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MASTER'S THESIS

Vietoris-Rips Complexes of the Circle and the Torus

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Abstract

Faculty of Science

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In this thesis we study the details of the computations by Adams and Adamaszek leading to the determination of the homotopy type of the Vietoris-Rips complexes of the circle. The result shows that the Vietoris-Rips complexes are (up to homotopy) either contractible, odd dimensional spheres or wedge sums of spheres of the same even dimension. Different kinds of products on graphs and simplicial complexes are also studied; this is done by introducing elements from discrete Morse theory for simplicial complexes. A relation between the product of clique complexes and the clique complex of a product is shown. The main contribution of the thesis is the determination (up to homotopy) of the Vietoris-Rips complexes of tori with the maximum metric using all the aforementioned results.

Acknowledgements

I would like to thank my advisor Andrés Ángel for his support, guidance and patience during this thesis project.

I also want to express my deepest gratitude to all my loved ones, especially for always being there whenever I needed it.

List of Symbols

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Introduction

The main object of study of this thesis is the Vietoris-Rips complex, denoted by $\mathbf{VR}_{<}(X;r)$. Given a metric space X and a fixed real number $r > 0$, the simplicial complex $\mathbf{VR}_{<}(X;r)$ has vertex set X and a finite subset of X of size $k+1$ determines a face of dimension k if the diameter of the set is strictly smaller than r. The set X can be either finite or infinite. When X is finite, the number of simplices of $\mathbf{VR}_{<}(X;r)$ grows exponentially as X increases in size.

The first thing we want to know about these objects is whether or not they keep some (topological) information about the space X we started with. For instance, we may suspect that taking sufficiently many points of our possibly infinite space X and r small enough, the simplicial complex associated to this input will preserve, at the very least, the homology of the space. Something a little bit stronger holds. The first result in this direction that we are aware of goes back to [Hausmann, 1995]. In that paper, Hausmann proves, among other things, that if the metric space is a compact Riemannian manifold and the distance is taken with respect to the infimum of the lenghts of curves connecting points, then, for r sufficiently small[1], there is a homotopy equivalence between the geometric realization of $\mathbf{VR}_{<}(X;r)$ and X. In particular, this says that a combinatorial object, $\mathbf{VR}_{<}(X;r)$, preserves all the information about X up to homotopy.

Hausmann also modifies the concept of deformation retraction (defining *crushing*) from X to $A \subset X$; he adds a condition on the homotopy taking the distance into account (intuitively, not allowing to increase distances between points as the space retracts). He says that a space is *crushable* if there is a crushing from the space to a point. A few things to have in mind is that every normed space is crushable, every crushable space is contractible and that not every contractible space is crushable. He showed that whenever there is a crushing from X to a subspace A of itself, the map $\mathbf{VR}(A;r) \to \mathbf{VR}(X;r)$ induced by the inclusion is a homotopy equivalence. Hausmann uses this to define a cohomology theory on compact metric spaces and leaves open questions about the kinds of spaces that arise as Vietoris-Rips complexes of manifolds. Three of these questions stand out. The first asks about the behaviour of our simplicial complex a r gets larger, since we already know what happens in the opposite case. The second question is also natural and asks the following: Once we know we can recover the space taking all the points of X, is it possible to do this taking only *finitely* many points? In other words, is it possible to find a finite subspace of X and an appropriate r

[1] This works, for example, when r is taken less than the convexity radius of the space.

so that the Vietoris-Rips complex associated to this subspace is homotopy equivalent to X? Lastly, it conjectures that the connectivity of the Vietoris-Rips complexes is a non-decreasing function of the parameter r until it is greater than the diameter of the space, in which case the complex is contractible.

Just a few years after the paper of Hausmann, a positive answer is given to the second question in general. In [Latschev, 2001] it is proved the stronger fact that for a closed Riemannian manifold X, there is a positive number ε' such that for any $0 < \varepsilon < \varepsilon'$ there is a $\delta > 0$ such that the geometric realization of any $\mathbf{VR}(Y;\varepsilon)$, where Y is a metric space with Gromov-Hausdorff distance to X less than δ, is homotopy equivalent to X. So, for instance, a finite ε-dense subset of Y can be taken as "representative" of X.

Despite this, the question about the spaces that arise as Vietoris-Rips complexes still remains, and little is known about it. Nevertheless, recently, it has been possible to determine all the Vietoris-Rips complexes (for any r, small and large) for the circle thought of as $\mathbf{R}/\mathbf{Z}$. In [Adamaszek and Adams, 2015], it is shown that the complexes behave in the expected way for r small, resembling the circle. It is also shown that the homotopy type of each complex lies in one of the following categories: a point, odd dimensional spheres, and wedge sums of spheres of the same even dimension. (This, in particular, gives positive answers to Hausmann's questions for the circle.)

The main contribution of this thesis is the computation of the homotopy type of the Vietoris-Rips complexes of tori with the maximum metric. We use a result of [Larrión, Pizaña, and Villarroel-Flores, 2013] to link points in tori with this metric and points in products of Vietoris-Rips complexes. Henry Adams in his Research Statement mentions that this extension of the result has already been obtained, but, as far as we know, it has not been published anywhere, hence we provide a (possibly different) proof here.

Here is a brief outline of this document.

In Chapter 1, we study the details of the proof that determines the homotopy type of the Vietoris-Rips complexes for the circle. First, the case when X is a finite subset of S^1 is considered. It is shown that it can even be reduced to the study of "evenly spaced" finite subsets. The homotopy type of these spaces is determined and it can be already foreseen the sort of spaces we expect for arbitrary subsets of the circle. Later, it is noted that the 1-skeleton of these simplicial complexes keep all the information of the complex up to homotopy. A numeric invariant is associated to these graphs allowing to control the homotopy type of the Vietoris-Rips complexes. Finally, it is found that the behaviour of the complexes changes according to the values of r taken; two different paths are necessary and are carried at the end of the chapter.

In Chapter 2, we study two different notions of products for simplicial complexes and for graphs from the topological viewpoint. A few elements of discrete Morse theory for simplicial complexes are needed and are developed within the chapter. The main result here will connect the topology of the clique complex of a special kind of product on graphs and the product of the clique complexes.

Finally, in Chapter 3, we determine up to homotopy the Vietoris-Rips complexes of tori with the maximum metric. The result we get is that these spaces are products of the Vietoris-Rips complexes of the factors. We do this by using our knowledge of the homotopy type of the Vietoris-Rips complexes studied in the first chapter and an observation about the products on graphs defined in the second chapter.

Chapter 1

Vietoris-Rips Complex of a Circle

In this chapter we study the details of the computation of the Vietoris-Rips complexes of the circle S^1 (thought of as $\mathbf{R}/\mathbf{Z}$) given in [Adamaszek and Adams, 2015; Adamaszek et al., 2016].

Definition 1. Let (X, d) be a metric space and $r > 0$. The simplicial complex $\mathbf{VR}_{<}(X;r)$ has vertex set X and the k-simplices are subsets $\{x_0, \dots, x_k\}$ of X of size $k+1$ such that $d(x_i, x_j) < r$ for all i, j. The simplicial complex $\mathbf{VR}_{\leqslant}(X;r)$ is defined analogously.

Theorem 2 (Adamaszek, Adams)**.** *If $X \subset S^1$ is dense (in particular when $X = S^1$) and $0 < r < \frac{1}{2}$, then*

$$\mathbf{VR}_{<}(X;r) \simeq S^{2l+1} \quad \textit{for} \quad \frac{l}{2l+1} < r \leqslant \frac{l+1}{2l+3},\ l = 0, 1, \dots.$$

Moreover, if $\frac{l}{2l+1} < r \leqslant r' \leqslant \frac{l+1}{2l+3}$, then the inclusion $\mathbf{VR}_{<}(X;r) \hookrightarrow \mathbf{VR}_{<}(X;r')$ is a homotopy equivalence.

Theorem 3 (Adamaszek, Adams)**.** *For $0 \leqslant r < \frac{1}{2}$, there is a homotopy equivalence*

$$\mathbf{VR}_{\leqslant}\left(S^1;r\right) \simeq \begin{cases} \bigvee^{\mathfrak{c}} S^{2l} & \textit{if} \quad r = \frac{l}{l+1} \\ S^{2l+1} & \textit{if} \quad \frac{l}{2l+1} < r < \frac{l+1}{2l+3},\ l = 0, 1, 2, \dots, \end{cases}$$

where $\mathfrak{c}$ is the cardinality of the continuum. Moreover, if $\frac{l}{2l+1} < r \leqslant r' \leqslant \frac{l+1}{2l+3}$, then the inclusion $\mathbf{VR}_{\leqslant}(X;r) \hookrightarrow \mathbf{VR}_{\leqslant}(X;r')$ is a homotopy equivalence.

In order to prove Theorem 2 and Theorem 3, the path we take starts by determining the homotopy types of the Čech complex of arcs in S^1 as in [Adamaszek et al., 2016].

Definition 4. For $n \in \mathbf{Z}_{\geqslant 1},\ k \in \mathbf{Z}_{\geqslant 0}$, let $\mathcal{N}(n,k)$ be the simplicial complexes with set of vertices $\{0, 1, \dots, n-1\}$ and maximal faces $\{i, i+1, \dots, i+k\}$ (where everything is taken modulo n) for $i = 0, 1, \dots, n-1$.

This complex $\mathcal{N}(n,k)$ is not arbitrary: it can be seen as Čech complex associated to the cover

$$\mathcal{U}_{n,k} = \left\{ \left[\frac{i}{n}, \frac{i+k}{n}\right] \subset S^1 : 0 \leqslant i \leqslant n-1 \right\} \tag{1.1}$$

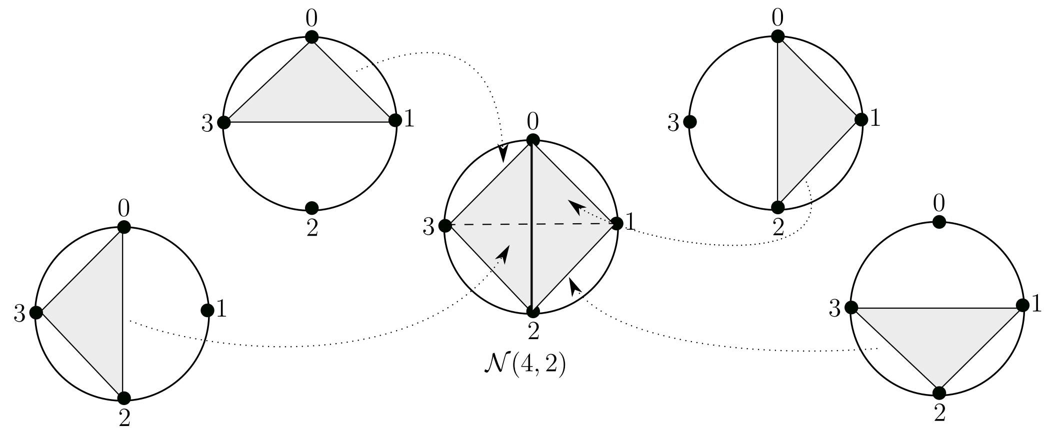

FIGURE 1.1: Visual representation $\mathcal{N}(4,2) \cong S^2$.

of S^1. This means that we are going to compute the homotopy type of the nerve complex of these evenly spaced arcs over S^1.

We note that $\mathcal{N}(n,0)$ is simply a set of n points with the discrete topology, and it will be convenient to express this as $\bigvee^{n-1} S^0$. Also, note that $\mathcal{N}(n,k) \simeq S^1$ if $1 \leqslant k < \frac{n}{2}$ as a consequence of the Nerve Lemma [Kozlov, 2008, Thm 15.21]. Recall that the Nerve Lemma says that whenever the nonempty intersections of the cover considered are contractible (and this is the case here by the restriction on k), then the complex is homotopy equivalent to the union of the elements of the cover, namely, S^1.

Another easy case where we can directly determine $\mathcal{N}(n,k)$ is when $k = n-2$. $\mathcal{N}(n, n-2)$ gives a simplicial complex with n vertices and such that every subset of $n-1$ vertices is a maximal face, so it is the boundary of the $(n-1)$-simplex, that is, the $(n-2)$-sphere. Finally, note that $\mathcal{N}(n,k)$ for $k \geqslant n-1$ is the whole $(n-1)$-simplex, which is a contractible space.

1.1 Reduction to Evenly Spaced Arcs

This section is not crucial for the determination of the Vietoris-Rips complexes for the circle, but it shows that the study of nerve complexes of arbitrary finite collections of arcs of S^1 can be reduced to the study of collections of evenly spaced arcs without losing any information about the homotopy types of the complexes. Hence, as a byproduct of the results of the following sections, the nerve complex of a finite collection of arcs in S^1 is also determined up to homotopy.

We start with the following observation.

Lemma 5. *Let $\mathcal{U}$ be an arbitrary finite collection of S^1 formed by arcs that are either open, closed or semi-open. Then there exists another finite collection $\mathcal{U}'$ formed solely by closed arcs $[a_i, b_i] \subset S^1$ such that no two of the a_i's or the b_i's are equal, and $\mathcal{N}(\mathcal{U}) \cong \mathcal{N}(\mathcal{U}')$.*

Proof. Let $(a,b) \in \mathcal{U}$. If there are no intervals of the form $(c,a]$, $[c,a]$ in $\mathcal{U}$, changing (a,b) to $[a,b)$ will have no consequence on $\mathcal{N}(\mathcal{U})$; otherwise, let $\varepsilon > 0$ small so that $a+\varepsilon$ is strictly less than the next endpoint of an arc going clockwise from a (this can be done since we work with finite coverings), and change (a,b) to $[a+\varepsilon,b)$. By symmetry, we can do the same for right endpoints of arcs in $\mathcal{U}$.

In order to avoid repetitions of the endpoints, we can do the following. If the left endpoints of m arcs are equal, say a, take ε small so that $a-\varepsilon$ is greater than the next endpoint of an arc going counterclockwise from a, and place the m left endpoints on $a-\frac{1}{m}\varepsilon$, $a-\frac{2}{m}\varepsilon, \ldots, a-\varepsilon$. The same can be done with right endpoints going clockwise. ■

Definition 6. Let K be a simplicial complex. The vertex $v \in K$ is dominated by the vertex $w \in K$ if any simplex $\sigma \in K$ that contains v satisfies that $\sigma \cup \{w\} \in K$.

Some notation: if we have two closed arcs $U_1 = [a_1,b_1]$ and $U_2 = [a_2,b_2]$, then $U_1 \leqslant U_2$ will mean that $a_1 \leqslant a_2 \leqslant b_1 \leqslant b_2 < a_1$. The following Lemma will allow us to reduce the nerve complexes $\mathcal{N}(\mathcal{U})$ to one of the form $\mathcal{N}(n,k)$ in the Theorem afterwards. We call the a_i's left endpoints and the b_i's right endpoints.

Lemma 7. *Let $\mathcal{U} = \{U_i = [a_i,b_i] \subset S^1 : i = 0,\ldots,n-1\}$ be a collection of n closed arcs, and let U_i, U_j in $\mathcal{U}$ such that*

1. *$U_i \subseteq U_j$,*
2. *$U_i \leqslant U_j$ and $b_k \notin [a_i,a_j)$ for all k, or*
3. *$U_j \leqslant U_i$ and $a_k \notin (b_j,b_i]$ for all k.*

Then the vertex i is dominated by the vertex j in $\mathcal{N}(\mathcal{U})$.

Proof. Let $\sigma \in \mathcal{N}(\mathcal{U})$ and $U_\sigma = \bigcap_{k\in\sigma} U_k$ with $U_\sigma \cap U_i \neq \emptyset$ (so that $i \in \sigma$). We must show that $j \in \sigma$, that is, $U_\sigma \cap U_i \cap U_j \neq \emptyset$.

- Assuming 1., $U_\sigma \cap U_i \cap U_j = U_\sigma \cap U_i \neq \emptyset$.
- Assume 2. Suppose that $U_\sigma \cap U_i \cap U_j = \emptyset$. Hence $U_\sigma \cap U_i \subseteq [a_i,a_j)$. Since $b_k \notin [a_i,a_j)$ for all k (in particular for $k \in \sigma\cup\{i\}$), we must have $a_j \in U_\sigma \cap U_i$ because the intersection of the U_k's lies in $[a_i,a_j)$. Therefore, $a_j \in [a_i,a_j)$. See Figure 1.2.
- Assuming 3., we conclude analogously to the second case.

■

Proposition 8. *Let $\mathcal{U}$ be a finite collection of arcs in S^1. If $\mathcal{N}(\mathcal{U})$ has no dominated vertices, then there is an isomorphism $\mathcal{N}(\mathcal{U}) \cong \mathcal{N}(n,k)$ for some $0 \leqslant k < n$.*

Proof. By Lemma 5, we can assume that $\mathcal{U}$ is a collection of closed arcs in S^1 with different endpoints, so let $\mathcal{U} = \{U_i = [a_i,b_i] \subseteq S^1 : i = 0,\ldots,n-1\}$. Since the left endpoints are all different, we can order the elements in $\mathcal{U}$ in such a way that $a_0 < \cdots < a_{n-1} < a_0$. We can further assume that no arc contains another because by Lemma 7(1.), there would be a dominated vertex that we

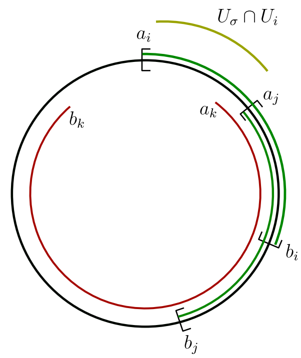

FIGURE 1.2: Proof Lemma 7

may take away without changing the homotopy type of the complex, so the right endpoints are also ordered by $b_0 < \cdots < b_n < b_0$.

Let us show that, in this setting, the set of endpoints alternate between right and left endpoints. If there were no right endpoint between a_i and a_{i+1}, the condition in Lemma 7(2.) would be satisfied, and the vertex $i+1$ of the complex would dominate vertex i. Since the numbers of left and right endpoints are the same, the endpoints must alternate, so there exists a constant $0 \leqslant k < n$ such that

$$a_0 < b_{-k} < a_1 < b_{-k+1} < a_2 < \cdots < b_{-k+n-1} < a_0.$$

Since $a_0 < b_{-k}$ implies that $a_k < b_0$, two arcs only intersect when their left endpoints are separated by at most k left endpoints of distance. This says that the maximal simplices of the complex $\mathcal{N}(\mathcal{U})$ are the nonempty intersections $\bigcap_{j=0}^{k} U_{i+j}$ for $i = 0, \ldots, n-1$. Therefore, $\mathcal{N}(\mathcal{U}) \simeq \mathcal{N}(n,k)$. ■

1.2 Homotopy Types of Nerves of Evenly Spaced Arcs

In the previous section we saw that in order to determine the homotopy types of the nerves $\mathcal{N}(\mathcal{U})$ we may simply restrict to the case where $\mathcal{U}$ is of the form $\mathcal{U}_{n,k}$. Now we show how to actually compute $\mathcal{N}(n,k) = \mathcal{N}(\mathcal{U}_{n,k})$ up to homotopy.

The next lemma will play a crucial role in the determination of the homotopy types of the $\mathcal{N}(n,k)$ in Proposition 12.

Lemma 9. *[Brown, 2006, Cor. 7.4.3] Let the space X be the union of closed subspaces X_1 and X_2 with intersection X_0 and such that the inclusions $X_0 \to X_1$, $X_0 \to X_2$ are cofibrations. Suppose that X_1 and X_2 are contractible. Then X is of the homotopy type of the suspension ΣX_0.*

We can use a theorem for gluing homotopy equivalences to prove Lemma 9.

Theorem 10. *[Brown, 2006, Thm. 7.4.3] Let X, Y be topological spaces, and $f : X \to Y$ a continuous map. Suppose that:*

1. *$X = X_1 \cup X_2$, $Y = Y_1 \cup Y_2$, where X_i, Y_i are closed subspaces, and $X_0 := X_1 \cap X_2$, $Y_0 := Y_1 \cap Y_2$.*
2. *$f(X_n) \subseteq Y_n$ for $n = 0, 1, 2$.*
3. *The restrictions $f^0 : X_0 \to Y_0$, $f^1 : X_1 \to Y_1$, $f^2 : X_2 \to Y_2$ are homotopy equivalences.*
4. *Each inclusion $X_0 \to X_1$, $X_0 \to X_2$, $Y_0 \to Y_1$, $Y_0 \to Y_2$ is a cofibration.*

Then f is a homotopy equivalence.

Proof. (*of Lemma 9*) Let $i_n : X_0 \to X_n$ be the inclusion for $n = 1, 2$. The maps i_n are null-homotopic because the spaces X_n are contractible. Let $H_n : X_0 \times [0,1] \to X_n$ for $n = 1, 2$ be the homotopies so that $H_n(x, 0) = *$, $H_n(x, 1) = i_n(x)$, where $*$ is a point.

Then i_n can be extended to a map f_i from a cone

$$CX_0 = \{(x,t) : x \in X_0, t \in [0,1]\}/\{(x,0) : x \in X_0\}$$

over X_0 to X_n for $n = 1, 2$, and these two maps f_1 and f_2 can be glued together to give a map $f : \Sigma X_0 \to X_1 \cup X_2 = X$.

The functions f_i are maps between contractible spaces (the cones and X_1, X_2), so they are homotopy equivalences. By hypothesis, the inclusions are cofibrations. Hence, by Theorem 10, the map f is a homotopy equivalence, i.e. $\Sigma X_0 \simeq X$. ∎

The way we will use Lemma 9 takes the following form.

Lemma 11. *Let K be a finite simplicial complex and let K_1, K_2 be contractible simplicial subcomplexes of K such that $K_1 \cup K_2 = K$. Then $K \simeq \Sigma(K_1 \cap K_2)$.*

Proof. This is simply a reformulation of Lemma 9 for simplicial complexes. The only thing that may require special attention is the hypothesis that the inclusions must be cofibrations, but this is indeed the case for finite simplicial complexes (see [Hatcher, 2001, Prop. 0.17]). ∎

The next proposition will give us a recurrence formula (modulo homotopy equivalence) to determine the homotopy type of $\mathcal{N}(n,k)$.

Proposition 12. *For $\frac{n}{2} \leqslant k < n$, there is a homotopy equivalence*

$$\mathcal{N}(n,k) \simeq \Sigma^2 \mathcal{N}(k, 2k-n).$$

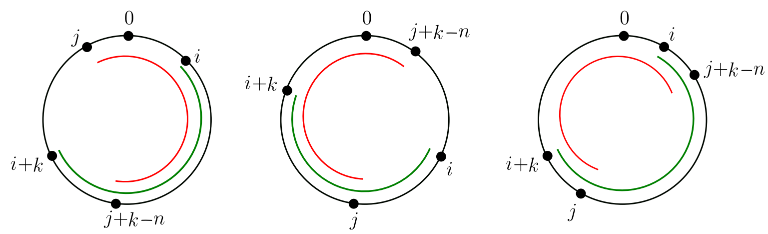

FIGURE 1.3: Possible intersections.

Proof. Let us denote by σ_i the maximal simplices $\{i, i+1, \ldots, i+k\}$ of $\mathcal{N}(n,k)$. Note that we can write $\mathcal{N}(n,k)$ as the union of two subcomplexes that are actually cones, namely,

$$\mathcal{N}(n,k) = \underbrace{\left(\bigcup_{i=0}^{n-k-2} \sigma_i\right)}_{K_1} \cup \underbrace{\left(\bigcup_{j=n-k-1}^{n-1} \sigma_j\right)}_{K_2} = K_1 \cup K_2,$$

where every maximal simplex carries all its subsimplices.

They are both cones because if $n-k-1 \leqslant j \leqslant n-1$, then $n-1 \leqslant j+k \leqslant n+k-1$, so every σ_j has $n-1$ as a vertex, and since $\frac{n}{2} \leqslant k < k+1$, $n < 2k+2$, so $n-k-2 < k$, which implies that every σ_i contains k as a vertex. Since we have K_1 and K_2 are contractible, by Lemma 11, $\mathcal{N}(n,k) \simeq \Sigma(K_1 \cap K_2)$, and

$$K_1 \cap K_2 = \{\sigma_i \cap \sigma_j : i = 0, 1, \ldots, n-k-2; j = n-k-1, \ldots, n-1\}.$$

Since $0 \leqslant i \leqslant n-k-2$ implies $k \leqslant i+k \leqslant n-2$, we get that $n-1$ is not a vertex of K_1, so the complex $K_1 \cap K_2$ has as vertices $\{0, 1, \ldots, n-2\}$. It is clear from Figure 1.3 that there are three types of intersections among the maximal simplices σ_i and σ_j:

1. If $0 \leqslant i \leqslant j+k-n \leqslant i+k < j \leqslant n-1$, then

$$\sigma_i \cap \sigma_j = \{i, \ldots, j+k-n\} \subseteq \sigma_0 \cap \sigma_{n-1} = \{0, 1, \ldots, k-1\},$$

   where the inclusion holds because the intersection lies on the "right side" of 0, the simplex σ_j starts before 0 and every simplex has $k+1$ vertices.

2. If $0 \leqslant j+k-n < i \leqslant j \leqslant i+k \leqslant n-2$, then

$$\sigma_i \cap \sigma_j = \{j, \ldots, i+k\} \subseteq \sigma_{n-k-2} \cap \sigma_{n-k-1} = \{n-k-1, \ldots, n-2\},$$

   because $0 \leqslant j+k-n$ implies $j \geqslant n-k > n-k-1$ and $i+k \leqslant n-2$.

3. If $i \leqslant j+k-n$ and $j \leqslant i+k$, then

$$\sigma_i \cap \sigma_j = \{i, \ldots, j+k-n\} \cup \{j, \ldots, i+k\}.$$

Hence the maximal simplices of $K_1 \cap K_2$ are the following:

$$\tau = \{0, \dots, k-1\}, \quad \tau' = \{n-k-1, \dots, n-2\},$$

$$\tau_{i,j} = \{i, \dots, j+k-n\} \cup \{j, \dots, i+k\} \text{ for } 0 \leqslant i \leqslant j+k-n,\ j \leqslant i+k \leqslant n-2.$$

We now want to express $K_1 \cap K_2$ as the union of two contractible subcomplexes to use Lemma 11 again. Let

$$K_1 \cap K_2 = \underbrace{\tau}_{K_1'} \cup \underbrace{\left(\tau' \cup \bigcup_{(i,j)\in\Lambda} \tau_{i,j}\right)}_{K_2'},$$

where every simplex carries all its subsimplices as before and $\Lambda = \{(i,j) : 0 \leqslant i \leqslant j+k-n,\ j \leqslant i+k \leqslant n-2\}$.

The subcomplex K_1' is contractible since it is the simplex of dimension k.

Let us now show that $K_2' \simeq \tau'$ (showing K_2' is contractible too) by a sequence of reductions. Define $K_2'(l)$ as the subcomplex of K_2' on the vertices $\{l, \dots, n-2\}$ for $l = 0, \dots, n-k-1$. Note that if $l \neq n-k-1$, then τ' cannot be a simplex of $K_2'(l)$ (hence does not contain l) and any maximal simplex $\tau_{i,j}$ in K_2' that has l as vertex in $K_2'(l)$ must satisfy $i \leqslant l \leqslant j+k-n$ and

$$\tau_{i,j} \cap \{l, \dots, n-2\} \subset \tau_{l,j} = \{l, \dots, j+k-n\} \cap \{j, \dots, l+k\},$$

that is, it is of the form $\tau_{l,j}$. This means that any simplex in $K_2'(l)$ that contains l also contains $l+k$, so $l+k$ dominates l in $K_2'(l)$ and we can take l away preserving the homotopy type: $K_2'(l) \simeq K_2'(l) \setminus \{l\} = K_2'(l+1)$. This gives

$$K_2' = K_2'(0) \simeq K_2'(1) \simeq \cdots \simeq K_2'(n-k-1) = \tau'.$$

Hence the maximal simplices of $K_1' \cap K_2'$ are

- $\tau \cap \tau' = \{n-k-1, \dots, k-1\}$
- $\tau \cap \tau_{0,j} = \{j, \dots, k-1\} \cup \{0, \dots, j+k-n\}$ for $n-k \leqslant j \leqslant k-1$.
- $\tau \cap \tau_{i,i+k} = \{i, \dots, i+2k-n\}$ for $0 \leqslant i \leqslant n-k-2$,

with vertex set $\{0, \dots, k-1\}$, but note that these sets are of the form $[i, i+(2k-n)] \mod k$, so $K_1 \cap K_2 = \mathcal{N}(k, 2k-n)$, and the result follows. ■

Recall the following fact about suspensions of wedge sums of spheres.

Proposition 13. *The suspension of a wedge sum of spheres is a wedge sum of spheres. More precisely, for nonnegative integers d_i,*

$$\Sigma\left(\bigvee_{i\in I} S^{d_i}\right) \simeq \bigvee_{i\in I} S^{d_i+1}.$$

We are ready to determine up to homotopy $\mathcal{N}(n,k)$.

Theorem 14. *Let* $0 \leqslant k \leqslant n-2$ *. Then*

$$\mathcal{N}(n,k) \simeq \begin{cases} \bigvee^{n-k-1} S^{2l} & \text{if} \quad \frac{k}{n} = \frac{l}{l+1}, \\ S^{2l+1} & \text{if} \quad \frac{l}{l+1} < \frac{k}{n} < \frac{l+1}{l+2}, \end{cases}$$

for some $l = 0, 1, 2, \ldots$

Proof. The proof is done by induction on l.

For $l = 0$, if $\frac{k}{n} = l$, then $k = 0$ and $\mathcal{N}(n, 0)$ is a discrete set of n points which is the same as $\bigvee^{n-1} S^0$. Now suppose that $\frac{k}{n} = \frac{l}{l+1}$ for some $l > 0$. We have that

$$\frac{2k-n}{k} - \frac{l-1}{l} = 2 - \frac{n}{k} - \frac{l-1}{l} = 2 - \frac{l+1}{l} - \frac{l-1}{l} = 0,$$

thus

$$\mathcal{N}(k, 2k-n) \simeq \bigvee^{k-(2k-n)-1} S^{2(l-1)} = \bigvee^{n-k-1} S^{2l-2}.$$

Therefore, by Proposition 12 and Proposition 13, we have

$$\mathcal{N}(n,k) \simeq \Sigma^2 \mathcal{N}(k, 2k-n) \simeq \bigvee^{n-k-1} S^{2l}.$$

Similarly, recall that if $1 \leqslant k < \frac{n}{2}$ (this is the case $l = 0$), then $\mathcal{N}(n,k) \simeq S^1$. Assume $\frac{l}{l+1} < \frac{k}{n} < \frac{l+1}{l+2}$ for some $l > 0$. Then

$$\frac{l}{l+1} - \frac{2k-n}{k} = \frac{l}{l+1} - 2 + \frac{n}{k} > \frac{l}{l+1} - 2 + \frac{l+2}{l+1} = 0$$

and

$$\frac{2k-n}{k} - \frac{l-1}{l} = 2 - \frac{n}{k} - \frac{l-1}{l} > 2 - \frac{l+1}{l} - \frac{l-1}{l} = 0,$$

hence $\frac{l-1}{l} < \frac{2k-n}{k} < \frac{l}{l+1}$. By induction hypothesis and Proposition 12,

$$\mathcal{N}(n,k) \simeq \Sigma^2 \mathcal{N}(k, 2k-n) \simeq \Sigma^2 S^{2(l-1)} \simeq S^{2l},$$

as we wanted to show. ■

1.3 Homotopy Equivalence Nerves and Cliques

Thus far we have only talked about Čech complexes when we are actually interested in Vietoris-Rips complexes. Just as we defined $\mathcal{N}(n,k)$ and this can be seen as Čech complexes of evenly spaced arcs, we now define $\overline{\mathcal{N}}(n,k)$, which can be seen as Vietoris-Rips complexes of some evenly spaced subsets of S^1.

Definition 15. The simplicial complex $\overline{\mathcal{N}}(n,k)$ has vertex set the collection $\mathcal{U}_{n,k}$ in 1.1, and there is a face $[i_0, \ldots, i_n]$ if $[\frac{i_r}{n}, \frac{i_r+k}{n}] \cap [\frac{i_s}{n}, \frac{i_r+k}{n}] \neq \emptyset$ for all $i_r, i_s \in \{i_0, \ldots, i_n\}$.

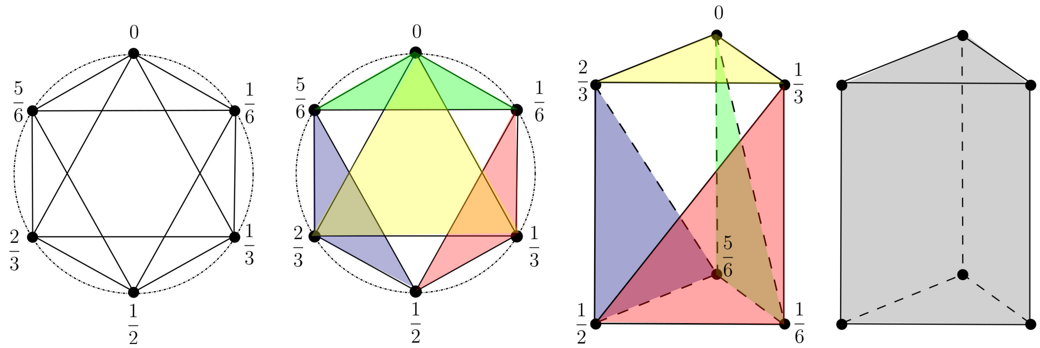

FIGURE 1.4: Visual representation $\mathbf{VR}_{\leqslant}\left(\{0, \frac{1}{6}, \dots, \frac{5}{6}\}; \frac{1}{3}\right) \simeq S^2$.

Alternatively, $\overline{\mathcal{N}}(n,k)$ is the maximal simplicial complex with 1-skeleton $\mathcal{N}(n,k)^{(1)}$. We note that $\overline{\mathcal{N}}(n,k) = \mathbf{VR}_{\leqslant}\left(\{0, \frac{1}{n}, \frac{2}{n}, \dots, \frac{n-1}{n}\}; \frac{k}{n}\right)$.

In this section the two kinds of complexes will be connected via a homotopy equivalence (for appropritate n and k) and we will be able to determine the homotopy type of the $\overline{\mathcal{N}}(n,k)$ by using Theorem 14. We will need the following result that is the version of Quillen's A theorem on conditions for a functor to induce a homotopy equivalence on classifying spaces applied to complexes.

Theorem 16. [Barmak, 2011] *Let $\varphi : K \to L$ be a simplicial map between two finite complexes. Suppose that the preimage of each closed simplex of L is contractible. Then φ is a simple homotopy equivalence.*

Recall that a simple homotopy equivalence is a homotopy equivalence that is induced by a sequence of elementary expansions and collapses. In particular φ preserves the homotopy type of the spaces.

We have the following result.

Theorem 17. *Let $n \geqslant 1$ and $k \geqslant 0$. Then $\overline{\mathcal{N}}(n+k,k) \simeq \mathcal{N}(n,k)$.*

Proof. The claim is true if $n-2 \leqslant k$ because in both simplicial complexes any two vertices form an edge, so we let $k \leqslant n-2$. Let us define

$$\varphi : \{0, 1, \dots, n+k-1\} \to \{0, 1, \dots, n-1\} \text{ by } \varphi(i) = i \pmod n.$$

This induces a surjective simplicial map $\varphi : \overline{\mathcal{N}}(n+k,k) \to \mathcal{N}(n,k)$. It is surjective because $[i, i+k] \in \overline{\mathcal{N}}(n+k,k) \mapsto [i, i+k] \in \mathcal{N}(n,k)$ for $0 \leqslant i < n$, and in general, for an arbitrary simplex $\sigma \in \overline{\mathcal{N}}(n+k,k)$, we have two cases for $\varphi(\sigma)$. Let $t := \min \sigma$ (in the order $0 \leqslant 1 \leqslant \dots \leqslant n+k-1$), so σ has no vertices in $\{0, 1, \dots, t-1\}$, and note that $\sigma \subseteq [t-k, t+k]$.

- $0 \leqslant t-k$: Since $\sigma \cap [0, 1, \dots, t-1] = \emptyset$, we have $\sigma \subseteq [t, t+k]$, so $\varphi(\sigma) \subseteq [t \pmod n, t+k \pmod n]$. Hence $\varphi(\sigma)$ is a face in $\mathcal{N}(n,k)$.
- $t-k < 0$: We have that

$$\sigma \subseteq [t-k \pmod{n+k}, n+k-1] \cup [t, t+k] = [t+n, n+k-1] \cup [t, t+k],$$

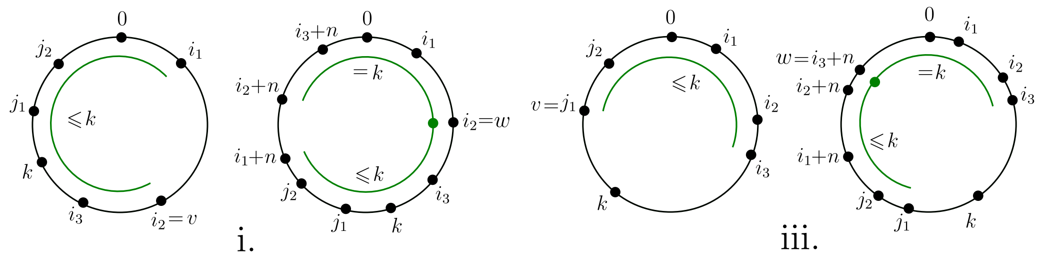

FIGURE 1.5: Cases i. and iii. for the choice of w given v.

hence, applying φ, we have

$$\varphi(\sigma) \subseteq [t, t+k] \cup [t, k-1] \subseteq [t, t+k],$$

and again $\varphi(\sigma)$ is a face in $\mathcal{N}(n,k)$.

Our goal now is to show that φ satisfies the hypothesis of Theorem 16 to prove that it is a homotopy equivalence. Let σ' be a simplex in $\mathcal{N}(n,k)$ with vertex set $\{i_1, \dots, i_s\} \cup \{j_1, \dots, j_t\}$ where $0 \leqslant i_1 < \cdots < i_s < k \leqslant j_1 < \cdots j_t \leqslant n-1$. We then have that $\sigma := \varphi^{-1}(\sigma')$ has vertex set $\{i_1, \dots, i_s\} \cup \{i_1+n, \dots, i_s+n\} \cup \{j_1, \dots, j_t\}$.

Let us show that there exists $w \in \sigma$ such that $\sigma \subseteq [w-k, w+k]$, so that σ is a cone with apex w and hence contractible. Recall that the maximal simplices of $\mathcal{N}(n,k)$ are of the form $[i, i+k]$. Therefore, there is $v \in \sigma'$ so that $\sigma' \subseteq [v, v+k]$. We have the following cases for w according to the choice of v.

i. For $v = i_q$ (for some $1 \leqslant q \leqslant s$) let $w = i_q$.

ii. For $v = j_1$ and $s = 0$ let $w = j_1$.

iii. For $v = j_1$ and $s > 0$ let $w = i_s + n$.

iv. We cannot have $v = j_q$ for $q \geqslant 2$ since j_{q-1} is not in the interval $[j_q, j_q+k]$.

See Figure 1.5 for a representation of the first and third cases.

This shows that $\varphi^{-1}(\sigma')$ is contractible for any simplex σ' in $\mathcal{N}(n,k)$. We conclude that $\overline{\mathcal{N}}(n+k,k) \simeq \mathcal{N}(n,k)$. ■

We already knew the two spaces were homotopy equivalent for an especific choice of n and k. Letting $n=4$ and $k=2$, we have $\mathcal{N}(4,2) \simeq S^2 \simeq \overline{\mathcal{N}}(4+2,2)$ (cf. Figure 1.1 and Figure 1.4).

We can now determine $\overline{\mathcal{N}}(n,k)$ up to homotopy.

Theorem 18. *Let* $0 \leqslant k \leqslant n-2$ *. Then*

$$\overline{\mathcal{N}}(n,k) \simeq \begin{cases} \bigvee^{n-2k-1} S^{2l} & \text{if} \quad \frac{k}{n} = \frac{l}{2l+1}, \\ S^{2l+1} & \text{if} \quad \frac{l}{2l+1} < \frac{k}{n} < \frac{l+1}{2l+3}, \end{cases}$$

for some $l = 0, 1, 2, \dots$

Proof. By Theorem 17, we have $\overline{\mathcal{N}}(n,k) \simeq \mathcal{N}(n-k,k)$. Now $k/(n-k) = l/(l+1)$ implies $k/n = l/(2l+1)$, and

$$\frac{l}{l+1} < \frac{k}{n-k} < \frac{l+1}{l+2} \quad \text{implies} \quad \frac{l}{2l+1} < \frac{k}{n} < \frac{l+1}{2l+3}.$$

The result follows now from Theorem 14. ■

See [Adamaszek, 2013] for an alternative way to arrive at this result.

1.4 Cyclic Graphs

In this section we define cyclic graphs (for which the 1-skeleton of Vietoris-Rips complexes $\mathbf{VR}(X;r)$ with $X \subset S^1$ are examples considering directed edges) and look at some of their properties. A particular invariant will be attached to these graphs, and this will allow us to control the behaviour of the complexes in terms of r.

A cyclic graph is a directed graph with a fixed cyclic order in its vertices, denoted by $v_0 \leqslant v_1 \leqslant \ldots \leqslant v_{n-1}$, in such a way that if there is an edge $v_i \to v_j$ for $i \neq j$, then either $j = i+1 \pmod{n}$ or there are edges $v_i \to v_{j-1}$ and $v_{i+1} \to v_j$ (where the indices are taken mod n). This in particular implies that if there is an edge between v_i and v_j, there are edges between v_i and v_k, and between v_k and v_j for all k between i and j.

As we mentioned earlier, the examples of cyclic graphs to have in mind are the ones associated to Vietoris-Rips complexes.

Definition 19. If $X \subseteq S^1$ is a finite set and $0 < r < \frac{1}{2}$, then we define the *directed* Vietoris-Rips graph $\overrightarrow{\mathbf{VR}}_{<}(X;r)$ as the graph with vertex set X and a directed edge $x_i \to x_j$ if the *clockwise* distance[1] $\vec{d}(x_i, x_j)$ between x_i and x_j is positive and strictly less than r. The graph $\overrightarrow{\mathbf{VR}}_{\leqslant}(X;r)$ is defined analogously.

Definition 20. Let G be a graph. The clique complex $\mathbf{Cl}(G)$ has vertex set G and a simplex for each complete subgraph of G.

Note that $\mathbf{VR}_{<}(X;r) = \mathbf{Cl}(\overrightarrow{\mathbf{VR}}_{<}(X;r))$ and $\mathbf{VR}_{\leqslant}(X;r) = \mathbf{Cl}(\overrightarrow{\mathbf{VR}}_{\leqslant}(X;r))$ when we forget the direction of the edges.

A specific family of cyclic graphs (to which every other cyclic graph will be reduced) is the following. Given integers n and k such that $0 \leqslant k \leqslant \frac{1}{2}n$, then the graph $\overrightarrow{C_n^k}$ has vertex set $\{0, \ldots, n-1\}$ and $i \to i+s$ for all $i = 0, \ldots, n-1$ and $s = 1, \ldots, k$. Note that

$$\overrightarrow{C_n^k} = \overrightarrow{\mathbf{VR}}_{\leqslant}\left(\left\{0, \frac{1}{n}, \ldots, \frac{n-1}{n}\right\}; \frac{k}{n}\right).$$

[1] For example, the clockwise distance in $S^1 = \mathbf{R}/\mathbf{Z}$ between $\frac{1}{4}$ and $\frac{1}{2}$ is $\frac{1}{4}$, and the clockwise distance between $\frac{1}{2}$ and $\frac{1}{4}$ is $\frac{3}{4}$.

The associated morphisms to cyclic graphs are called cyclic homomorphisms. Given cyclic graphs G and H, a cyclic homomorphism $f : G \to H$ is a homomorphism of directed graphs such that if $v_0 < \cdots < v_{n-1} \prec v_0$ in G, then $f(v_0) \leqslant \cdots \leqslant f(v_{n-1}) \preceq f(v_0)$ in H, and is not constant if G has a directed cycle. We note that he composition of two cyclic homomorphisms is a cyclic homomorphism, and that when we have a homomorphism of directed graphs $f : G \to \overrightarrow{\mathbf{VR}}_<(X;r)$ (or $\overrightarrow{\mathbf{VR}}_\leqslant(X;r)$), being a non-constant cyclic homomorphism is equivalent to having

$$\sum_{i=0}^{n-1} \vec{d}(f(v_i), f(v_{i+1})) = 1, \tag{1.2}$$

where $G = \{v_0, v_1, \ldots, v_n\}$.

In order to determine the homotopy type of the Vietoris-Rips complexes $\mathbf{VR}(X;r)$, we now define an invariant associated to cyclic graphs that will allow us to transform r into information about the 1-skeleton of the simplicial complex.

Definition 21. Let G be a cyclic graph. The winding fraction $\mathrm{wf}(G)$ of G is defined as

$$\mathrm{wf}(G) = \sup\left\{\frac{k}{n} : \text{ there is a cyclic homomorphism } \overrightarrow{C_n^k} \to G\right\}.$$

Remark 22. Note that having a cyclic homomorphism $f : G \to H$ immediately implies that $\mathrm{wf}(G) \leqslant \mathrm{wf}(H)$ because any cyclic homomorphism $\overrightarrow{C_n^k} \to G$ extends to one $\overrightarrow{C_n^k} \to H$.

We said that we wanted to control r in $\mathbf{VR}(X;r)$ in terms of the winding fraction of its 1-skeleton, so let us see now how the two are related and how to compute the winding fraction of $\overrightarrow{C_n^k}$.

Proposition 23. *Let X be a finite subset of S^1 and let $0 < r < 1$. Then*

1. $\mathrm{wf}(\overrightarrow{\mathbf{VR}}_<(X;r)) < r$,
2. $\mathrm{wf}(\overrightarrow{\mathbf{VR}}_\leqslant(X;r)) \leqslant r$,
3. $\mathrm{wf}(\overrightarrow{\mathbf{VR}}_<(X;r)) \leqslant \mathrm{wf}(\overrightarrow{\mathbf{VR}}_\leqslant(X;r))$.

Proof.

1. Let $f : \overrightarrow{C_n^k} \to \overrightarrow{\mathbf{VR}}_<(X;r)$ be a cyclic homomorphism with $k \geqslant 1$ (since if there are only maps with $k = 0$, the claim follows). Since $i \to i+k$ in $\overrightarrow{C_n^k}$

for $i = 0, \dots, n-1$, we have $\vec{d}(f(i), f(i+k)) < r$, hence

$$nr > \sum_{i=0}^{n-1} \vec{d}(f(i), f(i+k)) = \sum_{i=0}^{n-1} \sum_{j=i}^{i+k-1} \vec{d}(f(j), f(j+1))$$
$$= \sum_{j=0}^{n-1} \sum_{i=j+1-k}^{j} \vec{d}(f(j), f(j+1)) = \underbrace{\sum_{j=0}^{n-1} \vec{d}(f(j), f(j+1))}_{=1} \sum_{i=j+1-k}^{j} 1 = k,$$

by Equation 1.2. Therefore, $\frac{k}{n} < r$ and $\mathrm{wf}(\overrightarrow{\mathbf{VR}}_{<}(X;r)) < r$.

2. The same as 1. changing $\vec{d}(f(i), f(i+k)) < r$ by $\vec{d}(f(i), f(i+k)) \leqslant r$.
3. It is clear since we have an inclusion map $\overrightarrow{\mathbf{VR}}_{<}(X;r) \hookrightarrow \overrightarrow{\mathbf{VR}}_{\leqslant}(X;r)$. Inclusions are cyclic homomorphisms so we have the desired inequality by Remark 22. ■

Proposition 24. *The winding fraction* $\mathrm{wf}(\overrightarrow{C_n^k})$ *of* $\overrightarrow{C_n^k}$ *is* $\frac{k}{n}$.

Proof. The identity map $\overrightarrow{C_n^k} \to \overrightarrow{C_n^k}$ is a cyclic homomorphism, so, by definition of winding fraction, $\mathrm{wf}(\overrightarrow{C_n^k}) \geqslant \frac{k}{n}$. On the other hand, since $\overrightarrow{C_n^k}$ is a Vietoris-Rips graph with $r = \frac{k}{n}$, Proposition 23.2. gives $\mathrm{wf}(\overrightarrow{C_n^k}) \leqslant \frac{k}{n}$. ■

We already know the winding fractions of the graphs $\overrightarrow{C_n^k}$, but what about other cyclic graphs? It turns out that these $\frac{k}{n}$ are the only possible values for $\mathrm{wf}(G)$ for a cyclic graph G. To see this, we remove some vertices of the graphs and show that this removal does not affect the winding fraction. The vertices to be removed will be called dominated.[2]

A vertex v_i in a cyclic graph G with cyclic ordering $v_0 < \cdots < v_{n-1} < v_n$ is *dominated* by the vertex v_{i+1} if, for any vertex w in G, there is an edge $w \to v_i$ if and only if there is one $w \to v_{i+1}$.

Whenever we have a cyclic graph with a dominated vertex, we can take it away and obtain a new cyclic graph. Suppose G is a cyclic graph and v_i is dominated by v_{i+1}. Define the map $f : G \to G \setminus v_i$ by $f(v_j) = v_j$ if $j \neq i$, and $f(v_i) = v_{i+1}$. This map f is a cyclic homomorphism and clearly $G \setminus v_i \hookrightarrow G \xrightarrow{\psi} G \setminus v_i$ is the identity map, but we actually have more.

Proposition 25. *Let G be a cyclic graph, $v_i, v_{i+1} \in G$, with v_i dominated by v_{i+1}. Then $G \setminus v_i \hookrightarrow G$ and $\psi : G \to G \setminus v_i$ induce homotopy equivalences of clique complexes.*

Proof. We have $\mathbf{Cl}(G \setminus v_i) \hookrightarrow G \xrightarrow{\psi} \mathbf{Cl}(G \setminus v_i)$ is the identity map, so we only need to check $\mathbf{Cl}(G) \xrightarrow{\psi} \mathbf{Cl}(G \setminus v_i) \hookrightarrow \mathbf{Cl}(G)$ is homotopic to the identity on $\mathbf{Cl}(G)$.

[2] There is already a notion of dominated vertex in simplicial complexes, but it will be clear from the context to which one we refer.

By definition of cyclic graph, if we have an edge $v_i \to v_k$ for $k > i+1$, then there is an edge $v_{i+1} \to v_k$. This, together with the hypothesis that v_i is dominated by v_{i+1}, implies that every vertex w adjacent to v_i is adjacent to v_{i+1}. Hence, for every $\sigma = [v_{j_0}, \dots, v_{j_k}]$ in

$$\mathrm{lk}_{\mathbf{Cl}(G)}(v_i) = \{\sigma \in \mathbf{Cl}(G) : v_i \notin \sigma,\ \sigma \cup \{v_i\} \in \mathbf{Cl}(G)\}$$

we have edges v_{j_s} — v_i in G for $1 \leqslant s \leqslant k$, so there are edges v_{j_s} — v_{i+1} by the previous remark, which means that $[v_{i+1}, v_{j_1}, \dots, v_{j_s}] \in \mathrm{lk}_{\mathbf{Cl}(G)}(v_i)$. This implies that $\mathrm{lk}_{\mathbf{Cl}(G)}(v_i)$ is a cone with apex v_{i+1} and so $\mathbf{Cl}(G)$ can be ontained from $\mathbf{Cl}(G \setminus v_i)$ by attaching a cone (over the cone $\mathrm{lk}_{\mathbf{Cl}(G)}(v_i)$). Therefore $\mathbf{Cl}(G \setminus v_i) \hookrightarrow \mathbf{Cl}(G)$ is a homotopy equivalence. ■

Since we know that when we have a dominated vertex we may reduce the graph by deleting it, the natural question to ask is what happens when we delete all the dominated vertices. The next proposition gives an answer to this.

Proposition 26. *A cyclic graph without dominated vertices is isomorphic to $\overrightarrow{C_n^k}$ for some $0 \leqslant k < \frac{1}{2}n$.*

Proof. Let G be a cyclic graph without dominated vertices and with cycle order $v_0 < \cdots < v_{n-1} < v_0$. For every j there is a $k_j > 0$ such that $v_j \to v_{j+1}$, …, $v_j \to v_{j+k_j}$ and $v_j \not\to v_{j+k_j+1}$. Hence

$$(\{v_i\} \cup \{w : w \to v_i\}) \setminus \{w : w \to v_{i+1}\} = \{v_j : j + k_j = i\} =: N(v_i).$$

Note that $N(v_i) = \emptyset$ if and only if v_i is a dominated vertex. Writing $G = \bigcup_{i=0}^{n-1} N(v_i)$, since the $N(v_i)$ are all disjoint, we have $\sum_{i=0}^{n-1} |N(v_i)| = n$, so $|N(v_i)| = 1$ for all i. This implies that

$$|\{w : w \to v_i\}| = |\{w : w \to v_{i+1}\}| = \cdots = |\{w : w \to v_{i-1}\}| =: k.$$

Therefore $G \cong \overrightarrow{C_n^k}$ as we wanted to show. ■

Definition 27. We say that G dismantles to $\overrightarrow{C_n^k}$ if there is a sequence

$$G \to G_1 \to \cdots \to G_l \cong \overrightarrow{C_n^k} \tag{1.3}$$

where every map induces a homotopy equivalence at the level of clique complexes, and the composition $\overrightarrow{C_n^k} \hookrightarrow G \to \overrightarrow{C_n^k}$ is the identity.

As a consequence of Propositions 25 and 26, we have the following corollary.

Corollary 28. *Let G be a cyclic graph G. Then either $G \cong \overrightarrow{C_n^k}$ if G does not have any dominated vertices, or G dismantles to some $\overrightarrow{C_n^k}$.*

Therefore, the only homotopy types of $\mathbf{Cl}(G)$ for a cyclic graph G are the ones appearing in Theorem 18.

Remark 29. Note that the reduction in 28 shows that $\mathrm{wf}(G) = \frac{k}{n}$ by Remark 22 and the definition of winding fraction. Hence, the homotopy type of $\mathbf{Cl}(G)$ is determined by $\mathrm{wf}(G)$ subject to the same conditions for $\frac{k}{n}$ in Theorem 18.

1.5 Vietoris-Rips Circle

As can be noticed from a glimpse of Theorems 2 and 3, there is a difference in the behaviour of the Vietoris-Rips complexes when we take r from intervals of the form $(\frac{l}{2l+1}, \frac{l+1}{2l+3})$, and when we take r from the extremes of those intervals. Hence, this section is divided in two according to the values of r we take. The former values will be called generic and the latter ones will be called singular.

1.5.1 Generic Values

In this section the first thing we prove is a theorem of "stability" of the homotopy type of the clique complexes of cyclic graphs in the sense that whenever there is a cyclic homomorphism $f : G \to H$ and the winding fractions of G and H both lie within certain values, then f is a homotopy equivalence at the level of simplicial complexes (after taking clique complexes of the graphs). This is done by first considering the cases when G is a subgraph of H by taking away vertices and edges (this is Proposition 30.1. and 30.2.), and, after that, the general case (Proposition 30.3.).

Proposition 30. *Let G and H be cyclic graphs, v and e a vertex and an edge of H, respectively. Let $f : G \to H$ be a cyclic homomorphism and suppose also that*

$$\frac{l}{2l+1} < \mathrm{wf}(G) \leqslant \mathrm{wf}(H) < \frac{l+1}{2l+3} \tag{1.4}$$

for some $l \geqslant 0$. Then

1. *If $G = H \setminus v$ and f is the inclusion $H \setminus v \to H$, then f induces a homotopy equivalence of clique complexes.*
2. *Suppose $H \setminus e$ is a cyclic graph. If $G = H \setminus e$ and f is the inclusion $H \setminus v \to H$, then f induces a homotopy equivalence of clique complexes.*
3. *The map f induces a homotopy equivalence of clique complexes.*

Proof. 1. Let H_v be the subgraph of H induced by the set of vertices adjacent to v and write $\mathbf{Cl}(H) = \mathbf{Cl}(G) \cup (\mathbf{Cl}(H_v) * v)$ so that $\mathbf{Cl}(G) \cap (\mathbf{Cl}(H_v) * v) = \mathbf{Cl}(H_v)$. With this cover of $\mathbf{Cl}(H)$, we have the Mayer-Vietoris sequence (in reduced homology) given by

$$\cdots \to \widetilde{\mathrm{H}}_n(\mathbf{Cl}(H_v)) \to \widetilde{\mathrm{H}}_n(\mathbf{Cl}(G)) \oplus \widetilde{\mathrm{H}}_n(\mathbf{Cl}(H_v) * v) \to \widetilde{\mathrm{H}}_n(\mathbf{Cl}(H)) \to \widetilde{\mathrm{H}}_{n-1}(\mathbf{Cl}(H)) \to \cdots$$

By Remark 29 and 1.4, both complexes $\mathbf{Cl}(G)$ and $\mathbf{Cl}(H)$ are homotopy equivalent to S^{2l+1}, and $\mathbf{Cl}(H_v) * v$ is contractible, so the only non-zero groups in the sequence are

$$0 \to \widetilde{\mathrm{H}}_{2l+1}(\mathbf{Cl}(H_v)) \to \mathbf{Z} \to \mathbf{Z} \to \widetilde{\mathrm{H}}_{2l}(\mathbf{Cl}(H_v)) \to 0. \tag{1.5}$$

We know H_v is a cyclic graph, so, by Remark 30, the homology of $\mathbf{Cl}(H_v)$ is free and is not zero only in at most one dimension. If, for instance,

$$\widetilde{\mathrm{H}}_{2l+1}(\mathbf{Cl}(H_v)) \cong \mathbf{Z}^r,\ r > 0 \text{ and } \widetilde{\mathrm{H}}_{2l}(\mathbf{Cl}(H_v)) = 0,$$

the middle map in 1.5 is an isomorphism, but then there would be an injective map $\mathbf{Z}^r \to 0$, hence $r = 0$. Similarly assuming non-zero homology appears at dimension $2l$. We have $\widetilde{\mathrm{H}}_*(\mathbf{Cl}(H_v)) = 0$ and the middle map in 1.5 is an isomorphism. Since $f : \mathbf{Cl}(H) \to \mathbf{Cl}(G)$ induces isomorphisms in homology and $\mathbf{Cl}(G) \simeq \mathbf{Cl}(H) \simeq S^{2l+1}$, the map f is a homotopy equivalence by Whitehead's theorem.

2. Let H_e be the cyclic subgraph induced by the vertices adjacent to both a and b that the edge $a \xrightarrow{e} b$ connects. Then we write

$$\mathbf{Cl}(H) = \mathbf{Cl}(G) \cup (\mathbf{Cl}(H_e) * e)$$

with intersection $\mathbf{Cl}(H_e) * \{a, b\} = \Sigma\, \mathbf{Cl}(H_e)$. By Mayer-Vietoris, we get

$$0 \to \widetilde{\mathrm{H}}_{2l+1}(\Sigma\, \mathbf{Cl}(H_e)) \to \widetilde{\mathrm{H}}_{2l+1}(\mathbf{Cl}(G)) \to \widetilde{\mathrm{H}}_{2l+1}(\mathbf{Cl}(H)) \to \widetilde{\mathrm{H}}_{2l}(\Sigma\, \mathbf{Cl}(H_e)) \to 0,$$

but since $\widetilde{\mathrm{H}}_k(\mathbf{Cl}(H_e)) = \widetilde{\mathrm{H}}_{k+1}(\Sigma\, \mathbf{Cl}(H_e))$, we have

$$0 \to \widetilde{\mathrm{H}}_{2l}(\mathbf{Cl}(H_e)) \to \mathbf{Z} \to \mathbf{Z} \to \widetilde{\mathrm{H}}_{2l-1}(\mathbf{Cl}(H_e)) \to 0.$$

The same analysis as in Part 1. of this Proposition shows that f is a homotopy equivalence.

3. Let $f : V(G) \to V(H)$ be injective so that G can be seen as a subgraph of H. Then f is the composition of elementary cyclic homomorphisms $G \hookrightarrow G_1 \hookrightarrow \cdots \hookrightarrow H$ where G_{i+1} is obtained from G_i by adding a vertex or an edge. In this case Parts 1. and 2. apply, and hence f induces a homotopy equivalence on clique complexes.

Now let f arbitrary, $G \to \overrightarrow{C_n^k}$ and $H \to \overrightarrow{C_{n'}^{k'}}$ as in 28, and consider the composition

$$\overrightarrow{C_n^k} \xrightarrow{\simeq} G \xrightarrow{f} H \xrightarrow{\simeq} \overrightarrow{C_{n'}^{k'}},$$

where $\simeq$ means that it induces a homotopy equivalence on clique complexes. If we proved the composition induces a homotopy equivalence on clique complexes, the map f would do too, so it is enough to prove the claim for an arbitrary cyclic homomorphism $f : \overrightarrow{C_n^k} \to \overrightarrow{C_{n'}^{k'}}$ for $\frac{l}{2l+1} < \frac{k}{n} \leqslant \frac{k'}{n'} < \frac{l+1}{2l+3}$. This is what we do now.

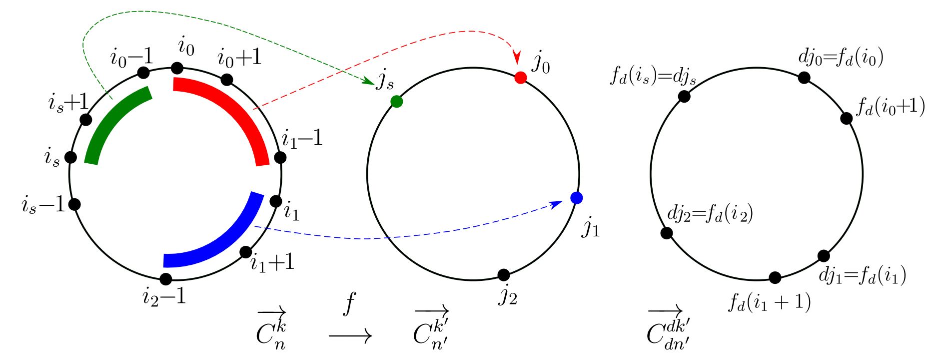

FIGURE 1.6: Definition of the function f_d.

If for some $d > 0$ we managed to create cyclic homomorphisms f_d, τ and γ so that we have a commutative diagram of the form

$$\begin{array}{ccccc} \overrightarrow{C_n^k} & \xrightarrow[\simeq]{f_d} & \overrightarrow{C_{dn'}^{dk'}} & \xleftarrow[\simeq]{\gamma} & \overrightarrow{C_{n'}^{k'}} \\ & \searrow^{f} & \downarrow \tau & \| & \\ & & \overrightarrow{C_{n'}^{k'}} & & \end{array}$$

then f would have the required form if τ does, and τ does because γ does. The maps γ and τ are actually easy to define once we have d: set $\gamma(i) = di$ and $\tau(j) = \left\lfloor \frac{j}{d} \right\rfloor$. The tricky one is f_d.

See Figure 1.6 for a guide for the definition of the function f_d below.

Let $f(\overrightarrow{C_n^k}) = \{j_0 < \cdots < j_s < j_0\} \subseteq \overrightarrow{C_{n'}^{k'}}$ for some $1 \leqslant s \leqslant n-1$. The preimage of each j_q by f is an interval $(\mod n)$ so we can relabel the vertices so that $f^{-1}(j_q) = \{i_q, i_q+1, \ldots, i_{q+1}-1\}$. Let $d := \max\{|f^{-1}(j_q)| : 0 \leqslant q \leqslant s\}$. Define

$$f_d : \overrightarrow{C_n^k} \to \overrightarrow{C_{dn'}^{dk'}} \quad \text{by} \quad i \mapsto dj_q + \vec{d}_n(i_q, i) \quad \text{for} \quad i \in f^{-1}(j_q).$$

Let us see that f_d is a graph homomorphism, that is, for $0 \leqslant i \leqslant n-1$, we must have $\vec{d}_{dn'}(f_d(i), f_d(i+k)) \leqslant dk'$.

- If $i, i+k \in f^{-1}(j_q)$, then $\vec{d}_{dn'}(f_d(i), f_d(i+k)) \leqslant d \leqslant dk'$.
- If $i \in f^{-1}(j_q)$ and $i+k \in f^{-1}(j_{q'})$, then $\vec{d}_{n'}(j_q, j_{q'}) \leqslant k'$ because f is a graph homomorphism.
  - If $\vec{d}_{n'}(j_q, j_{q'}) < k'$, then $\vec{d}_{dn'}(f_d(i), f_d(i+k)) \leqslant \vec{d}_{nd'}(dj_q, dj_{q'} + d) \leqslant dk'$.

– If $\vec{d}_{n'}(j_q, j_{q'}) = k'$ so that $j_{q'} = j_q + k'$, then

$$\begin{aligned}\vec{d}_{dn'}(f_d(i), f_d(i+k)) &= dk' + \vec{d}_n(i_q, i+k) - \vec{d}_n(i_q, i)\\ &= dk' + \vec{d}_n(i+k) - \vec{d}_n(i_q, i_{q'})\\ &= dk' + k - \vec{d}_n(i_q, i_{q'}) \leqslant dk',\end{aligned}$$

where the last inequality holds because otherwise $\vec{d}_n(i_q - 1, i_{q'}) \leqslant k$ and hence $\vec{d}_{n'}(f(i_q - 1), f(i_{q'})) = \vec{d}_{n'}(j_{q-1}, j_q) > k'$ contradicting the fact that f is a homomorphism.

Finally, note that within each $f^{-1}(j_q)$ the map f_d preserves the cyclic ordering, so f_d is a cyclic homomorphism. ■

We now have to make the transition from finite subsets to arbitrary subsets of the circle. To do this, we first assign to the geometric realization of $\mathbf{VR}(X;r)$ the weak topology with respect to subcomplexes induced by finite subsets of X. That is, if $F(X)$ denotes the poset of all finite subsets of X ordered by inclusion, then

$$\mathbf{VR}(X;r) = \operatorname*{colim}_{Y \text{ in } F(X)} \mathbf{VR}(X;r).$$

Given a finite subset Y_0 of X, the poset of all finite subsets of X containing Y_0 is cofinal in $F(X)$ and is denoted by $F(X;Y0)$. Hence we also have

$$\mathbf{VR}(X;r) = \operatorname*{colim}_{Y \text{ in } F(X;Y_0)} \mathbf{VR}(X;r). \tag{1.6}$$

We will reduce 1.6 further (modulo homotopy equivalences) by taking homotopy colimits instead of simply colimits. This can be done by the Projection Lemma (see [Welker, Ziegler, and Živaljević, 1999]) that says that this can be done when the maps in the diagram are closed cofibrations, which is the case here.

We extend the definition of winding fraction to arbitrary sets by setting

$$\mathrm{wf}(\overrightarrow{\mathbf{VR}}_<(X;r)) = \sup\{\mathrm{wf}(\overrightarrow{\mathbf{VR}}_<(Y;r)) : Y \subseteq X,\ Y \text{ finite}\},$$

and similarly for $\mathrm{wf}(\overrightarrow{\mathbf{VR}}_\leqslant(X;r))$.

Proposition 31. *Suppose that X is a non-empty subset of S^1 and $0 < r < \frac{1}{2}$. If, for some $l = 0, 1, \ldots$, either*

1. $\frac{l}{2l+1} < \mathrm{wf}(\overrightarrow{\mathbf{VR}}_<(X;r)) < \frac{l+1}{2l+3}$, *or*
2. $\mathrm{wf}(\overrightarrow{\mathbf{VR}}_<(X;r)) = \frac{l+1}{2l+3}$ *and* $\mathrm{wf}(\overrightarrow{\mathbf{VR}}_<(Y;r)) \neq \mathrm{wf}(\overrightarrow{\mathbf{VR}}_<(X;r))$ *for all finite subsets Y of X,*

is true, then $\mathbf{VR}_<(X;r) \simeq S^{2l+1}$. If $r' \geqslant r$ and conditions 1. or 2. hold with the same l for r', the inclusion $\overrightarrow{\mathbf{VR}}_<(X;r) \hookrightarrow \overrightarrow{\mathbf{VR}}_<(X;r')$ is a homotopy equivalence.

Proof. Conditions 1. and 2. are imposed to justify the existence of a finite subset Y_0 of X such that if $Y \in F(X;Y_0)$, then $\frac{l}{2l+1} < \mathrm{wf}(\mathbf{VR}(Y;r)) < \frac{l+1}{2l+3}$. Hence the hypothesis of Proposition 30 are satisfied, and we have

$$\mathbf{VR}(X;r) \simeq \operatorname*{hocolim}_{Y \text{ in } F(X;Y_0)} \mathbf{VR}(Y;r) \simeq S^{2l+1}.$$

The last claim also follows from Proposition 30 because then we have that

$$\overrightarrow{\mathbf{VR}}_<(Y;r) \hookrightarrow \overrightarrow{\mathbf{VR}}_<(Y;r')$$

is a homotopy equivalence for all $Y \in F(X;Y_0)$. Therefore, the same is true for $\overrightarrow{\mathbf{VR}}_<(X;r) \hookrightarrow \overrightarrow{\mathbf{VR}}_<(X;r')$ by the Homotopy Lemma in [Welker, Ziegler, and Živaljević, 1999]. ■

Lemma 32. *Let X be a finite subset of S^1 and $0 < r < \frac{1}{2}$. If for any two consecutive points v_i and v_{i+1} of X, we have $\vec{d}(v_i, v_{i+1}) < 2\varepsilon$ for some $\varepsilon > 0$, then $\mathrm{wf}(\overrightarrow{\mathbf{VR}}_<(X;r)) > r - 2\varepsilon$.*

Proof. Assume $r - 2\varepsilon > 0$ (otherwise it is trivial). Choose another $\varepsilon' < \varepsilon$ such that $\vec{d}(v_i, v_{i+1}) < 2\varepsilon'$. We are going to find a cyclic homomorphism $\varphi : \overrightarrow{C_n^k} \to \overrightarrow{\mathbf{VR}}_<(X;r)$ (which implies $\mathrm{wf}(\overrightarrow{\mathbf{VR}}_<(X;r)) \geqslant \frac{k}{n}$) for any $\frac{k}{n} < r - 2\varepsilon'$ so that $\mathrm{wf}(\overrightarrow{\mathbf{VR}}_<(X;r))$ must be greater than or equal to $r - 2\varepsilon'$, and hence greater than $r - 2\varepsilon$.

For every $0 \leqslant i \leqslant n-1$ define $\varphi(i) = x_i \in X$ as the nearest point to $\frac{i}{n}$. Hence φ preserves the cyclic ordering and is a graph homomorphism because

$$d(x_i, x_{i+k}) \leqslant d(x_i, \tfrac{i}{n}) + d(\tfrac{i}{n}, \tfrac{i+k}{n}) + d(\tfrac{i+k}{n}, x_{i+k}) < 2\varepsilon' + \tfrac{k}{n} < r,$$

so we have the homomorphism φ required and the result follows. ■

We are finally ready to prove Theorem 2.

Theorem 1. *If $X \subset S^1$ is dense (in particular when $X = S^1$) and $0 < r < \frac{1}{2}$, then*

$$\mathbf{VR}_<(X;r) \simeq S^{2l+1} \quad \textit{for} \quad \frac{l}{l+1} < r \leqslant \frac{l+1}{2l+3},\ l = 0, 1, \ldots .$$

Moreover, if $\frac{l}{2l+1} < r \leqslant r' \leqslant \frac{l+1}{2l+3}$, then the inclusion $\mathbf{VR}_<(X;r) \hookrightarrow \mathbf{VR}_<(X;r')$ is a homotopy equivalence.

Proof. We first note that $\mathrm{wf}(\overrightarrow{\mathbf{VR}}_<(X;r)) = r$. This follows from Lemma 32 as we can find, for any $\varepsilon > 0$, a finite subset of X such that the distance between consecutive vertices is less than 2ε, so $\mathrm{wf}(\overrightarrow{\mathbf{VR}}_<(X;r)) \geqslant r$. On the other hand, Proposition 23 gives the reverse inequality and also shows the suppremum is not attained by any finite subset Y of X.

Condition $\frac{l}{l+1} < r \leqslant \frac{l+1}{2l+3}$ then transforms in either Condition 1. or 2. of Proposition 31 and the result follows. ■

1.5.2 Singular Values

Now that we have already dealt with Vietoris-Rips complexes $\mathbf{VR}(X;r)$ for r within intervals of the form $(\frac{l}{l+1}, \frac{l+1}{2l+3})$, we have to analyze what happens when r is equal to a extreme of those intervals and the winding fraction is attained by some finite subset Y of X.

The idea of the proof is the following. First it is shown that if $r = \frac{l}{2l+1}$ for some $l = 1, \ldots$, then $\mathbf{VR}_{\leqslant}(X;r)$ is simply-connected, $\widetilde{\mathrm{H}}_*(\mathbf{VR}_{\leqslant}(X;r))$ is torsion-free and $\widetilde{\mathrm{H}}_n(\mathbf{VR}_{\leqslant}(X;r)) = 0$ if $n \neq 2l$. If we somehow knew that $\widetilde{\mathrm{H}}_{2l}(\mathbf{VR}_{\leqslant}(X;r))$ is a free abelian group, then we would have

$$\mathbf{VR}_{\leqslant}(X;r) \simeq M(\bigoplus^{\kappa} \mathbf{Z}, 2l) \simeq \bigvee^{k} S^{2l} \tag{1.7}$$

by the uniqueness up to homotopy of (CW-complex) Moore spaces $M(G,n)$, consequence of Whitehead's Theorem (see [Hatcher, 2001, Cor. 4.33]), for $n > 1$. Our goal then is to prove that $\widetilde{\mathrm{H}}_{2l}(\mathbf{VR}_{\leqslant}(X;r))$ is free abelian by studying its generators (seen as images of certain homology classes from other spaces induced by cyclic homomorphisms). Finally, a counting argument will show that $\kappa = \mathfrak{c}$.

We start by defining the classes we will use to describe generators for the aforementioned homology group. Recall that the cross-polytope K_n in $\mathbf{R}^n$ is the convex hull of the set of points

$$\{\pm e_1, \pm e_2, \ldots, \pm e_n\} \subset \mathbf{R}^n,$$

where $e_1 = (1, 0, \ldots, 0), e_2 = (0, 1, \ldots, 0)$, and so on. Consider

$$\overrightarrow{C^{2l}_{2(2l+1)}} = \overrightarrow{\mathbf{VR}}_{\leqslant}\left(\left\{0, \frac{1}{4l+2}, \frac{2}{4l+2}, \ldots, \frac{1}{2}, \ldots, \frac{4l+1}{4l+2}\right\}; \frac{l}{2l+1}\right).$$

Let us note that any two vertices in $\overrightarrow{C^{2l}_{2l+1}}$ are adjacent except for antipodal vertices, that is, elements of the form

$$\left\{\frac{i}{4l+2}, \frac{i+2l+1}{4l+2}\right\}.$$

Identifying

$$\frac{i}{4l+2} \mapsto e_i \quad \text{and} \quad \frac{i+2l+1}{4l+2} \mapsto -e_i,$$

we see that the boundary of the cross-polytope K_{2l+1} is equal to the clique complex of $\overrightarrow{C^{2l}_{2(2l+1)}}$.[3]

[3] This, in particular, gives another way to visualize $\mathbf{VR}_{\leqslant}\left(\{0, \frac{1}{6}, \ldots, \frac{5}{6}\}; \frac{1}{3}\right) \simeq S^2$ as the octahedron in $\mathbf{R}^3$ (cf. Figure 1.4).

In $\widetilde{\mathrm{H}}_{2l}(\mathbf{Cl}(\overrightarrow{C_{2(2l+1)}^{2l}}))$ we fix the homology class

$$\begin{aligned}\iota_{2l} &= (-1)^{\frac{l(l+3)}{2}}([0]-[2l+1])\wedge([1]-[2l+2])\wedge\cdots\wedge([2l]-[4l+1]) \\ &= [0,2,\ldots,4l]-[1,3,\ldots,4l+1]\pm\cdots.\end{aligned} \tag{1.8}$$

Let G be a cyclic graph. We say a homology class $0\neq\alpha\in\widetilde{\mathrm{H}}_{2l}(\mathbf{Cl}(G))$ is cross-polytopal if it is the image $f_*(\iota_{2l})$ of a map induced by a cyclic homomorphism $f:\overrightarrow{C_{2(2l+1)}^{2l}}\to G$.

The next proposition describes all the cyclic homomorphisms from injective homomorphisms $\overrightarrow{C_{2(2l+1)}^{2l}}\to\overrightarrow{C_n^k}$. It is enough to consider injective homomorphisms because a homology class of degree $2l$ in a clique complex must have as underlying vertex set at least $4l+2$ vertices, as can be seen in [Kahle, 2009, Lem. 5.3].

Proposition 33. *Let $d\geqslant 1$.*

1. *Every cyclic homomorphism $\theta:\overrightarrow{C_{2l+1}^{l}}\to\overrightarrow{C_{d(2l+1)}^{dl}}$ is of the form $\theta_a(i)=a+di \pmod{d(2l+1)}$ for some $a=0,\ldots,d(2l+1)-1$.*
2. *Every injective cyclic homomorphism $\alpha:\overrightarrow{C_{2(2l+1)}^{2l}}\to\overrightarrow{C_{d(2l+1)}^{dl}}$ is of the form $\alpha_{a,b}$ defined by*

$$\alpha_{a,b}(i)=\begin{cases}a+d\frac{i}{2} \pmod{d(2l+1)} & i \text{ even}\\ b+d\frac{i-1}{2} \pmod{d(2l+1)} & i \text{ odd},\end{cases}$$

for some $a=0,\ldots,d(2l+1)-1$ and $b=a+1,\ldots,a+d-1$

Proof.

1. For $l=0$, we simply have a point on the domain and the result follows, so let $l>0$. Since θ is a cyclic homomorphism, we have

$$\vec{d}_{d(2l+1)}(\theta(i),\theta(i+l))\leqslant dl, \tag{1.9}$$

which implies, using an analogue to Equation 1.2,

$$(2l+1)(dl)\geqslant\sum_{i=0}^{2l}\vec{d}_{d(2l+1)}(\theta(i),\theta(i+l))=l\sum_{i=0}^{2l}\vec{d}_{d(2l+1)}(\theta(i),\theta(i+1))=ld(2l+1),$$

hence 1.9 is an equality for all i. Note also that $\vec{d}_{d(2l+1)}(\theta(i),\theta(i+1))$ equals

$$n-\vec{d}_{d(2l+1)}(\theta(i+1),\theta(i+l+1))-\vec{d}_{d(2l+1)}(\theta(i+l+1),\theta(i))=n-2k=d.$$

Defining $a:=\theta(0)$ and from the fact that $\theta(i+1)=\theta(i)+d$, the result follows.

2. Note that $\overrightarrow{C_{2(2l+1)}^{2l}}$ can be written as the union of two copies of $\overrightarrow{C_{2l+1}^{l}}$ with the vertex sets $\{0,2,\ldots,4l\}$ and $\{1,3,\ldots,4l+1\}$. Defining $a:=\alpha(0)$ and $b:=\alpha(1)$, we have by Part 1. that $\alpha(2i)=a+di$ and $\alpha(2i+1)=b+di$. Since α is an injective

cyclic homomorphism, we have $a < b < a+d$ (this is $\alpha(0) < \alpha(1) < \alpha(2)$), so any option for a is possible, but b must be in the set $\{a+1, \ldots, a+d-1\}$. ■

The following Proposition will show (among other things) that the homology of clique complexes of cyclic graphs dismantling to a $\overrightarrow{C_{d(2l+1)}^{dl}}$, are generated by cross-polytopal classes.

Proposition 34. *Let G be a cyclic graph dismantling to $\overrightarrow{C_{d(2l+1)}^{dl}}$. Then $\widetilde{\mathrm{H}}_{2l}(\mathbf{Cl}(G)) = \mathbf{Z}^{d-1}$ has a basis $\{e_1, \ldots, e_{d-1}\}$ such that all the cross-polytopal elements in $\widetilde{\mathrm{H}}_{2l}(\mathbf{Cl}(G))$ are $\pm e_1, \ldots, \pm e_{d-1}$, $e_i - e_j$ for $1 \leqslant i, j \leqslant d-1$ and $i \neq j$, for a total of $d(d-1)$ cross-polytopal elements in $\widetilde{\mathrm{H}}_{2l}(\mathbf{Cl}(G))$.*

Proof. First, note that if $n = d(2l+1)$ and $k = dl$, then $n - 2k - 1 = d - 1$, so, by Theorem 18, we have that $\widetilde{\mathrm{H}}_{2l}(\mathbf{Cl}(G)) = \mathbf{Z}^{d-1}$.

Assume that the proposition is true for $\overrightarrow{C_{d(2l+1)}^{dl}}$. Then, if G is a cyclic graph that dismantles to $\overrightarrow{C_{d(2l+1)}^{dl}}$, we have cyclic homomorphisms

$$\overrightarrow{C_{d(2l+1)}^{dl}} \xrightarrow{f} G \xrightarrow{g} \overrightarrow{C_{d(2l+1)}^{dl}}$$

that compose to the identity and induce homotopy equivalences at the level of clique complexes (see Corollary 28). Hence

$$\widetilde{\mathrm{H}}_{2l}(\mathbf{Cl}(\overrightarrow{C_{d(2l+1)}^{dl}})) \xrightarrow{\cong} \widetilde{\mathrm{H}}_{2l}(\mathbf{Cl}(G)) \xrightarrow{\cong} \widetilde{\mathrm{H}}_{2l}(\mathbf{Cl}(\overrightarrow{C_{d(2l+1)}^{dl}})),$$

and this shows both that $f_*(\{e_1, \ldots, e_{d-1}\})$ is a basis (of cross-polytopal classes, by definition) for $\widetilde{\mathrm{H}}_{2l}(\mathbf{Cl}(G))$, and that $\pm f_*(e_i)$, $f_*(e_i) - f_*(e_j)$ for all $i \neq j$, are cross-polytopal classes. Furthermore, if $\alpha \in \widetilde{\mathrm{H}}_{2l}(\mathbf{Cl}(G))$ is cross-polytopal, so is $\beta := g_*(\alpha)$, but then $g_*(\alpha) = \beta = g_* f_*(\beta)$ shows that $\alpha = f_*(\beta)$, so α is a cross-polytopal class of the required form.

By the previous argument, it is then enough to show the proposition for $G = \overrightarrow{C_{d(2l+1)}^{dl}}$. If σ is an oriented simplex, let $\sigma^\vee$ denote the cochain assigning 1 to σ, -1 to σ with the opposite direction, and 0 to any other simplex. For $a = 0, \ldots, n-1$, let

$$\gamma_a := [a, a+d, \ldots, a+2ld].$$

We have γ_a is a maximal simplex in $\overrightarrow{C_{d(2l+1)}^{dl}}$, so $[\beta_a] := [\gamma_a^\vee]$ is a cohomology class in $\widetilde{\mathrm{H}}^{2l}(\mathbf{Cl}(\overrightarrow{C_{d(2l+1)}^{dl}}))$. Now let, from 1.8 and Proposition 33.2,

$$[\alpha_{a,b}] := (\alpha_{a,b})_*(\iota_{2l}) = [a, a+d, \ldots, a+2ld] - [b, b+d, \ldots, b+2ld] \pm \cdots .$$

It follows that $[\beta_i]([\alpha_{j,d}]) = \delta_{i,j}$ for $1 \leqslant i, j \leqslant d-1$, so $\{[\alpha_{1,d}], \ldots, [\alpha_{d-1,d}]\}$ and $\{[\beta_1], \ldots, [\beta_{d-1}]\}$ are (dual) basis for $\widetilde{\mathrm{H}}_{2l}(\mathbf{Cl}(\overrightarrow{C^{dl}_{d(2l+1)}}))$ and $\widetilde{\mathrm{H}}^{2l}(\mathbf{Cl}(\overrightarrow{C^{dl}_{d(2l+1)}}))$, respectively. Note that

$$\begin{aligned}[\alpha_{a+d,b+d}] &= [a+d, a+2d, \ldots, a+(2l+1)d] - [b+d, b+2d, \ldots, b+(2l+1)d] \pm \cdots \\ &= (-1)^{2l}[a, a+d, \ldots, a+2ld] + (-1)^{2l}[b, b+d, \ldots, b+2ld] \pm \cdots = [\alpha_{a,b}]\end{aligned}$$

hence every cross-polytopal class is of the form $[\alpha_{a,b}]$ for $0 \leqslant a \leqslant d-1$ and $a+1 \leqslant b \leqslant a+d-1$ (recall Proposition 33.2). We now show the relationships between the $\alpha_{a,b}$. For every $[v] \in \widetilde{\mathrm{H}}_{2l}(\mathbf{Cl}(\overrightarrow{C^{dl}_{d(2l+1)}}))$ we have

$$[v] = \sum_{i=1}^{d-1} [\beta_i]([v]) \cdot [\alpha_{i,d}] \tag{1.10}$$

We consider the following cases:

- $a = 0, 1 \leqslant b \leqslant d-1:$ Then $[v] = [\alpha_{a,b}]$ in 1.10 shows that $[\alpha_{0,b}] = -[\alpha_{b,d}]$.
- $1 \leqslant a < b \leqslant d-1:$ Then $[v] = [\alpha_{a,b}]$ in 1.10 shows that $[\alpha_{a,b}] = [\alpha_{a,d}] - [\alpha_{b,d}]$ since $[\beta_a]([\alpha_{a,b}]) = 1$ and $[\beta_b]([\alpha_{a,b}]) = -1$.
- $1 \leqslant a \leqslant d-1, b = d:$ Then $[\alpha_{a,b}] = [\alpha_{a,d}]$, so $[\alpha_{a,b}]$ is a generator.
- $1 \leqslant a \leqslant d-1, d+1 \leqslant b \leqslant a+d-1:$ Then $1 \leqslant b-d < a \leqslant d-1$ and $[v] = [\alpha_{a,b}]$ in 1.10 shows that $[\alpha_{a,b}] = [\alpha_{a,d}] - [\alpha_{b,d}]$ since $[\beta_a]([\alpha_{a,b}]) = 1$ and $[\beta_{b-d}]([\alpha_{a,b}]) = -1$ (this by a similar computation to the one that showed that $[\alpha_{a,b}] = [\alpha_{a+d,b+d}]$).

Taking $e_i := \alpha_{i,d}$ for $i = 1, \ldots, d-1$, the result follows. ■

We are now ready to prove Theorem 3.

Theorem 2. *For $0 \leqslant r < \frac{1}{2}$, there is a homotopy equivalence*

$$\mathbf{VR}_{\leqslant}\left(S^1; r\right) \simeq \begin{cases} \bigvee^{\mathfrak{c}} S^{2l} & \text{if} \quad r = \frac{l}{l+1} \\ S^{2l+1} & \text{if} \quad \frac{l}{2l+1} < r < \frac{l+1}{2l+3},\ l = 0, 1, 2, \ldots, \end{cases}$$

where $\mathfrak{c}$ is the cardinality of the continuum.

Moreover, if $\frac{l}{2l+1} < r \leqslant r' \leqslant \frac{l+1}{2l+3}$, the inclusion $\mathbf{VR}_{\leqslant}(X; r) \hookrightarrow \mathbf{VR}_{\leqslant}(X; r')$ is a homotopy equivalence.

Proof. We note, as we did in the proof of Theorem 1., that $\mathrm{wf}(\mathbf{VR}_{\leqslant}\left(S^1; r\right)) = r$, and in this case the supremum is attained. For $\frac{l}{2l+1} < r < \frac{l+1}{2l+3}$, the result follows from Theorem 1. If $r = \frac{l}{2l+1}$, then the supremum is attained by the a finite subset Y_0 of S^1 (given by the vertex set of a regular $(2l+1)$-gon). Hence, for any finite set $Y_0 \subseteq Y \subseteq S^1$ we have $\mathrm{wf}(\mathbf{VR}_{\leqslant}(Y; r)) = \frac{l}{2l+1}$, so $\mathbf{VR}_{\leqslant}(Y; r)$ is a finite wedge of spheres of the same dimension $2l$ (see Remark 29). Since

$$\mathbf{VR}_{\leqslant}\left(S^1;r\right) = \operatorname*{colim}_{Y \text{in} F(S^1;Y_0)} \mathbf{VR}_{\leqslant}\left(Y;r\right), \tag{1.11}$$

we have $\mathbf{VR}_{\leqslant}\left(S^1;r\right)$ is simply-connected, with torsion-free homology and trivial except (at most) in degree $2l$. By the comments at the beginning of this (sub-)section, it remains to show $\widetilde{\mathrm{H}}_{2l}(\mathbf{VR}_{\leqslant}\left(S^1;r\right))$ is free abelian to have 1.7.

We extend the definition of cross-polytopal classes. A non-trivial homology class in $\widetilde{\mathrm{H}}_{2l}(\mathbf{VR}_{\leqslant}(X;r))$ is called cross-polytopal if it is the image of a cross-polytopal class in $\widetilde{\mathrm{H}}_{2l}(\mathbf{VR}_{\leqslant}(Y;r))$ (induced by an inclusion) for some finite Y. By 1.11, we have $\widetilde{\mathrm{H}}_{2l}(\mathbf{VR}_{\leqslant}\left(S^1;r\right))$ is generated by cross-polytopal classes.

Let us consider the family $\mathcal{B}$ of independent sets $B \subset \widetilde{\mathrm{H}}_{2l}(\mathbf{VR}_{\leqslant}\left(S^1;r\right))$ formed only by cross-polytopal clases. We have $\mathcal{B} \neq \emptyset$ and for every chain $\{B_i\}_{i\in I}$ the set $\bigcup_{i\in I} B_i$ is in $\mathcal{B}$. By Zorn's lemma, we have a maximal set $B \in \mathcal{B}$.

If $\langle B\rangle \neq \widetilde{\mathrm{H}}_{2l}(\mathbf{VR}_{\leqslant}\left(S^1;r\right))$, there is a cross-polytopal class $v \notin \langle B\rangle$. The set $B \cup \{v\}$ is not independent, so $\lambda v \in \langle\{v_1,\dots,v_k\}\rangle$ for some $\lambda \neq 0$ and some $k \geqslant 1$ such that $\{v_1,\dots,v_k\} \subset B$. By definition of cross-polytopal classes in $\widetilde{\mathrm{H}}_{2l}(\mathbf{VR}_{\leqslant}(X;r))$, we can find a finite subset $Y \subset X$ (dismantling to $\overrightarrow{C_{d(2l+1)}^{dl}}$), and each class $v, v_1, \dots, v_k$ has a corresponding cross-polytopal representative

$$v', v_1', \dots, v_k' \in \widetilde{\mathrm{H}}_{2l}(\mathbf{VR}_{\leqslant}(Y;r)) = \mathbf{Z}^{d-1}.$$

By Proposition 34, we know that

$$\{v', v_1', \dots, v_k'\} \subseteq \{\pm e_1, \dots, \pm e_{d-1}\} \cup \{e_i - e_j : 1 \leqslant i,j \leqslant d-1\},$$

and we have $\lambda v' \in \langle v_1', \dots, v_k'\rangle$, hence $v' \in \langle v_1', \dots, v_k'\rangle$, by Lemma 35. Therefore, $v \in B$, a contradiction. Thus, we must have that $\widetilde{\mathrm{H}}_{2l}(\mathbf{VR}_{\leqslant}\left(S^1;r\right))$ is free abelian (with basis B).

So far, then, we have $\mathbf{VR}_{\leqslant}\left(S^1;\frac{l}{2l+1}\right) \simeq \bigvee^{\kappa} S^{2l}$ for some κ. Let us see that $\kappa = \mathfrak{c}$. By 1.11, we have that κ is at most $\mathfrak{c}$ because each $\mathbf{VR}_{\leqslant}(Y;r)$ has finitely many simplices of dimension $2l$. On the other hand, let us consider

$$Y_t = \left\{\frac{i}{2l+1}, t + \frac{i}{2l+1} : i = 0,\dots,2l\right\} \subset S^1$$

for $0 < t < \frac{l}{2l+1}$. (Imagine infinitely many rotations of a $2l+1$-gon.) Hence each composition

$$j_t : \overrightarrow{C_{2(2l+1)}^{2l}} = \mathbf{VR}_{\leqslant}\left(Y_t;\frac{l}{2l+1}\right) \hookrightarrow \mathbf{VR}_{\leqslant}\left(S^1;\frac{l}{2l+1}\right)$$

gives a cross-polytopal class

$$\alpha_t = (j_t)_*(\iota_{2l}) = \left[0, \frac{2}{2l+1}, \dots, \frac{2l}{2l+1}\right] - \left[t, t+\frac{i}{2l+1}, \dots, t+\frac{2l}{2l+1}\right] \pm \cdots .$$

These classes are independent because each simplex

$$\left[t, t+\frac{i}{2l+1}, \dots, t+\frac{2l}{2l+1}\right]$$

is a maximal face of $\mathbf{VR}_{\leqslant}\left(S^1; \frac{l}{2l+1}\right)$ and appears only in the support of α_t, and not in α_s for $s \neq t$. Hence $\widetilde{\mathrm{H}}_{2l}(\mathbf{VR}_{\leqslant}\left(S^1; \frac{l}{2l+1}\right))$ contains a free abelian group of rank $\mathfrak{c}$. This shows that

$$\mathbf{VR}_{\leqslant}\left(S^1; \frac{l}{l+1}\right) \simeq \bigvee^{\mathfrak{c}} S^{2l},$$

as we wanted. ■

In the previous theorem we used the following fact.

Lemma 35. *Let $V = \{e_1, \dots, e_n\}$ be a basis for the free abelian group $\mathbf{Z}^n$ and define*

$$\widetilde{V} := V \cup \{e_i - e_j : 1 \leqslant i < j \leqslant n\}.$$

If $v, v_1, \dots, v_k \in \widetilde{V}$ and $\lambda v \in \langle v_1, \dots, v_k\rangle$ for some $\lambda \neq 0$, then $v \in \langle v_1, \dots, v_k\rangle$.

A proof can be found in Appendix A. of [Adamaszek and Adams, 2015].

Chapter 2

Products of Graphs and Clique Complexes

In this chapter we introduce the essential concepts from Discrete Morse Theory for simplicial complexes and two different kinds of products on simplicial complexes to show that the clique complex of a special product of graphs is simple-homotopic to the product of the clique complexes. We follow [Larrión, Pizaña, and Villarroel-Flores, 2013] for the main result of this chapter (Theorem 46) and [Jonsson, 2007] for the background.

2.1 Basics of Discrete Morse Theory

Let S be a finite set and let $\mathcal{S}$ be a finite family of subsets of S; the sets used in this chapter are vertices of simplicial complexes and the families will be taken from the faces of those complexes. A matching on $\mathcal{S}$ is a collection $\mathcal{M}$ of pairs $\{\sigma, \tau\}$ such that $\sigma, \tau \in \mathcal{S}$ with the condition that no element of $\mathcal{S}$ is in more than one pair of $\mathcal{M}$. Any other element that is not part of a pair in $\mathcal{M}$ will be called critical.

The most important kind of matchings that will be considered are element matchings. A matching $\mathcal{M}$ is an element matching if any pair in $\mathcal{M}$ is of the form $\{\sigma - x, \sigma + x\}$ for some $\sigma \subseteq S$ and $x \in S$; in the case σ is a face of a complex and x is a vertex, $\sigma + x$ and $\sigma - x$ denote $\sigma \cup \{x\}$ and $\sigma \setminus \{x\}$, respectively.

We construct directed graphs out of element matchings on finite sets in the following way. Fix an element matching $\mathcal{M}$ on a collection $\mathcal{S}$, and let $D(\mathcal{S}, \mathcal{M})$ be the directed graph with vertex set $\mathcal{S}$ and a directed edge $\sigma \to \tau$ if and only if one of the following is true:

1. $\{\sigma, \tau\} \in \mathcal{M}$ and $\tau = \sigma + x$ for some $x \notin \sigma$.
2. $\{\sigma, \tau\} \notin \mathcal{M}$ and $\sigma = \tau + x$ for some $x \notin \tau$.

The directed graph $D(\mathcal{S}, \mathcal{M})$ is just the Hasse diagram associated to the collection $\mathcal{S}$ ordered by inclusion, but in this case whenever there is a pair in the element

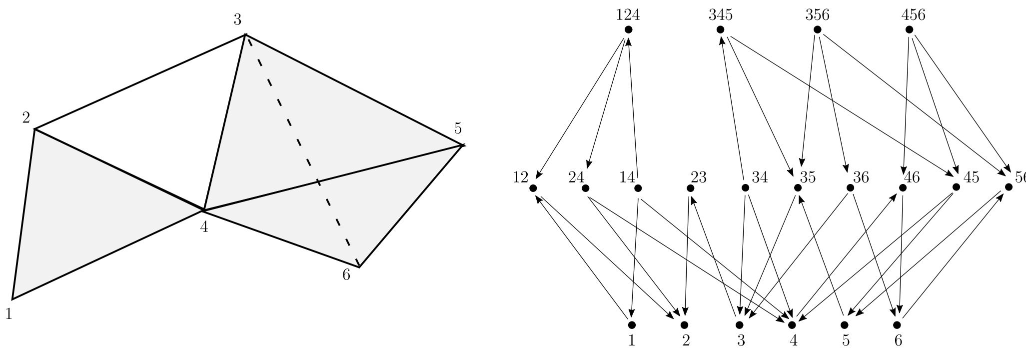

FIGURE 2.1: Simplicial complex and its associated directed graph $D(\mathcal{S}, \mathcal{M})$ for the element matching $\mathcal{M}$.

matching $\mathcal{M}$, the edge goes from the smaller set towards the bigger set (Condition 1), and for pairs not in the element matching the arrow goes in the opposite direction (Condition 2).

For example, let us consider the simplicial complex K with vertex set $S = \{1, 2, \ldots, 6\}$ and maximal faces $\{124, 23, 345, 456, 356\}$. Let $\mathcal{S}$ be the collection of all the faces of K and let the element matching be

$$\mathcal{M} = \{\{6, 56\}, \{5, 35\}, \{4, 46\}, \{3, 23\}, \{14, 124\}, \{34, 345\}, \{1, 12\}\}$$

The simplicial complex K and its directed graph $D(\mathcal{S}, \mathcal{M})$ for the element matching $\mathcal{M}$ are both shown in Figure 2.1.

If there is a directed path from σ to τ in $D(\mathcal{S}, \mathcal{M})$, then we denote it by $\sigma \rightsquigarrow \tau$. If we have collections of simplices $\mathcal{V}$ and $\mathcal{W}$, then $\mathcal{V} \rightsquigarrow \mathcal{W}$ will mean that there are elements $\sigma \in \mathcal{V}$ and $\tau \in \mathcal{W}$ such that $\sigma \rightsquigarrow \tau$. On the other hand, $\mathcal{V} \not\rightsquigarrow \mathcal{W}$ means that no such directed path exists.

There is an easy way to produce element matchings. For any fixed vertex v in a simplicial complex K, the collection

$$\mathcal{M}(v) = \{\{\sigma - v, \sigma + v\} : \sigma + v \in K\}$$

defines an element matching on K.

Finally, the last concept we need is that we will call a matching $\mathcal{M}$ acyclic if $D(\mathcal{S}, \mathcal{M})$ has no directed cycles.

Remark 36. Any cycle in a directed graph $D(\mathcal{S}, \mathcal{M})$ corresponding to an element matching $\mathcal{M}$ is of the form $(\sigma_0, \tau_0, \sigma_1, \tau_1, \ldots, \sigma_{r-1}, \tau_{r-1})$ with $r > 1$ such that $\sigma_i, \sigma_{i+1} \subseteq \tau_i$ and $\{\sigma_i, \tau_i\} \in \mathcal{M}$. See [Shareshian, 2001].

Lemma 37. *[Larrión, Pizaña, and Villarroel-Flores, 2013, Lemma 2.1] Let K be a simplicial complex with a vertex v so that $\sigma + v \in K$ for all $\sigma \in K$, i.e K is a cone with apex v. Then the element matching $\mathcal{M}(v)$ is an acyclic matching on K with no critical simplices.*

Proof. It is clear that it has no critical simplices because if σ is any simplex in K either $v \in \sigma$ or $v \notin \sigma$. In the former case, $\{\sigma - v, \sigma + v\} = \{\sigma - v, \sigma\} \in \mathcal{M}$, and in the latter, $\{\sigma - v, \sigma + v\} = \{\sigma, \sigma + v\} \in \mathcal{M}$, since by hypothesis $\sigma + v \in K$ for all $\sigma \in K$. Hence every simplex σ is in a pair.

Suppose that $\mathcal{M}(v)$ is not an acyclic matching. By Remark 36, there exists a path of the form

$$(\sigma_0, \tau_0, \sigma_1, \tau_1, \ldots, \sigma_{r-1}, \tau_{r-1})$$

with $\sigma_i, \sigma_{i+1} \subseteq \tau_i$ and $\sigma_i, \tau_i \in \mathcal{M}(v)$. By the definition of $\mathcal{M}(v)$, we have that $\{\sigma_i, \tau_i\} = \{\sigma'_i - v, \sigma'_i + v\}$ for some simplex σ'_i in K, so the cycle is of the form

$$(\sigma'_0 - v, \sigma'_0 + v, \sigma'_1 - v, \sigma'_1 + v, \ldots, \sigma'_{r-1} - v, \sigma'_{r-1} + v).$$

The condition $\sigma_i, \sigma_{i+1} \subseteq \tau_i$ implies that $\sigma'_0 - v, \sigma'_1 - v \subseteq \sigma'_0 + v$, which means that $\sigma'_1 \subseteq \sigma'_0$. Inductively, we get $\sigma'_0 \supseteq \sigma'_1 \supseteq \sigma'_2 \supseteq \cdots \supseteq \sigma'_{r-1} \supseteq \sigma'_0$, so $\sigma'_0 = \sigma'_1 = \cdots = \sigma'_{r-1} =: \sigma$. Hence the cycle is actually of the form

$$(\sigma - v, \sigma + v, \sigma - v, \sigma + v, \ldots, \sigma - v, \sigma + v),$$

and this is only possible for the trivial path. ■

Lemma 38. *[Jonsson, 2007, Lemma 4.2] Let K be a simplicial complex and let $f : K \to Q$ be a poset map, where Q is an arbitrary poset. For $q \in Q$, let $\mathcal{M}_q$ be an acyclic matching on $f^{-1}(q)$. Let*

$$\mathcal{M} := \bigcup_{q \in Q} \mathcal{M}_q.$$

Then $\mathcal{M}$ is an acyclic matching on K.

Proof. It is clear that the disjoint union of the element matchings will get us an element matching $\mathcal{M}$. Let us see that it is acyclic. By Remark 36, if we have a directed cycle in $D(K, \mathcal{M})$, it is of the form $(\sigma_0, \tau_0, \sigma_1, \tau_1, \ldots, \sigma_{r-1}, \tau_{r-1})$ with $\sigma_i, \sigma_{i+1} \subseteq \tau_i$ and $\sigma_i, \tau_i \in \mathcal{M}$. Since $\mathcal{M} = \bigcup_q \mathcal{M}_q$, we may choose $q_0, \ldots, q_{r-1}$ with $\sigma_k, \tau_k \in f^{-1}(q_k)$ (so $f(\sigma_k) = f(\tau_k)$) for $k = 0, \ldots, r-1$. All the vertices of the directed cycle are actually in only one $\mathcal{M}_q$, which is a contradiction because all the $\mathcal{M}_q$ are acyclic. Indeed, since f is a poset map, we have

$$q_{k+1} = f(\sigma_{k+1}) \leqslant f(\tau_k) = f(\sigma_k) = q_k.$$

Inductively, we get $q_k \leqslant q_{k'}$ for any k, k', so $q_0 = q_1 = \cdots = q_{r-1} =: q$ since Q is a poset. ■

Theorem 39. *[Jonsson, 2007, Lemma 4.4] Suppose that K_0 is a subcomplex of K such that $K_0 \not\rightsquigarrow K \setminus K_0$ and such that all critical faces belong to K_0. Then it is possible to collapse K to K_0. In particular, K and K_0 are homotopy equivalent.*

Proof. Let us note that if we have faces $\tau \in K \setminus K_0$ and $\sigma \in K_0$ such that $\{\sigma, \tau\} \in \mathcal{M}$, $\tau \subset \sigma$ cannot happen because τ would be an element of K_0, so it must necessarily happen that $\sigma \subset \tau$ and then $\sigma \rightsquigarrow \tau$, which would imply $K_0 \rightsquigarrow K \setminus K_0$, contrary to the hypothesis.

We now show that we may reduce K to a subcomplex with less number of vertices still containing K_0. Choose a face $\sigma \in K \setminus K_0$ so that no edge of $D(K, \mathcal{M})$ ends in σ, which exists because otherwise we would end up with a directed cycle ■

2.2 Products (Graphs, Complexes)

In this section we define and show their relationships of the different kinds of products between graphs and between simplicial complexes that we will need.

When it comes to graphs, we will use the following two products. Let G_1 and G_2 be any two graphs. We denote by $V(G_i)$ the set of vertices of the graph G_i.

Definition 40. The boxtimes product graph $G_1 \boxtimes G_2$ has as set of vertices $V(G_1) \times V(G_2)$, and for two distinct vertices (a_i, b_i), $i = 1, 2$, of $G_1 \boxtimes G_2$ we have

$$(a_1, b_1) — (a_2, b_2) \text{ in } G_1 \boxtimes G_2 \iff \begin{cases} a_1 = a_2 \text{ and } b_1 — b_2 \text{ in } G_2, \text{ or} \\ b_1 = b_2 \text{ and } a_1 — a_2 \text{ in } G_1, \text{ or} \\ a_1 — a_2 \text{ in } G_1 \text{ and } b_1 — b_2 \text{ in } G_2. \end{cases}$$

Related to the boxtimes product of graphs, we have another product, the ordered product.

Definition 41. Let $\leqslant_i$ be a partial order on $V(G_1)$ so that $a_1 — a_2$ in G_1 implies that either $a_1 \leqslant_1 a_2$ or $a_2 \leqslant_1 a_1$. Similarly take $\leqslant_2$ for $V(G_2)$. The ordered product graph $G_1 \times_{\leqslant} G_2$ has as set of vertices $V(G_1) \times V(G_2)$, and

$$(a_1, b_1) — (a_2, b_2) \text{ in } G_1 \times_{\leqslant} G_2 \iff \begin{cases} (a_1, b_1) — (a_2, b_2) \text{ in } G_1 \boxtimes G_2 \text{ and either} \\ (a_1, b_1) \leqslant (a_2, b_2) \text{ or } (a_2, b_2) \leqslant (a_1, b_1), \end{cases}$$

where $(a_1, b_1) \leqslant (a_2, b_2)$ means that $a_1 \leqslant_1 b_1$ and $b_1 \leqslant_2 b_2$.

It is clear that in general $G_1 \times_{\leqslant} G_2$ will have fewer edges than $G_1 \boxtimes G_2$.

See Figure 2.2 for an example of the two products just defined for the graphs G_1 and G_2 over three vertices as shown in the figure with associated orders $\leqslant_i$ for G_i defined by $a \leqslant_1 b$, $a \leqslant_1 c$, $3 \leqslant_2 1$.

We also want to talk about products of simplicial complexes, and this is an analogous situation to the one we had with graphs. Let K_1 and K_2 be two finite simplicial complexes.

Definition 42. The categorical product $K_1 \otimes K_2$ of K_1 and K_2 is the simplicial complex with vertex set $V(K_1) \otimes V(K_2)$ and we have a face σ in the product $K_1 \otimes K_2$ if and only if $\pi_1(\sigma)$ and $\pi_2(\sigma)$ are faces in K_1 and K_2, respectively.

Definition 43. The second product will require a partial order on $V(K_1)$ and $V(K_2)$ in such a way that every simplex is totally ordered. Then, the ordered product $K_1 \otimes_{\leqslant} K_2$ of K_1 and K_2 has vertex set $V(K_1) \times V(K_2)$ and the simplices

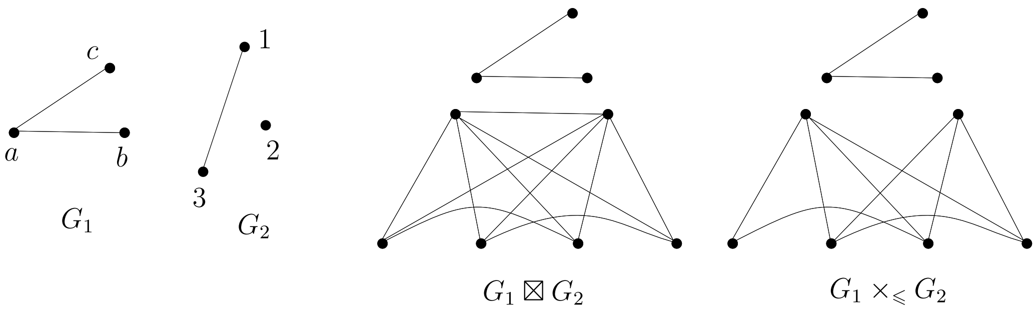

FIGURE 2.2: Example of box and ordered product for the two finite graphs G_1 and G_2 (on the left).

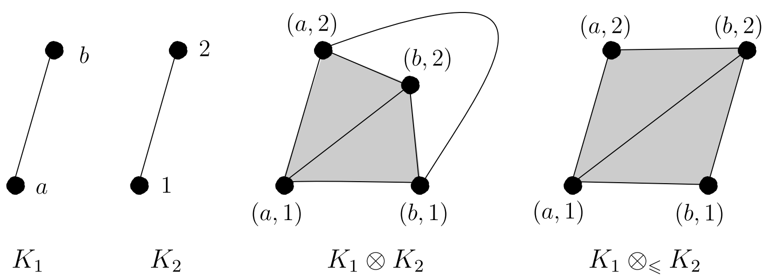

FIGURE 2.3: Products on simplicial complexes.

are the totally ordered subsets σ of $V(K_1) \times V(K_2)$ such that $\pi_1(\sigma)$ and $\pi_2(\sigma)$ are simplices of K_1 and K_2, respectively.

For example, let $K_1 = K_2$ be two 1-simplices with vertex sets $\{a, b\}$ and $\{1, 2\}$ and natural orders $a < b$, $1 < 2$. The two products for simplicial complexes of K_1 and K_2 are shown in Figure 2.3.

Lemma 44. *Let G_1 and G_2 be two graphs with partial orders $\leqslant_1$ and $\leqslant_2$ as above. Then*

1. $\mathbf{Cl}(G_1 \boxtimes G_2) = \mathbf{Cl}(G_1) \otimes \mathbf{Cl}(G_2)$.
2. $\mathbf{Cl}(G_1 \times_\leqslant G_2) = \mathbf{Cl}(G_1) \otimes_\leqslant \mathbf{Cl}(G_2)$.

Proof.

1. We have that $\sigma = \{(a_1, b_1), \ldots, (a_n, b_n)\}$ is a simplex in $\mathbf{Cl}(G_1 \boxtimes G_2)$ if and only if $(a_i, b_i) \to (a_j, b_j)$ in $G_1 \boxtimes G_2$ for all distinct $i, j \in \{1, \ldots, n\}$, if and only if either $a_i = a_j$, $b_i = b_j$, $a_i \to a_j$ (in G_1) or $b_i \to b_j$ (in G_2) for any i, j, if and only if $\{a_1, \ldots, a_n\} = \pi_1(\sigma)$ is a complete subgraph of G_1 and $\{b_1, \ldots, b_n\} = \pi_2(\sigma)$ is a complete subgraph of G_2, if and only if σ is a simplex in $\mathbf{Cl}(G_1) \otimes \mathbf{Cl}(G_2)$.
2. We have that $\sigma = \{(a_1, b_1), \ldots, (a_n, b_n)\}$ is a simplex in $\mathbf{Cl}(G_1 \times_\leqslant G_2)$ if and only if $(a_i, b_i) \to (a_j, b_j)$ in $G_1 \times_\leqslant G_2$ for all distinct $i, j \in \{1, \ldots, n\}$, if and only if either $a_i = a_j$, $b_i = b_j$, $a_i \to a_j$ (in G_1) or $b_i \to b_j$ (in G_2) for any i, j and either $(a_i, b_i) \leqslant (a_j, b_j)$ or $(a_j, b_j) \leqslant (a_i, b_i)$, if and only if $\{a_1, \ldots, a_n\} = \pi_1(\sigma)$ is a complete subgraph of G_1, $\{b_1, \ldots, b_n\} = \pi_2(\sigma)$ is a complete

subgraph of G_2 and both $\pi_1(\sigma)$ and $\pi_2(\sigma)$ are totally ordered (because any pair is comparable), if and only if σ is a simplex in $\mathbf{Cl}(G_1) \otimes_{\leqslant} \mathbf{Cl}(G_2)$. ∎

Theorem 45. *[Eilenberg and Steenrod, 1964] Let K_1 and K_2 be simplicial complexes. The geometric realization of $K_1 \otimes_{\leqslant} K_2$ is homotopy equivalent to the produt of the geometric realizations $|K_1| \times |K_2|$.*

2.3 Clique Complexes of Products

In this section we present a theorem relating the simplicial complexes obtained by taking the clique complex of a product of graphs and the product of the clique complexes taken separately.

Theorem 46. *[Larrión, Pizaña, and Villarroel-Flores, 2013, Thm. 4.2] Let G_1 and G_2 be any two finite graphs. Then the complexes $\mathbf{Cl}(G_1 \boxtimes G_2)$ and $\mathbf{Cl}(G_1 \times_{\leqslant} G_2)$ are simple-homotopic, and so the space $|\mathbf{Cl}(G_1 \boxtimes G_2)|$ is homotopic to the product space $|\mathbf{Cl}(G_1)| \times |\mathbf{Cl}(G_2)|$.*

Proof. Let us denote the vertex $(a, b) \in A \times B$ by ab. Let us first show that

$$\mathbf{Cl}(G_1 \boxtimes G_2) \simeq_s \mathbf{Cl}(G_1 \times_{\leqslant} G_2),$$

where $\simeq_s$ denote simple homotopy equivalence.

The idea is that we can take out those extra faces that we get from the graph $G_1 \boxtimes G_2$ that might not exist in $G_1 \times_{\leqslant} G_2$ (recall Figure 2.2). In order to do that, we need an acyclic matching on $\mathbf{Cl}(G_1 \boxtimes G_2)$ and so we define the poset

$$Q = \{(ab, a'b') \in (A \times B) \times (A \times B) : a > a', b < b'\}$$

subject to the lexicographic order, namely, the order defined by

$$(ab, a'b') > (cd, c'd') \iff \begin{cases} a > c, \text{ or} \\ a = c \text{ and } b > d, \text{ or} \\ a = c, b = d \text{ and } a' > c', \text{ or} \\ a = c, b = d, a' = c' \text{ and } b' > c'. \end{cases}$$

The important thing to note is that a simplex σ is in $\mathbf{Cl}(G_1 \boxtimes G_2) \setminus \mathbf{Cl}(G_1 \times_{\leqslant} G_2)$ if and only there are $ab, a'b'$ in σ such that $(ab, a'b') \in Q$.

Let us define a poset map $f : \mathbf{Cl}(G_1 \boxtimes G_2) \setminus \mathbf{Cl}(G_1 \times_{\leqslant} G_2) \to Q$ by declaring

$$f(\sigma) = \max\{(ab, a'b') \in Q : \{ab, a'b'\} \subseteq \sigma\},$$

which is well-defined because the graphs considered are finite, and it is indeed a poset map because an inclusion of simplices $\sigma \subseteq \tau$ implies that any pair obtained in σ can be equally obtained in τ and hence the maximum over τ is greater than or equal to the one over σ.

We now want to define acyclic matchings on $f^{-1}(ab, a'b')$ and this will be done with the help of Lemma 37. First we need to choose a vertex v and show that the pairs of the matching (which are of the form $\{\sigma - v, \sigma + v\}$) are always elements of $f^{-1}(ab, a'b')$ (in particular, they come from complete subgraphs of $G_1 \boxtimes G_2$).

Let v be $a'b$ and $\sigma \in f^{-1}(ab, a'b')$. Let us denote by π_1 and π_2 the projection from $V(G_1 \boxtimes G_2) = V(G_1) \times V(G_2)$ onto $V(G_1)$ and $V(G_2)$, respectively. Since $f(\sigma) = (ab, a'b')$, $\{ab, a'b'\} \subseteq \sigma$, so $a' \in \pi_1(\sigma)$ and $b \in \pi_2(\sigma)$. Hence we have

$$\sigma - a'b \subseteq \sigma + a'b \subseteq \pi_1(\sigma) \times \pi_2(\sigma),$$

and since $\pi_1(\sigma)$ and $\pi_2(\sigma)$ are complete subgraphs of G_1 and G_2, respectively, $\sigma - a'b$ and $\sigma + a'b$ are complete, i.e., they are simplices in $\mathbf{Cl}(G_1 \boxtimes G_2)$. We must show $f(\sigma - a'b) = (ab, a'b') = f(\sigma + a'b)$.

We cannot have that $a'b \in \{ab, a'b'\}$ because we would have $a = a'$ or $b = b'$, contradicting the definition of Q. But, of course, if $a'b \notin \{ab, a'b'\}$, then $f(\sigma - a'b) = (ab, a'b')$.

Now, to show the second equality, let us note that if $(xy, a'b) \in Q$ for some $xy \in \sigma$ (so $x > a'$ and $y < b$), then this is less than or equal to $(ab, a'b')$, showing that the latter is the maximum obtained from $\sigma + a'b$. If $x > a$, then $(xy, ab) > (ab, a'b') = f(\sigma)$, which is absurd because $(ab, a'b')$ is the maximum among pairs in σ. If $x \leqslant a$, then $(xy, a'b) < (ab, a'b')$, as required. On the other hand, if $(xy, a'b) \in Q$ and $xy \in \sigma$, then $a > a'$ implies $(ab, a'b') > (a'b, xy)$. The only possibility is that $f(\sigma + a'b) = (ab, a'b')$.

This shows that $\mathcal{M}(a'b)$ is an acyclic element matching on $f^{-1}(ab, a'b')$ with no critical simplices by Lemma 37, and we can collect these matchings to get one

$$\mathcal{M} = \bigcup_{(ab,a'b')\in Q} \mathcal{M}(a'b)$$

on $\mathbf{Cl}(G_1 \boxtimes G_2) \setminus \mathbf{Cl}(G_1 \times_{\leqslant} G_2)$ by Lemma 38. This element matching $\mathcal{M}$ has no critical simplices because

$$\mathbf{Cl}(G_1 \boxtimes G_2) \setminus \mathbf{Cl}(G_1 \times_{\leqslant} G_2) = \bigcup_{(ab,a'b')\in Q} f^{-1}(ab, a'b')$$

and we do not have any critical simplex in $f^{-1}(ab, a'b')$. Since there are no pairs $\{\sigma, \tau\}$ with $\sigma \in \mathbf{Cl}(G_1 \times_{\leqslant} G_2)$ and $\tau \in \mathbf{Cl}(G_1 \boxtimes G_2)$, we have that

$$\mathbf{Cl}(G_1 \times_{\leqslant} G_2) \not\leftrightarrow \mathbf{Cl}(G_1 \boxtimes G_2),$$

which by Theorem 39 implies that $\mathbf{Cl}(G_1 \boxtimes G_2)$ collapses to $\mathbf{Cl}(G_1 \times_{\leqslant} G_2)$.

Finally, for the second claim, we have

$$\begin{aligned} |\mathbf{Cl}(G_1 \boxtimes G_2)| &\simeq |\mathbf{Cl}(G_1 \times_{\leqslant} G_2)| \\ &= |\mathbf{Cl}(G_1) \otimes_{\leqslant} \mathbf{Cl}(G_2)| \simeq |\mathbf{Cl}(G_1)| \times |\mathbf{Cl}(G_2)|, \end{aligned}$$

where the first homotopy equivalence is given by the first claim, the second equality is Lemma 44.2 and the last homotopy equivalence is Theorem 45. ■

Chapter 3

The Vietoris Rips Complex of a Torus

In this chapter we use the results from the two previous chapters on products of clique complexes to show that the homotopy type of the Vietoris-Rips complex of tori with the maximum metric is completely determined by the Vietoris-Rips complex of the circle as shown in Theorem 2, namely, it is the product of the simplicial complexes.

3.1 Products Vietoris-Rips Over Finite Sets

Analogously to the way Adamaszek et al. computed the homotopy type of the Vietoris-Rips complexes of the circle first by dealing with *finite* subsets of S^1, in this section we show that when we do this on a product with the appropriate metric, the expression is the best one could possibly hope for.

Let us start with the following observation justifying the introduction of the box-product for finite graphs in Chapter 2. Recall that if K is a simplicial complex, then $K^{(1)}$ denotes its 1-skeleton.

Lemma 47. *Let X and Y be two finite sets with distance functions d_1 and d_2, respectively. Let $X \times Y$ have the maximum metric*

$$d((a_1, b_1), (a_2, b_2)) := \max\{d_1(a_1, a_2), d_2(b_1, b_2)\}.$$

Then

$$\mathbf{VR}_< (X \times Y; r)^{(1)} = \mathbf{VR}_< (X; r)^{(1)} \boxtimes \mathbf{VR}_< (Y; r)^{(1)} .$$

Proof. We may consider the 1-skeleton of a simplicial complex as a graph and so the boxproduct on graphs from Chapter 2 makes sense. The vertex set of the two graphs is equal to $X \times Y$ (in the boxproduct it is simply the product of the vertex sets, which are X and Y).

Let us denote by a — b adjacency in a graph between the vertices a and b.

Now, we have (a_1, b_1) — (a_2, b_2) in $\mathbf{VR}_<(X \times Y; r)$ if and only if $d((a_1, b_1)) < r$, $d_1(a_1, a_2) < r$ and $d_2(b_1, b_2) < r$, which means that a_1 — a_2 in $\mathbf{VR}_<(X; r)$ and b_1 — b_2 in $\mathbf{VR}_<(Y; r)$.

On the other hand, (a_1, b_1) — (a_2, b_2) in $\mathbf{VR}_<(X; r)^{(1)} \boxtimes \mathbf{VR}_<(Y; r)^{(1)}$ if and only if either $a_1 = a_2$ and b_1 — b_2 (so $d_1(a_1, a_2) = 0$ and $d_2(b_1, b_2) < r$), $b_1 = b_2$ and a_1 — a_2 (so $d_1(a_1, a_2) < r$ and $d_2(b_1, b_2) = 0$), or a_1 — a_2 and b_1 — b_2 (so $d_1(a_1, a_2) < r$ and $d_2(b_1, b_2) < r$), and any of these cases imply that $d((a_1, a_2), (b_1, b_2)) < r$. ■

From the previous lemma we obtain immediately the following.

Proposition 48. *Let X and Y be two finite sets with distance functions d_1 and d_2, and let $X \times Y$ have the maximum metric. Then*

$$\mathbf{VR}_<(X \times Y; r) \simeq \mathbf{VR}_<(X; r) \times \mathbf{VR}_<(Y; r).$$

Proof. For any set X we have that $\mathbf{VR}_<(X; r) = \mathbf{Cl}(\mathbf{VR}_<(X; r)^{(1)})$. So by Lemma 47 and Theorem 46 we get

$$\begin{aligned} \mathbf{VR}_<(X \times Y; r) &= \mathbf{Cl}(\mathbf{VR}_<(X \times Y; r)^{(1)}) \\ &= \mathbf{Cl}(\mathbf{VR}_<(X; r)^{(1)} \boxtimes \mathbf{VR}_<(Y; r)^{(1)}) \\ &\simeq \mathbf{Cl}(\mathbf{VR}_<(X; r)^{(1)}) \times \mathbf{Cl}(\mathbf{VR}_<(Y; r)^{(1)}) \\ &= \mathbf{VR}_<(X; r) \times \mathbf{VR}_<(Y; r), \end{aligned}$$

as we wanted to show. ■

Remark 49. Note that neither in Lemma 47 nor in Proposition 48 we paid special attention to the strict inequality $<$ considered in the Vietoris-Rips complexes. Indeed, the same is true when we use $\leqslant$ instead, that is, for the appropriate metrics, we also have

$$\mathbf{VR}_\leqslant(X \times Y; r)^{(1)} = \mathbf{VR}_\leqslant(X; r)^{(1)} \boxtimes \mathbf{VR}_\leqslant(Y; r)^{(1)}$$

and

$$\mathbf{VR}_\leqslant(X \times Y; r) \simeq \mathbf{VR}_\leqslant(X; r) \times \mathbf{VR}_\leqslant(Y; r).$$

Remark 50. Another observation we can make is that the fact that we are dealing with the product of *two* spaces is not essential and can be extended to any finite number of factors. For finite metric spaces $X_1, \ldots, X_n$ we have that

$$\mathbf{VR}_<\left(\prod_{k=1}^{n} X_k; r\right) \simeq \prod_{k=1}^{n} \mathbf{VR}_<(X_k; r),$$

where $\prod_{k=1}^{n} X_k$ is considered with the maximum metric.

As before, the same applies with $\leqslant$.

3.2 Colimits and Products

In the previous section we considered Vietoris-Rips complexes of finite subsets of products. In order to extend this to a space like the circle we need a (co)limit process. Since we want to do this for a product of spaces, this requires the interchange of a limit (specifically, a product) with a colimit. In this section we show the relevant result that will allow us to perform this interchange if we consider appropriate categories for the diagrams involved. A reference for the results in this section is [Mac Lane, 1998].

Recall that a complete (resp. cocomplete) category is one in which all small limits (resp. colimits) exist, such as **Set** or **Top**. It is bicomplete if it is both complete and cocomplete. Let $\mathcal{X}$ be a bicomplete category, $\mathcal{P}$ and $\mathcal{J}$ categories, and F a bifunctor $\mathcal{P} \times \mathcal{J} \to \mathcal{X}$. We want to consider $\mathcal{P}$ for diagrams of limits and $\mathcal{J}$ for diagrams of colimits, and we would like to have an isomorphism

$$\operatorname*{colim}_j \ \lim_p \ F(p,j) \cong \lim_p \ \operatorname*{colim}_j \ F(p,j).$$

Unfortunately, this is not always the case. For instance, take $\mathcal{P} = \mathcal{J} = \{\bullet, \bullet\}$, the category with 2 objects and morphisms only the identities. Define

$$F : \mathcal{P} \times \mathcal{J} \to \mathbf{Set}$$

as the constant functor sending everything to a one point set $\{*\}$. In this setting, we would like an isomorphism in **Set** between a set with 2 points and one with 4, which is impossible.

In general, nonetheless, there is a canonical map

$$\kappa : \operatorname*{colim}_j \ \lim_p \ F(p,j) \longrightarrow \lim_p \ \operatorname*{colim}_j \ F(p,j)$$

that we will now describe.

By definition of limit, for j fixed and an arrow $p \to p'$, we have a cone of the form

$$\begin{array}{ccc} \lim_p \ F(p,j) & \xrightarrow{v_p} & F(p,j) \\ & \searrow^{v_{p'}} & \downarrow \\ & & F(p',j) \end{array}$$

and similarly for p fixed we have a map $F(p,j) \longrightarrow \operatorname*{colim}_j F(p,j)$. These two things combined give us a commutative diagram

$$\begin{array}{ccccc} \lim_p \ F(p,j) & \xrightarrow{v_p} & F(p,j) & \xrightarrow{\mu_p} & \operatorname*{colim}_j F(p,j) \\ & \searrow^{v_{p'}} & \downarrow & & \downarrow \\ & & F(p',j) & \xrightarrow[\mu_{p'}]{} & \operatorname*{colim}_j F(p',j) \end{array}$$

We thus have a cone by composition

$$\begin{array}{ccc} \lim\limits_{p} F(p,j) & \longrightarrow & \operatorname*{colim}\limits_{j} F(p,j) \\ & \searrow & \downarrow \\ & & \operatorname*{colim}\limits_{j} F(p',j) \end{array}$$

which by the universal property gives a map $\alpha_j : \lim\limits_{p} F(p,j) \longrightarrow \lim\limits_{p} \operatorname*{colim}\limits_{j} F(p,j)$.

Similarly, for an arrow $j \to j'$, we have a cocone

$$\begin{array}{ccc} \lim\limits_{p} F(p,j) & \longrightarrow & \lim\limits_{p} \operatorname*{colim}\limits_{j} F(p,j) \\ \downarrow & \nearrow & \\ \lim\limits_{p} F(p,j') & & \end{array}$$

which by the universal property gives the map we want

$$\kappa : \operatorname*{colim}_{j} \lim_{p} F(p,j) \longrightarrow \lim_{p} \operatorname*{colim}_{j} F(p,j).$$

The following diagram summarizes the construction of κ.

$$\begin{array}{ccccc} F(p,j) & \longleftarrow & \lim\limits_{p} F(p,j) & \longrightarrow & \operatorname*{colim}\limits_{j} \lim\limits_{p} F(p,j) \\ \downarrow & & \downarrow & \swarrow & \\ \operatorname*{colim}\limits_{j} F(p,j) & \longleftarrow & \lim\limits_{p} \operatorname*{colim}\limits_{j} F(p,j) & & \end{array}$$

We now want to construct an inverse arrow for k, but, as we mentioned earlier, this can not always be done. Hence we narrow down the choices for our categories $\mathcal{P}$ and $\mathcal{J}$. We are going to take $\mathcal{J}$ as a finite category and $\mathcal{P}$ a filtered category (defined below), and this will be more than enough for our purposes.

A filtered category resembles the notion of directed set. We say that a category is filtered is the following two conditions hold

1. For any two objects $i, j \in \mathcal{J}$, there is a third object $k \in \mathcal{J}$ and arrows $i \to k$ and $j \to k$, and
2. For any two parallel arrows $u, v : i \rightrightarrows j$ in $\mathcal{J}$, there is an object k in $\mathcal{J}$ and an arrow $w : j \to k$ such that $wu = wv$.

Hence, for instance, if we have a directed set S, we can construct a category $\mathcal{S}$ with one object for each point in P and an arrow $p \to q$ if $p \leqslant q$ in S. Then $\mathcal{S}$ is a filtered category.

We then have the following property.

Here and from now on, we may restrict to compactly generated spaces and retopologize when necessary, recalling that there is always a weak homotopy equivalence between a space and its associated k-space, as this operation is right adjoint to the inclusion functor. A reference for this is [Strickland, 2009].

Proposition 51. *If the category $\mathcal{P}$ is finite and $\mathcal{J}$ is a small, filtered category, then for any bifunctor $F : \mathcal{P} \times \mathcal{J} \to$ **Top**, we have that the canonical map*

$$\kappa : \operatorname*{colim}_j \lim_p F(p,j) \longrightarrow \lim_p \operatorname*{colim}_j F(p,j)$$

is an isomorphism[1].

Proof. We are going to construct an inverse arrow η for κ. We know an explicit way to construct colimits in **Top**: these take the form, for p fixed,

$$\operatorname*{colim}_j F(p,j) = \left(\coprod_j F(p,j) \right) / \sim,$$

where $\sim$ is the equivalence relation making two elements $x_j \in F(p,j)$ and $x_{j'} \in F(p,j')$ if and only there exist an object k and arrows $u : j \to k$, $u' : j' \to k$ in $\mathcal{J}$ such that $F(p,-)(u)x = F(p,-)(u')x'$.

Let us denote by (x,j) the equivalence class of $x \in F(p,j)$, and note that the property of $\mathcal{J}$ being filtered allows us to easily describe elements in $\operatorname{colim}_j F(p,j)$ because for finitely many elements $(x_1,j_1),\dots,(x_m,j_m)$ we can find an index k and arrows $j_r \to k$ such that $(x_r,j_r) = (y_r,k)$ (we take elements in pairs) for $r = 1,\dots,m$, and, if there are two of them equal, say (y_1,k) and (y_2,k), there is an arrow $k \to k'$ so that $(y_1,k) = (y,k') = (y_2,k)$.

Let $G(p) := \operatorname*{colim}_j F(p,j)$. We know the form of limits in **Top** too. Since $\mathcal{P}$ is finite, we have that

$$\lim_p G(p) = \left(\prod_{r=1}^{n} G(p_r) \right) / \sim,$$

so an element z of $\lim_p G(p)$ can be described with (the equivalence class of) a tuple $z = (z_{p_1},\dots,z_{p_n})$, where each z_{p_r} is a element of $G(p_r)$. This implies that $z_{p_r} = (x_{p_r}, j_{p_r})$ for some $j_{p_r} \in \mathcal{J}$ and $x_{p_r} \in F(p_r,j_r)$, and, as we said before, we can find an index k so that

$$(z_{p_1},\dots,z_{p_n}) = ((x_{p_1},k),\dots,(x_{p_n},k)),$$

and this is an element (x,k) of $\lim_p G(p)$ where $x = (x_{p_1},\dots,x_{p_n})$.

Hence we define $\eta : \lim_p \operatorname*{colim}_j F(p,j) \longrightarrow \operatorname*{colim}_j \lim_p F(p,j), \quad z \mapsto [(x,k)]$.

[1] **Top** is bicomplete, so all these objects exist.

In this explicit context, the canonical map that we defined earlier does the following. For x in $\operatorname{colim}_j \lim_p F(p,j)$,

$$x = (x, j') = ((x_{p_r})_{r\in\mathcal{P}}, j') \mapsto ((x_{p_r}, j'))_{r\in\mathcal{P}} \mapsto ([x_{p_r}, j'])_{r\in\mathcal{P}}$$

in $\lim_p \operatorname{colim}_j F(p,j)$, and no choices were made.

It is clear that η and κ are inverses of each other. ■

Remark 52. We note that Proposition 51 may not be true if $\mathcal{P}$ is not finite. Let $\mathcal{J}$ and $\mathcal{P}$ be the categories obtained from the posets of natural numbers $\mathbf{N} = \{0, 1, 2, \ldots\}$, the former from the usual order of $\mathbf{N}$ and the latter with the trivial order $n \leqslant n$ for all $n \in \mathbf{N}$. Let us define the bifunctor

$$F : \mathcal{P} \times \mathcal{J} \to \mathbf{Top}, \quad (n, m) \mapsto F(n, m) = [m] = \{0, 1, \ldots, m\}$$

with the discrete topology, and if $m \to m'$ in $\mathcal{J}$, the arrow $[m] \to [m']$ is the inclusion. Then we have that

$$\lim_{n \text{ in } \mathcal{P}} \operatorname*{colim}_{m \text{ in } \mathcal{J}} F(n,m) = \lim_{n \text{ in } \mathcal{P}} \operatorname*{colim}_{m \text{ in } \mathcal{J}} [m] = \lim_{n \text{ in } \mathcal{P}} \mathbf{N} = \mathbf{N}^{\mathbf{N}},$$

but, on the other hand,

$$\operatorname*{colim}_{m \text{ in } \mathcal{J}} \lim_{n \text{ in } \mathcal{P}} F(n,m) = \operatorname*{colim}_{m \text{ in } \mathcal{J}} \lim_{n \text{ in } \mathcal{P}} [m] = \operatorname*{colim}_{m \text{ in } \mathcal{J}} [m]^{\mathbf{N}} = \bigcup_{m\in\mathbf{N}} [m]^{\mathbf{N}}.$$

The canonical map in this case is the inclusion

$$\kappa : \bigcup_{m\in\mathbf{N}} [m]^{\mathbf{N}} \hookrightarrow \mathbf{N}^{\mathbf{N}}$$

and it is not an isomorphism because it is not even surjective. Moreover, we know that $\mathbf{N}^{\mathbf{N}} \cong \mathbf{R} \setminus \mathbf{Q}$ with the subspace topology from the usual topology on $\mathbf{R}$ (via continued fractions), so $\mathbf{N}^{\mathbf{N}}$ is not σ-compact (i.e., it cannot be written as a countable union of compact sets) as a consequence of Baire theorem. On the other hand, each $[m]^{\mathbf{N}}$ is compact by Tychonoff's theorem, so $\bigcup_{m\in\mathbf{N}}[m]^{\mathbf{N}}$ is σ-compact. This shows that actually no isomorphism between the two spaces is possible.

3.3 Vietoris-Rips Complexes of a Product

We finally have all the tools we need to show that the Vietoris-Rips complex of a product is the product of the Vietoris-Rips complexes.

As before, let $F(X)$ be the poset of all finite subsets of X ordered by inclusion and let us also denote by $F(X)$ the associated category to the poset. Also, as mentioned earlier, $F(X)$ is filtered because the poset $(F(X), \subseteq)$ can be seen as a directed set because whenever $U, U' \subseteq F(X)$, we have $U \leqslant U \cup U'$ and $U' \leqslant U \cup U'$.

Let π_i denote the projection from $A \times B$ onto the i-th coordinate.

Theorem 53. *Let S^1 be the circle thought of as $\mathbf{R}/\mathbf{Z}$ with its metric inherited from the usual in $\mathbf{R}$, and let the torus $T \subseteq \mathbf{R}^4$ have the maximum metric. Then*

$$\mathbf{VR}_< (T; r) \simeq \mathbf{VR}_< \left(S^1; r\right) \times \mathbf{VR}_< \left(S^1; r\right)$$

for any $r > 0$.

Proof. Let us note that in the poset $(F(S^1 \times S^1), \subseteq)$ the subposet $(G(S^1 \times S^1), \subseteq)$, where

$$G(S^1 \times S^1) = \{X \times Y : X \in F(S^1), Y \in F(S^1)\},$$

is cofinal because if $Z \in F(S^1 \times S^1)$, then $Z \subseteq \pi_1(Z) \times \pi_2(Z)$, and clearly $\pi_1(Z) \times \pi_2(Z) \in G(S^1 \times S^1)$.

Hence

$$\begin{aligned}
\mathbf{VR}_< (T; r) = \mathbf{VR}_< \left(S^1 \times S^1; r\right) &= \operatorname*{colim}_{Y \text{ in } F(S^1 \times S^1)} \mathbf{VR}_< (Y; r) \\
&= \operatorname*{colim}_{X \times Z \text{ in } F(S^1 \times S^1)} \mathbf{VR}_< (X \times Z; r) \\
&\simeq \operatorname*{colim}_{X \times Z \text{ in } F(S^1 \times S^1)} \mathbf{VR}_< (X; r) \times \mathbf{VR}_< (Z; r) \\
&= \operatorname*{colim}_{X \times Z \text{ in } F(S^1 \times S^1)} \mathbf{VR}_< (\pi_1(X \times Z); r) \times \mathbf{VR}_< (\pi_2(X \times Z); r) \\
&\cong \left(\operatorname*{colim}_{X \times Z \text{ in } F(S^1 \times S^1)} \mathbf{VR}_< (\pi_1(X \times Z); r) \right) \\
&\qquad \times \left(\operatorname*{colim}_{X \times Z \text{ in } F(S^1 \times S^1)} \mathbf{VR}_< (\pi_2(X \times Z); r) \right) \\
&= \left(\operatorname*{colim}_{X \text{ in } F(S^1)} \mathbf{VR}_< (X; r) \right) \times \left(\operatorname*{colim}_{Z \text{ in } F(S^1)} \mathbf{VR}_< (Z; r) \right) \\
&= \mathbf{VR}_< \left(S^1; r\right) \times \mathbf{VR}_< \left(S^1; r\right),
\end{aligned}$$

where we have used the earlier note in the proof in the third equality, Proposition 48 for the homotopy equivalence and Proposition 51 for the homeomorphism when we interchanged the colimit and the product of the Vietoris-Rips complexes. ■

The proposition works too for $\leqslant$ instead of $<$; that is, we also have

$$\mathbf{VR}_\leqslant (T; r) \simeq \mathbf{VR}_\leqslant \left(S^1; r\right) \times \mathbf{VR}_\leqslant \left(S^1; r\right).$$

Once we know we can express the Vietoris Rips complex of the torus in terms of the Vietoris-Rips of the circle, we arrive at the main theorem.

Let $T^s = S^1 \times \cdots \times S^1$ (s times).

Theorem 54. *Let T^s have the maximum metric where each S^1 has the metric from $\mathbf{R}$ viewed as $\mathbf{R}/\mathbf{Z}$. Let $0 \leqslant r \leqslant \frac{1}{2}$. Then*

$$\mathbf{VR}_{<}(T^s; r) \simeq \prod_{k=1}^{s} S^{2l+1} \quad \textit{for} \quad \frac{l}{l+1} < r \leqslant \frac{l+1}{2l+3}$$

for $l = 0, 1, 2, \ldots$, and

$$\mathbf{VR}_{\leqslant}(T^s; r) \simeq \begin{cases} \prod_{k=1}^{s} \bigvee^{\mathfrak{c}} S^{2l} & \textit{if} \quad r = \frac{l}{l+1} \\ \prod_{k=1}^{s} S^{2l+1} & \textit{if} \quad \frac{l}{2l+1} < r < \frac{l+1}{2l+3}, \end{cases}$$

for $l = 0, 1, 2, \ldots$, where $\mathfrak{c}$ is the cardinality of the continuum.

Proof. Consequence of Theorem 2, Theorem 3 and Theorem 53. ∎

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