When is a differentiable manifold the boundary of an orbifold?

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Abstract. The aim of this short communication is to review some classical results on cobordism of manifolds and discuss recent extensions of this theory to orbifolds. In particular, we present an answer to the question, *When is differentiable a manifold the boundary of an orbifold?* in the oriented case and in the unoriented case when we restrict to isotropy groups of odd order.

Introduction

When is a differentiable manifold the boundary of another differentiable manifold? This question was solved by R.Thom [**17**] in the fifties, the necessary and sufficient condition is the vanishing of certain characteristic numbers, invariants defined by evaluating characteristic classes of the tangent bundle on the fundamental class of the manifold. His proof is one of the cornerstones of algebraic topology [**2**] and shows the powerful tools that homotopy theory gives to the study of the geometry of manifolds.

Orbifolds, originally introduced as V-manifolds by I. Satake [**15**], and named this way by W. Thurston [**18**], are useful generalizations of manifolds, locally they look like the quotient of euclidean space by the action of a finite group. Their study lies at the intersection of many different areas of mathematics and they appear naturally in many situations such as moduli problems, noncommutative geometry and foliation theory. The local character of the definition of orbifolds allows many constructions that can be applied to manifolds to be extended to orbifolds, and it is natural to ask, *When is an orbifold the boundary of another orbifold?* Key ingredients of Thom's proof do not seem to extend naturally to orbifolds, and the interplay between geometry and homotopy theory, in this case, is still not understood.

A complete solution to this question is still open, but some partial results are known. In the oriented case K. Druschel [**9**] defines generalized Pontrjagin numbers that determine when a multiple of an oriented orbifold is a boundary. In [**4**] the author provides similar characteristic numbers that determine when an orbifold with isotropy groups of odd order is a boundary.

The purpose of this short communication is to present an answer to the more restricted question, *When is a differentiable manifold the boundary of an orbifold?*

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In this case the constructions in [**9**] and [**4**] take a simpler form and the following answer can be given.

An oriented manifold is the boundary of an oriented orbifold precisely when all Pontrjagin numbers vanish.

A differentiable manifold is the boundary of an orbifold with only odd singularities precisely when all Stiefel-Whitney numbers vanish.

The character of this paper is expository and is organized as follows. In section one we review some of the classical theory for manifolds, and state the results of R. Thom [**17**] and C.T.C Wall [**19**] on the cobordism rings. In section two we discuss the necessary background on orbifolds. Section three is the main part of this article, we give a proof that if an oriented differentiable manifold is the boundary of an oriented orbifold then some multiple of the manifold bounds. This easily follows from the existence of a rational fundamental class for oriented orbifolds and the fact that Pontrjagin classes determine the cobordism class of a manifold up to torsion.

Also we present the proof that if a manifold is the boundary of an orbifold with only odd singularities then it is actually the boundary of a manifold. This also follows from the existence of $\mathbb{Z}_2$-fundamental classes for this type of orbifolds, but instead we present a direct geometric proof that constructs the bounding manifold out of the orbifold.

Throughout this paper manifold will mean compact differentiable manifold.

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This paper is dedicated on her birthday to Nora Raggio, for her support and love.

1. Cobordism of manifolds

In this section we present the basic results on the classical theory of cobordism of manifolds. We give the definition of the cobordism groups, characteristic numbers and state the main results of R. Thom and C.T.C Wall that led to a description of the cobordism ring of manifolds. They provide a complete answer to the question, *When is a manifold the boundary of another manifold?* The main references for this section are [**16**], [**13**] and [**19**].

1.1. Unoriented cobordism. We say that two closed n-dimensional manifolds, M_1^n, M_2^n are cobordant if there exists an $n+1$-dimensional manifold with boundary W^{n+1} such that the boundary is the disjoint union of M_1^n and M_2^n.

$$\partial W^{n+1} = M_1^n \sqcup M_2^n$$

The cobordism relation is an equivalence relation on the class of differentiable manifolds, we denote by $[M]$ the corresponding equivalence class, and denote by $\mathfrak{N}_n$ the set of equivalence classes of n-dimensional manifolds. This set can be endowed with a group operation given by disjoint union and the empty set as identity. We call $\mathfrak{N}_n$ the cobordism group of n dimensional manifolds. This is the central object of our study and for its determination more structure on these groups turns out to be of outmost importance, the cartesian product of manifolds induces a ring structure on the graded vector space

$$\mathfrak{N}_* = \bigoplus_n \mathfrak{N}_n$$

Observe that $\mathfrak{N}_*$ is a $\mathbb{Z}_2$-algebra, i.e. every element has order two, because two copies of a manifold M are the boundary of the manifold $M \times I$.

Remark 1.1. We have translated the question, *When is a manifold M the boundary of another manifold?* into the algebraic statement: determine when the class $[M]$ is zero in the cobordism ring $\mathfrak{N}_*$.

In [**17**] R. Thom completely determines this algebra, $\mathfrak{N}_*$ is the polynomial algebra over $\mathbb{Z}_2$ generated by elements in dimensions not of the form $2^j - 1$. He provides complete invariants that determine when two manifolds are cobordant.

1.2. Characteristic numbers. We will now introduce invariants of the cobordism class of a manifold, these invariants are defined in terms of the differentiable structure of the manifold. Given an n-dimensional differentiable manifold M, the tangent bundle $TM \to M$ is a rank n dimensional real vector bundle over M. To such vector bundle you can assign certain cohomology classes $\omega_i(TM) \in H^i(M;\mathbb{Z}_2)$, called Stiefel-Whitney classes [**13**]. Here $H^i(M;\mathbb{Z}_2)$ denotes the singular cohomology of M with $\mathbb{Z}_2$ coefficients.

Recall that any closed n-dimensional manifold has a $\mathbb{Z}_2$-fundamental class, i.e. a top homology class that restricts at any point $x \in M$ to the non-zero class of

$$H_n(M, M - \{x\}; \mathbb{Z}_2) \cong \mathbb{Z}_2$$

Now given any partition $I = (i_1, \cdots, i_m)$ of n, the cup product

$$\omega_I(TM) = \omega_{i_1}(TM) \cup \cdots \cup \omega_{i_m}(TM)$$

is a top cohomology class. By evaluating on the fundamental class of M, we obtain elements of $\mathbb{Z}_2$ that are called the *Stiefel-Whitney numbers* associated to the partition I.

$$\langle \omega_I(TM), [M] \rangle \in \mathbb{Z}_2$$

Here $[M]$ denotes the fundamental class, and $\langle , \rangle$ is the Kronecker pairing between singular cohomology and homology.

These numbers turn out to be invariants of the cobordism class, to see this, it is enough to prove that if M is the boundary of W then all its Stiefel-Whiteney numbers are zero. Suppose that $M^n = \partial W^{n+1}$ and let $I = (i_1, \cdots, i_m)$ be a partition of n. Denote by $\iota : M \to W$ the inclusion of M into W as the boundary, by the collaring theorem there is a neighborhood of M in W that is diffeomorphic to $M \times [0,1)$ therefore $\iota^*(TW) = TM \oplus \mathbb{R}$, the sum of the tangent bundle with a trivial one.

$$\begin{aligned}
\langle \omega_I(TM), [M] \rangle &= \langle \omega_I(TM \oplus \mathbb{R}), [M] \rangle && \text{by stability} \\
&= \langle \omega_I(\iota^*(TW)), [M] \rangle && \text{by the above remark} \\
&= \langle \iota^* \omega_I(TW), [M] \rangle && \text{by naturality} \\
&= \langle \omega_I(TW), \iota_*[M] \rangle && \text{by functoriality of the pairing}
\end{aligned}$$

Now, in the long exact sequence on homology with $\mathbb{Z}_2$-coefficients of the pair (W, M)

$$\cdots \to H_{n+1}(M) \overset{\iota_*}{\to} H_{n+1}(W) \to H_{n+1}(W, M) \overset{\partial}{\to} H_n(M) \overset{\iota_*}{\to} H_n(W) \to \cdots$$

we have that $\iota_* \partial = 0$, but W is a manifold with boundary and it has a fundamental class $[W] \in H_{n+1}(W, M)$ such that $\partial[W] = [M]$, therefore $\iota_*([M]) = 0$. This shows that the Stiefel-Whitney numbers are zero.

EXAMPLE 1.2. For real projective spaces, see [**13**]

$$\omega_I(T\mathbb{RP}^n) = \binom{n+1}{i_1} \cdots \binom{n+1}{i_k} \mod 2$$

in particular for the trivial partition of $n = 2k$, $I = (2k)$ we have

$$\omega_{(2k)}(T\mathbb{RP}^{2k}) \neq 0$$

i.e. $\mathbb{RP}^{2k}$ is not the boundary of another manifold.

R. Thom's proof starts by realizing that the cobordism groups are the stable homotopy groups of a spectrum, now called a Thom spectrum and denoted by MO. Thus the problem of determining when a manifold is the boundary of another manifold has been translated into a homotopy theory problem: determine when a map from a sphere into the spectrum MO is null-homotopic.

The calculation of the stable homotopy groups of MO is still not trivial and was accomplished by R.Thom in [**17**]. A corollary of his work, is that unorientably, a manifold bounds if and only if all its Stiefel-Whitney numbers are zero. His proof also determines completely the cobordism ring, it is the polynomial algebra $\mathfrak{N}_* \cong \mathbb{Z}_2[x_i \mid i \neq 2^j - 1]$. On even dimensions, generators can be taken to be the real projective spaces, odd dimensional generators were given by A. Dold [**7**] soon after R.Thom's paper.

1.3. Oriented cobordism. Observe that an orientation of a manifold with boundary induces an orientation of the boundary by choosing a normal unit vector field on the boundary. By always considering the outward normal vector, the boundary of an oriented manifold is also an oriented manifold, therefore it is possible to also talk about cobordism of such objects. We say M_1^n, M_2^n closed oriented n-dimensional manifolds are cobordant if there exists an $n+1$-dimensional oriented manifold with boundary W^{n+1}, such that the boundary is the disjoint union of M_1^n and $-M_2^n$.

$$\partial W^{n+1} = M_1^n \cup -M_2^n$$

where $-M_2^n$ is just M_2^n with the reverse orientation.

Just as before, cobordism is an equivalence relation on the class of n-dimensional oriented manifolds, the set of equivalence classes is denoted by Ω_n, and is a group under disjoint union. The graded vector space

$$\Omega_* = \bigoplus_n \Omega_n$$

is a graded-commutative ring, the oriented cobordism ring. Given an oriented manifold we will use the linear structure on the tangent bundle to define invariants that determine the oriented cobordism class, at least up to torsion. They are defined in similar way as the Stiefel-Whitney numbers, but now homology and cohomology are with integer coefficients.

Recall that an oriented closed n-dimensional manifold has a fundamental class $[M] \in H_n(M;\mathbb{Z})$ that is, a top homology class that restricts at each point to the class

$$H_n(M, M - \{x\}; \mathbb{Z}) \cong \mathbb{Z}$$

given by the orientation.

To the tangent bundle $TM \to M$, we can associate certain characteristic classes $p_i(TM) \in H^{4i}(M;\mathbb{Z})$ called Pontrjagin classes, see [**14**] and [**13**]. To define invariants of the cobordism class we evaluate these classes at the fundamental class.

Given any partition $I = (i_1, \cdots, i_m)$ of n, the cup product $p_I(TM) = p_{i_1} \cup \cdots \cup p_{i_m}(TM)$ is an n-dimensional cohomology class that when we evaluate on the fundamental class

$$\langle p_I(TM), [M] \rangle$$

gives an integer. The same proof as with Stiefel-Whitney classes shows that these are also cobordism invariants. Note that by dimensional reasons these numbers are always zero unless n is divisible by four.

Example 1.3. For even complex projective spaces, see [**13**]

$$p_I(T\mathbb{CP}^{2n}) = \binom{2n+1}{i_1} \cdots \binom{2n+1}{i_k}$$

if $i_1 + \cdots i_k = n$ and zero otherwise. In particular they are not the boundary of an oriented manifold.

In the same manner, we can identify the oriented cobordism ring with the stable homotopy groups of another spectrum, now denoted MSO. In this case the homotopy problem turns out to be even harder, and R. Thom managed to calculate the stable homotopy groups of MSO after tensoring with the rationals, therefore giving a complete description of the ring $\Omega_* \otimes \mathbb{Q}$. It is a polynomial algebra over the rationals generated by classes on dimensions multiple of 4. Generators can be taken to be the complex projective spaces on even dimensions.

$$\mathbb{Q}[\mathbb{CP}^2, \mathbb{CP}^4, \cdots]$$

R. Thom's proof shows also that Pontrjagin numbers completely determine an element of $\Omega_* \otimes \mathbb{Q}$. Therefore a multiple of a manifold bounds if and only if all its Pontrjagin numbers vanish.

Later work of J. Milnor [**13**] and C.T.C Wall [**19**] settled the complete calculation of the torsion, and gave an algebraic description of the oriented cobordism ring. The main results contained in those papers can be summarized as follows,

- Ω_* has no odd torsion.
- Ω_* has no element of order 4.
- The torsion free part of Ω_* is a polynomial ring.
- Two oriented manifolds are cobordant if and only if their Pontrjagin and Stiefel-Whiteny numbers are the same.

Therefore,

An oriented manifold is the boundary of an oriented manifold precisely when all Pontrjagin and Stiefel-Whitney numbers vanish.

2. Orbifolds

Orbifolds were first introduced by I. Satake [**15**] in the fifties as generalizations of smooth manifolds that allow mild singularities. Since then, orbifolds have became a subject of study of their own, see [**1**] for example, and their use has transcended mathematics: nowaday orbifolds are used in string theory and crystallography. In this section we follow the classical perspective on orbifolds by defining orbifold charts and atlases akin to the way manifolds are defined. We define the analogs of vector bundles and fiber bundles, discuss orientations and fundamental classes. The main references for this section are [**15**],[**1**],and [**11**].

2.1. Charts. Let X be a paracompact Hausdorff topological space. Now we introduce the analog of charts for manifolds, an *n-dimensional orbifold chart* on X is a triple $(\overline{U}, G, U)$, where $\overline{U}$ is a connected manifold, G is a finite group acting on $\overline{U}$, and U is an open subset of X, homeomorphic to $\overline{U}/G$.

To be able to glue charts, we need to specify when two charts are compatible. First, *an embedding of charts* $(\overline{U}, G, U) \hookrightarrow (\overline{V}, H, V)$ is a differentiable embedding $\overline{U} \hookrightarrow \overline{V}$ that is equivariant with respect to a monomorphism $G \hookrightarrow H$, that preserves the kernel of the actions.

Two charts $(\overline{U}, G, U) \hookrightarrow (\overline{V}, H, V)$ are compatible if for every point in $U \cap V$, there exists a chart $(\overline{W}, K, W)$ with embeddings of charts

$$\begin{array}{ccc} (\overline{U}, G, U) & & (\overline{V}, H, V) \\ & \nwarrow \qquad \nearrow & \\ & (\overline{W}, K, W) & \end{array}$$

As with manifolds, an *orbifold atlas* on X is a family of charts $(\overline{U}_\alpha, G_\alpha, U_\alpha)$, that is compatible and covers X. An *orbifold structure* on X is just an equivalence class of orbifold atlases, where two atlases are equivalent if there is a zig-zag of common refinements. We will denote an orbifold structure on X by calligraphic letters, like $\mathcal{X}$, and the topological space X will be called the underlying space and will be denoted by $|\mathcal{X}|$.

Example 2.1. A manifold M with an action of a finite group G gives rise to an orbifold structure that we will denote by $[M/G]$, it has an atlas with only one chart. In a more general vein, if a compact Lie group acts differentiably on a manifold with finite stabilizers, then the quotient space can be endowed with an orbifold structure. Charts can be constructed with the slice theorem for differentiable actions.

Example 2.2. For a finite group G, the orbifold $[*/G]$ is a zero dimensional orbifold, the underlying space has only one point.

Given a point $x \in X$, take a chart $(\overline{U}, G, U)$ around x, let $\overline{x} \in \overline{U}$ be a lift of x. Then we define the *isotropy group* of x to be,

$$G_x = \{g \in G \mid g\overline{x} = \overline{x}\}$$

This group is well defined up to isomorphism and the restriction of the action of the isotropy group on any chart gives a well defined representation of the isotropy group that we call the *local representation* at x. A point is called *non-singular* if G_x is trivial, and *singular* otherwise. The set $\Sigma\mathcal{X} = \{x \in X \mid G_x \neq \{1\}\}$ is called the *singular locus* of X. In general the singular locus is not an orbifold.

REMARK 2.3. The isotropies give a stratification of $|\mathcal{X}|$.

EXAMPLE 2.4. Consider the action of S^1 on $\mathbb{C}^2 - \{0\}$ given by $\lambda(z_1, z_2) = (\lambda^2 z, \lambda z)$, it is a an action with finite stabilizers, and therefore the underlying space, which topologically is just a sphere, has an orbifold structure coming from this action, it has one singular point with isotropy group $\mathbb{Z}_2$, but it can be seen that is not the quotient of a manifold by a finite group (even though it is a quotient by a compact Lie group).

We say that an orbifold is *effective* if we can find an atlas for which all the local groups act effectively. On the other side, we say that an orbifold is *purely ineffective* if all the local groups act trivially.

REMARK 2.5. Extra care should be taken when working with orbifolds that are not effective, for these the definition that we gave is not the best one and the languange of groupoids provide a much superior definition. See [**1**] and [**11**].

2.2. Orbibundles. A *vector orbibundle* $\mathcal{V}$ over an orbifold Q, is an orbifold $\mathcal{V}$ that has an atlas of the form $(\overline{U}_\alpha \times \mathbb{R}^n, G_\alpha)$, where $(\overline{U}_\alpha, G_\alpha)$ is an atlas for $\mathcal{Q}$, all the actions are linear on the second component, and we require also the embeddings between the charts to be linear on the second component.

We have an induced map on the underlying spaces $p : |\mathcal{W}| \to |\mathcal{Q}|$, which in general is not a vector bundle, the fibers of a point are topological spaces homeomorphic to $\mathbb{R}^n/G_x$.

EXAMPLE 2.6. If G is a finite group, and M is a manifold with an action of G, then a G-bundle E gives an orbivector bundle $[E/G]$ over $[M/G]$. In particular for a representation V of G, we have an orbivector bundle $[V/G]$ over $[*/G]$.

In general, given a manifold F and a group acting effectively on F, we define a *fiber orbibundle* with structure group G in a similar way, by requiring an atlas of the form $(\overline{U}_\alpha \times F, G_\alpha)$, where the actions and embeddings on the second component are through elements of G. For example given an orbivector bundle $\mathcal{V}$, we can form the projectivization $\mathbb{RP}(\mathcal{V})$, charts are of the form $(\overline{U}_\alpha \times \mathbb{RP}^{n-1}, G_\alpha)$. Note that the projectivization can have more singularities coming from elements fixing set-wise (and not necessarily point-wise) a line.

EXAMPLE 2.7. Let ρ be the representation of $\mathbb{Z}_2$ on $\mathbb{R}^2$ given by multiplying by -1, the linear orbifold $[\mathbb{R}^2/\mathbb{Z}_2]$ has only one singular point. The projectivization $\mathbb{RP}(\rho \oplus \mathbb{R})$ is the orbifold $[\mathbb{RP}^2/\mathbb{Z}_2]$. Its singular locus consists of one point and a copy of S^1. The local representation at the isolated point is ρ, and the local representation at points on S^1 is given by complex conjugation.

A *local orientation* of an orbifold is a choice of an orientation at each point that makes the action of G_x orientation preserving, this induces orientations on all smaller charts. This is equivalent to identifying G_x with a subgroup of $SO(n)$. As with manifolds an *orientation of an orbifold* is just a choice of local orientations in such a way that the transition functions are orientation preserving. We say that an orbifold is *locally orientable* if all the local groups act preserving the orientation.

Locally orientable orbifolds share many of the properties that manifolds have, in particular a locally orientable n-dimensional orbifold is a *rational homology manifold* i.e. for every $x \in |\mathcal{X}|$

$$H_n(|\mathcal{X}|, |\mathcal{X}| - \{x\}; \mathbb{Q}) = \mathbb{Q}$$

therefore a closed oriented orbifold of dimension n has a fundamental class in $H_n(|\mathcal{X}|;\mathbb{Q})$. This is not true working with integer coefficients and not even with $\mathbb{Z}_2$ coefficients, the order of the groups and the local actions can introduce torsion in $H_n(|\mathcal{X}|,|\mathcal{X}|-\{x\};\mathbb{Z})$.

To an orbifold $\mathcal{X}$ we can associate a topological space $B\mathcal{X}$ called the *classifying space of the orbifold*, see [**11**],[**1**]. For example for a global quotient $\mathcal{X}=[M/G]$ with G a finite group acting on M, $B\mathcal{X}$ is homotopy equivalent to the homotopy orbit space $EG\times_G M$. A key feature of this classifying space construction, is that if $\mathcal{V}$ is an orbibundle on $\mathcal{X}$, then $B\mathcal{V}\to B\mathcal{X}$ is an honest vector bundle, and therefore we can talk about characteristic classes, now elements of $H^*(B\mathcal{X})$.

To an orbifold $\mathcal{X}$ we have associated two topological spaces, the underlying space $|\mathcal{X}|$ and the classifying space $B\mathcal{X}$; there is a map $B\mathcal{X}\to|\mathcal{X}|$, but in general this map is far from being a homotopy equivalence.

Example 2.8. For a finite group G, $B[*/G]$ is homotopy equivalent to BG the classifying space for principal G-bundles [**13**], but $|[*/G]|$ is just a point.

Over the rationals these spaces are the same, at least homologically i.e.

$$H_*(|\mathcal{X}|;\mathbb{Q})\cong H_*(B\mathcal{X};\mathbb{Q})$$

Remark 2.9. Therefore for an oriented orbifold we can talk at the same time about a fundamental class and characteristic classes, at least rationally.

3. Cobordism of orbifolds

In her thesis, K. Druschel [**8**] started the study of orbifold cobordism by introducing a complete set of invariants that determine the oriented cobordism class up to torsion. To study the torsion, K. Druschel in [**8**] considers cobordism with restrictions on the set of local groups and how they fit into a commutative diagram to show that every two and three dimensional effective oriented orbifold bounds.

By restricting the type of singularities that we allow we get different cobordism groups and the study of how these groups relate when we allow more singularities is the main theme of [**8**] and [**5**] where the interplay between different cobordism groups is codified algebraically by a commutative diagram and a spectral sequence, respectively. K.Druschel's result on two and three dimensional orbifolds is explained in [**5**] as the collapse of this spectral sequence.

Generalizing classical constructions from equivariant cobordism, such as family of subgroups and fixed point homomorphisms, in [**4**] a framework to study cobordism groups with restricted singularities is introduced. For example, restricting the isotropy to be only groups of odd order, we get a non-trivial cobordism ring $\mathfrak{N}^{odd}_{*,orb}$ for which a complete description in terms of bordism theory is given.

In this section we present a complete answer to the question *When is a differentiable manifold the boundary of an orbifold?*, for orbifolds with isotropy groups of odd order and oriented orbifolds. The proofs are self-contained and are adaptations of the main techniques of [**9**] and [**4**].

3.1. Orbifolds with boundary.

As with manifolds we can talk about *orbifolds with boundary*, now the charts correspond to open sets in $\mathbb{R}^{n-1}\times[0,\infty)$ with a finite group acting linearly on it. An orbifold has a well defined boundary, just by taking the restriction of the action to the boundary on each chart. But some care should be taken: being a point on the boundary is not a condition that can be checked topologically on the underlying space. For manifolds the boundary was just the set of points for which

$$H_n(M, M - \{x\}) = 0.$$

EXAMPLE 3.1. Consider the interval $I = [-1, 1]$ with the action of $\mathbb{Z}_2$ by multypling by -1. The quotient $[I/\mathbb{Z}_2]$ is an orbifold with boundary,the underlying space is an interval, but the boundary is just one point.

EXAMPLE 3.2. Consider S^1 the complex numbers of norm 1, and the $\mathbb{Z}_2$ action given by complex conjugation, the quotient $[S^1/\mathbb{Z}_2]$ is an orbifold without boundary, but the underlying space is homeomorphic to a closed interval.

The assidous reader will realize that in both examples the action is non-orientable and has codimension one fixed points. As before we can say that two orbifolds $\mathcal{Q}_1, \mathcal{Q}_2$ are cobordant if there exists an $n+1$-dimensional orbifold with boundary $\mathcal{W}^{n+1}$ such that

$$\partial \mathcal{W}^{n+1} = \mathcal{Q}_1 \sqcup \mathcal{Q}_2$$

without restricting the types of orbifolds that we allow, this definition is vacuous, every orbifold is the boundary of another orbifold. The boundary of the orbifold $[I/\mathbb{Z}_2]$ is only one point, from this easy observation it follows that all orbifolds are boundaries.

REMARK 3.3. $\mathcal{Q} \times [I/\mathbb{Z}_2]$ is an orbifold with boundary precisely $\mathcal{Q}$.

Since any manifold is naturally an orbifold, we have a well defined map $\mathfrak{N}_* \to \mathfrak{N}^{odd}_{*,orb}$ and $\Omega_* \to \Omega_{*,orb}$, and the question, *When is a differentiable manifold the boundary of an orbifold?*, can be recasted algebraically

REMARK 3.4. Determine the kernels of the map $\mathfrak{N}_* \to \mathfrak{N}^{odd}_{*,orb}$ and $\Omega_* \to \Omega_{*,orb}$

3.2. The unoriented case. As we have seen before, without imposing any restriction on the type of orbifolds that we allow, any orbifold is the boundary of another orbifold, simply consider $\mathcal{X} \times [I/\mathbb{Z}_2]$. A good class of orbifolds for which the theory is not trivial is the class of orbifolds with only odd singularities. For this class of orbifolds in [**4**] it is shown that the long exact sequences that appear when we increase the allowed singularities split. The splitting has a very clear geometric construction, it is the blow-up along the singular set.

Let us see know that if a manifold is the boundary of an orbifold with only odd singularities then it is actually the boundary of a manifold. The proof will give a way to construct the bounding manifold out of the orbifold.

Suppose that M is an n-dimensional manifold that is the boundary of an $n+1$-dimensional orbifold $\mathcal{W}$ with only isotropies of odd order. Since $\mathcal{W}$ is compact, let H be a group of isotropy of maximal order. The points of $\mathcal{W}$ that have isotropy H form a suborbifold, it is called the H-singular set.

$$\mathcal{W}^H = \{x \in |\mathcal{W}| \mid G_x \cong H\}$$

$\mathcal{W}^H$ is a purely ineffective orbifold, on each connected component the action of H on a local chart is a fixed representation of H. Since $\mathcal{W}^H$ is a suborbifold of $\mathcal{W}$, there is a normal orbibundle $\nu \to \mathcal{W}^H$. For orbifolds the tubular neighborhood theorem also holds, and therefore we can identify a tubular neighborhood with the total space of the normal orbibundle.

Consider, $\mathbb{RP}(\nu \oplus \mathbb{R})$, the projectivization of the orbibundle $\nu \oplus \mathbb{R}$. This also can be thought as the orbibundle that one gets by identifying antipodal points on the disk orbibundle of $\nu \oplus \mathbb{R}$.

Since we are assuming that all groups are of odd order, an element fixes a line setwise if and only if it fixes the line pointwise, therefore $\mathbb{RP}(\nu \oplus \mathbb{R})$ has the same isotropy than $\mathcal{W}$.

The singular sets of $\mathbb{RP}(\nu \oplus \mathbb{R})$ and $\mathcal{W}$ can be identified, as well as their respective tubular neighborhoods. We can take the connected sum of $\mathcal{W}$ and $\mathbb{RP}(\nu \oplus \mathbb{R})$ along $\mathcal{W}^H$, the resulting orbifold has strictly less isotropy groups. Since the boundary of $\mathcal{W}$ is a manifold and $\mathbb{RP}(\nu \oplus \mathbb{R})$ has no boundary, this process does not alter the boundary. We can iterate this construction, and since in each step we are reducing the number of singularities, this process is finite. At the end we have a manifold whose boundary is precisely M, therefore

THEOREM 3.5. *The kernel of the map $\mathfrak{N}_* \to \mathfrak{N}^{odd}_{*,orb}$ is zero.*

3.3. The oriented case. As with manifolds we can also talk about cobordism between oriented orbifolds. The cobordism ring of oriented orbifolds $\Omega_{*,orb}$ was studied in [**9**], where $\Omega_{*,orb} \otimes \mathbb{Q}$ was determined.

Let us proceed now and see that if an oriented manifold bounds an oriented orbifold, then actually some multiple of it bounds a manifold (i.e. is a torsion element of Ω_*) This now easily follows from the existence of fundamental classes for oriented orbifolds (over the rationals) and Pontrjagin classes for orbibundles.

Suppose M^n is an oriented manifold that is the boundary of an oriented orbifold $\mathcal{W}$. We want to see that the Pontrjagin numbers of M are zero, which shows that M is a torsion class of Ω_n. The proof is a slight modification of the one given before for the vanishing of the Stiefel-Whitney numbers.

Call $\iota : M \to \mathcal{W}$ the inclusion, this map induces a map between the classifying spaces $B\iota : BM \to B\mathcal{W}$, since M is a manifold, the projection $BM \to M$ is a homotopy equivalence that has the property that $p^*(TM) \cong BTM$. Denote by $[BM] = p_*^{-1}([M])$ the preimage of the fundamental class of M.

Also, $\mathcal{W}$ is an orbifold and therefore has a tangent orbibundle $T\mathcal{W}$, this is not a vector bundle, but $BT\mathcal{W} \to B\mathcal{W}$ is an honest vector bundle and even more, $B\iota^*(BT\mathcal{W}) = BTM \oplus \mathbb{R}$.

$$\begin{aligned}
\langle p_I(TM),[M]\rangle &= \langle p_I(TM), p_*[BM]\rangle \\
&= \langle p^*p_I(TM),[BM]\rangle && \text{by functoriality of the pairing} \\
&= \langle p_I(p^*TM),[BM]\rangle && \text{by naturality} \\
&= \langle p_I(BTM),[BM]\rangle \\
&= \langle p_I(BTM\oplus\mathbb{R}),[BM]\rangle && \text{by stability} \\
&= \langle p_I(B\iota^*(BT\mathcal{W})),[BM]\rangle && \text{by the above remark} \\
&= \langle B\iota^* p_I(TW),[BM]\rangle && \text{by naturality} \\
&= \langle p_I(TW), B\iota_*[BM]\rangle && \text{by functoriality of the pairing}
\end{aligned}$$

As before by looking at the long exact sequence of the pair $(B\mathcal{W}, BM)$ we see that $B\iota_*[BM] = 0$. Therefore all the Pontrjagin numbers of M are zero, i.e. M is a torsion element of Ω_*.

In [**3**], Anderson showed that torsion elements of Ω_* may be represented as sums of classes of manifolds of the form

$$V^n = \mathbb{RP}(\lambda \oplus \mathbb{R}^{2k+1})$$

the projectivization of the direct sum of a line bundle λ and a trivial $(2k+1)$-bundle, these manifolds have orientation reversing involutions coming from the bundle involution on λ.

REMARK 3.6. If an orbifold $\mathcal{X}$ admits an orientation reversing involution (actually any orientation reversing periodic map) then it is the boundary of an oriented orbifold. Just consider the orbifold $[\mathcal{X} \times I]/\mathbb{Z}_2$, where $\mathbb{Z}_2$ acts on $\mathcal{X}$ by the orientation reversing involution, and on $I = [-1, 1]$ by multiplying by -1.

By the preceeding remark and Anderson's result, all the torsion elements of Ω_* are the boundary of oriented orbifolds.

THEOREM 3.7. *The kernel of the map $\Omega_* \to \Omega_{*,orb}$ is precisely the torsion of Ω_**

The idea of using orientation reversing involutions to construct bounding orbifolds is the key geometric argument used in [**8**] and [**5**], where it is used to show that all two and three dimensional oriented orbifolds bound.

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