

A survey on orbifold string topology

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ABSTRACT. The aim of this short communication is to review some classical results on string topology of manifolds and discuss recent extensions of this theory to orbifolds. In particular, we review the relation between the loop homology of the classifying space of the orbifold and the Hochschild cohomology of dg-ring naturally associated to the orbifold.

Introduction

String topology is the study of the topological properties of to the space of smooth loops LM on a closed and oriented manifold M . In particular algebraic structures defined on the homology of the loop space. The area started with the paper [5] by Chas and Sullivan where they defined an intersection product in the homology of the free loop space

$$H_p(LM) \otimes H_q(LM) \rightarrow H_{p+q-n}(LM),$$

having total degree $-n$.

Orbifolds, originally known as V -manifolds by I. Satake [18], and named this way by W. Thurston [19], are a generalizations of manifolds, locally they look like the quotient of euclidean space by the action of a finite group. They appear naturally in many areas such as moduli problems, noncommutative geometry and topology. It is natural to ask, *What is the string topology of an orbifold?* In this survey article we give different answers in the case where the orbifold is a global quotient.

The character of this paper is expository and is organized as follows. In section one we review some of the classical theory for manifolds, and state the results of Chas-Sullivan [5] and Cohen-Jones [7] on the string topology of manifolds. In section two we discuss the necessary background on orbifolds. In section three, we review the construction of [14] of the string topology of quotient orbifolds over the rationals and the construction of [3] of the string topology of quotient orbifolds over the integers. In section four we specialize to the case of $[*/G]$ and give geometric descriptions of the string topology product on the cohomology $H^*(LBG)$.

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1. String topology of manifolds

In this section we present the basic results on the classical theory of string topology of manifolds. We give the definition of the string topology product using intersection theory on loop spaces, and the description at the level of spectra of this product.

1.1. Intersection theory. From the point of view of algebraic topology, the most fundamental property of closed oriented manifold of finite dimension n is Poincare duality, which asserts that the bilinear pairing giving by cup product and evaluating on the fundamental class is a non-degenerate pairing

$$\begin{aligned} H^{n-p}(M) \times H^p(M) &\rightarrow \mathbf{k} \\ (\alpha, \beta) &\rightarrow \langle \alpha \cup \beta, [M] \rangle \end{aligned}$$

where homology and cohomology are taken with values in a field \mathbf{k} . By the universal coefficients theorem $H_p(M) \cong (H^p(M))^*$ and we have that the Poincare duality pairing identifies homology with cohomology

$$P.D : H_p(M) \cong (H^p(M))^* \cong H^{n-p}(M).$$

Since the cohomology of space is a ring, we have that the homology $H_{*+n}(M)$ (shifted by the dimension) of a closed oriented manifold has a ring structure \star (grading preserving because of the shift).

$$\begin{array}{ccc} H_p(M) \otimes H_q(M) & \xrightarrow{\star} & H_{p+q-n}(M) \\ \downarrow P.D \otimes P.D & & \uparrow P.D^{-1} \\ H^{n-p}(M) \otimes H^{n-q}(M) & \xrightarrow{\cup} & H^{2n-p-q}(M) \end{array}$$

This ring structure is called *the intersection ring of the manifold*. This ring structure can be interpreted in terms of intersection theory because when two homology classes $\sigma \in H_p(M), \tau \in H_q(M)$ can be represented by submanifolds ¹ $P^p \subseteq M$ and $Q^q \subseteq M$ respectively, then the product is represented by the transverse intersection $P \cap Q$.

The intersection ring can be realized at the level of spaces by a Thom-Pontrjagin collapse map. Take the diagonal embedding $\Delta : M \rightarrow M \times M$ which has a tubular neighborhood ν_Δ homeomorphic to the total space of the normal bundle, which is TM . Consider the projection map that collapses the complement of the tubular neighborhood to a point

$$\tau_\Delta : M \times M \rightarrow (M \times M)/\nu_\Delta^c$$

$(M \times M)/\nu_\Delta^c$ is homeomorphic to the Thom space M^{TM} of the normal bundle (the one point compactification). Thus we have a Thom-Pontrjagin collapse map

$$\tau_\Delta : M \times M \rightarrow M^{TM}$$

¹We say that a homology class $\sigma \in H_p(M)$ can be represented by submanifold if there exists a p -dimensional closed oriented submanifold P , such that σ is the pushforward of the fundamental class of P under the inclusion map

which induces on homology a map that composed with the Thom isomorphism and the Kunneht map

$$H_p(M) \otimes H_q(M) \longrightarrow H_{p+q}(M \times M) \xrightarrow{\tau_{\Delta*}} H_{p+q}(M^{TM}) \xrightarrow{Thom} H_{p+q-n}(M)$$

realizes the intersection product.

The key fact of what we did is that if a map $e : P \rightarrow Q$ is an embedding with finite dimensional normal bundle, then we have a Thom-Pontrjagin collapse map $Q \rightarrow P^{\nu(e)}$ from Q to the Thom space of the normal bundle $\nu(e) \rightarrow P$. If the normal bundle is orientable after applying the Thom isomorphism we obtain a wrong way or umkehr map $e_! : H_*(Q) \rightarrow H_{*-\dim \nu(e)}(P)$.

1.2. String topology of manifolds. Let LM be the space of piecewise smooth maps from S^1 to M , the free loop space. Since LM is infinite dimensional and we do not have Poincare duality we do not expect to have a non-trivial ring structure on the homology. But still, Chas and Sullivan defined a "loop product" in the homology of the free loop space of a closed oriented n -dimensional manifold

$$\circ : H_p(LM) \otimes H_q(LM) \rightarrow H_{p+q-n}(LM).$$

Let $Map(8, M)$ be the space of piecewise smooth maps from the figure 8 to M . This space fits into a pullback square

$$\begin{array}{ccc} Map(8, M) = LM \times_M LM & \longrightarrow & LM \times LM \\ \downarrow & & \downarrow ev \times ev \\ M & \xrightarrow{\Delta} & M \times M \end{array}$$

Since the evaluation at zero $ev : LM \rightarrow M$ can be considered as a fiber bundle of infinite dimensional manifolds, the map $Map(8, M) \rightarrow LM \times LM$ is an embedding of finite codimension, its normal bundle is given by the inverse image of the normal bundle to $M \rightarrow M \times M$. Thus we have a Thom-Pontrjagin collapse map $\tau_{\Delta} : LM \times LM \rightarrow Map(8, M)^{ev^*(TM)}$.

There is a concatenation map $Map(8, M) \rightarrow LM$ that extends the usual Pontrjagin product $\Omega M \times \Omega M \rightarrow \Omega M$. Composing the Kunneht map with the map induced by the Thom-Pontrjagin collapse map, together with the Thom isomorphism and the map induced by the concatenation map gives the Chas-Sullivan loop product.

$$\begin{array}{ccc} H_*(LM) \otimes H_*(LM) & \rightarrow & H_*(LM \times LM) \\ & & \downarrow \tau_{\Delta*} \\ & & H_*(Map(8, M)^{ev^*(TM)}) \xrightarrow{Thom} H_{*-n}(Map(8, M)) \rightarrow H_{*-n}(LM) \end{array}$$

1.3. String Topology spectra. Cohen and Jones [7] realized homotopically this operation, by constructing a ring spectrum

$$LM^{-TM} \wedge LM^{-TM} \rightarrow LM^{-TM},$$

where LM^{-TM} is the Thom spectrum of a virtual bundle over the loop space.

The Poincare duality identification is realized at the level of stable homotopy theory by the composition of the Thom isomorphism for stable normal bundle $-TM \rightarrow M$ (a virtual bundle of dimension $-n$)

$$H_p(M^{-TM}) \cong H_{p+n}(M)$$

and the Atiyah duality $M^{-TM} \simeq F(M_+, S)$, that identifies M^{-TM} as the Spanier-Whitehead dual of M_+^2 . Therefore $H_*(M^{-TM}) \cong H_*(F(M_+, S)) \cong H^{-*}(M)$.

The classical construction of the Thom spectrum M^{-TM} is as follows, given an embedding $e : M \hookrightarrow \mathbb{R}^k$, let $\nu(e) \rightarrow M$ the normal bundle of the embedding and denote by $M^{\nu(e)}$ its Thom space. Define the spectrum

$$M^{-TM} := \Sigma^{-k} M^{\nu(e)}$$

as the k -th de-suspension of the Thom space $M^{\nu(e)}$.

The diagonal embedding $\Delta : M \rightarrow M \times M$ has a Thom-Pontrjagin collapse map $M \times M \rightarrow M^{TM}$ that can be twisted by the bundle $-2TM$ to obtain a map spectra

$$\Delta^* : M^{-TM} \wedge M^{-TM} = (M \times M)^{-2TM} \rightarrow M^{TM-2TM} = M^{-TM}$$

that is associative up to homotopy.

THEOREM 1.1 (Atiyah Duality). *There exists a ring spectrum structure on the Thom spectrum M^{-TM} which induced (after applying the Thom isomorphism) the intersection product on homology.*

In a similar fashion, we can twist the concatenation map $Map(8, M) \rightarrow LM$ by the virtual bundle $ev^*(-TM)$ to obtain a map of spectra $Map(8, M)^{ev^*(-TM)} \rightarrow LM^{ev^*(-TM)}$, and the Thom-Pontrjagin collapse map $\tau_\Delta : LM \times LM \rightarrow Map(8, M)^{ev^*(TM)}$ by $ev^*(-2TM)$ to get a map of spectra

$$LM^{ev^*(-TM)} \wedge LM^{ev^*(-TM)} \rightarrow Map(8, M)^{ev^*(-TM)}.$$

Composing these maps we get a map of spectra

$$LM^{ev^*(-TM)} \wedge LM^{ev^*(-TM)} = (LM \times LM)^{ev^*(-2TM)} \rightarrow Map(8, M)^{ev^*(-TM)} \rightarrow LM^{ev^*(-TM)}$$

that is associative up to homotopy.

THEOREM 1.2 ([7]). *There exists a ring spectrum structure on the Thom spectrum $LM^{ev^*(-TM)}$ which induces the Chas-Sullivan product on homology.*

In the same paper Cohen and Jones give a cosimplicial model of LM^{-TM} which is used to prove that for a simply connected manifold the string topology is related with Hochschild cohomology of the cochains of the manifold.

Recall that for an associative k -algebra A and a A -bimodule M , we have the Hochschild homology and cohomology of A with coefficients in M are

$$HH_*(A, M) \cong \mathrm{Tor}_{A \otimes A^{op}}^*(A, M)$$

$$HH^*(A, M) \cong \mathrm{Ext}_{A \otimes A^{op}}^*(A, M)$$

and these functors can be extended to the category of differential graded algebras. This extension can be done by using the bar construction for differential graded algebras to obtain complexes

²For a space X , the disjoint union of X with a point is denoted by X_+ . For a space X the collection of unbased maps from X to a spectrum Y is denoted by $F(X_+, Y)$.

$$\begin{aligned} A \otimes_{A^e}^L M &= B(A) \otimes_{A^e}^L M \\ \mathcal{R}Hom_{A^e}(A, M) &= Hom_{A^e}(B(A), M) \end{aligned}$$

that calculate the Hochschild homology and cohomology of the dga A with coefficients in the dg-bimodule M .

$$\begin{aligned} HH_*(A, M) &= H^*(A \otimes_{A^e}^L M) \\ HH^*(A, M) &= H_*(\mathcal{R}Hom_{A^e}(A, M)) \end{aligned}$$

The cosimplicial model of LM^{-TM} is used to prove,

THEOREM 1.3 ([7]). *For M a simply connected, oriented, closed manifold, there is a quasi-isomorphism*

$$C_*(LM^{-TM}) \xrightarrow{\cong} \mathcal{R}Hom_{C^*(M)^e}(C^*(M), C^*(M))$$

that induces an isomorphism of rings

$$H_{*+n}(LM) \cong HH^*(C^*(M), C^*(M))$$

2. Orbifolds

Orbifolds were first introduced by I. Satake [18] in the fifties as generalizations of smooth manifolds that allow mild singularities. In this section we follow the classical perspective on orbifolds by defining orbifold charts and atlases akin to the way manifolds are defined.

2.1. Charts. Let X be a paracompact Hausdorff topological space. An n -dimensional orbifold chart on X is a triple (\bar{U}, G, U) , where \bar{U} is a connected manifold, G is a finite group acting on \bar{U} , and U is an open subset of X , homeomorphic to \bar{U}/G .

As with manifolds, an orbifold atlas on X is a family of charts $(\bar{U}_\alpha, G_\alpha, U_\alpha)$, that is compatible and covers X . An orbifold structure on X is just an equivalence class of orbifold atlases, where two atlases are equivalent if there is a zig-zag of common refinements.

EXAMPLE 2.1. *A manifold M with an action of a finite group G gives rise to an orbifold structure that we will denote by $[M/G]$.*

EXAMPLE 2.2. *For a finite group G , the orbifold $[*/G]$ is a zero dimensional orbifold, the underlying space has only one point.*

Given a point $x \in X$, take a chart (\bar{U}, G, U) around x , let $\bar{x} \in \bar{U}$ be a lift of x . Then we define the isotropy group of x to be,

$$G_x = \{g \in G \mid g\bar{x} = \bar{x}\}$$

This group is well defined up to isomorphism. A point is called *non-singular* if G_x is trivial, and *singular* otherwise.

EXAMPLE 2.3. *Consider the action of S^1 on $\mathbb{C}^2 - \{0\}$ given by $\lambda(z_1, z_2) = (\lambda^2 z_1, \lambda z_2)$, it is an action with finite stabilizers, and therefore the underlying space, which topologically is just a sphere, has an orbifold structure coming from this action, it has one singular point with isotropy group \mathbb{Z}_2 , but it can be seen that is not the quotient of a manifold by a finite group (even though it is a quotient by a compact Lie group).*

To an orbifold \mathcal{X} we can associate a topological space $B\mathcal{X}$ called the *classifying space of the orbifold*, see [16],[1]. For example for a global quotient $\mathcal{X} = [M/G]$ with G a finite group acting on M , $B\mathcal{X}$ is homotopy equivalent to the homotopy orbit space $EG \times_G M$, where $EG \rightarrow BG$ is the universal principal G -bundle.

To an orbifold \mathcal{X} we have associated two topological spaces, the underlying space $|\mathcal{X}|$ and the classifying space $B\mathcal{X}$; there is a map $B\mathcal{X} \rightarrow |\mathcal{X}|$, but in general this map is far from being a homotopy equivalence.

3. String topology of orbifolds

3.1. Rational string topology. In the case of global quotient orbifolds of the form $[M/G]$ for G finite group, Lupercio and Uribe [13] constructed the loop groupoid $[P_G M/G]$ as the natural model for the free loop space of the orbifold, whose homotopic quotient turns out to be homotopy equivalent to the space of free loops of the homotopy quotient $M \times_G EG$, i.e.

$$P_G M \times_G EG \simeq L(M \times_G EG).$$

Lupercio, Xicoténcatl and Uribe [14] showed that the rational homology of the free loop space of an oriented orbifold

$$H_*(\mathcal{L}(M \times_G EG); \mathbb{Q}) \cong H_*(P_G M; \mathbb{Q})^G$$

could also be endowed with a ring structure; the authors called this structure with the name **orbifold string topology**.

Let X be a topological space and G a discrete group acting on X . Denote by $[X/G]$ the topological groupoid whose objects are X and whose morphisms are $X \times G$ with $s(x, g) = x$ and $t(x, g) = xg$.

The loops on $[X/G]$ can be understood as the groupoid whose objects are the functors from the groupoid $[\mathbf{R}/\mathbf{Z}]$ to $[X/G]$ and whose morphisms are given by natural transformations. More explicitly, we have

DEFINITION 3.1. *The loop groupoid for $[X/G]$ is the groupoid $[P_G X/G]$ whose space of objects is*

$$P_G X := \bigsqcup_{g \in G} P_g X \times \{g\} \quad \text{with} \quad P_g X := \{f : [0, 1] \rightarrow X \mid f(0)g = f(1)\}$$

and which are endowed with the G -action

$$\begin{aligned} G \times P_G X &\rightarrow P_G X \\ ((f, k), g) &\mapsto (fg, g^{-1}kg). \end{aligned}$$

Now, for $t \in \mathbb{R}$ let $e_t : P_g M \rightarrow M$ be the evaluation of a map at the time t : $e_t(f) := f(t)$. Consider the space of composable maps

$$P_g M \times_{1 \times 0} P_h M = \{(\phi, \psi) \in P_g M \times P_h M \mid \phi(1) = \psi(0)\}$$

and note that they fit into the pullback square

$$\begin{array}{ccc} P_g M \times_{1 \times 0} P_h M & \longrightarrow & P_g M \times P_h M \\ \downarrow e_0 \circ \pi_1 & & \downarrow e_0 \times e_0 \\ M & \xrightarrow{\Delta_g} & M \times M \end{array}$$

where the downward arrow $e_0 \circ \pi_1$ is the evaluation at zero of the first map.

The normal bundle of the upper horizontal map becomes isomorphic to the pullback under $e_0 \circ \pi_1$ of the normal bundle of the map Δ_g and therefore we can construct the Thom-Pontrjagin collapse maps $P_g M \times P_h M \rightarrow P_g M_{1 \times 0} P_h M^{\pi_1^* e_0^* N_g}$, and obtain a map of spectra $P_g M^{-e_0^* TM} \wedge P_h M^{-e_0^* TM} \rightarrow P_g M_{1 \times 0} P_h M^{-\pi_1^* e_0^* TM}$. The concatenation of paths in $P_g M_{1 \times 0} P_h M$ produces a map $P_g M_{1 \times 0} P_h M \rightarrow P_{gh} M$ which induces the map of spectra $P_g M_{1 \times 0} P_h M^{-\pi_1^* e_0^* TM} \rightarrow P_{gh} M^{-e_0^* TM}$ that defines the map $\mu_{g,h} : P_g M^{-e_0^* TM} \wedge P_h M^{-e_0^* TM} \rightarrow P_{gh} M^{-e_0^* TM}$. Define

$$P_G M^{-TM} := \bigsqcup_{g \in G} P_g M^{-TM}$$

which by assembling the maps $\mu_{g,h}$ define a map that is associative up to homotopy.

$$\mu : P_G M^{-TM} \wedge P_G M^{-TM} \rightarrow P_G M^{-TM}.$$

THEOREM 3.2 ([14]). *$P_G M^{-TM}$ is a ring spectrum with a G -action in the homotopy category, taking homology, we get a G -module ring $H_*(P_G M^{-TM}; \mathbb{Z})$, the induced ring structure on the invariant set*

$$H_*(P_G M^{-TM}; \mathbb{Q})^G \cong H_{*+n}(L(M \times_G EG); \mathbb{Q})$$

is an invariant of the orbifold $[M/G]$.

This ring was called in [14] the **orbifold string topology ring**.

3.2. Hochschild cohomology of global quotient orbifolds. In the case that M is simply connected, Angel, Backelin and Uribe in [3] showed that the orbifold string topology ring of a global quotient can be recovered as the Hochschild cohomology ring of the dga $C^*(M; \mathbb{Q}) \# G$, i.e. there is an isomorphism of rings

$$HH^*(C^*(M; \mathbb{Q}) \# G, C^*(M; \mathbb{Q}) \# G) \cong H_*(P_G M^{-TM}; \mathbb{Q})^G$$

where $C^*(M) \# G$ is the convolution differential graded algebra,

$$C^*(M) \# G := C^*(M) \otimes \mathbb{Q}G \quad (\alpha, g) \cdot (\beta, h) := (\alpha \cdot g^* \beta, gh).$$

This isomorphism is obtained by carefully decomposing the Hochschild cohomology ring into smaller parts, which leads to the ring isomorphism

$$HH^*(C^*(M; \mathbb{Q}) \# G, C^*(M; \mathbb{Q}) \# G) \cong \text{Ext}_{\mathbb{Q}G}^*(\mathbb{Q}, C_*(P_G M^{-TM}; \mathbb{Q})).$$

The decomposition works also over the integers and starts by taking any space X and giving an explicit cofibrant replacement for $C^*(X) \# G$ as a $(C^*(X) \# G)^e$ -module, together with an explicit diagonal map. It is shown in [3] that if we consider $C^*(X) \# G$ as a G -module-dg-ring by the action $g \cdot (x \otimes h) \mapsto g(x) \otimes ghg^{-1}$, then we have that

THEOREM 3.3. *For any space X there are isomorphisms of graded groups*

$$HH_*((C^*(X) \# G, C^*(X) \# G) \cong \text{Tor}_{\mathbb{Z}G}^*(\mathbb{Z}, C^*(X) \overset{L}{\otimes}_{C^*(X)^e} C^*(X) \# G)$$

$$HH^*(C^*(X) \# G, C^*(X) \# G) \cong \text{Ext}_{\mathbb{Z}G}^*(\mathbb{Z}, \mathcal{R}\text{Hom}_{C^*(X)^e}(C^*(X), C^*(X) \# G))$$

defined in an appropriate way such that the second isomorphism becomes one of graded rings.

Using a cosimplicial description for the space $P_G X$, in [3] it is proved that

THEOREM 3.4. *If X is simply connected space and G is finite, then there is an equivariant quasi-isomorphism*

$$C^*(X) \otimes_{C^*(X)^e}^L C^*(X) \# G \xrightarrow{\sim} C^*(P_G X)$$

that induces the equivariant isomorphism

$$\mathrm{Tor}_{C^*(X)^e}(C^*(X), C^*(X) \# G) \cong H^*(P_G X).$$

Whenever M is a closed oriented manifold of dimension l , Atiyah duality gives a homomorphism of graded groups

$$C^{-*}(M) \rightarrow C_*(M^{-TM})$$

that induces an isomorphism in homology. By using the multiplicative properties of Atiyah duality and a cosimplicial model of $P_G M$ in [3] it is proved that

THEOREM 3.5. *If M is a simply connected closed oriented manifold and G is finite, then there is an equivariant quasi-isomorphism*

$$C_*(P_G M^{-TM}) \xrightarrow{\sim} \mathcal{R}\mathrm{Hom}_{C^*(M)^e}(C^*(M), C^*(M) \# G)$$

Using the theorem 3.3 and the previous theorem we obtain that

$$HH^*(C^*(M; \mathbb{Q}) \# G, C^*(M; \mathbb{Q}) \# G) \cong \mathrm{Ext}_{\mathbb{Q}G}^*(\mathbb{Q}, C_*(P_G M^{-TM}; \mathbb{Q})).$$

and since rationally taking invariants is exact.

$$HH^*(C^*(M; \mathbb{Q}) \# G, C^*(M; \mathbb{Q}) \# G) \cong H_*(P_G M^{-TM}; \mathbb{Q})^G$$

Therefore when M is simply connected, it is natural to consider the ring

$$HH^*(C^*(M) \# G, C^*(M) \# G)$$

as the **orbifold string topology ring with integer coefficients**

3.3. Orbifold string topology pro-spectrum. We would like to have a topological construction associated to the Hochschild cohomology of $C^*(M) \# G$. Since

$$\mathrm{Ext}_{\mathbb{Z}G}^*(\mathbb{Z}, C_*(P_G M^{-TM})) \cong H^*\left((C^*(EG) \otimes C_*(P_G M^{-TM}))^G\right).$$

has a mixture of chain with cochain complexes, we need to introduce some sort of Poincare duality for EG .

Let $EG_1 \subset \cdots \subset EG_n \subset EG_{n+1} \subset \cdots \subset EG$ be a finite dimensional closed manifold approximation of the universal G -principal bundle $EG \rightarrow BG$. Consider the maps

$$P_G M \times_G EG_n \xrightarrow{e_t} M \times_G EG_n$$

where by abuse of notation we denote the evaluation at the time t of a pair $(f, \lambda) \in P_G M \times_G EG_n$ also by e_t . Let us call $M_n = M \times_G EG_n$.

Take the Thom spectra formed by pulling back the negative of the tangent bundle of M_n under e_0

$$(P_G M \times_G EG_n)^{-e_0^* T M_n}.$$

The inclusions $EG_n \subset EG_{n+1} \subset \cdots \subset EG$ induce an inverse system of maps

$$(P_G M \times_G EG_n)^{-e_0^* T(M_n)} \leftarrow (P_G M \times_G EG_{n+1})^{-e_0^* T(M_{n+1})} \leftarrow \cdots$$

that defines a pro-ring spectrum called $\mathcal{L}(M \times_G EG)^{-T(M \times_G EG)}$. The homology of this pro-spectrum

$$H_*^{\text{pro}}(\mathcal{L}(M \times_G EG)^{-T(M \times_G EG)}) := \lim_{\leftarrow n} H_*((P_G M \times_G EG_n)^{-e_0^* T(M \times_G EG_n)})$$

is the orbifold string topology ring of the orbifold $[M/G]$ that in the simply connected case recovers the Hochschild cohomology of $C^*(M)\#G$.

THEOREM 3.6 ([3]). *If M is simply connected and G is finite*

$$HH^*(C^*(M)\#G, C^*(M)\#G) \cong H_*^{\text{pro}}\left(\mathcal{L}(M \times_G EG)^{-T(M \times_G EG)}\right).$$

4. String topology of point orbifolds

When $M = *$ the orbifold string topology of $[*/G]$ is precisely the pro-ring spectrum LBG^{-TBG} which has been studied in [10],[11] and [12] for general compact Lie groups G . In the case of a finite group G , the loop groupoid $\mathbb{L}[*/G]$ is the groupoid $[G^{ad}/G]$ that is the translation groupoid associated to the action of G on itself by conjugation, since G is a discrete group we have an equivalence of groupoids

$$[G^{ad}/G] \cong \bigsqcup_{(g)} [*/C(g)]$$

where the union runs over the conjugacy classes of elements in G and $C(g)$ is the centralizer of g in G . Rationally, since all centralizers are finite, we have $H_*(P_G M^{-TM}; \mathbb{Q})^G \cong \mathbb{Q}G$ concentrated in degree zero and where the ring product described in section 3.1 is just the product of the group ring.

Approximate EG by $EG_1 \subset \dots \subset EG_n \subset EG_{n+1} \subset \dots \subset EG$ finite dimensional closed manifolds. By definition $BG_n := EG_n/G$ and the pro-ring spectrum LBG^{-TBG} is therefore the inverse system formed by the spaces $(G^{ad} \times_G EG_n)^{-TBG_n}$.

The space $G^{ad} \times_G EG_n$ decomposes into a disjoint union of spaces $\bigsqcup_{(g)} EG_n/C(g)$ and the tangent space $T(G^{ad} \times_G EG_n)$ is isomorphic to $\bigsqcup_{(g)} T(EG_n/C(g))$. Since any model of EG is also a model of EH for any subgroup $H \subseteq G$ we can take $EG_1 \subset \dots \subset EG_n \subset EG_{n+1} \subset \dots$ to be an approximation of each $EC(g)$. Therefore we have that the pro-ring spectrum LBG^{-TBG} is formed by the spaces

$$\bigsqcup_{(g)} BC(g)_n^{-TBC(g)_n}$$

The homology of LBG^{-TBG} is therefore the inverse limit of $\bigoplus_{(g)} H_*((BC(g)_n)^{-TBC(g)_n})$, which is by Atiyah Duality the inverse system of $\bigoplus_{(g)} H^*(BC(g)_n)$, that is just $\bigoplus_{(g)} H^*(BC(g))$. To get the ring structure we have to understand the map induced by the correspondance:

$$EG_n/C(g) \times EG_n/C(h) \xleftarrow{\Delta} EG_n/(C(g) \cap C(h)) \longrightarrow EG_n/C(gh)$$

First the Thom-Pontrjagin construction applied to Δ

$$H_*(EG_n/C(g) \times EG_n/C(h)) \rightarrow H_{*-k_n}(EG_n/(C(g) \cap C(h)))$$

with $k_n = \dim(EG_n)$, is Poincaré dual to the pull-back map in cohomology. Second, the natural map in homology

$$H_*(EG_n/(C(g) \cap C(h))) \rightarrow H_*(EG_n/C(gh))$$

is Poincaré dual to the push-forward map in cohomology

$$H^*(EG_n/(C(g) \cap C(h))) \rightarrow H^*(EG_n/C(gh))$$

that defines the induction map

$$H^*(B(C(g) \cap C(h))) \rightarrow H^*(BC(gh)).$$

Therefore the ring structure in $H_*^{\text{pro}}(\mathcal{L}BG^{-TBG})$ is obtained by taking classes in $H^*(BC(g))$ and $H^*(BC(h))$, pulling them back to $H^*(B(C(g) \cap C(h)))$ and then pushing them forward to $H^*(BC(gh))$. This procedure is what is known as the pull-push formalism. For a finite abelian group, this is just $H^*(BG) \otimes \mathbb{Z}G$.

For a finite group G theorem 3.6 gives an isomorphism of rings

$$HH^*(\mathbb{Z}G, \mathbb{Z}G) \cong H_*^{\text{pro}}(\mathcal{L}BG^{-TBG}).$$

The prospectrum LBG^{-TBG} is Spanier-Whitehead dual (in the sense of Christensen and Isaksen [8]) to a spectrum LBG^{-ad} . When G is finite this is just the suspension spectrum of the space LBG and therefore we have an isomorphism of groups

$$H_*^{\text{pro}}(\mathcal{L}BG^{-TBG}) \cong H^*(LBG).$$

There is a product on the cohomology of LBG that can be described by a coproduct at the level of spectra defined using a transfer map of the multiplication of the group on the suspension spectrum of LBG . This coproduct is described as follows:

For any topological group G , the space of free loops on the classifying space is homotopically equivalent to the Borel construction of the action of G on itself by conjugation $LBG \cong EG \times_G G$.

Furthermore when G is finite the multiplication map $\mu : G \times G \rightarrow G$ is a (trivial) covering space and defines a stable transfer map $m_! : G_+ \rightarrow G \wedge G_+$. This transfer map is equivariant and induces a map on the suspension spectrum of the Borel construction $EG \times_G G \rightarrow EG \times_G (G \times G)$.

Composing the map $EG \times_G G \rightarrow EG \times_G (G \times G)$ with the natural map $EG \times_G (G \times G) \rightarrow (EG \times_G G) \times (EG \times_G G)$ we obtain a map $EG \times_G G \rightarrow (EG \times_G G) \times (EG \times_G G)$, that induces on cohomology a product which makes the isomorphism

$$H_*^{\text{pro}}(\mathcal{L}BG^{-TBG}) \cong H^*(LBG).$$

an isomorphism of rings.

In [6], Chateur and Menichi proved that when taken with coefficients in a field the homology of LBG is a co-unital non unital homological conformal field theory, and the cohomology $H^*(LBG)$ is a Batalin-Vilkovisky algebra structure. In particular they give another way to describe the product on $H^*(LBG)$. Consider the correspondance

$$LBG \leftarrow \text{Map}(\Theta, BG) \rightarrow LBG \times LBG$$

where Θ is the topological space obtained by glueing two circles along a common interval and the left map restricts to the outer circle, and the right map restricts to the upper and lower circle respectively.

The map $\text{Map}(\Theta, BG) \rightarrow LBG$ is a fibration with fiber ΩBG , which is homotopically equivalent to G , that is finite. For fibrations with homotopically finite fiber Dwyer [9] has defined a transfer map on homology and cohomology. Using this transfer map we obtain a map $H^*(\text{Map}(\Theta, BG)) \rightarrow H^*(LBG)$.

Composing the map $H^*(\text{Map}(\Theta, BG)) \rightarrow H^*(LBG)$ with the natural map $H^*(LBG) \otimes H^*(LBG) \rightarrow H^*(\text{Map}(\Theta, BG))$ we obtain a product

$$H^*(LBG) \otimes H^*(LBG) \rightarrow H^*(LBG)$$

that in [2] it is shown to define a ring structure isomorphic to the orbifold string topology of $[*/G]$.

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