Talk 6: Totaro's spectral sequence

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Abstract. The main source for this talk is [**2**]. The goal was the study of a spectral sequence that abuts to the cohomology of the configuration spaces $F(X, n)$ of a space X, where $F(X, n)$ is the set of n distinct points in X.

The spectral sequence is the Leray spectral sequence for the inclusion $F(X, n) \subset X^n$. For orientable manifolds the E_2 term and the first non-trivial differential can be expliciteley identified.

In the case of a projective algebraic variety, the spectral sequence has only one non-trivial differential if the coefficients are taken in a field of characteristic zero and we can determine from the spectral sequence the rational cohomology ring of the configuration space of n-tuples of distinct points in X. The answer depends only on the cohomology ring of X.

In this talk I described the paper of Burt Totaro "Configuration spaces of algebraic varieties", where the Leray spectral sequence for the inclusion $F(X, n) \subset X^n$ is studied. This spectral sequence converges to the cohomology of the ordered configuration spaces with integral coefficients. It was described earlier by Cohen and Taylor in [**1**] algebraically by a filtration of a dga that calculates the cohomology of the configuration spaces. Taking field coefficients,

Theorem 1. *Let X be an oriented manifold. Then there is a spectral sequence of S_n-algebras converging to $H^*(F(X, n); \mathbb{K})$. The E_2 term is the quotient of the graded-commutative $\mathbb{K}$-algebra*

$$H^*(X^n; \mathbb{K})[A_{a,b}] \quad \textit{for } 1 \le a \ne b \le n$$

where $H^i(X^n, \mathbb{K})$ has degree $(i, 0)$ and the $A_{a,b}$ are of degree $(0, m-1)$, subject to the relations,

$$A_{a,b} = (-1)^m A_{b,a} \tag{0.1}$$

$$A_{a,b}^2 = 0 \tag{0.2}$$

$$A_{a,b}A_{a,c} + A_{b,c}A_{b,a} + A_{c,a}A_{c,b} = 0 \textit{ for } k < j < i \tag{0.3}$$

$$p_a^*(x)A_{a.b} = p_b^*(x)A_{a,b} \textit{ for } a \ne b, x \in H^*(X; \mathbb{K}) \tag{0.4}$$

(0.5)

The first non-trivial differential is given by

$$dA_{a,b} = p_{a,b}^*(\Delta)$$

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where $\Delta \in H^m(X^2;\mathbb{K})$ is the diagonal class. The action of S_n is induced from the action on $H^*(X^n;\mathbb{K})$ and $\sigma A_{a,b} = A_{\sigma(a),\sigma(b)}$.

$(n-1)(m-1)$	$H^0(X)\otimes\mathbb{Z}$	$H^1(X)\otimes\mathbb{Z}$	$\cdots$	$H^p(X)\otimes\mathbb{Z}$
$(n-2)(m-1)$	$\bigoplus_J H^0(X^2_J)\otimes\mathbb{Z}^{c_J}$	$\bigoplus_J H^1(X^2_J)\otimes\mathbb{Z}^{c_J}$	$\cdots$	$\bigoplus_J H^p(X^2_J)\otimes\mathbb{Z}^{c_J}$
$\vdots$	$\vdots$	$\vdots$		$\vdots$
$r(m-1)$	$\bigoplus_J H^0(X^{n-r}_J)\otimes\mathbb{Z}^{c_J}$	$\bigoplus_J H^1(X^{n-r}_J)\otimes\mathbb{Z}^{c_J}$	$\cdots$	$\bigoplus_J H^p(X^{n-r}_J)\otimes\mathbb{Z}^{c_J}$
$\vdots$	$\vdots$	$\vdots$		$\vdots$
$(m-1)$	$\bigoplus_J H^0(X^{n-1}_J)$	$\bigoplus_J H^1(X^{n-1}_J)$	$\cdots$	$\bigoplus_J H^p(X^{n-1}_J)$
0	$H^0(X^n)$	$H^1(X^n)$	$\cdots$	$H^p(X^n)$
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The Leray spectral sequence associated to a continuous map $f : X \to Y$, can be seen as a special case of the Grothendieck spectral sequence for the derived functor of the composition of two left exact functors. The global section functors Γ and the direct image f_*, $Sheaves(X) \xrightarrow{f_*} Sheaves(Y) \xrightarrow{\Gamma} Abelian$. In our case for the inclusion $F(X,n) \subset X^n$, and the locally constant sheaf $\underline{\mathbb{Z}}$ on $F(X,n)$, it is a spectral sequence,

$$H^p(X^n; R^q j_*\underline{\mathbb{Z}}) \Rightarrow H^{p+q}(F(X,n),\mathbb{Z})$$

converging to the cohomology of the configuration space. The $E_2^{p,q}$ is the cohomology of the n-fold product with coefficients in the higher direct images of the locally constant sheaf $\underline{\mathbb{Z}}$ under the inclusion $j : F(X,n) \subset X^n$.

The higher direct image sheaf $R^q j_*\underline{\mathbb{Z}}$ is the sheafication of the presheaf

$$U \to H^q(U \cap F(X,n);\mathbb{Z})$$

and then when X is a manifold, we use the local structure of X to calculate the stalk. For $x \in X^n$, suppose (after permutation of indices) that x has the form $x = (x_1, \ldots, x_1, \ldots, x_s, \ldots, x_s)$ with $x_1, \ldots, x_s \in X$ and $\sum i_j = n$, then the stalk at x of the higher direct image sheaf $R^q j_*\underline{\mathbb{Z}}$ is,

$$H^q(F(T_{x_1}X, i_1) \times \ldots \times F(T_{x_s}X, i_s);\mathbb{Z})$$

and therefore we are led to the study of the cohomology of products of ordered configuration spaces of euclidean spaces. To state the main results let us introduce some notation, recall that a partition I of a set $\{1,\ldots,n\}$ is a non-empty collection of subsets of $\{1,\ldots,n\}$ that are disjoint and whose union is $\{1,\ldots,n\}$.

Definition 2. To a partition I into k blocks, ($|I| = k$), we associate the diagonal subspace $X^k_I \subseteq X^n$,

$$X^k_I := \{(x_1,\ldots,x_n) \in X^n \mid x_i = x_j \text{ if } i,j \text{ are in the same block of } I\ \}$$

Definition 3. We say that a partition J refines a partition I ($J \leq I$) if for every $A \in J$ there exists $B \in I$ with $A \subseteq B$.

By the results of Cohen we know that the cohomology algebra of ordered configuration spaces of n points in $\mathbb{R}^m$ is concentrated on multiples of $m-1$ and the top dimensional cohomology group is equal to $\mathbb{Z}^{(n-1)!}$ in dimension$(n-1)(m-1)$, similarly for a product of the form

$$F(R^m, i_1) \times \cdots \times F(R^m, i_k)$$

the cohomology is zero except in dimensions divisible by $m-1$. Each factor $F(\mathbb{R}^m, i_s)$ has top non-zero cohomology in dimension $(i_s - 1)(m-1)$ and therefore for the product $F(R^m, i_1) \times \cdots \times F(R^m, i_k)$ the top non-zero cohomology is in dimension

$$\sum_s (i_s - 1)(m-1) = (n-k)(m-1)$$

and is $\mathbb{Z}^{c_I}$ where, $c_I := \prod_{A \in I}(|A| - 1)! = \prod_s (i_s - 1)!$ Suppose that I is a partition with k blocks of sizes $i_1, \ldots, i_k$, and J a refinement of I with blocks of sizes $j_1, \ldots, j_{n-r}$. We have a natural restriction map,

$$\prod_s^k F(X, i_s) \to \prod_s^{n-r} F(X, j_s)$$

By adding the induced maps on cohomology over all refinements,

Lemma 4. *For $0 \leq r \leq n-k$ we have an isomorphism,*

$$\bigoplus_J H^{r(m-1)}(F(\mathbb{R}^m, j_1) \times \cdots \times F(\mathbb{R}^m, j_{n-r}); \mathbb{Z}) \cong H^{r(m-1)}(F(R^m, i_1) \times \cdots \times F(R^m, i_k); \mathbb{Z})$$

where the sum is over all partitions of J of $\{1, ..., n\}$ with $n-r$ blocks such that $J \leq I$.

This lemma shows that the $r(m-1)$ dimensional classes are pulled back from classes that are top dimensional for some refinement of I. For example, a basis for $H^{r(m-1)}(F(\mathbb{R}^m, n); \mathbb{Z})$ is given by monomials,

$$A_{a_1,b_1} \cdots A_{a_r,b_r}, a_1 < \cdots < a_r \text{ and } b_s < a_s, 1 \leq s \leq r$$

and we can define a partition J of $\{1, \ldots, n\}$ with $n-r$ blocks by,

$$a_s \sim b_s$$

then it is clear that the element $A_{a_1,b_1} \cdots A_{a_r,b_r}$ lies in the image of the map

$$H^{r(m-1)}(F(\mathbb{R}^m, j_1) \times \cdots \times F(\mathbb{R}^m, j_{n-r}); \mathbb{Z}) \to H^{r(m-1)}(F(R^m, n); \mathbb{Z})$$

From this lemma follows that the higher direct images sheaves are sums of locally constant sheaves supported on the diagonals and if we assume that X is oriented we have an isomorphism of sheaves,

$$R^{r(m-1)} j_* \underline{\mathbb{Z}} \cong \sum_{|J| = n-r} \mathbb{Z}^{c_J}_{X_J^{n-r}}$$

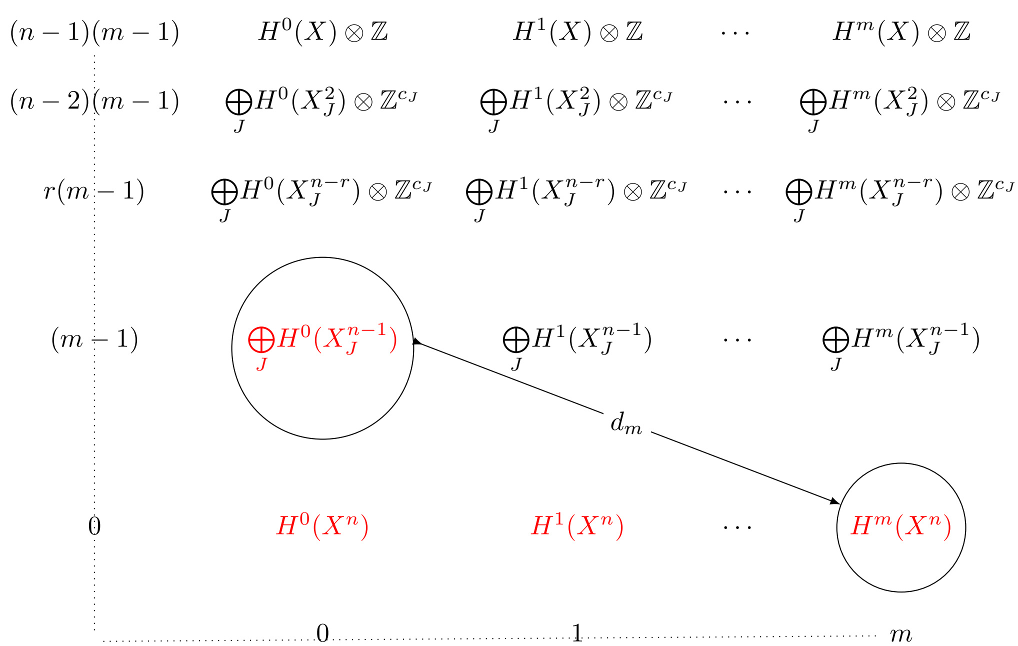
Since $H^i(X^n; \mathbb{Z}^{c_J}_{X_J^{n-r}}) \cong H^i(X_J^{n-r}; \mathbb{Z})$ and cohomology commutes with direct sums, we obtain the following description of the E_2 term of the Leray spectral sequence with $\mathbb{Z}$ coefficients.

$$E_2^{p,r(m-1)} \cong \bigoplus_J H^p(X_J^{n-r};\mathbb{Z}) \otimes \mathbb{Z}^{c_J}$$

To describe the first non-trivial differential note that the E_2 term is generated as an algebra by the first row and the the group in the position $(1, m-1)$. The differential d_m is zero on the bottom row, by dimensional reasons. Therefore it is determined by the map

$$d_m : \bigoplus_{|J|=n-1} H^0(X_J^{n-1};\mathbb{Z}) \to H^m(X^n;\mathbb{Z})$$

which is the sum of the Gysin maps in cohomology associated to the inclusion $X_J^{n-1} \subseteq X^n$.



For X smooth projective variety of complex dimension l, when we take the coefficients to be a field of characteristic zero, this is the only non-trivial differential of the spectral sequence. Even more the cohomology of the configuration space $F(X,n)$ is isomorphic to the cohomology of the dga $E_2 \otimes \mathbb{Q}$ with the differential d_{2l}.

References

[1] Cohen, F. R. and Taylor, L. R., *Configuration spaces: applications to Gelfand-Fuks cohomology*, Bull. Amer. Math. Soc., 84, 1978 , 134–136.

[2] Totaro, Burt., *Configuration spaces of algebraic varieties*, Topology, 35,1996, 1057–1067.