $\mathbb{Z}/p$ in the obvious way, then we have

$$H_i((S^d)^p; \mathbb{Z}/p) \cong \begin{cases} \mathbb{Z}/p, & i = 0, pd, \\ N^{\oplus \frac{1}{p}\binom{p}{i/d}}, & i = d, 2d, \dots, (p-1)d, \\ 0, & \text{otherwise.} \end{cases} \tag{3.3.5}$$

The long exact sequence in homology of the pair $((S^d)^p, L)$ gives isomorphisms

$$H_{jd}((S^d)^p, L; \mathbb{Z}/p) \cong H_{jd}((S^d)^p; \mathbb{Z}/p) \tag{3.3.6}$$

for $j = 2, 3, \dots, p$, and an exact sequence

$$0 \to H_{d+1}((S^d)^p, L) \to H_d(L) \xrightarrow{\iota_*} H_d((S^d)^p) \to H_d((S^d)^p, L) \to 0, \tag{3.3.7}$$

where ι_* is the map induced by the inclusion $L \hookrightarrow (S^d)^p$. We have that $H_i((S^d)^p, L; \mathbb{Z}/p) = 0$ in any other case. If x is the generator of $H_d(L; \mathbb{Z}/p)$ and x_i the generator of $H_d((S^d)^p; \mathbb{Z}/p)$ carried by the i-th copy of S^d in $(S^d)^p$, then we have that $\iota_*(x) = x_1 + \cdots + x_p$. Thus, ι_* is injective, $H_{d+1}((S^d)^p, L; \mathbb{Z}/p) = 0$ and

$$\begin{aligned} H_d((S^d)^p, L) &\cong H_d((S^d)^p)/im\ \iota_* \\ &= \langle x_1, \dots, x_p \rangle / \langle x_1 + \cdots + x_p \rangle \\ &=: M. \end{aligned}$$

Thus, we have that

$$H_i((S^d)^p, L; \mathbb{Z}) = \begin{cases} M, & i = d, \\ N^{\oplus \frac{1}{p}\binom{p}{i/d}}, & i = 2d, \dots, (p-1)d, \\ \mathbb{Z}/p, & i = pd, \\ 0, & \text{otherwise.} \end{cases} \tag{3.3.8}$$

and by theorem 3.3.7:

$$H^i(X_{d,p}; \mathbb{Z}) = \begin{cases} \mathbb{Z}/p, & i = 0, \\ N^{\oplus \frac{1}{p}\binom{p}{i/d}}, & i = d, \dots, (p-2)d, \\ M, & t = (p-1)d, \\ 0, & \text{otherwise.} \end{cases} \tag{3.3.9}$$

(note that $\binom{p}{i/d} = \binom{p}{p-i/d}$).

The Serre spectral sequence of the fibration $X_{d,p} \to (X_{d,p})_{\mathbb{Z}/p} \to B\mathbb{Z}/p$

The E_2-page of the Serre spectral sequence of the fibration $X_{d,p} \to (X_{d,p})_{\mathbb{Z}/p} \to B\mathbb{Z}/p$ is given by:

$$E_2^{s,t} = H^s(\mathbb{Z}/p; H^t(X_{d,p}; \mathbb{Z}/p)). \tag{3.3.10}$$

We have that $H^t(X_{d,p}; \mathbb{F}_p) \neq 0$ only for $t = 0, d, 2d, \ldots, (p-1)d$. For $t = 0$, $H^0(X_{d,p}; \mathbb{Z}/p) = \mathbb{Z}/p$ so that $E_2^{*,0} = H^*(\mathbb{Z}/p; \mathbb{Z}/p) = \mathbb{Z}/p[a,b]/\langle a^2 \rangle$. It can be proved that

$$H^s(\mathbb{Z}/p; N) = \begin{cases} \mathbb{Z}/p, & s = 0, \\ 0, & s > 0. \end{cases} \tag{3.3.11}$$

For $t = d, 2d, \ldots, (p-2)d$ we have

$$E_2^{s,t} = H^s(\mathbb{Z}/p; N^{\oplus \frac{1}{p}\binom{p}{t/p}}) = \begin{cases} (\mathbb{Z}/p)^{\frac{1}{p}\binom{p}{t/p}}, & s = 0, \\ 0, & s > 0. \end{cases} \tag{3.3.12}$$

We also have $E_2^{s,(p-1)d} = H^s(\mathbb{Z}/p; M)$. Otherwise, $E_2^{s,t} = 0$. The E_2-term of the spectral sequence is shown in figure 3.2.

The element $b^{r(p-1)/2}$ lives in $E_2^{r(p-1),0} = H^{r(p-1)}(\mathbb{Z}/p; \mathbb{Z}/p)$. With our description of $E_2^{*,*}$ we see that there are only two differentials that might hit elements in $E_2^{r(p-1),0}$, that are $d_{r(p-1)} : E_2^{0,r(p-1)-1} \to E_2^{r(p-1),0}$ and $d_{(p-1)d+1} : E_2^{(r-d)(p-1)-1,d(p-1)} \to E_2^{r(p-1),0}$.

Proposition 3.3.8. *If the two differentials $d_{r(p-1)} : E_2^{0,r(p-1)-1} \to E_2^{r(p-1),0}$ and $d_{(p-1)d+1} : E_2^{(r-d)(p-1)-1,d(p-1)} \to E_2^{r(p-1),0}$ are zero, then $b^{r(p-1)/2}$ does not belong to $Ind_{\mathbb{Z}/p,\mathbb{F}_p} X_{d,p}$, that is, $Ind_{\mathbb{Z}/p,\mathbb{F}_p} S(\Delta^r)^{\perp} \not\subseteq Ind_{\mathbb{Z}/p,\mathbb{F}_p} X_{d,p}$. In particular, there is no $\mathbb{Z}/p$-equivariant map $X_{d,p} \to S(\Delta^r)^{\perp}$.*

If $r \leq d$, then $E_2^{(2r-d)(p-1)-1,d(p-1)} = 0$. On the other hand, $E_2^{0,r(p-1)-1} \neq 0$ implies that $r(p-1) - 1 = jd$ for some $j = 1, 2, \ldots, p-1$. The result is that if $r \leq d$ and $r(p-1)$ is not of the form $jd + 1$ for any $j = 1, 2, \ldots, p-1$, then

$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\cdots$	$\vdots$	$\cdots$
$(p-1)d$	$E_2^{0,(p-1)d}$	$E_2^{1,(p-1)d}$	$E_2^{2,(p-1)d}$	$E_2^{3,(p-1)d}$	$E_2^{4,(p-1)d}$	$E_2^{5,(p-1)d}$	$\cdots$	$E_2^{\frac{r(p-1)}{2},(p-1)d}$	$\cdots$
$(p-2)d$	$N^{\frac{p-1}{2}}$	0	0	0	0	0	$\cdots$	0	$\cdots$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\cdots$	$\vdots$	$\cdots$
$2d$	$N^{\frac{p-1}{2}}$	0	0	0	0	0	$\cdots$	0	$\cdots$
d	N	0	0	0	0	0	$\cdots$	0	$\cdots$
0	$\langle 1\rangle$	$\langle a\rangle$	$\langle b\rangle$	$\langle ab\rangle$	$\langle b^2\rangle$	$\langle ab^2\rangle$	$\cdots$	$\langle b^{\frac{r(p-1)}{2}}\rangle$	$\cdots$
	0	1	2	3	4	5	$\cdots$	$\frac{r(p-1)}{2}$	$\cdots$

FIGURE 3.2. E_2-page of the Serre spectral sequence

the two differentials above are zero. This implies that $b^{r(p-1)/2}$ is not in the index $Ind_{\mathbb{Z}/p,\mathbb{Z}/p}X_{d,p}$. Thus we have:

Proposition 3.3.9. *If $r \leq d$ and $r(p-1)$ is not of the form $jd+1$ for any $j = 1, 2, \ldots, p-1$, then there is no $\mathbb{Z}/p$-equivariant map $X_{d,p} \to S(\Delta^r)^\perp$.*

Theorem 3.3.2 follows from proposition 3.3.6 and proposition 3.3.9 by taking $r = (p-1)/2$.

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