ORBIFOLD COBORDISM

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Abstract. In this paper we undertake the study of different cobordism groups of orbifolds, our main results present decompositions of the unoriented cobordism ring of orbifolds with isotropy groups of odd order, and the rational oriented cobordism ring of effective orbifolds; these rings are expresed in terms of bordism rings of classifying spaces of groups.

Introduction

Orbifolds, originally introduced as V-manifolds by Satake, and named this way by Thurston, are generalizations of manifolds, locally they look like euclidean space mod the action of a finite group. Their geometrical and topological properties are just started to being studied, and the purpose of this paper is to contribute to the study of orbifolds by providing a description of several cobordism rings of orbifolds.

In [Dru94] K. Druschel started the study of the cobordism groups of oriented effective orbifolds by introducing a complete set of invariants that determine the cobordism class up to torsion. These generalized Pontrjagin numbers are used to prove that any odd dimensional oriented effective orbifold rationally bounds and unlike in the manifold case there is a $4k+2$ dimensional oriented effective orbifold that does not bound rationally. To study the torsion, K. Druschel in [Dru00] consider cobordisms with restrictions on the set of local groups and how they fit into a commutative diagram to show that every two and three dimensional effective oriented orbifold bounds.

The main results of this paper are,

Theorem 1. *Denote by* $\mathfrak{N}^{odd}_{*,orb}$ *the cobordism ring of effective orbifolds with isotropy groups of odd order, then*

$$\mathfrak{N}^{odd}_{*,orb} \cong \mathfrak{N}_* \bigoplus_{H\ odd} \bigoplus_{Rep_*(H)} \mathfrak{N}_{\mathbf{degree}(\rho)}(B\Gamma_{H,\rho})$$

where the sum extends over all faithfull representations of all groups of of odd order.

Theorem 2. *Denote by* $\Omega_{*,orb}$ *the cobordism ring of effective oriented orbifolds, then*

$$\Omega_{*,orb} \otimes \mathbb{Q} \cong \Omega_* \otimes \mathbb{Q} \bigoplus_{H finite} \bigoplus_{Rep^+_*(H)} \Omega_{\mathbf{degree}(\rho),t}(B\Gamma_{H,\rho}) \otimes \mathbb{Q}$$

where the sum extends over all faithfull orientation preserving representations of finite groups.

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Where given a representation $\rho : H \to O(n)$ of a finite group H, the degree of ρ is the multiplicity of the trivial representation in ρ and $\Gamma_{H,\rho}$ is the group of local automorphisms of the representation. For effective orbifolds, $\Gamma_{H,\rho} \cong N_{O(n-\mathbf{degree}(\rho))}\rho(H)/\rho(H)$.

The starting point of the analysis is that if we take a maximal group H among the isotropy groups of an orbifold $\mathcal{Q}$, then the singular set $\mathcal{Q}^H$ is a compact closed manifold and the normal orbibundle gives rise to a principal bundle with fiber $\mathbb{R}^k/H$ and structure group $N_{O(n-k)}H/H$, where k is the degree of the representation.

Now if this orbibundle bounds, then a simple geometric construction gives a cobordism with an orbifold with strictly less isotropy, these cobordisms can be taken to have the same isotropy groups as the original one. By means of a classifying map we can identify this cobordism group with a more familiar twisted bordism group

$$\Omega_{k,t}(BN_{O(n-k)}H/H)$$

Working over the rationals, the bordism spectral sequence collapses and we have

$$\Omega_{k,t}(BN_{O(n-k)}H/H) \otimes \mathbb{Q} \cong \bigoplus_j \Omega_j \otimes \mathbb{Q} \otimes_{\mathbb{Q}} H_{k-j}(BN_{O(n-k)}H/H; \widehat{Q})$$

where the local coefficients come for the universal bundle over $BN_{O(n-k)}H/H$. This isomorphism allows her to define generalized Pontrjagin numbers.

Now, if H is not maximal among the isotropy groups, the singular set is not in general a suborbifold and one of the main points of K. Druschel construction [Dru94] is a suborbifold Q_H that is used to define these generalized characteristic numbers.

Now if all generalized characteristic numbers are zero, then we take the maximal isotropy group, and since the orbibundle rationally bounds, we get a cobordism with an orbifold with less isotropy, since the characteristic numbers are cobordism invariants, for this new orbifold all characteristic numbers are zero too, we proceed inductively and get that the a multiple of the orbifold is cobordant with a manifold which has all Pontrjagin numbers zero, and therefore bounds rationally.

Adapting techniques from equivariant cobordism to orbifolds we can recover K.Druschel resuts on oriented orbifolds and obtain a similar decomposition in the unoriented case.

In the early sixties Conner and Floyd demonstrated the effectiveness of bordism methods in the analysis of transformation groups. In their monograph [CF64] and in a later paper [CF66] they introduced a framework for the study of actions of finite groups on compact manifolds, the basic idea is to analyze the cobordism groups by the information of the fixed points, their main calculational tools are families of subgroups and fixed point homomorphisms. Our techniques are adaptations to the orbifold setting of these ideas.

This paper is organized as follows. In section one we describe the necessary background on orbifolds. Section two introduces the main calculational tools, family of local representations and cobordism of orbifolds with restricted local representations. In the spirit of [CF66] these cobordism groups with restricted local representations fit into a long exact sequence. By adding extra conditions on our families, the relative term in this long exact sequence can be computed in terms of usual bordism theory. In section three, a spectral sequence coming from a filtration of a family is introduced.

In section four we construct an splitting of the long exact sequence in the unoriented context, it is given by the projectivisation of the normal bundle plus a trivial one. Theorem 1 follows from this splitting.

Section six applies our machinery to the rational oriented cobordism ring of complex type, in a similar vein as in section four we construct an splitting and obtain theorem 2.

There are interesting geometric consequences of our results, in particular we can give a partial answer to the question, When does a manifold bound an orbifold?.

1. Preliminaries

Manifolds are useful generailzations of orbifolds that have been used in geometry and topology. The building blocks of orbifolds are representations.

Notation 1. *Given an n-dimensional representation (H, ρ) of a finite group H we denote by $V(\rho)$, $\mathbb{R}^n$ with the action given by ρ. $S(\rho), D(\rho)$ the corresponding sphere and disk. S^ρ the one compactification of the representation. The representation ρ determines a homomorphism $\rho : H \to O(n)$. well defined up to conjugation in $O(n)$ and if no confusion arise we will identify H with a subgroup of $O(n)$.*

Given a local representation (H, ρ), the degree, $\mathbf{degree}(\rho)$, is the multiplicity of the trivial representation i.e. the dimension of the subspace fixed by H. We have a decomposition into a sum $\hat{\rho} \oplus \underbrace{id \oplus \cdots \oplus id}_{\mathbf{degree}(\rho)}$, where $\hat{\rho}$ has no trivial summands and id denotes the trivial 1-dimensional representation. The decomposition $\rho \cong \hat{\rho} \oplus id \oplus \cdots \oplus id$, induces a factorization

$$\begin{array}{ccc} H & & O(n - \mathbf{degree}(\rho)) \\ & & \\ & & O(n) \end{array}$$

where $O(n - \mathbf{degree}(\rho)) \to O(n)$ is the natural inclusion. Using this factorization, we define $C_{O(n-\mathbf{degree}(\rho))}H$ to be the centralizer in $O(n - \mathbf{degree}(\rho))$ of the image of H and $N_{O(n-\mathbf{degree}(\rho))}H$ to be the corresponding normalizer.

Let Q be a paracompact Hausdorff topological space.

Definition 1. *A n-dimensional orbifold chart on Q is a triple $(\overline{U}, G, U)$, where $\overline{U}$ is a connected manifold, G is a finite group acting on $\overline{U}$ and U is an open subset of Q, homeomorphic with $\overline{U}/G$. We call π the natural projection $\pi : \overline{U} \to \overline{U}/G \cong U$*

Definition 2. *An embedding of charts*

$$(\overline{U}, G, U) \hookrightarrow (\overline{V}, H, V)$$

is a differentiable embedding $\overline{U} \hookrightarrow \overline{V}$ that is equivariant with respect to a monomorphism $G \hookrightarrow H$, that preserves the kernel of the actions.

Definition 3. *Two charts* $(\overline{U}, G, U)$, $(\overline{V}, H, V)$ *are compatible if for every point in* $U \cap V$, *there exists a chart* $(\overline{W}, K, W)$ *with embeddings of charts*

$$(\overline{U}, G, U) \qquad\qquad (\overline{V}, H, V)$$

$$(\overline{W}, K, W)$$

Definition 4. *An orbifold atlas on* Q *is a family of charts* $(\overline{U}_\alpha, G_\alpha, U_\alpha)$, *that is compatible and cover* Q. *An orbifold structure on* X *is just an equivalence class of orbifold atlases, where two atlases are equivalent if they have a common refinement. We will denote an orbifold structure on* Q *by calligraphic letters, like* $\mathcal{Q}$, *and the topological space* Q, *will be called the underlying space*

Example 1. *A manifold* M *with an action of a finite group* G *gives rise to an orbifold that we will denote by* $[M/G]$. *More generally, if a compact Lie group acts differentiably on a manifold with finite stabilizers the quotient space can be endowed with an orbifold structure. Charts can be constructed with the slice theorem for differentiable actions* [Bre72].

Example 2. *Let* d *be a positive integer. Consider the action of* S^1 *on* $S^3 \subseteq \mathbb{C}^2 - \{0\}$ *given by* $\lambda(z_1, z_2) = (\lambda^d z, \lambda z)$, *it is a an action with finite stabilizers, and therefore the underlying space, which topologically is just a sphere, has an orbifold structure coming from this action, it has one singular point with isotropy group* $\mathbb{Z}_d$. *It can be seen that is not the quotient of a manifold by a finite group (even though it is a quotient by a compact Lie group). It is called the weighted projective space* $W(d, 1)$.

Definition 5. *We say that an orbifold is effective if we can find an atlas for which all the local groups act effectively. On the other side, we say that an orbifold is purely ineffective if all the local groups act trivially.*

The following is a list of important technical facts about effective orbifolds.

Remark 1.

- *Any orbifold has an atlas consisting of linear charts, by which we mean, charts of the form* $(\mathbb{R}^n, G, U)$, *where* G *acts on* $\mathbb{R}^n$ *via an orthogonal representation* $G \subset O(n)$.
- *Given two embedding of charts* $\lambda, \mu : (\overline{U}, G, U) \hookrightarrow (\overline{V}, H, V)$, *there exists a unique* $h \in H$ *such that* $\mu = h \cdot \lambda$.
- *And if* $\lambda(\overline{U}) \cap h \cdot \lambda(\overline{U}) \neq \emptyset$ *then actually* h *is in the image of the monomorphism* $G \hookrightarrow H$.
- *Given a connected open subset* $V \subseteq U$ *of a chart* $(\overline{U}, G, U)$, *let* $\overline{V}$ *be a connected component of* $\pi^{-1}(V)$ *and consider the group*
$$H = \{g \in G \mid g(\overline{V}) = \overline{V}\}$$
The triple $(\overline{V}, H, V)$ *is an orbifold chart and the natural inclusion* $\overline{V} \hookrightarrow \overline{U}$ *gives an embedding of charts.*

Given a point $x \in X$, take a chart $(\overline{U}, G, U)$ around x, let $\overline{x} \in \overline{U}$ be a lift of x. Then we define the isotropy group of x to be,

$$G_x = \{g \in G \mid g\overline{x} = \overline{x}\}$$

Different lifts of x give conjugate subgroups of G and different choices of charts give isomorphic groups. The restriction of the action of the isotropy group on any chart gives a well defined representation of the isotropy group, (G_x, ρ_x), that we call the local representation at x.

A local orientation of an orbifold is a choice of an orientation of $V(\rho_x)$ that makes the action of G_x orientation preserving, this induces orientations of all smaller charts. This is equivalent to identifying G_x with a subgroup of $SO(n)$. As with manifolds an orientation is just a choice of local orientations in such a way that the transition functions are orientation preserving.

2. Families of local representations

In the early sixties Conner and Floyd demonstrated the effectiveness of bordism methods in the analysis of transformation groups. In their monograph [CF64] and in a later paper [CF66], they introduced a framework for the study of actions of finite groups on compact manifolds, the basic idea is to analyze the cobordism groups by the information of the fixed points, their main calculational tools are families of subgroups and fixed point homomorphisms.

In this section we adapt these techniques and introduce families of local representations, cobordism groups of orbifolds with restricted local representations and a long exact sequence relating them. This is much in the spirit of [CF66] and [Kos78].

As with manifolds, we can talk about *orbifolds with boundary*, where the charts correspond to open sets in $\mathbb{R}^{n-1} \times [0, \infty)$ with a finite group acting linearly on it. An orbifold has a well defined boundary, just by taking the restriction of the action to the boundary on each chart. But some care should be taken: being a point on the boundary is not a condition that can be checked topologically on the underlying space. For manifolds the boundary was just the set of points for which

$$H_n(M, M - \{x\}) = 0.$$

Example 3. *Consider the interval $I = [-1, 1]$ with the action of $\mathbb{Z}_2$ by multiplying by -1. The quotient $[I/\mathbb{Z}_2]$ is an orbifold with boundary, the underlying space is an interval, but the boundary is just one point.*

Example 4. *Consider S^1 the complex numbers of norm 1, and the $\mathbb{Z}_2$ action given by complex conjugation, the quotient $[S^1/\mathbb{Z}_2]$ is an orbifold without boundary, but the underlying space is homeomorphic to a closed interval.*

Recall that a family of subgroups of G is a collection of subgroups that is closed under subconjugation. Two families $\mathcal{F}' \subseteq \mathcal{F}'$ are adjacent if $\mathcal{F} - \mathcal{F}'$ consists of a single conjugacy class of subgroups.

Definition 6. *A family, $\mathcal{F}$, of local representations, is a collection of finite dimensional representations of finite groups that satisfies the following conditions*

i) Is closed under isomorphism of representations.

ii) If $(G, \rho) \in \mathcal{F}$ is a representation of the finite group G, then all the representations (G_x, ρ_x) of the isotropy groups for $x \in [V(\rho)/G]$ belong in $\mathcal{F}$.

iii) $(G, \rho) \in \mathcal{F}$ if and only if $(G, \rho \oplus id) \in \mathcal{F}$, where $V \oplus id$ is adding a trivial representation.

We define a partial order between representations,

$$(H, \tau) \leq (G, \rho)$$

if (H, τ) appears among the local representations of of the linear orbifold $[V(\rho)/G]$. In particular H is isomorphic to a subgroup of G, and $\rho \mid H = \tau$.

Remark 2. *We will require that the family $\mathcal{F}$ satisfies the following finiteness condition, the length of $\leq$-chains is bounded. This will allow us to make inductive arguments.*

Remark 3. *The intersection of families is a family, and every collection of representations is contained in a family, therefore we can talk about the family generated by a set of representations.*

Example 5. *The emptyset is a family, given a finite group G the family of finite dimensional representations of subgroups will be denoted by $\{G\}$. Given a representation ρ and a family $\mathcal{F}$, there exists a minimal family containing both, we denote it by $\mathcal{F}\langle\rho\rangle$, in particular there is a minimal family associated to a representation, $\langle\rho\rangle$*

We do not require the families to be closed under restrictions for subgroups, just under the partial order described above, this is enough for our puposes and makes the calculations easier.

By property iii) a family $\mathcal{F}$ has elements in each dimension, we denote by $\mathcal{F}(n)$ the n-dimensional elements of $\mathcal{F}$.

Definition 7. *Given two families of local representations $\mathcal{F} \supseteq \mathcal{F}'$, we say that $\mathcal{Q}$ is an $(\mathcal{F}, \mathcal{F}')$-orbifold if every local representation of $\mathcal{Q}$ belong to $\mathcal{F}$, and all local representations of points on the boundary belong to $\mathcal{F}'$.*

Condition *iii)* gives us that if $\mathcal{Q}$ is an $\mathcal{F}$-orbifold, then $\mathcal{Q} \times I$ is also, and that the boundary of an $(\mathcal{F}, \mathcal{F}')$-orbifold is a $\mathcal{F}'$-orbifold. For example if $[V(\rho)/G]$ is a $\mathcal{F}$-orbifold then $[D(\rho)/G]$ and $[S(\rho)/G]$ are also.

Given a local representation (H, ρ), the degree is the multiplicity of the trivial representation i.e the dimension of the subspace fixed by H, disregarding these part obtain a faithful representation. We define the group of local automorphisms of (H, ρ) to be

$$\Gamma_{(H,\rho)} = \{(\alpha, \sigma) \in N_{O(n-\mathbf{degree}(\rho))}\rho(H) \times Aut(H) \mid C_\alpha \cdot \rho = \rho \cdot \sigma\}/\Delta(H)$$

$\Delta : H \to N_{O(n-\mathbf{degree}(\rho))}\rho(H) \times Aut(H)$ is the map given by

$$h \longrightarrow (\rho(h), C_h)$$

C_h is conjugation by the element h.

In the case that the representation is faithful, ρ is an isomorphism with its image and we can recover σ from α, therefore $\Gamma_{(H,\rho)}$ is isomorphic to $N_{O(n-\mathbf{degree}(\rho))}H/H$.

Definition 8. *We say that, $\mathcal{Q}^n$, an $(\mathcal{F}, \mathcal{F}')$-orbifold bounds if there exists, $\mathcal{W}^{n+1}$, an $(\mathcal{F}, \mathcal{F})$-orbifold with $\mathcal{Q}_1^n$, an embedded, $(\mathcal{F}, \mathcal{F}')$-orbifold, such that $\mathcal{Q}_1^n$ and $\mathcal{Q}^n$ are orbifold diffeomorphic, and such that for all $x \in \partial W^{n+1} \backslash \mathcal{Q}_1^n$, $(G_x, \rho_x) \in \mathcal{F}'$. If the orbifolds are oriented we require $\mathcal{Q}_1^n$ to have the induced orientation from $\mathcal{W}^{n+1}$.*

We say $\mathcal{Q}^n$ is bordant to $\mathcal{Q}_1^n$ if the disjoint union $\mathcal{Q}^n \cup -\mathcal{Q}_1^n$ bords.

The necessary tools from differential topology, like the existence of collaring neighborhoods and glueing by orientation reversing diffeomorphisms of open sets and straghtening angles, are available, and, as in the equivariant case we have that cobordism defines an equivalence relation on the class of $(\mathcal{F}, \mathcal{F}')$-orbifolds.

Definition 9. *We define $\mathcal{O}_n(\mathcal{F}, \mathcal{F}')$ to be the group of classes of n-dimensional orbifolds under the operation of disjoint union.*

If $\mathcal{F}'$ is empty, $\mathcal{O}_n(\mathcal{F},\mathcal{F}')$ is the group of bordism classes of closed orbifolds where all local representations are in $\mathcal{F}$, we denote this group by $\mathcal{O}_n(\mathcal{F})$, the $\mathcal{F}$-orbifolds. In particular if $\mathcal{F}$ is the trivial group, we have the bordism classes of all manifolds.

The group,

$$\mathcal{O}_*(\mathcal{F}) = \bigoplus \mathcal{O}_n(\mathcal{F})$$

can be given the structure of a ring if we add a multiplicative condition on the family of representations. We denote by $\mathcal{O}_*$ the cobordism ring of all orbifolds.

Every $\mathcal{F}'$-orbifold is an $\mathcal{F}$-orbifold, and similarly every $\mathcal{F}$-orbifold is an $(\mathcal{F},\mathcal{F}')$-orbifold. Together with taking boundary we have the natural maps,

$$\begin{aligned}
\mathbb{I} : \mathcal{O}_n(\mathcal{F}') &\longrightarrow \mathcal{O}_n(\mathcal{F}) \\
\mathbb{L} : \mathcal{O}_n(\mathcal{F}) &\longrightarrow \mathcal{O}_n(\mathcal{F},\mathcal{F}') \\
\partial : \mathcal{O}_n(\mathcal{F},\mathcal{F}') &\longrightarrow \mathcal{O}_{n-1}(\mathcal{F}')
\end{aligned}$$

which fit in a long exact sequence

$$(*) \qquad \cdots \longrightarrow \mathcal{O}_n(\mathcal{F}') \xrightarrow{\mathbb{I}} \mathcal{O}_n(\mathcal{F}) \xrightarrow{\mathbb{L}} \mathcal{O}_n(\mathcal{F},\mathcal{F}') \xrightarrow{\partial} \mathcal{O}_{n-1}(\mathcal{F}') \xrightarrow{\mathbb{I}} \cdots$$

The proof is based on fact that an $\mathcal{F}$-orbifold $\mathcal{Q}^n$ is zero in $\mathcal{O}_n(\mathcal{F},\mathcal{F}')$ if it is bordant to an $\mathcal{F}'$-orbifold or more generally the following lemma, which follows by considering $\mathcal{Q}^n \times I$.

Lemma 1. *Let $\mathcal{Q}^n$ be an $(\mathcal{F},\mathcal{F}')$-orbifold, let $\mathcal{Q}_1^n$ be a compact orbifold, regularly embedded in the interior of $\mathcal{Q}^n$. Suppose that all the isotropy groups of points $x \in \mathcal{Q}^n \backslash \mathcal{Q}_1^n$ belong to $\mathcal{F}'$. Then $\mathcal{Q}^n = \mathcal{Q}_1^n$ in $\mathcal{O}_n(\mathcal{F},\mathcal{F}')$.*

Also for three families of representations $\mathcal{F}_1 \subseteq \mathcal{F}_2 \subseteq \mathcal{F}_3$, We have the corresponding long exact sequence of a triple.

(**)

$$\cdots \longrightarrow \mathcal{O}_n(\mathcal{F}_2,\mathcal{F}_1) \longrightarrow \mathcal{O}_n(\mathcal{F}_3,\mathcal{F}_1) \longrightarrow \mathcal{O}_n(\mathcal{F}_3,\mathcal{F}_2) \longrightarrow \mathcal{O}_{n-1}(\mathcal{F}_2,\mathcal{F}_1) \longrightarrow \cdots$$

Now suppose that (H,ρ) is an element of $\mathcal{F}$, we denote by $\mathcal{Q}^\rho$, the ρ-singular set of $\mathcal{Q}$

$$\mathcal{Q}^\rho = \{x \in Q \mid (H,\rho) \le (G_x,\rho_x)\}$$

In general $\mathcal{Q}^\rho$ is not necessarily an orbifold, due to the fact that there could be different subgroups of G_x isomorphic to H that are conjugate in $O(n)$ and therefore in general $\mathcal{Q}^\rho$ is a union of suborbifolds. But when ρ is maximal in the local representations of $\mathcal{Q}$,

$$\mathcal{Q}^\rho = \{x \in Q \mid (H,\rho) \simeq (G_x,\rho_x)\}$$

with the following charts; for $x \in \mathcal{Q}^\rho$ and linear chart $(\overline{U}_x, G_x)$ of $\mathcal{Q}$, take as chart for $\mathcal{Q}^\rho$ around x, $(\overline{U}_x^{G_x}, G_x)$ with G_x acting trivially. The change of coordinate maps for $\mathcal{Q}^\rho$ are given by restriction of the overlap maps for $\mathcal{Q}$.

If $\mathcal{Q}$ is compact, the maximality of ρ implies that $\mathcal{Q}^\rho$ is also compact. Now we will define the normal orbivector bundle $\nu\rho \to \mathcal{Q}^\rho$. Given $x \in \mathcal{Q}^\rho$, and a linear chart $(\overline{U}_x, G_x)$ of $\mathcal{Q}$ around x, then

$$T\overline{U}_x \mid_{\overline{U}_x^{G_x}} = T\overline{U}_x^{G_x} \oplus V_x$$

Where V_x is the normal bundle of $\overline{U}_x^{G_x}$ in $\overline{U}_x$, since we are using linear charts we can identify V_x with the trivial bundle $\overline{U}_x^{G_x} \times \overline{U}_x^{G_x\perp}$. Note that this bundle has an action of G_x that covers the trivial action on $\overline{U}_x^{G_x}$. We can patch this bundles using the overlap maps from $\mathcal{Q}$ to get an orbivector bundle.

Given $x_1, x_2 \in \mathcal{Q}^\rho$, take linear charts $(\overline{U}_1, G_{x_1})$ and $(\overline{U}_2, G_{x_2})$ of $\mathcal{Q}$, the charts of $\mathcal{Q}^\rho$ are given by $\overline{U}_i^{G_{x_i}}$. Fix embeddings $\tau_i : \overline{U}_i \to \mathbb{R}^n$ that identify G_{x_i} with a subgroup of $O(n)$, in particular these maps become G_{x_i}-equivariant. Since $x_i \in \mathcal{Q}^\rho$, G_{x_i} and H are conjugate in $O(n)$, let $r_i \in O(n)$ be such that

$$r_i G_{x_i} r_i^{-1} = H$$

For convenience lets denote by $\mathbb{R}^k$ the fixed points of $\mathbb{R}^n$ under the action of H and by $\mathbb{R}^{n-k}$ its orthogonal complement. Note the k is the degree of ρ.

The map $r_i\tau_i : \overline{U}_i \to \mathbb{R}^n$ identifies $\overline{U}_i^{G_{x_i}}$ with $\mathbb{R}^k$ and $\overline{U}_i^{G_{x_i}\perp}$ with $\mathbb{R}^{n-k}$, because for $h \in H$, $r_i^{-1} h r_i \in G_{x_i}$ and if $v \in \overline{U}_i^{G_{x_i}}$

$$\begin{aligned} h r_i \tau_i(v) &= r_i(r_i^{-1} h r_i)\tau_i(v) \\ &= r_i \tau_i((r_i^{-1} h r_i) v) \\ &= r_i \tau_i(v) \end{aligned}$$

Using this trivializations it can be seen that if ϕ is an overlap map between $\overline{U}_1$ and $\overline{U}_2$ then the transition functions are given by

$$r_2\tau_2(d_x\phi)\tau_1^{-1} r_1^{-1} : \mathbb{R}^{n-k} \to \mathbb{R}^{n-k}$$

Observe that this functions depend on the overlap map ϕ, but any other overlap map is equal to $g_2 \phi g_1^{-1}$, for some $g_2 \in G_{x_2}$, $g_1 \in G_{x_1}$.

Lets see now that this transition functions belong to $N_{O(n-k)}H$, if $h \in H$ then

$$\begin{aligned} r_2\tau_2(d_u\phi)\tau_1^{-1} r_1^{-1} h r_1 \tau_1 (d_u\phi)^{-1} \tau_2^{-1} r_2^{-1} &= r_2\tau_2(d_u\phi)\tau_1^{-1}(r_1^{-1} h r_1)\tau_1(d_u\phi)^{-1}\tau_2^{-1} r_2^{-1} \\ &= r_2\tau_2(d_u\phi)\tau_1^{-1}(g)\tau_1(d_u\phi)^{-1}\tau_2^{-1} r_2^{-1} \\ &= r_2\tau_2(d_u\phi) g (d_u\phi)^{-1}\tau_2^{-1} r_2^{-1} \end{aligned}$$

Now, by the compatibility of the charts of $\mathcal{Q}$, the overlap maps, ϕ, are equivariant with respect to an homomorphism $\lambda : G_{x_1} \to G_{x_2}$ of the isotropy groups, therefore

$$\phi \cdot g = \lambda(g) \cdot \phi$$

and taking the differential and using that the charts are linear,

$$d_x\phi \cdot g = \lambda(g) \cdot d_x\phi$$

which gives,

$$\begin{aligned} r_2\tau_2(d_x\phi)\tau_1^{-1} r_1^{-1} h r_1 \tau_1 (d_x\phi)^{-1}\tau_2^{-1} r_2^{-1} &= r_2\tau_2(d_x\phi) g (d_x\phi)^{-1}\tau_2^{-1} r_2^{-1} \\ &= r_2\tau_2\lambda(g)\tau_2^{-1} r_2^{-1} \\ &= r_2\lambda(g) r_2^{-1} \end{aligned}$$

which is in H. For an overlap map ϕ the transition functions are given by $\tau_\phi = r_2\tau_2(d_x\phi)\tau_1^{-1} r_1^{-1}$, any other overlap map is of the form $g_2\phi g_1^{-1}$, therefore the transition function is $g_2 \tau_\phi g_1^{-1}$, which is a well defined element of $N_{O(n-k)}H/H$.

Notice that if $(\overline{U}_1, G_1)$ is a linear chart for $\mathcal{Q}$, then locally ν^ρ is modelled on

$$\overline{U}_1^{G_{x_1}} \times_{G_{x_1}} \mathbb{R}^{n-k}$$

with the action of G_{x_1} given by

$$g(u,v) = (u, r_1 g r_1^{-1} v)$$

and $r_1 g r_1^{-1} \in H$, but the transition functions lie in $N_{O(n-k)}H$.

Therefore, we have proved,

Theorem 3 (Reduction to the singular set)**.** *Given an n-dimensional compact orbifold $\mathcal{Q}$, and a maximal element, (H,ρ) of the local representations, the ρ-singular set, $\mathcal{Q}^\rho$ is a compact suborbifold of $\mathcal{Q}$ and the normal bundle $\nu\rho \to \mathcal{Q}^\rho$ is an orbivector bundle with structural group $N_{O(n-\mathbf{degree}(\rho))}H$. Even more is a purely ineffective orbifold* [HM04] *with trivial stabilizer H.*

The following theorem follows from the previous theorem, the existence of tubular neighborhoods and lemma 1. It is the key result of this section.

Theorem 4. *Suppose that $\mathcal{F} \supset \mathcal{F}'$ are families which differ at dimension n by the representation (H,ρ), i.e. $\mathcal{F}(n) - (H,\rho) = \mathcal{F}'(n)$. Then the oriented cobordism group $\mathcal{O}_n(\mathcal{F},\mathcal{F}')$ is isomorphic to the cobordism group of orbibundles over purely ineffective orbifolds. Where the orbibundles have an oriented total space, the fiber representation is the non-trivial summand of ρ and the ρ-singular set is the zero section.*

Proof. Let $\mathcal{Q} \in \mathcal{O}_n(\mathcal{F},\mathcal{F}')$, by the adjacency condition, (H,ρ) is maximal between the local representations, consider $\mathcal{Q}^\rho$, this is a compact manifold by maximality of (H,ρ) of dimension k, and since (H,ρ) is not in $\mathcal{F}'$, $\mathcal{Q}^\rho$ has no boundary. We have the normal bundle ν^ρ over it.

A cobordism $\mathcal{W}^{n+1}$ between orbifolds $\mathcal{Q}_1^n$ and $\mathcal{Q}_2^n$, induces a cobordism $\mathcal{W}^H$ between the ρ-singular sets $\mathcal{Q}_1^\rho$ and $\mathcal{Q}_2^\rho$. Even more the normal orbibundle $\nu_\rho \to \mathcal{W}^\rho$ restricts to the boundary to the respective normal orbibundles of $\mathcal{Q}_1^\rho$ and $\mathcal{Q}_2^\rho$. □

For a local represenation (H,ρ) not necessarily maximal, we can still define

$$\mathcal{Q}^\rho = \{x \in \mathcal{Q} \mid (H,\rho) \simeq (G_x,\rho_x)\}$$

and we have a similar statement, but we due to the lack of maximality we do not have compactness.

Now, in presence of maximality, since $\mathcal{Q}^\rho$ is purely ineffective orbifold, the underlying space is a manifold, The transition functions of the orbivector bundle ν_ρ are elements of $N_{O(n-k)}H$ defined up to elements of H, now $N_{O(n-k)}H$ acts on $\mathbb{R}^{n-k}$ and $N_{O(n-k)}H/H$ acts effectively on $\mathbb{R}^{n-k}/H$, therefore we can use the same set of transition functions, seen as elements of $N_{O(n-k)}H/H$, to construct a $N_{O(n-k)}H/H$-principal bundle.

Remark 4. *The orbibundle $\nu^\rho \to \mathcal{Q}^\rho$ is determined by the the associated principal $N_{O(n-k)}H/H$-bundle over the underlying space and the principal H-bundle $N_{O(n-\mathbf{degree}(\rho))}H \to N_{O(n-\mathbf{degree}(\rho))}H/H$.*

If the original orbifold was oriented, then we do not necessarily have that the singular set is oriented, or that the normal bundle is oriented, we only have that $\mathcal{Q}^\rho \bigoplus \nu\rho$ is oriented, which boils down to the fact that the transition functions are orientation preserving. The transition functions for the normal bundle and the overlap maps for the singular set come from restriction this transition functions to subspaces and that is were the orientability can be lost. However, in the case when the representations have degree zero, the normal bundle is oriented, and the structure group is trivial.

We introduce now a twisted form of bordism.

Definition 10. *Given a topological space X and a character $\rho : \pi_1(X) \to \mathbb{Z}_2$ the twisted bordism group $\Omega_{k,t}(X)$ is defined to be the cobordism group of pairs (M, f), where M is a closed compact k-dimensional manifold and $f : M \to X$ is a continuous map such that the following diagram is commutative*

$$\begin{array}{ccc} \pi_1(M) & \xrightarrow{\omega} & \mathbb{Z}_2 \\ & {\scriptstyle f_*}\searrow \quad \nearrow{\scriptstyle \rho} & \\ & \pi_1(X) & \end{array}$$

where ω is the orientation character of M.

In the special case that $X = BK$, the classifying space of a linear group, and ρ is the character induced from the determinant map, the commutativity of this diagram is equivalent to require that the total space of the corresponding vector bundle over M is oriented.

Now we come to the main results of this section, when the families are close enough the relative term in the Conner and Floyd sequence becomes computable in terms of the bordism group of the classifying space of the normalizer of the groups of $\mathcal{F} - \mathcal{F}'$. The following results are also true in the unoriented case,

Theorem 5. *Suppose that $\mathcal{F} \supset \mathcal{F}'$ are families which differ at dimension n by the representation (H, ρ), i.e $\mathcal{F}(n) - (H, \rho) = \mathcal{F}'(n)$. Then for the oriented cobordism group, we have*

$$\mathcal{O}_n(\mathcal{F}, \mathcal{F}') \simeq \Omega_{\mathbf{degree}(\rho),t}(BN_{O(n-\mathbf{degree}(\rho))}H/H)$$

Proof. Let $\mathcal{Q} \in \mathcal{O}_n(\mathcal{F}, \mathcal{F}')$, by the adjacency condition, (H, ρ) is maximal between the local representations , consider $\mathcal{Q}^\rho$, this is a compact manifold by maximality of (H, ρ) of dimension $\mathbf{degree}(\rho)$, and since (H, ρ) is not in $\mathcal{F}'$, $\mathcal{Q}^\rho$ has no boundary. We have the fiber bundle ν^ρ over it. The associated principal $N_{O(n-\mathbf{degree}(\rho))}H/H$-bundle over $\mathcal{Q}^\rho$ has a classifying map

$$\nu^\rho/H : \mathcal{Q}^\rho \longrightarrow BN_{O(n-\mathbf{degree}(\rho))}H/H$$

A cobordism, $\mathcal{W}^{n+1}$ between orbifolds $\mathcal{Q}_1^n$ and $\mathcal{Q}_2^n$, induces a cobordism, $\mathcal{W}^H$ between the ρ-singular sets, $\mathcal{Q}_1^\rho$ and $\mathcal{Q}_2^\rho$. Even more the normal orbibundle $\nu^\rho \to \mathcal{W}^\rho$ restricts to the boundary to the respective normal orbibundles of $\mathcal{Q}_1^\rho$ and $\mathcal{Q}_2^\rho$. And we have,

$$T\mathcal{Q}|_{\mathcal{Q}^\rho} = T\mathcal{Q}^\rho \oplus \nu^\rho$$

which has an inherited orientation from the orientation of $\mathcal{Q}$.

Therefore giving a well defined homomorphism

$$\alpha : \mathcal{O}_n(\mathcal{F}, \mathcal{F}') \longrightarrow \Omega_{k,t}(BN_{O(n-\mathbf{degree}(\rho))}H/H)$$

This homomorphism is onto. Given a compact closed $\mathbf{degree}(\rho)$-manifold M and a map $f : M \to BN_{O(n-\mathbf{degree}(\rho))}H/H$, then the total space of the disk bundle of the associated orbivector bundle, is an orbifold with boundary, since $(H, \rho) \in \mathcal{F}$, by definition of family, all the local representations are also in $\mathcal{F}$, also since H acts without any nontrivial fixed points on $\mathbb{R}^{n-\mathbf{degree}(\rho)}$ then the boundary has all its isotropy groups in $\mathcal{F}(n) - (H,\rho) = \mathcal{F}'(n)$. The ρ-singular set is precisely M and the normal orbibundle is the one we started with. Cobordant pairs (M_i, f_i) give rise to cobordant elements of $\mathcal{O}_n(\mathcal{F},\mathcal{F}')$.

Lets see that the map is one-to-one, suppose that $\mathcal{Q} \in \mathcal{O}_n(\mathcal{F},\mathcal{F}')$, and that $\alpha(\mathcal{Q}) = 0$. Then there exists an $\mathbf{degree}(\rho)+1$-manifold $\mathcal{W}$ and an orbivector bundle $\eta \to \mathcal{W}$ such that $\partial\mathcal{W} = \mathcal{Q}^\rho$ and $\eta|_{\mathcal{Q}^\rho} = \nu\rho$. Now consider the orbifold that its obtained by attaching the total space of the disk orbibundle $D\eta$ to $\mathcal{Q}\times I$ by identifying a closed tubular neighborhood of $\mathcal{Q}^\rho \times 1$ with the disk bundle of $\nu\rho$ which is $D\eta|_{\mathcal{Q}^\rho}$. This orbifold has boundary $\mathcal{Q}\cup\mathcal{Q}'$. $\mathcal{Q}'$ is just $\mathcal{Q}$ minus an open neighborhood of the ρ-singular set union with the sphere bundle of the orbibundle η. Thus its isotropy groups are in $\mathcal{F}(n) - (H,\rho) = \mathcal{F}'(n)$. □

If $N_{O(n-\mathbf{degree}(\rho))}H \leq SO(n-\mathbf{degree}(\rho))$ then, $\nu\rho$ is an oriented orbivector bundle and then from the orientation of $\mathcal{Q}$ we get an orientation of $\mathcal{Q}^\rho$ thus

$$\mathcal{O}_n(\mathcal{F},\mathcal{F}') \simeq \Omega_{\mathbf{degree}(\rho)}(BN_{O(n-\mathbf{degree}(\rho))}H/H)$$

and we get the usual cobordism group with oriented manifolds.

By the Thom-Pontrjagin theorem we have that in this case

$$\mathcal{O}_n(\mathcal{F},\mathcal{F}') \simeq \Omega_{\mathbf{degree}(\rho)}(BN_{O(n-\mathbf{degree}(\rho))}H/H) \simeq \pi_{\mathbf{degree}(\rho)}(MSO \wedge BN_{O(n-\mathbf{degree}(\rho))}H/H_+)$$

We call two families $\mathcal{F}' \subseteq \mathcal{F}$ adjacent if they differ by at most one representation in each dimension. By condition iii) of definition 6, there is a k-dimensional representation (H,ρ) such that $\mathcal{F}(n) - (H, \rho \oplus (n-k)id) = \mathcal{F}'(n)$ for all $n \geq k$.

In a more general way, since for different maximal representations the singular sets are closed disjoint sets, we can find disjoint tubular neighborhoods, and as before, identify the relative cobordism group as a sum of usual bordism groups of spaces. We say that two non-empty families $\mathcal{F} \supset \mathcal{F}'$ are quasi-adjacent if every element in $\mathcal{F} - \mathcal{F}'$ is maximal in $\mathcal{F}$.

Theorem 6. *Suppose that $\mathcal{F} \supset \mathcal{F}'$ are quasi-adjacent. Then*

$$\mathcal{O}_n(\mathcal{F},\mathcal{F}') \simeq \sum \Omega_{\mathbf{degree}(\rho_j),t}(BN_{O(n-\mathbf{degree}(\rho_j))}H_j/H_j)$$

where the sum is over all isomorphism classes of elements (H_j,ρ_j) of $\mathcal{F}(n)$ not in $\mathcal{F}'(n)$.

3. A spectral sequence

If $\{e\} = \mathcal{F}_0 \subseteq \mathcal{F}_1 \subseteq \cdots \subseteq \mathcal{F}_n = \mathcal{F}$ is a sequence of families, the corresponding long exact sequences can be interpreted as an exact couple, giving a spectral sequence converging to $\mathcal{O}_*(\mathcal{F})$ with the E^1 term given by,

$$E^1_{p,q} = \mathcal{O}_{p+q}(\mathcal{F}_p, \mathcal{F}_{p-1})$$

The filtration of $\mathcal{O}_*(\mathcal{F})$ is given,

$$F_p\mathcal{O}_n(\mathcal{F}) := im\,(\mathcal{O}_n(\mathcal{F}_p) \longrightarrow \mathcal{O}_n(\mathcal{F}))$$

and the differentials can be identified with the boundary map. The $E^1_{p,q}$ terms are computable (at least is a non-equivariant problem now) if the families are quasi-adjacent.

The easiest way to construct this kind of filtrations is at each step only add representations of a new fixed group or use the canonical filtration given by the order $\leq$.

We define the level of an element of a family of representation inductively, given a n-dimensional representation, the $\mathbf{level}_n$ of a maximal element is zero (and therefore > -1).

We say that the $\mathbf{level}_n(G,\rho) > i$, if there is an element $(H,\tau) \in \mathcal{F}$ with $\mathbf{level}_n > i-1$ such that $(G,\rho) \leq (H,\tau)$ but $(G,\rho) \ncong (H,\tau)$. We say $\mathbf{level}_n(G,\rho) = i$ if $\mathbf{level}_n(G,\rho) > i-1$ but $\mathbf{level}_n(G,\rho) \not> i$.

Remark 5. *By induction, it is easily seen that* $\boldsymbol{level}_n(G,\rho) \geq i$ *implies* $\boldsymbol{level}_n(G,\rho \oplus id) \geq i$*, but not necessarily the converse. To construct families satisfying condition iii) of definition 6 we take the maximum over all dimensions.*

$$\boldsymbol{level}\ (G,\rho) = \max_{k\in\mathbb{Z}} \boldsymbol{level}_{\dim\rho+k}(G,\rho\oplus kid)$$

By construction $\mathbf{level}\ (G,\rho) = \mathbf{level}\ (G,\rho\oplus id)$ and it can be checked that if $(G,\rho) \leq (H,\tau)$ then $\mathbf{level}\ (G,\rho) \geq \mathbf{level}\ (H,\tau)$. Therefore, given a family of local representations $\mathcal{F}$, we can use the levels to define a filtration

$$\mathcal{F}^i = \{(G,\rho) \in \mathcal{F} \mid \ \mathbf{level}\ (G,\rho) \geq i\}$$

This gives a decreasing sequence of families of local representations

$$\cdots \subset \mathcal{F}^r \subset \mathcal{F}^{r-1} \subset \cdots \subset \mathcal{F}^1 \subset \mathcal{F}^0 = \mathcal{F}$$

where all the elements in $\mathcal{F}^i - \mathcal{F}^{i+1}$ are maximal in $\mathcal{F}^i$. These elements correspond to the elements of the family of level i.

This filtration is finite because we require our families to have a bound on the lenght of $\leq$-chains.

Theorem 7. *Given a family of local representations $\mathcal{F}$ there is an spectral sequence converging to $\mathcal{O}_*(\mathcal{F})$. The E^1 term is given by*

$$E^1_{p+q} \cong \mathcal{O}_{p+q}(\mathcal{F}^p,\mathcal{F}^{p+1}) \simeq \sum \Omega_{\mathbf{degree}(\rho_j),t}(BN_{O(n-\mathbf{degree}(\rho_j))}H_j/H_j)$$

where the sum extend over all representations H_j of level p and of dimension $p+q$.

4. Unoriented cobordism

First note that every orbifold is the (non-oriented) boundary of an orbifold, just consider $\mathcal{Q}\times[I/\mathbb{Z}_2]$, with $\mathbb{Z}_2$ acting on I by reflection along the origin, and note that the boundary of $[I/\mathbb{Z}_2]$ is just a point.

Therefore it is natural to study the cobordism groups of orbifolds with restricted singularities, in particular with only local groups of odd order.

Given a representation (H,ρ), we want to consider the projectivisation of $\mathbb{RP}(\rho\oplus id)$, the natural action of H on this manifold is effective if the representation is faithful. The local groups of this orbifold are given by the old ones and if $v \in S(\rho)$.

$$G_{[v]} = \left\{ g\in H \;\middle|\; \begin{array}{ll} \text{gv} = & \text{v} \\ \text{gv} = & \text{-v} \end{array} \right\}$$

$G_{[v]}$ contains G_v as a normal subgroup of index at most 2. If there is an $h \in H$ of order 2 that sends a point v to its antipodal $-v$, then $h \in N_H G_v$ and

$$G_{[v]} \cong G_v \rtimes \mathbb{Z}_2 \tag{1}$$

where $\mathbb{Z}_2$ acts on G_v by conjugation of h.

If there if no element of H that sends a point to its antipodal, then

$$G_{[v]} \cong G_v$$

Theorem 8. *Suppose that $\mathcal{F} \supset \mathcal{F}'$ are quasi-adjacent families which differ by representations of groups of odd order. Then the long exact sequence*

$$\cdots \longrightarrow \mathcal{O}_n(\mathcal{F}') \longrightarrow \mathcal{O}_n(\mathcal{F}) \longrightarrow \mathcal{O}_n(\mathcal{F},\mathcal{F}') \longrightarrow \mathcal{O}_{n-1}(\mathcal{F}') \longrightarrow \cdots \tag{2}$$

splits

Proof. By quasi-adjacency, we can identify the relative term with the sum of cobordism group of orbibundles over purely ineffective orbifolds. Given such, we want to consider the projectivisation of the bundle $E \oplus \mathbb{R}$. Locally the projectivisation of an orbibundle can be described as follows, the orbibundle has fibers of the form $[\mathbb{R}^n/G_x]$, where we can assume that the action of G_x is linear, and we can form a bundle with fibers $[\mathbb{RP}^n/G_x]$.

Now, given an orbivector bundle E, consider the projectivisation $\mathbb{RP}(E \oplus \mathbb{R})$, with the trivial action on $\mathbb{R}$. The fibers of this bundle can be also thought as the fibers of the disk bundle when you identify antipodal points in each sphere. The local representations for points not coming from the sphere bundle are the local representations of the original orbibundle, for points in the sphere bundle, the isotropy group is the set of elements that fix a point or send it to its antipodal.

Now if the groups are odd then they cannot contain a subgroup of index 2, thus we have only the old isotropies of elements in the sphere.

Therefore if we start with an element of $\mathcal{O}_n(\mathcal{F},\mathcal{F}')$, by maximality we can think it as an orbibundle over a purely ineffective orbifold, each fiber is of the form $[\mathbb{R}^{n-k}/H]$ and only the zero section has isotropy H, by the previous discussion the projectivisition is also an $\mathcal{F}$-orbifold. By construction the projectivisation, contains a copy the disk bundle of E and the complement has isotropy in $\mathcal{F}'(n)$. By lemma 1 this two elements represent the same element of $\mathcal{O}_n(\mathcal{F},\mathcal{F}')$, this map gives the desired splitting of the sequence. □

Note that the splitting and exactness, implies that if $\mathcal{Q} \in \mathcal{O}_*(\mathcal{F})$, then $\mathcal{Q} + \mathbb{RP}(\nu_\rho \oplus \mathbb{R})$ is actually in the image of the map $\mathcal{O}_*(\mathcal{F}') \to \mathcal{O}_*(\mathcal{F})$ and is uniquely determined, i.e the connected sum along an open neighborhood of the H-singular set of $\mathcal{Q}$ and the projectivisation $\mathbb{RP}(\nu_\rho \oplus \mathbb{R})$ is an $\mathcal{O}_*(\mathcal{F}')$-orbifold

Given a family $\mathcal{F}$, the levels give a canonical filtration by quasi-adjacent families,

$$\{e\} = \mathcal{F}^r \subset \mathcal{F}^{r-1} \subset \cdots \subset \mathcal{F}^0 = \mathcal{F}$$

combining the previous result with theorem 6 we obtain,

Theorem 9. *For $\mathcal{F}$ consisting of groups of odd order, $\mathcal{O}_*(\mathcal{F})$ decomposes into a sum of bordism groups*

$$\mathcal{O}_*(\mathcal{F}) \cong \bigoplus_i \mathcal{O}_*(\mathcal{F}^i, \mathcal{F}^{i+1}) \oplus \mathcal{O}(\mathcal{F}^r) \cong \mathfrak{N}_* \bigoplus \mathfrak{N}_{k_j}(BN_{O(*-k_j)}H_j/H_j) \tag{3}$$

where the sum extends over all representations in $\mathcal{F}$, k_j is the degree of the representation.

As corollary we have

Corollary 1. *For any family consisting of groups of odd order, the map $\mathfrak{N}_* \longrightarrow \mathcal{O}_*(\mathcal{F})$ has a splitting, in particular is injective.*

The projection map $\mathcal{O}_*(\mathcal{F}) \longrightarrow \mathfrak{N}_*$ can be described geometrically as follows, take a maximal element of the local representations, form the projectivisation of the normal bundle plus a trivial one and make the connected sum, now you have an orbifold with less singularites and repeat the process. You end up with a connected sum of your orbifold and projective orbibundles, for example if you start with a Riemann surface with n-singular points, then you end up with the connected sum of the Riemann surface and n copies of $\mathbb{RP}^2$.

Now given two families $\mathcal{F}' \subseteq \mathcal{F}$ consisting of groups of odd order, each group $\mathcal{O}_*(\mathcal{F}')$, $\mathcal{O}_*(\mathcal{F})$ decomposes into a direct sum

$$\mathfrak{N}_* \bigoplus \mathfrak{N}_{k_j}(BN_{O(*-k_j)}H_j/H_j)$$

The natural map $\mathcal{O}_*(\mathcal{F}') \to \mathcal{O}_*(\mathcal{F})$ correspond to the inclusion

$$\mathfrak{N}_* \bigoplus_{H_j \in \mathcal{F}'} \mathfrak{N}_{k_j}(BN_{O(*-k_j)}H_j/H_j) \to \mathfrak{N}_* \bigoplus_{H_j \in \mathcal{F}} \mathfrak{N}_{k_j}(BN_{O(*-k_j)}H_j/H_j)$$

we have inclusions $\mathcal{F}'^i \subseteq \mathcal{F}^i$ of the canonical filtrations by levels. and therefore maps of the short exact sequences

$$\begin{array}{ccccccccc} 0 & & \mathcal{O}_*(\mathcal{F}'^{i+1}) & & \mathcal{O}_*(\mathcal{F}'^{i}) & & \mathcal{O}_*(\mathcal{F}'^{i}, \mathcal{F}'^{i+1}) & & 0 \\ \\ 0 & & \mathcal{O}_*(\mathcal{F}^{i+1}) & & \mathcal{O}_*(\mathcal{F}^{i}) & & \mathcal{O}_*(\mathcal{F}^{i}, \mathcal{F}^{i+1}) & & 0 \end{array}$$

For some r, $\{e\} = \mathcal{F}'^r = \mathcal{F}^r$, and $\mathcal{O}_*(\mathcal{F}'^{r-1}, \mathcal{F}'^r)$ injects into $\mathcal{O}_*(\mathcal{F}^{r-1}, \mathcal{F}^r)$.

Corollary 2. *For any two families, $\mathcal{F}' \subseteq \mathcal{F}$ consisting of groups of odd order, the map $\mathcal{O}(\mathcal{F}') \to \mathcal{O}(\mathcal{F})$ is injective i.e if an orbifold bounds (with groups of odd order) then it actually bounds an orbifold without an increase of the isotropy.*

Given this we can now take an increasing sequence of families that will exhaust all representations of all groups of odd order and taking the direct limit we have a similar decomposition.

Theorem 10. *Denote by $\mathcal{O}_*^{odd}$ the cobordism ring of orbifolds with isotropy groups of odd order, then*

$$\mathcal{O}_*^{odd} \cong \mathfrak{N}_* \bigoplus_{H \ odd} \bigoplus_{Rep_*(H)} \mathfrak{N}_{\deg}(BN_{O(*-\mathbf{degree})}H/H)$$

where the sum extends over all groups of of odd order and all faithful representations. This is an isomorphism of $\mathfrak{N}_*$-modules.

Corollary 3. *If a manifold is the boundary of an orbifold with only odd singularities then it is actually the boundary of another manifold.*

4.1. **Invariants of unoriented cobordism.** From our previous work, the cobordism ring decomposes as a sum of terms of the form $\mathfrak{N}_*(BN_{O(n-k_j)}H_j/H_j)$ and each element is represented by the projectivisation of an orbibundle, we now take a closer look to these terms.

The sequence of groups

$$1 \to H_j \to N_{O(n-k_j)}H_j \to N_{O(n-k_j)}H_j/H_j \to 1$$

gives rise to a fibration

$$BH_j \to BN_{O(n-k_j)}H_j \to BN_{O(n-k_j)}H_j/H_j$$

but now $\mathfrak{N}_*(BH_j, \mathbb{Z}_2) = 0$ because H_j is of odd order, therefore an application of the Serre spectral sequence gives that the natural projection

$$\mathfrak{N}_*(BN_{O(n-k_j)}H_j/H_j) \to \mathfrak{N}_*(BN_{O(n-k_j)}H_j)$$

is an isomorphism.

$$\mathfrak{N}_* \otimes_{\mathbb{Z}_2} H_*(BN_{O(n-k_j)}H_j; \mathbb{Z}_2)$$

Since the Atiyah-Hirzebruch spectral sequence for unoriented bordism collapses, we have then that (not naturally)

$$\mathfrak{N}_*(BN_{O(n-k_j)}H_j/H_j) \cong \mathfrak{N}_* \otimes_{\mathbb{Z}_2} H_*(BN_{O(n-k_j)}H_j/H_j; \mathbb{Z}_2)$$

Using this we can rewrite theorem 10 as

$$\mathcal{O}_*^{odd} \cong \mathfrak{N}_* \bigoplus_{H \text{ odd}} \bigoplus_{Rep_*(H)} \mathfrak{N}_* \otimes_{\mathbb{Z}_2} H_{k_j}(BN_{O(*-k_j)}H_j; \mathbb{Z}_2)$$

Theorem 11. *$\mathcal{O}_*^{odd}$ and $\mathcal{O}_*(\mathcal{F})$, for any family of groups of odd order, are free $\mathfrak{N}_*$-modules.*

This isomorphism will allow us to define characteristic numbers, that will determine the cobordism class of an orbifold. This numbers are defined by taking cohomology classes in $H^{k_j}(BN_{O(n-k_j)}H_j; \mathbb{Z}_2)$, pulling them back to the singular set, taking classes in the cohomology of the singular set, doing the cup product and evaluating at the fundamental class.

4.2. **Examples.** Since all two dimensional faithful representations are given by cyclic groups and have degree zero, then

$$\mathfrak{N}_{2,orb}^{odd} \cong \mathfrak{N}_2 \bigoplus_{\mathbb{Z}_n} \mathfrak{N}_0(BN_{O(2)}\mathbb{Z}_n/\mathbb{Z}_n) \cong \bigoplus_n \mathbb{Z}_2$$

where the generators can be taken to be the orbifold real projective spaces $\mathbb{RP}^2/\mathbb{Z}_{p^i}$. All these orbifolds are locally oriented and all bound when we allow cobordisms that involve groups of even order. In the three dimensional case, the only odd groups that can appear as isotropy groups of three dimensional orbifolds are the cyclic groups, thus

$$\mathfrak{N}_{3,orb}^{odd} \cong \bigoplus_{\mathbb{Z}_n} \mathfrak{N}_1(BN_{O(2)}\mathbb{Z}_n/\mathbb{Z}_n)$$

An application of the Serre spectral sequence shows that each of these groups is isomorphic to $\mathfrak{N}_1(BO(2))$, which is $\mathbb{Z}_2$, therefore

$$\mathfrak{N}_{3,orb}^{odd} \cong \bigoplus_n \mathbb{Z}_2$$

5. Rational oriented orbifold cobordism

In this section we will get a decomposition of the rational cobordism group of oriented orbifolds of complex type in term of the bordism ring of classifying space of groups. The decomposition is similar to the one obtained in the unoriented case for orbifolds with only groups of odd order.

The basic idea is to prove that the spectral sequence associated to the canonical filtratrion of a family of local representation collapses at the E^1 term, i.e. the long exact sequences split. Recall that in the spectral sequence, after identifying the E^1

term with a cobordism group of orbibundles, the d^1 differential is simply given by taking the sphere bundle.

There are several equivalent statements:

(1) For any family $\mathcal{F}$ of local representations, the spectral sequence collapses at the E^1-level after tensoring with the rationals.
(2) For any two adjacent families $\mathcal{F}' \subseteq \mathcal{F}$ the long exact sequence splits after tensoring with the rationals.
(3) For any two families $\mathcal{O}(\mathcal{F}') \otimes \mathbb{Q} \to \mathcal{O}(\mathcal{F}) \otimes \mathbb{Q}$ is injective.
(4) Some multiple of the sphere orbibundle, of an orbibundle over a purely ineffective orbifold such that the fiber representation contains no copies of the trivial representation, bounds without any increase of local representations.
(5) An oriented orbifold bounds (orientably) another orbifold if and only if some multiple bounds another orbifold without increasing the local representations.

Before proving these statements, let us consider some special cases:

Lemma 2. *If an oriented manifold bounds an oriented orbifold, then actually some multiple of it bounds a manifold (i.e. is a torsion element of Ω_*)*

This easily follows from the existence of fundamental classes for oriented orbifolds (over the rationals) and Pontrjagin classes for orbibundles.

Suppose M^n is an oriented manifold that is the boundary of an oriented orbifold $\mathcal{W}$. We want to see that the Pontrjagin numbers of M are zero, which shows that M is a torsion class of Ω_n.

To an orbifold $\mathcal{Q}$ we can associate a topological space $B\mathcal{Q}$ called the *classifying space of the orbifold* [Moe02],[ALR07]. For example, for a global quotient $\mathcal{Q} = [M/G]$ with G a finite group acting on M, $B\mathcal{Q}$ is homotopy equivalent to the homotopy orbit space $EG\times_G M$. A key feature of this classifying space construction, is that if $\mathcal{V}$ is an orbibundle on $\mathcal{Q}$, then $B\mathcal{V} \to B\mathcal{Q}$ is an honest vector bundle. Therefore we can talk about characteristic classes, now elements of $H^*(B\mathcal{Q})$.

To an orbifold $\mathcal{Q}$ we have associated two topological spaces, the underlying space Q and the classifying space $B\mathcal{Q}$. There is a map $B\mathcal{Q} \to Q$, but in general this map is far from being a homotopy equivalence.

Example 6. *For a finite group G, $B[*/G]$ is homotopy equivalent to BG the classifying space for principal G-bundles, but $|[*/G]|$ is just a point.*

Over the rationals these spaces are the same, at least homologically, see [ALR07], that is

$$H_*(Q;\mathbb{Q}) \cong H_*(B\mathcal{Q};\mathbb{Q}).$$

Remark 6. *For an oriented orbifold we can therefore talk at the same time about a fundamental class and characteristic classes, at least rationally.*

Call $\iota : M \to \mathcal{W}$ the inclusion, this map induces a map between the classifying spaces $B\iota : BM \to B\mathcal{W}$; since M is a manifold, the projection $BM \to M$ is a homotopy equivalence that has the property that $p^*(TM) \cong BTM$. Denote by $[BM] = p_*^{-1}([M])$ the preimage of the fundamental class of M.

Also, $\mathcal{W}$ is an orbifold and therefore has a tangent orbibundle $T\mathcal{W}$. This is not a vector bundle, but $BT\mathcal{W} \to B\mathcal{W}$ is an honest vector bundle and even more,

$B\iota^*(BT\mathcal{W}) = BTM \oplus \mathbb{R}$.

$$\begin{array}{rcll}
\langle p_I(TM), [M]\rangle & = & \langle p_I(TM), p_*[BM]\rangle & \\
& = & \langle p^* p_I(TM), [BM]\rangle & \text{by functoriality of the pairing} \\
& = & \langle p_I(p^*TM), [BM]\rangle & \text{by naturality of Pontrjagin classes} \\
& = & \langle p_I(BTM), [BM]\rangle & \\
& = & \langle p_I(BTM \oplus \mathbb{R}), [BM]\rangle & \text{by stability of Pontrjagin classes} \\
& = & \langle p_I(B\iota^*(BT\mathcal{W})), [BM]\rangle & \text{by the above remark} \\
& = & \langle B\iota^* p_I(TW), [BM]\rangle & \text{by naturality of Pontrjagin classes} \\
& = & \langle p_I(TW), B\iota_*[BM]\rangle & \text{by functoriality of the pairing}
\end{array}$$

And as in the manifold case, by looking at the long exact sequence of the pair $(B\mathcal{W}, BM)$ we see that $B\iota_*[BM] = 0$. Therefore all the Pontrjagin numbers of M are zero, i.e. is a torsion element of Ω_*.

Lemma 3. *Suppose that ν is an orbibundle over purely ineffective orbifold with isotropy group H and oriented total space. Suppose that fiber representation contains no non-trivial fixed points, then some multiple of the sphere orbibundle bounds a manifold.*

Proof. The hypothesis assures us that the sphere orbibundle is an oriented manifold, but clearly the sphere orbibundle is the boundary of the disk orbibundle and therefore by the previous result, some multiple of it bounds an oriented manifold. □

Now we will proceed to generalize the previous result to orbibundles where the fiber representation contains non-trivial fixed points, this is the central proof of this section. For this, we will need to consider the complexification $\mathbb{CP}(\nu \oplus \mathbb{C})$ of an complex orbibundle ν. Each fiber of the complexification is of the form $\mathbb{CP}(\rho \oplus \mathbb{C})$ for ρ a representation of complex type.

Let us anayze the singularities of $\mathbb{CP}(\rho \oplus \mathbb{C})$. Since (ρ, H) is a representation of complex type of dimension k i.e. $k = 2l$ and H is conjugate (inside $O(k)$) to a subgroup of $U(l)$, in particular, up to conjugation, $U(1) \leq C_{O(k)}H$. The centralizer of H inside the orthogonal group $O(k)$.

Consider the action of $h \in H$ on the complex projective space $\mathbb{CP}^{n+l}$ that is induced by the linear action on $\underbrace{\mathbb{C} \oplus \cdots \oplus \mathbb{C}}_{n+1} \oplus \mathbb{C}^l$, given by

$$h(z_0, \ldots, z_n, \overrightarrow{z_{n+1}}) = (z_0, \ldots, z_n, h\overrightarrow{z_{n+1}})$$

This action descends to the complex projective space since the representation is of complex type. Also it is effective and preserves orientation if the original action was. We will denote the corresponding orbifold $\mathbb{CP}^{n+\rho}$. Similarly we can define $\mathbb{RP}^{n+\rho}$ and $\mathbb{HP}^{n+\rho}$ for real and symplectic type representations. Note that $\mathbb{RP}^{n+\rho}$ is just $\mathbb{RP}(\mathbb{R}^n \oplus \rho)$, the (real) projectivization of the orbibundle $\gamma_{n-1} \otimes \rho$ over $\mathbb{RP}^{n-1}$, a special case of the construction used in the unoriented case.

For the oriented cobordism ring calculations we will only need the real and complex versions of this constuction.

We need to understand the local representations of these (complex) projective spaces. The points of the form $[z_0 : z_1 : \cdots : z_n : \overrightarrow{z_{m+1}}]$ with some $z_i \neq 0$ $(0 \leq i \leq n)$ have local representations ρ (when $\overrightarrow{z_{n+1}} = 0$) and $\rho_v < \rho$ (if $\overrightarrow{z_{n+1}} = v$ for some $v \neq 0$).

The points of the form $[0:0:\cdots:0:\overrightarrow{z_{n+1}}]$ have local groups

$$G_{[0:0:\cdots:0:v]} = \{g \in H\, |gv = \lambda v \text{ for some } \lambda\}$$

And the local representation is multiplication by $\overline{\lambda}$ on $\mathbb{C}^l$. (Note that λ is a root of unity.) Since every finite subgroup of the multiplicative group of a field is cyclic, the local representation of $G_{[0:0:\cdots:0:v]}$ arises by a homomorphism $G_{[0:0:\cdots:0:v]} \to \mathbb{Z}_d$ with $\mathbb{Z}_d$ acting by multiplication by ζ_d, a primitive d^{th}-root of unity. The group $G_{[0:0:\cdots:0:v]}$ is an extension of $\mathbb{Z}_d$ by G_v

$$1 \to G_v \to G_{[0:0:\cdots:0:v]} \to \mathbb{Z}_d \to 1.$$

Note that this representations are not faithful.

The fixed points with representation ρ_v are $\mathbb{CP}^n$'s with normal orbibundle $\gamma_n \otimes \rho_v$, the tensor product of the canonical line bundle γ_n over $\mathbb{CP}^n$ and the representation ρ_v.

We need to analyze what happens when we decompose the representations into irreducible ones. Given a representation (H, ρ), write $\rho = \rho_1^{m_1} \oplus \cdots \oplus \rho_l^{m_l}$ into a sum of irreducible representations. This decomposition induces a decomposition of the centralizer of ρ in $O(\dim \rho)$

$$C_{O(\dim(\rho))} \cong \times C_{O(n_i m_i)}(\rho_i^{m_i})$$

where $n_i = \dim \rho_i$. By Schur's lemma we have

$$C_{O(m\dim(\rho))}(\rho^m) \cong Sp^{\dim \rho}(m)$$

$$C_{O(m\dim(\rho))}(\rho^m) \cong U^{\dim \rho}(m)$$

$$C_{O(m\dim(\rho))}(\rho^m) \cong O^{\dim \rho}(m)$$

respectively, depending whether the representation is of symplectic, complex or of real type.

Where $O^{\dim(\rho)}(m) \cong O(m)$ sits inside $O(m\dim(\rho))$ by blocks, i.e. the embedding induced from

$$\mathbb{R}^m \to \mathbb{R}^{m\dim(\rho)}$$

$$(x_1, \cdots, x_m) \to (\underbrace{x_1, \cdots, x_1}_{\dim(\rho)}, \underbrace{x_2, \cdots, x_2}_{\dim(\rho)}, \cdots, \underbrace{x_m, \cdots, x_m}_{\dim(\rho)})$$

and similarly for $Sp^{\dim(\rho)}(m)$ and $U^{\dim(\rho)}(m)$ using the identifications $\mathbb{H} \cong \mathbb{R}^4$ and $\mathbb{C} \cong \mathbb{R}^2$.

Remark 7. *If an irreducible representation is symplectic or complex, it has dimension even. Only real representations can have odd dimension.*

Remark 8. *if* $\dim(\rho)$ *is even, then* $C_{m\dim(\rho)}(\rho^m) \leq SO(m\dim(\rho))$.

Remark 9. *if* ρ *is an irreducible representation of real type, then the centralizer is just* $\mathbb{Z}_2$.

Lemma 4. *Suppose that* ν *is a complex orbibundle over a purely ineffective orbifold with isotropy group* H *and oriented total space. Suppose that the only points with isotropy* H *are the points in the zero section.Then some multiple of the sphere orbibundle bounds an orbifold of complex type with only local representations of proper subgroups of* H.

Proof. By looking at the different connected components, we can assume that the fiber representation is the same at every point, let us call it (H, ρ). The cobordism of orbibundles over a purely ineffective orbifold with isotropy group H and fiber representation ρ can be identified with the twisted bordism of the classifying space of of the Weyl group

$$\Omega_{*,t}(BN_{O(dim\rho)}H/H)$$

Rationally, since H is finite, we have that this group is isomorphic to

$$\Omega_{*,t}(BN_{O(dim\rho-\mathbf{degree}(\rho))}H)\otimes\mathbb{Q}$$

By the transfer map, this is isomorphic to a subgroup of

$$\Omega_{*,t}(BC_{O(dim\rho-\mathbf{degree}(\rho))}H)\otimes\mathbb{Q}$$

But this latter group represents, see [CF66], the cobordism group of H-equivariant bundles (with oriented total space) over trivial H-manifolds. Now if $\rho = \rho_1^{m_1} \oplus \cdots \oplus \rho_l^{m_l}$ is the decomposition into irreducible representations. Then

$$C_{O(\dim(\rho))} \cong \times C_{O(n_i m_i)}(\rho_i^{m_i})$$

and by the Kunneth theorem

$$\Omega_{*,t}(BC_{O(dim\rho)}H)\otimes\mathbb{Q} \cong \bigotimes \Omega_{*,t}(C_{O(n_i m_i)}(\rho_i^{m_i}))\otimes\mathbb{Q}.$$

Geometrically the tensor product corresponds to external product of bundles. But given two bundles for which the lemma is true, one can use the rational null bordisms provided by the lemma for each one, to construct ambient manifolds such that some multiples of the given H bundles occur as the normal bundles to the fixed point set. Forming the product of these manifolds and removing a tubular neighborhood of the fix point set constructs a rational null bordism of the sphere bundle of the external product of the bundles. Therefore it is enough to prove it for powers of irreducible representations.

The proof will be done by induction on the complexity of the group. The base case is representations of $\mathbb{Z}_d$. Note that when d is divisible by 2, there could be orbibundles with non-orientable base, but by taking the pullback to the orientation cover we can assume that we have oriented bases. Let us focus on the complex case. It is enough to prove assuming that the fiber representation is some multiple of an irreducible one. Since the irreducible representations of cyclic groups are given by multiplication by $\zeta_d^j = e^{\frac{2\pi i j}{d}}$, we can assume that the representation is given by the action of $\mathbb{Z}_d$ by multiplying ζ_d^j on $\mathbb{C}^m$, where j is some integer.

By the splitting principle, the inclusion of the maximal torus $(S^1)^m \to C_{O(dim\rho)}\mathbb{Z}_d$ induces a surjective map

$$\Omega_*(B(S^1)^m)\otimes\mathbb{Q} \to \Omega_*(BC_{O(dim\rho)}\mathbb{Z}_d)\otimes\mathbb{Q}$$

By the Kunneth-formula for bordism, the left hand side is the tensor product of $\Omega_*(B(S^1))$, which is a free Ω_*-module on generators the canonical bundles over complex projective spaces. Geometrically these generators represent the $\mathbb{Z}_d$-equivariant bundle $\gamma_n \oplus \mathbb{C}^{m-1}$ over $\mathbb{CP}^n$ with the action given by multiplication by ζ_d^j on the fibers. The last calculations say that every such orbibundle over purely ineffective orbifolds is a linear combination (over $\Omega_* \otimes \mathbb{Q}$) of external products of sums canonical and trivial bundles with $\mathbb{Z}_d$-actions. Notice that the $\mathbb{Z}_d$-bundle $\gamma_n \oplus \mathbb{C}^{m-1}$ over $\mathbb{CP}^n$ is the external product of the canonical bundle γ_n over $\mathbb{CP}^n$ and the trivial bundle $\mathbb{C}^{m-1} \otimes \rho$ over a point.

But $\mathbb{CP}^{n+\zeta_d^j}$ contains a singular point with local representation given by multiplication by ζ_d^{-j} on $\mathbb{C}^{m+1}$, a copy of $\mathbb{CP}^n$ with normal equivariant bundle given by $\gamma_n \otimes \zeta_d^j$ and if $(j,d) \neq 1$, points with cyclic singularities but with strictly smaller isotropy groups. Let B_d the d-branched cover of S^2 along d-points together with the deck translation which rotates the fixed points through an angle $\frac{2\pi j}{d}$. We want to glue d^{n+1} copies of $\mathbb{CP}^{n+\zeta_d}$ and one copy of B^{n+1}, the latter has d^{n+1} points fixed under the action of $\mathbb{Z}_d$ that have local representation ζ_d^j. By identifying (via an orientation reversing diffeomorphism) the corresponding normal bundles to the isolated points, we obtain a closed orbifold $d^{n+1}\mathbb{CP}^{n+\zeta_d} \cup B^{n+1}$, for which the normal bundle to the fixed point (d^{n+1} copies of $\mathbb{CP}^n$) is d^{n+1} copies of $\gamma_n \otimes \zeta_d^j$. Observe that there are no more fixed points. By removing the disk bundle of $\gamma_n \otimes \zeta_d^j$ we obtain an orbifold with boundary with strictly smaller isotropy groups. Therefore d^{n+1} copies of the sphere bundle are the boundary of an orbifold with strictly smaller isotropy groups. Which proves the case for orbibundles with local group $\mathbb{Z}_d$.

The general case for representations of a finite group H, is similarly done. Write $\rho = \rho_1^{m_1} \oplus \cdots \oplus \rho_l^{m_l}$, then as perviously explained, every orbibundle ν with fiber representation ρ over a purely ineffective orbifold is the external product of H-equivariant bundles over trivial H-manifolds.

$$\nu = \nu_1 \boxtimes \cdots \boxtimes \nu_l$$

where each ν_i has fiber representation $\rho_i^{m_i}$.

Now consider the complex projectivizations $\mathbb{CP}(\nu_i \oplus \mathbb{C})$. Each one is an H-manifold that contains a copy of the disk bundle of ν_i and singularitites with isotropy groups $G_{[v:0]}$.

Let us suppose that in some $\mathbb{CP}(\nu_i \oplus \mathbb{C})$ we have that $G_{[v:0]} = H$. Then the corresponding representation $\rho_i^{m_i}$ contains a copy of the representation given by the homomorphism $G_{[v:0]} \to \mathbb{Z}_d$. Since we assumed that ρ_i was an irreducible representation, we have that ρ_i is precisely the representation given by the homomorphism $G_{[v:0]} = H \to \mathbb{Z}_d$ on $\mathbb{C}^m$. The normal bundle to the points with isotropy $G_{[v:0]} = H$ can be considered as a $\mathbb{Z}_d$-bundle. We already know that the sphere bundle of an $\mathbb{Z}_d$-bundle over a trivial $\mathbb{Z}_d$-manifold bounds a $\mathbb{Z}_d$-manifold with strictly smaller local representations. By using the homomorphism $G_{[0:v]} = H \to \mathbb{Z}_d$ we can consider it as an H-manifold. By removing from $\mathbb{CP}(\nu_i \oplus \mathbb{C})$ the disk bundle to the points with isotropy $G_{[v:0]} = H$ and glueing it we obtain a closed H-manifold, let us call it $\mathcal{W}_i$. This H-manifold contains a copy of the the disk bundle of ν_i and singularities with isotropy groups strictly smaller than H.

Now consider the product of the $\mathbb{CP}(\nu_i \oplus \mathbb{C})'s$, when $G_{[v:0]}$ is a proper subgroup of H, and $\mathcal{W}_i's$ when $G_{[v:0]} = H$. This is a closed H-manifold that contains a copy of the external product $\nu = \nu_1 \boxtimes \cdots \boxtimes \nu_l$ and strictly smaller isotropy groups. By removing the disk bundle of the external product we obtain an H-manifold that bounds the sphere bundle, which concludes the proof.

□

From this we can also construct a splitting of the long exact sequence when our families satisfy a stronger condition. We now require our families to be closed under subgroups, instead of condition $ii)$ of definition 6, we require ii') If $(G,\rho) \in \mathcal{F}$ is a representation of the finite group G, then all the representations (H,σ) of the subgroups H of G belong in $\mathcal{F}$.

Theorem 12. *Suppose that $\mathcal{F} \supset \mathcal{F}'$ are quasi-adjacent families that satisfy the property above. Then the rational long exact sequence*

(4)
$$\cdots \longrightarrow \mathcal{O}_n(\mathcal{F}') \otimes \mathbb{Q} \longrightarrow \mathcal{O}_n(\mathcal{F}) \otimes \mathbb{Q} \longrightarrow \mathcal{O}_n(\mathcal{F},\mathcal{F}') \otimes \mathbb{Q} \longrightarrow \mathcal{O}_{n-1}(\mathcal{F}') \otimes \mathbb{Q} \longrightarrow \cdots$$

splits.

Proof. By quasi-adjacency, we can identify the relative term with a sum of cobordism group of orbibundles over purely ineffective orbifolds, where the total space of the orbibundle is oriented and the fiber is the nontrivial part of the representation ρ.

Given such orbibundle ν, the sphere bundle $S(\nu)$ is an orbifold with isotropies in $\mathcal{F}'$. By the previous lemma, this orbifold is the boundary of an orbifold (the disk bundle $D(\nu)$) and of another orbifold $\mathcal{W}$ with isotropies in $\mathcal{F}'$ such that $\partial\mathcal{W} = mS(\nu)$ with m some natural number. Consider the orbifold

$$\mathcal{V} = mD(\nu) \cup_{S(\nu)} \mathcal{W}$$

This is a closed orbifold with isotropies in $\mathcal{F}$ that contains m copies of the disk bundle of ν and the complement has isotropy in $\mathcal{F}(n) - (H,\rho) = \mathcal{F}'(n)$. By lemma 1 in $\mathcal{O}_n(\mathcal{F},\mathcal{F}')$ this element is equal to $m\nu$. The assignment

$$\nu \to \frac{\mathcal{V}}{m}$$

is the desired splitting. □

Now we can do the same thing that we did before, extend the previous result to adjacent families, use the spectral sequence associated to the canonical filtration given by the levels to get a decomposition of the cobordism groups of oriented orbifolds with restricted isotropies and then use an exhausting sequence of families to get the analog of theorem 10.

Theorem 13.

$$\mathcal{O}_* \otimes \mathbb{Q} \cong \Omega_* \otimes \mathbb{Q} \bigoplus_{H finite} \bigoplus_{Rep_*^+(H)} \Omega_{\mathbf{degree}(\rho),t}(BN_{O(n-\mathbf{degree}(\rho))}H/H) \otimes \mathbb{Q}$$

where the sum extends over all finite groups and all representations which are effective and orientation preserving.

As in the unoriented case, there is a projection map

$$\mathcal{O}_* \otimes \mathbb{Q} \to \Omega_* \otimes \mathbb{Q}$$

which assigns an oriented manifold to an oriented orbifold. Invariants of orbifolds can be constructed by composing with ring homomorphism $\Omega_* \otimes \mathbb{Q} \to \mathbb{Q}$

Since rationally the Atiyah-Hirzebruch spectral sequence for oriented bordism collapses, we have then that

$$\Omega_*(BN_{O(n-k_j)}H_j/H_j) \otimes \mathbb{Q} \cong \Omega_* \otimes \mathbb{Q} \otimes_{\mathbb{Q}} H_*(BN_{O(n-k_j)}H_j/H_j;\mathbb{Q})$$

Using this we can rewrite theorem 13 as

Corollary 4 (Druschel)**.**

$$\mathcal{O}_* \otimes \mathbb{Q} \cong \Omega_* \otimes \mathbb{Q} \bigoplus_H \bigoplus_j \Omega_* \otimes \mathbb{Q} \otimes_{\mathbb{Q}} H_{k_j - *}(BN_{O(n-k_j)}H/H; \mathbb{Q}_\rho)$$

where the local coefficients come from the universal bundle over $BN_{O(n-k)}H/H$ and the sum is over all finite groups and faithful representations that are oriented.

Since a submodule of a free module is also free we also have that for any family of representations that are oriented $\mathcal{O}_*(\mathcal{F}) \otimes \mathbb{Q}$ is a free $\Omega_* \otimes \mathbb{Q}$-module.

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