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Hochschild cohomology and string topology of global quotient orbifolds

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Abstract

Let M be a connected, simply connected, closed and oriented manifold, and G be a finite group acting on M by orientation-preserving diffeomorphisms. In this paper, we show an explicit ring isomorphism between the orbifold string topology of the orbifold $[M/G]$ and the Hochschild cohomology of the dg-ring obtained by performing the smash product between the group G and the singular cochain complex of M.

1. *Introduction*

String topology stands for the study of the topological properties associated to the space of smooth free loops $\mathcal{L}M$ on a closed and oriented manifold M of dimension d.

The starting point of string topology was the paper [**3**] by Chas and Sullivan where the authors discovered an intersection product in the homology of the free loop space

$$H_p(\mathcal{L}M) \otimes H_q(\mathcal{L}M) \longrightarrow H_{p+q-d}(\mathcal{L}M),$$

having total degree $-d$, which together with the degree 1 operator $H_*(\mathcal{L}M) \to H_{*+1}(\mathcal{L}M)$ induced by the circle action on the loops, endowed the homology of the free loop space with the structure of a Batalin–Vilkovisky algebra.

Cohen and Jones [**5**] developed a homotopical theoretic realization of string topology, by endowing the Thom spectrum $\mathcal{L}M^{-TM}$ with the structure of a ring spectrum

$$\mathcal{L}M^{-TM} \wedge \mathcal{L}M^{-TM} \longrightarrow \mathcal{L}M^{-TM}$$

that allowed them to show that, at the level of homology, the intersection product of Chas and Sullivan can be recovered by the product in homology induced by the ring spectrum $\mathcal{L}M^{-TM}$. In the same paper Cohen and Jones furthermore showed, generalizing results of Jones [**15**], that, in the case when M is simply connected, there is a ring isomorphism between the homology of the ring spectrum $\mathcal{L}M^{-TM}$ and the Hochschild cohomology of the singular cochains of M

$$H_*(\mathcal{L}M^{-TM}) \cong HH^*(C^*(M), C^*(M)).$$

In the case of global quotient orbifolds of the form $[M/G]$ for a G finite group, Lupercio and the third author [**18**] constructed the loop groupoid $[P_G M/G]$ as the natural free loop space of the orbifold, whose homotopic quotient turns out to be homotopy equivalent to the space of free loops of the homotopy quotient $M \times_G EG$, that is,

$$P_G M \times_G EG \simeq \mathcal{L}(M \times_G EG);$$

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and with this equivalence at hand, Lupercio, Xicoténcatl and the third author [**19**] showed that the homology of the free loop space of the orbifold

$$H_*(\mathcal{L}(M \times_G EG);\mathbb{Q}) \cong H_*(P_GM;\mathbb{Q})^G$$

could also be endowed with the structure of a Batalin–Vilkovisky algebra; the authors coined this structure with the name *orbifold string topology*.

But, can the orbifold string topology be defined over the homology with integer coefficients? And, is there any relation between the orbifold string topology ring and the Hochschild cohomology of some specific dg-ring? This paper is devoted to positively answering these two questions.

Let us start with a brief description of the answer to the second question, as its solution leads the way to solving the first. From the isomorphism showed by Cohen and Jones between the string topology ring of a manifold and the Hochschild cohomology of the singular cochains, one is tempted to try to show that the orbifold string ring should be isomorphic to the ring

$$HH^*(C^*(M \times_G EG), C^*(M \times_G EG));$$

but due to convergence issues (the Eilenberg–Moore spectral sequence does not converge in general [**6**]), one cannot show using standard cosimplicial methods that indeed this ring recovers the orbifold string ring, and even worse, in some cases we show (see Section 7) that this ring does not give the appropriate ring structure on the homology of the free loop space of $[M/G]$. Instead we consider the dg-ring $C^*(M)\#G$ defined as the smash product of G with the singular cochains $C^*(M) = C^*(M;\mathbb{Z})$ of M, and we compare its Hochschild cohomology with the homology ring of the ring spectrum P_GM^{-TM} that was constructed in [**19**].

In the case where M is simply connected and connected, we find that the orbifold string topology ring can be recovered as the Hochschild cohomology ring of $C^*(M;\mathbb{Q})\#G$, that is, there is an isomorphism of rings

$$HH^*(C^*(M;\mathbb{Q})\#G, C^*(M;\mathbb{Q})\#G) \cong H_*(P_GM^{-TM};\mathbb{Q})^G.$$

This isomorphism is obtained by carefully decomposing the Hochschild cohomology ring into smaller parts, which leads to the ring isomorphism

$$HH^*(C^*(M)\#G, C^*(M);\#G) \cong \mathrm{Ext}^*_{\mathbb{Z}G}(\mathbb{Z}, C_*(P_GM^{-TM})).$$

With the previous isomorphism at hand, it was clear that in order to get a topological counterpart to the Hochschild cohomology of $C^*(M)\#G$, it was necessary to introduce some sort of Poincaré dual to the universal principal G bundle EG. Fortunately, the spaces EG can be approximated by finite-dimensional manifolds EG_n with free G actions, which together with the S-duality identification

$$C^*(EG_n) \simeq C_{-*}(EG_n^{-TEG_n})$$

allow us to construct a pro-ring spectrum whose homology

$$H_*^{\mathrm{pro}}(\mathcal{L}(M \times_G EG)^{-T(M\times_G EG)}) := \lim_{\leftarrow n} H_*((P_GM \times_G EG_n)^{-e_0^*T(M\times_G EG_n)})$$

turned out to be isomorphic to the Hochschild cohomology of $C^*(M)\#G$

$$HH^*(C^*(M)\#G, C^*(M)\#G) \cong H_*^{\mathrm{pro}}\left(\mathcal{L}(M \times_G EG)^{-T(M\times_G EG)}\right).$$

This is the main theorem of the paper (Theorem 5.3).

Because of this last isomorphism, we define the orbifold string topology ring with integer coefficients to be the ring

$$H_*^{\mathrm{pro}}(\mathcal{L}(M \times_G EG)^{-T(M\times_G EG)}),$$

and in this way we answer also the first question.

The first five sections are devoted to showing the main theorem (Theorem 5.3). After that we have some simple applications of the main theorem and in Section 7 we explain why Hochschild cohomology is not preserved under groupoid equivalence.

When we started this project, we found that the literature about Hochschild cohomology of dg-rings was a little disperse; it tended to either be based on formulas whose categorical meaning were not properly explained or gave much more sophisticated expositions than was required for our purposes. Therefore, we decided to give a detailed and elementary description of the homological aspects of Hochschild cohomology that both describe the concrete formulas needed to compute things as well as the interpretation of Hochschild cohomology as Ext-groups in the derived category of dg-modules. This also clarifies the relationships between algebraic constructions associated to $C^*(M)\#G$ and topological constructions that deal with free loop spaces.

The layout of the paper is as follows. In Section 2, we consider the smash product of a discrete group G with a dg-ring $\mathcal{A}$ and we give an explicit resolution of $\mathcal{A}\#G$ as an $\mathcal{A}\#G^e$-module that allows us to decompose the complex that produces the Hochschild cohomology of $\mathcal{A}\#G$ into the composition of two functors

$$\mathcal{RH}om_{\mathbb{Z}G}(\mathbb{Z}, \mathcal{RH}om_{\mathcal{A}^e}(\mathcal{A}, \mathcal{A}\#G)).$$

In Section 3, we construct cosimplicial spaces $\mathbb{P}_g M$ whose total spaces realize the orbifold loops $P_g M$, and with this identification at hand we construct a quasi-isomorphism

$$C^*(M) \overset{L}{\otimes}_{C^*(M)^e} C^*(M)\#G \simeq C^*(P_G M).$$

In Section 4, we construct cosimplicial spectra $\mathfrak{P}_g M$ whose total spectra realize the spectra $P_g M^{-TM}$, and this moreover permits one to construct a quasi-isomorphism

$$\mathcal{H}om_{C^*(M)^e}(B(C^*(M)), C^*(M)\#G) \simeq C_*(P_G M^{-TM}),$$

where $B(C^*(M))$ denotes the Bar construction of $C^*(M)$. And in Section 5, we prove the main theorem of the paper (Theorem 5.3) by constructing a pro-ring spectrum for the orbifold $[M/G]$ whose homology ring is isomorphic to the Hochschild cohomology of $C^*(M)\#G$.

In Section 6, we show some applications of the main theorem, and in Section 7 we show why the Hochschild cohomology is not an invariant of the orbifold by presenting two equivalent groupoids with different Hochschild cohomologies. We complete with three appendices; in Appendix A we give the preliminaries on derived categories of dg-modules over dg-rings and we describe several equivalent ways in which the Hochschild cohomology of a dg-ring $\mathcal{A}$ can be defined. Appendix B is devoted to explaining the sign notation that is used in the Bar construction, and Appendix C is devoted to showing that all the constructions performed in this paper for dg-rings that are free with respect to $\mathbb{Z}$ can be applied to the dg-ring of singular cochains $C^*(M)$ on a manifold that in general is not free over $\mathbb{Z}$. This third appendix puts onsolid ground the results of Sections 4 and 5.

2. *Hochschild cohomology for the smash product of a group and a dg-ring*

In this section, we shall describe how to calculate the Hochschild cohomology for the smash product of a group and a dg-ring. The main construction of this section consists of an explicit cofibrant replacement from which the results that we claim follow clearly, and which gives an alternative explanation of the results of Sanada [**24**] over the Hochschild cohomology of crossed products over commutative rings.

In this section, G is a finitely generated discrete group.

Let $\mathcal{A} = (A, d_{\mathcal{A}})$ be a dg-ring together with a group homomorphism $\sigma : G \to \mathrm{Aut}_{\mathrm{dg-Rings}}(A)$. We write $ga := g(a) := \sigma(g)(a)$ for $a \in A$, $g \in G$. We refer to such an $\mathcal{A}$ as a G-module dg-ring (this name is a dg-analogue of the term module algebra).

Definition 2.1. Let $\mathcal{A}$ be a G-module dg-ring. The smash product gives a dg-ring that is a principal object of study in this paper

$$\mathcal{A}\#G := \mathcal{A} \otimes_{\mathbb{Z}} \mathbb{Z}G,$$

with multiplication given on generators by

$$(x \otimes g)(y \otimes h) := xg(y) \otimes gh,$$

and the differential is $d_{\mathcal{A}} \otimes 1$.

Note that $\mathcal{A}$ is a sub-dg-ring of $\mathcal{A}\#G$ by the map $x \mapsto x \otimes 1_G$, and $\mathbb{Z}G$ is also a sub-dg-ring of $\mathcal{A}\#G$ via the map $g \mapsto 1_{\mathcal{A}} \otimes g$.

In this section, we will construct an explicit cofibrant replacement for $\mathcal{A}\#G$ as an $(\mathcal{A}\#G)^e$-module, which together with an explicit diagonal map, will be the main tools to show that if we consider $\mathcal{A}\#G$ as a G-module-dg-ring by the action $g \cdot (x \otimes h) \mapsto g(x) \otimes ghg^{-1}$, then we have the following theorem.

Theorem 2.2. *There are isomorphisms of graded groups*

$$\begin{aligned} HH_*(\mathcal{A}\#G, \mathcal{A}\#G) &\cong \mathrm{Tor}^*_{\mathbb{Z}G}(\mathbb{Z}, \mathcal{A} \overset{L}{\otimes}_{\mathcal{A}^e} \mathcal{A}\#G), \\ HH^*(\mathcal{A}\#G, \mathcal{A}\#G) &\cong \mathrm{Ext}^*_{\mathbb{Z}G}(\mathbb{Z}, \mathcal{RH}om_{\mathcal{A}^e}(\mathcal{A}, \mathcal{A}\#G)) \end{aligned}$$

defined in an appropriate way such that the second isomorphism becomes one of graded rings.

The explicit ring structure on the right-hand side of the second isomorphism will be explained in the proof of the theorem, and it is further emphasized in Paragraphs 2.3.1 and 2.3.2. This ring structure will be of use when we study the string topology for orbifolds and its relation with Hochschild cohomology of the dg-rings of singular cochains.

2.1. *Cofibrant replacement for $\mathcal{A}\#G$*

Let us consider the cofibrant replacements constructed via the Bar construction (see Paragraph A.3.1) for the dg-ring $\mathcal{A}$ as an $\mathcal{A}^e$-module and for the ring $\mathbb{Z}G$ as a $\mathbb{Z}G^e$-module

$$B(\mathcal{A}) \xrightarrow{\epsilon} \mathcal{A} \quad B(\mathbb{Z}G) \to \mathbb{Z}G.$$

If we consider the isomorphism

$$\begin{aligned} (\mathbb{Z}G)^{k+2} &\to (\mathbb{Z}G)^{k+2}, \\ (g_0|g_1|\cdots|g_{k+1}) &\longmapsto (g_0|g_0g_1|g_0g_1g_2|\cdots|g_0\cdots g_kg_{k+1}), \end{aligned}$$

then, by the transportation of structures, we can change $B(\mathbb{Z}G)$ by an alternative cofibrant replacement $\bar{B}(\mathbb{Z}G)$ of $\mathbb{Z}G$ defined as follows: as a $\mathbb{Z}$-graded module we have that

$$\bar{B}(\mathbb{Z}G) = \bigoplus_{k=0}^{\infty} (\mathbb{Z}G)^{k+2}[k],$$

the differential becomes

$$\delta(h_0|\cdots|h_{k+1}) = \sum_{j=0}^{k} (-1)^j (h_0|\cdots|\widehat{h_j}|\cdots|h_{k+1}),$$

the $\mathbb{Z}G^e$-module structure is

$$(g \otimes k)(h_0|\cdots|h_{k+1}) = (gh_0|gh_1|gh_2|\cdots|gh_{k+1}k),$$

and the $\mathbb{Z}G^e$-module homomorphism $\bar{\epsilon} : \bar{B}(\mathbb{Z}G) \to \mathbb{Z}G$ is

$$\bar{\epsilon}(h_0|h_1) = h_1 \quad \text{and} \quad \bar{\epsilon}(h_0|\cdots|h_{k+1}) = 0 \quad \text{for } k > 1.$$

Note that, for $\bar{B}(\mathbb{Z}G)$, the diagonal map defined in (A.4) becomes

$$\begin{aligned} \bar{B}(\mathbb{Z}G) &\longrightarrow \bar{B}(\mathbb{Z}G) \otimes_{\mathbb{Z}G} \bar{B}(\mathbb{Z}G) \\ (h_0|\cdots|h_{k+1}) &\longmapsto \sum_{j=0}^{k} (h_0|\cdots|h_j|1) \otimes_{\mathbb{Z}G} (h_j|h_{j+1}|\cdots|h_{k+1}). \end{aligned} \tag{2.1}$$

Lemma 2.3. *The tensor product $B(\mathcal{A}) \otimes_{\mathbb{Z}} \bar{B}(\mathbb{Z}G)$ can be endowed with the structure of an $\mathcal{A}\#G^e$-module structure, thus making*

$$\epsilon \otimes \bar{\epsilon} : B(\mathcal{A}) \otimes_{\mathbb{Z}} \bar{B}(\mathbb{Z}G) \to \mathcal{A}\#G,$$

a cofibrant replacement for $\mathcal{A}\#G$ as an $\mathcal{A}\#G^e$-module.

Proof. Let us denote the elements in $\mathcal{A}\#G^e$ by $(a \otimes g|b \otimes k)$, where $a \otimes g$ and $b \otimes k$ belong to $\mathcal{A}\#G$. The $\mathcal{A}\#G^e$-module structure of $B(\mathcal{A}) \otimes_{\mathbb{Z}} \bar{B}(\mathbb{Z}G)$ is defined as

$$\begin{aligned} &(a \otimes g|b \otimes k)((x_0|\cdots|x_{k+1}) \otimes (h_0|\cdots|h_{l+1})) \\ &\quad = (ag(x_0)|g(x_1)|\cdots|g(x_k)|g(x_{k+1})gh_{l+1}(b)) \otimes (gh_0|gh_1\cdots|gh_l|gh_{l+1}k). \end{aligned}$$

It is a simple calculation to show that indeed the previous structure is an $\mathcal{A}\#G^e$-module structure on $B(\mathcal{A}) \otimes \bar{B}(\mathbb{Z}G)$.

To show that $\epsilon \otimes \bar{\epsilon} : B(\mathcal{A}) \otimes_{\mathbb{Z}} \bar{B}(\mathbb{Z}G) \to \mathcal{A}\#G$ is a morphism of $\mathcal{A}\#G^e$-modules, we need only to concentrate our attention on the elements in $\mathcal{A}^2 \otimes_{\mathbb{Z}} \mathbb{Z}G^2$ and this follows from the commutativity of the following diagram:

$$\begin{tikzcd} (x_0|x_1) \otimes (h_0, h_1) \arrow[r, "(a\otimes g|b\otimes k)"] \arrow[d, "\epsilon\otimes\bar{\epsilon}"] & (ag(x_0)|g(x_1)gh_1(b)) \otimes (gh_0|gh_1k) \arrow[d, "\epsilon\otimes\bar{\epsilon}"] \\ (x_0x_1|h_1) \arrow[r, "(a\otimes g|b\otimes k)"] & (ag(x_0x_1)gh_1(b)|gh_1k). \end{tikzcd}$$

The fact that the map $\epsilon \otimes \bar{\epsilon}$ is a quasi-isomorphism follows from the fact that ϵ and $\bar{\epsilon}$ are quasi-isomorphisms. Now we are left to prove the cofibrant condition. For this, consider a filtration defined for $q \geqslant 0$

$$F^{2q} := \bigoplus_{p=0}^{q} \bigoplus_{j=0}^{p} ((\mathcal{A} \otimes \mathcal{A}^{\otimes j} \otimes \mathcal{A}) \otimes (\mathbb{Z}G)^{p-j+2})[p]$$

and

$$F^{2q+1} := F^{2q} \oplus \bigoplus_{p=0}^{q} ((\mathcal{A} \otimes W^{p+1} \otimes \mathcal{A}) \otimes (\mathbb{Z}G)^{q-p+2})[q+1],$$

where W^p is the $\mathcal{A}^e$-submodule of $\mathcal{A}^p$ defined as the kernel of the differential $d : \mathcal{A}^p \to \mathcal{A}^{p+1}$.

The subquotients of the filtration are isomorphic to the $\mathcal{A}\#G$-modules

$$F^{2q+1}/F^{2q} \cong \bigoplus_{p=0}^{q} ((\mathcal{A} \otimes W^{p+1} \otimes \mathcal{A}) \otimes (\mathbb{Z}G)^{q-p+2})[q+1],$$

$$F^{2q}/F^{2q-1} \cong \bigoplus_{j=0}^{q} ((\mathcal{A} \otimes \mathcal{A}^{\otimes j}/W^j \otimes \mathcal{A}) \otimes (\mathbb{Z}G)^{q-j+2})[q],$$

where in both cases the induced differential is only different from zero on the components of $\mathcal{A}$ on the far left and on the far right. As the $\mathbb{Z}G^e$-modules $(\mathbb{Z}G)^{q-p+2}$ are all free, it follows that the subquotients $F^{2q+1}/F^{2q}, F^{2q}/F^{2q-1}$ are summands of a direct sum of shifted copies of $\mathcal{A}\#G^e$, therefore $B(\mathcal{A})\otimes_{\mathbb{Z}}\bar{B}(\mathbb{Z}G)$ is cofibrant. □

Now let us define a diagonal map for $B(\mathcal{A})\otimes_{\mathbb{Z}}\bar{B}(\mathbb{Z}G)$:

$$\begin{aligned} B(\mathcal{A})\otimes_{\mathbb{Z}}\bar{B}(\mathbb{Z}G) &\longrightarrow B(\mathcal{A})\otimes_{\mathbb{Z}}\bar{B}(\mathbb{Z}G)\otimes_{\mathcal{A}\#G} B(\mathcal{A})\otimes_{\mathbb{Z}}\bar{B}(\mathbb{Z}G),\\ (x_0|\cdots|x_{k+1})\otimes(h_0|\cdots|h_{l+1}) &\longmapsto \sum_{i=0}^{k}\sum_{j=0}^{l}(x_0|\cdots|x_i|1)\otimes(h_0|\cdots|h_j|1)\\ &\qquad \otimes_{\mathcal{A}\#G}(1|x_{i+1}|\cdots|x_{k+1})\otimes(h_j|h_{j+1}|\cdots|h_{l+1}) \end{aligned} \tag{2.2}$$

that is just the juxtaposition of the diagonals maps for $B(\mathcal{A})$ and $\bar{B}(\mathbb{Z}G)$ defined in (A.4) and (2.1), respectively. It follows then that the diagonal map for $B(\mathcal{A})\otimes_{\mathbb{Z}}\bar{B}(\mathbb{Z}G)$ satisfies the hypothesis for Proposition A.7 and therefore it will induce the ring structure on the Hochschild cohomology of $\mathcal{A}\#G$.

We are now ready to prove the main theorem of this section.

2.2. *Proof of Theorem* 2.2

We have that $B(\mathcal{A})\otimes_{\mathbb{Z}}\bar{B}(\mathbb{Z}G)\to\mathcal{A}\#G$ is a cofibrant replacement for $\mathcal{A}\#G$ as an $\mathcal{A}\#G^e$-module. Therefore, we have the isomorphisms

$$HH_*(\mathcal{A}\#G,\mathcal{A}\#G)=H^*((B(\mathcal{A})\otimes\bar{B}(\mathbb{Z}G))\otimes_{\mathcal{A}\#G^e}\mathcal{A}\#G)$$

and

$$HH^*(\mathcal{A}\#G,\mathcal{A}\#G)=H^*\mathcal{H}om_{\mathcal{A}\#G^e}(B(\mathcal{A})\otimes\bar{B}(\mathbb{Z}G),\mathcal{A}\#G).$$

Let us consider the sub-dg-rings $\mathbb{Z}G^e\subset\mathcal{A}\#G^e$ with $g\otimes k\mapsto(1\otimes g|1\otimes k)$ and $\mathcal{A}^e\subset\mathcal{A}\#G^e$ with $x_0\otimes x_1\mapsto(x_0\otimes 1|x_1\otimes 1)$, and note that as such $\mathcal{A}^e$ and $\mathbb{Z}G^e$ generate the dg-ring $\mathcal{A}\#G^e$. The sub-dg-ring $\mathcal{A}^e$ acts trivially on the component $\bar{B}(\mathbb{Z}G)$ of $B(\mathcal{A})\otimes\bar{B}(\mathbb{Z}G)$, therefore we have an isomorphism

$$(B(\mathcal{A})\otimes\bar{B}(\mathbb{Z}G))\otimes_{\mathcal{A}\#G^e}\mathcal{A}\#G\cong\bar{B}(\mathbb{Z}G)\otimes_{\mathbb{Z}G^e}(B(\mathcal{A})\otimes_{\mathcal{A}^e}\mathcal{A}\#G),$$

where the induced action of $g\otimes k\in\mathbb{Z}G^e$ into $B(\mathcal{A})$ is diagonal on g and trivial on k, that is,

$$(g\otimes k)(x_0|\cdots|x_{k+1})\longmapsto(g(x_0)|\cdots|g(x_{k+1})).$$

The previous argument applies also for the $\mathcal{H}om$ functor and therefore we have

$$\mathcal{H}om_{\mathcal{A}\#G^e}(B(\mathcal{A})\otimes\bar{B}(\mathbb{Z}G),\mathcal{A}\#G)\cong\mathcal{H}om_{\mathbb{Z}G^e}(\bar{B}(\mathbb{Z}G),\mathcal{H}om_{\mathcal{A}^e}(B(\mathcal{A}),\mathcal{A}\#G)).$$

Let us now see more carefully the functors

$$\bar{B}(\mathbb{Z}G)\otimes_{\mathbb{Z}G^e}\_\quad\text{and}\quad\mathcal{H}om_{\mathbb{Z}G^e}(\bar{B}(\mathbb{Z}G),\_).$$

Note that the action of the elements of the form $(1\otimes k)$ is by multiplication on the right, and this action is free on $\bar{B}(\mathbb{Z}G)$. Note also that we could generate the ring $\mathbb{Z}G^e$ by the subrings $1\otimes\mathbb{Z}G$ and the image of the diagonal homomorphism $\Delta:\mathbb{Z}G\to\mathbb{Z}G^e$, $\Delta(g)=g\otimes g^{-1}$. Therefore, we could restrict our attention to the subcomplex $\bar{B}_G(\mathbb{Z})$ of $\bar{B}(\mathbb{Z}G)$

$$\bar{B}_G(\mathbb{Z})=\{(h_0|\cdots|h_{k+1})\in\bar{B}(\mathbb{Z}G)|h_{k+1}=1\}$$

consisting of the elements that end in 1, disregarding the module structure of the subring $1\otimes\mathbb{Z}G$ and considering it as a $\mathbb{Z}G$-module given by the action induced by its image under the diagonal map in $\mathbb{Z}G^e$.

Note that the $\mathbb{Z}G$ module structure $\bar{B}_G(\mathbb{Z})$ becomes a diagonal action

$$\begin{aligned} g\cdot(h_0|\cdots|h_k|1) &:= (g\otimes g^{-1})(h_0|\cdots|h_k|1)\\ &= (gh_0|\cdots|gh_k|g1g^{-1})\\ &= (gh_0|\cdots|gh_k|1), \end{aligned}$$

and that $\bar{B}_G(\mathbb{Z})$ becomes a cofibrant replacement for $\mathbb{Z}$ as a trivial $\mathbb{Z}G$-module.

Thus, we have the isomorphisms of complexes

$$\bar{B}(\mathbb{Z}G)\otimes_{\mathbb{Z}G^e}(B(\mathcal{A})\otimes_{\mathcal{A}^e}\mathcal{A}\#G)\cong\bar{B}_G(\mathbb{Z})\otimes_{\mathbb{Z}G}(B(\mathcal{A})\otimes_{\mathcal{A}^e}\mathcal{A}\#G),$$
$$\mathcal{H}om_{\mathbb{Z}G^e}(\bar{B}(\mathbb{Z}G),\mathcal{H}om_{\mathcal{A}^e}(B(\mathcal{A}),\mathcal{A}\#G))\cong\mathcal{H}om_{\mathbb{Z}G}(\bar{B}_G(\mathbb{Z}),\mathcal{H}om_{\mathcal{A}^e}(B(\mathcal{A}),\mathcal{A}\#G)),$$

where the $\mathbb{Z}G$-module structure of $B(\mathcal{A})$ is given by the diagonal action and the $\mathbb{Z}G$-module structure of $\mathcal{A}\#G$ is given by the natural action on $\mathcal{A}$ and by conjugation on G, that is,

$$g\cdot(x\otimes k)=(1\otimes g)(x\otimes k)(1\otimes g^{-1})=(g(x)\otimes gkg^{-1}).$$

We can therefore endow $\mathcal{H}om_{\mathcal{A}^e}(B(\mathcal{A}),\mathcal{A}\#G)$ with the structure of a $\mathbb{Z}G$-module as follows: for $f\in\mathcal{H}om_{\mathcal{A}^e}(B(\mathcal{A}),\mathcal{A}\#G)$ and $g\in G$ we have

$$(gf)(x_0|\cdots|x_{k+1}):=g^{-1}(f(g(x_0)|\cdots|g(x_{k+1}))). \tag{2.3}$$

Hence, we can conclude that there are isomorphisms

$$\begin{aligned} HH_*(\mathcal{A}\#G,\mathcal{A}\#G) &= H^*(\bar{B}_G(\mathbb{Z})\otimes_{\mathbb{Z}G}(B(\mathcal{A})\otimes_{\mathcal{A}^e}\mathcal{A}\#G))\\ &= \mathrm{Tor}^*_{\mathbb{Z}G}(\mathbb{Z},\mathcal{A}\overset{L}{\otimes}_{\mathcal{A}^e}\mathcal{A}\#G),\\ HH^*(\mathcal{A}\#G,\mathcal{A}\#G) &= H^*\mathcal{H}om_{\mathbb{Z}G}(\bar{B}_G(\mathbb{Z}),\mathcal{H}om_{\mathcal{A}^e}(B(\mathcal{A}),\mathcal{A}\#G))\\ &= \mathrm{Ext}^*_{\mathbb{Z}G}(\mathbb{Z},\mathcal{R}\mathcal{H}om_{\mathcal{A}^e}(\mathcal{A},\mathcal{A}\#G)). \end{aligned}$$

2.2.1. We are now left to describe the dg-ring structure of

$$\mathcal{H}om_{\mathbb{Z}G}(\bar{B}_G(\mathbb{Z}),\mathcal{H}om_{\mathcal{A}^e}(B(\mathcal{A}),\mathcal{A}\#G))$$

that will induce the ring structure on the Hochschild cohomology. The dg-ring structure is induced from the diagonal map for $B(\mathcal{A})\otimes_{\mathbb{Z}}\bar{B}(\mathbb{Z}G)$ that was defined in (2.2) and from this it follows that, for

$$\phi,\psi\in\mathcal{H}om_{\mathbb{Z}G}(\bar{B}_G(\mathbb{Z}),\mathcal{H}om_{\mathcal{A}^e}(B(\mathcal{A}),\mathcal{A}\#G)),$$

we have that

$$\begin{aligned} &\phi\cdot\psi(h_0|\cdots|h_k|1)(x_0|\cdots|x_{l+1})\\ &\quad\sum_{i=0}^{k}\sum_{j=0}^{l}(\phi(h_0|\cdots|h_i|1)(x_0|\cdots|x_j|1))(\psi(h_i|\cdots|h_k|1)(1|x_{j+1}|\cdots|x_{l+1})). \end{aligned} \tag{2.4}$$

This ends the proof of Theorem 2.2. □

Note that the $\mathbb{Z}G$-module structure on $\mathcal{A}\#G$ is given by $g\cdot(x\otimes k)=(g(a),gkg^{-1})$. Therefore, we can split $\mathcal{A}\#G$ into $\mathbb{Z}G$-modules in the following way

$$\mathcal{A}\#G\cong\bigoplus_{T\in[G]}\mathcal{A}_T\quad\text{with}\quad\mathcal{A}_T=\bigoplus_{h\in T}\mathcal{A}_h, \tag{2.5}$$

where $[G]$ is the set of conjugacy classes of elements in G and $\mathcal{A}_h$ is the subset of $\mathcal{A}\#G$ of elements of the form $x\otimes h$,

$$\mathcal{A}_h:=\{x\otimes h\in\mathcal{A}\#G|x\in\mathcal{A}\}. \tag{2.6}$$

Let us choose one element of G for each conjugacy class in $[G]$, and let us denote this set of representatives by $\langle G\rangle$. Then we have the following corollary.

Corollary 2.4. *There are isomorphisms of graded groups*

$$HH_*(\mathcal{A}\#G, \mathcal{A}\#G) \cong \bigoplus_{g\in\langle G\rangle} \mathrm{Tor}_{\mathbb{Z}C_g}(\mathbb{Z}, B(\mathcal{A})\otimes_{\mathcal{A}^e}\mathcal{A}_g),$$
$$HH^*(\mathcal{A}\#G, \mathcal{A}\#G) \cong \bigoplus_{g\in\langle G\rangle} \mathrm{Ext}_{\mathbb{Z}C_g}(\mathbb{Z}, \mathcal{H}om_{\mathcal{A}^e}(B(\mathcal{A}), \mathcal{A}_g)),$$

where C_g *is the centralizer of* g *in* G *and* $\mathcal{A}_g$ *is viewed as a* $\mathbb{Z}C_g$ *module.*

Proof. Both isomorphisms are proved by the same argument; let us prove the second one. As a $\mathbb{Z}G$-module has the isomorphism

$$\mathcal{H}om_{\mathcal{A}^e}(B(\mathcal{A}), \mathcal{A}\#G) \cong \bigoplus_{T\in[G]} \mathcal{H}om_{\mathcal{A}^e}(B(\mathcal{A}), \mathcal{A}_T),$$

therefore we have that

$$\begin{aligned}\mathrm{Ext}^*_{\mathbb{Z}G}(\mathbb{Z}, \mathcal{H}om_{\mathcal{A}^e}(B(\mathcal{A}), \mathcal{A}\#G)) &\cong \bigoplus_{T\in[G]} \mathrm{Ext}^*_{\mathbb{Z}G}(\mathbb{Z}, \mathcal{H}om_{\mathcal{A}^e}(B(\mathcal{A}), \mathcal{A}_T))\\ &\cong \bigoplus_{g\in\langle G\rangle} \mathrm{Ext}^*_{\mathbb{Z}C_g}(\mathbb{Z}, \mathcal{H}om_{\mathcal{A}^e}(B(\mathcal{A}), \mathcal{A}_g)).\end{aligned}$$ □

Corollary 2.5. *If* $\mathcal{A}$ *is a free* $\mathbb{Z}G$*-module-dg-ring, then we have isomorphisms of graded groups*

$$HH_*(\mathcal{A}\#G, \mathcal{A}\#G) \cong \bigoplus_{g\in\langle G\rangle} H^*((B(\mathcal{A})\otimes_{\mathcal{A}^e}\mathcal{A}_g)^{C_g}),$$
$$HH^*(\mathcal{A}\#G, \mathcal{A}\#G) \cong \bigoplus_{g\in\langle G\rangle} H^*((\mathcal{H}om_{\mathcal{A}^e}(B(\mathcal{A}), \mathcal{A}_g))^{C_g}).$$

Proof. The isomorphisms follow from Corollary 2.4 and the fact that for free modules the invariants and the coinvariants are isomorphic. □

2.3. *Further results*

We end this section with some results that follow from the proof of Theorem 2.2.

2.3.1. The diagonal map of $B(\mathcal{A})$ defined on (A.4) endows the complex

$$\mathcal{H}om_{\mathcal{A}^e}(B(\mathcal{A}), \mathcal{A}\#G)$$

with the structure of a dg-ring that is moreover a $\mathbb{Z}G$-module dg-ring. Following the notation of (2.6), if we take two functions

$$\phi \in \mathcal{H}om_{\mathcal{A}^e}(\mathcal{A}^{k+2}[k], \mathcal{A}_g), \quad \psi \in \mathcal{H}om_{\mathcal{A}^e}(\mathcal{A}^{l+2}[l], \mathcal{A}_h),$$

then their product $\phi\cdot\psi$ lives in $\mathcal{H}om_{\mathcal{A}^e}(\mathcal{A}^{k+l+2}[k+l], \mathcal{A}_{gh})$ and is defined as

$$\begin{aligned}(\phi\cdot\psi)(a_0|\cdots|a_{k+l+2}) &= (-1)^{|\psi|\varepsilon_k}(\phi_g(a_0|\cdots|a_k|1)\otimes g)(\psi_h(1|a_{k+1}|\cdots|a_{k+l+1})\otimes h)\\ &= (-1)^{|\psi|\varepsilon_k}\phi_g(a_0|\cdots|a_k|1)g(\psi_h(1|a_{k+1}|\cdots|a_{k+l+1}))\otimes gh,\end{aligned}$$

where we use the convention

$$\phi(\mathbf{a}) = \sum_g (\phi_g(\mathbf{a})\otimes g).$$

The $\mathbb{Z}G$-module structure of $\mathcal{H}om_{\mathcal{A}^e}(B(\mathcal{A}), \mathcal{A}\#G)$ is defined in (2.3) and it is easy to check that, for $k\in G$, we $(k\phi)(k\psi) = k(\phi\cdot\psi)$.

We therefore have that $\mathrm{Ext}^*_{\mathcal{A}^e}(\mathcal{A}, \mathcal{A}\#G)$ becomes a $\mathbb{Z}G$-module ring.

2.3.2. Whenever W is a $\mathbb{Z}G$-module ring, the complex $\mathcal{H}om_{\mathbb{Z}G}(\bar{B}_G(\mathbb{Z}), W)$ can be endowed with the structure of a dg-ring with a product structure induced by the formula in (2.4), that is, for $\alpha, \beta \in \mathcal{H}om_{\mathbb{Z}G}(\bar{B}_G(\mathbb{Z}), W)$ we have that

$$\alpha \cdot \beta \ (h_0|\cdots|h_l|1) = \sum_{j=0}^{l} \alpha(h_0|\cdots|h_j|1)\beta(h_j|\cdots|h_l|1).$$

This dg-ring structure makes $\mathrm{Ext}^*_{\mathbb{Z}G}(\mathbb{Z}, W)$ into a ring. In the case where $W = \mathbb{Z}$, the ring $\mathrm{Ext}^*_{\mathbb{Z}G}(\mathbb{Z}, \mathbb{Z})$ is isomorphic to the ring $H^*(BG, \mathbb{Z})$.

2.3.3. If we filter both complexes

$$\bar{B}_G(\mathbb{Z}) \otimes_{\mathbb{Z}G} (B(\mathcal{A}) \otimes_{\mathcal{A}^e} \mathcal{A}\#G),$$
$$\mathcal{H}om_{\mathbb{Z}G}(\bar{B}_G(\mathbb{Z}), \mathcal{H}om_{\mathcal{A}^e}(B(\mathcal{A}), \mathcal{A}\#G))$$

by the degree of the elements in $\bar{B}_G(\mathbb{Z})$, then we get spectral sequences which abuts to the Hochschild homology and cohomology, respectively, and whose second page is given by

$$E^2 = \mathrm{Tor}^*_{\mathbb{Z}G}(\mathbb{Z}, \mathrm{Tor}_{\mathcal{A}^e}(\mathcal{A}, \mathcal{A}\#G)) \Rightarrow HH_*(\mathcal{A}\#G, \mathcal{A}\#G)$$

and

$$E^2 = \mathrm{Ext}^*_{\mathbb{Z}G}(\mathbb{Z}, \mathrm{Ext}_{\mathcal{A}^e}(\mathcal{A}, \mathcal{A}\#G)) \Rightarrow HH^*(\mathcal{A}\#G, \mathcal{A}\#G).$$

In the case of the Hochschild cohomology ring, the spectral sequence is a sequence of rings where

$$\mathrm{Ext}^*_{\mathbb{Z}G}(\mathbb{Z}, \mathrm{Ext}_{\mathcal{A}^e}(\mathcal{A}, \mathcal{A}\#G))$$

has the ring structure explained in Paragraphs 2.3.1 and 2.3.2.

3. *Loops on quotient spaces*

This section is devoted to showing the relation between the complexes

$$B(\mathcal{A}) \otimes_{\mathcal{A}^e} \mathcal{A}\#G \quad \text{and} \quad \mathcal{H}om_{\mathcal{A}^e}(B(\mathcal{A}), \mathcal{A}\#G),$$

and the cochains and chains, respectively, for the loops on the groupoid $[X/G]$ whenever we take $\mathcal{A} = C^*(X)$.

3.1. *Loops on* $[X/G]$

Let X be a connected CW-complex of finite type and G be a discrete group acting on X. Denote by $[X/G]$ the action groupoid whose objects are X and whose morphisms are $X \times G$ with $s(x, g) = x$ and $t(x, g) = xg$.

The loops on $[X/G]$ can be understood as the groupoid whose objects are the functors from the groupoid $[\mathbb{R}/\mathbb{Z}]$ to $[X/G]$ and whose morphisms are given by natural transformations. More explicitly, we have the following definition.

Definition 3.1. The loop groupoid $\mathsf{L}[X/G]$ for $[X/G]$ is the action groupoid $[P_GX/G]$ whose space of objects is

$$P_GX := \bigsqcup_{g\in G} P_gX \times \{g\} \quad \text{with} \quad P_gX := \{f : [0,1] \to X | f(0)g = f(1)\},$$

and which are endowed with the G-action

$$\begin{aligned} G \times P_G X &\longrightarrow P_G X, \\ ((f,k),g) &\longmapsto (fg, g^{-1}kg). \end{aligned}$$

In [19, Theorem 2.3], it is proved that, for G a finite group, there exists a natural weak homotopy equivalence

$$\mathcal{L}(EG \times_G X) \to EG \times_G P_G X,$$

between the free loop space of the homotopy quotient $EG \times_G X$ and the homotopy quotient of the loop groupoid. This proof can be easily generalized to the case where G is discrete.

In this section, we will consider $P_G X$ as a G-space and will not work with its homotopy quotient. Now we will give a cosimplicial description for the spaces $P_g X$.

3.2. *Cosimplicial description for $P_g X$*

Let us start by recalling a cosimplicial construction of the space of paths of a topological space X that was done in [15] (we will use the properties of simplicial and cosimplicial spaces that are developed in [26] and in [2], respectively). Take the category Δ whose objects are the finite ordered sets $\mathfrak{n} = \{0, 1, \dots, n\}$ and whose morphisms $\Delta(\mathfrak{n}, \mathfrak{m})$ are the order-preserving maps $s\colon \mathfrak{n} \to \mathfrak{m}$. The morphisms of Δ are generated by the following.

(1) The face maps $\delta_i \in \Delta(\mathfrak{n}-1, \mathfrak{n})$, $0 \leqslant i \leqslant n$; the unique order-preserving map whose image does not contain i.
(2) The degeneracy maps $\sigma_i \in \Delta(\mathfrak{n}+1, \mathfrak{n})$, $0 \leqslant i \leqslant n$; the unique surjective order-preserving map that repeats i.

These generators satisfy the usual cosimplicial relations. We define a simplicial object in a category $\mathfrak{C}$ to be a contravariant functor $\Delta \to \mathfrak{C}$ and a cosimplicial object in $\mathfrak{C}$ to be a covariant functor $\Delta \to \mathfrak{C}$.

Now let us define the simplicial sets λ^n, where $\lambda^n(\mathfrak{m}) := \Delta(\mathfrak{m}, \mathfrak{n})$ with the natural coface and codegeneracy maps induced by Δ. The geometric realization $|\lambda^n|$ of the simplicial set λ^n is homeomorphic to the n-simplex Δ_n. In particular when $n = 1$, the simplicial space λ^1 has as geometric realization the 1-simplex Δ_1. One could think of it as $\lambda^1(\mathfrak{n}) = \mathfrak{n}+2$ with cofaces $\lambda^1(\mathfrak{n}) \to \lambda^1(\mathfrak{n}-1)$ and codegeneracies $\lambda^1(\mathfrak{n}-1) \to \lambda^1(\mathfrak{n})$ given by order-preserving maps that send 0 to 0, and $n+2$ to $n+1$ in the former case and $n+1$ to $n+2$ in the latter.

Consider the cosimplicial space $\mathbb{P}X := X^{\lambda^1}$ defined by taking the maps from the simplicial set λ^1 to X. Then we have that

$$\mathbb{P}X(\mathfrak{n}) := \mathrm{Map}(\lambda^1(\mathfrak{n}), X) \cong X^{n+2},$$

and the cosimplicial structure maps are the ones induced by the simplicial structure of λ^1. We have the tautology [2, Proposition 5.1].

Lemma 3.2. *There is a natural homeomorphism between the space of paths*

$$PX := Map(|\lambda^1|, X) = Map(\Delta_1, X),$$

and the total space $|\mathbb{P}X|$ of the cosimplicial space $\mathbb{P}X$.

Consider $\bar{\mathbb{P}}_g X$ the subcosimplicial space of $\mathbb{P}X$ defined by the spaces

$$\bar{\mathbb{P}}_g X(\mathfrak{n}) := \{(x_0, \dots, x_{n+1}) \in \mathbb{P}X(\mathfrak{n})\colon x_0 g = x_{n+1}\} \cong X^{n+1}.$$

Clearly, the coface and codegeneracy maps are well defined in $\bar{\mathbb{P}}_g X$ and it follows that the total space of $\bar{\mathbb{P}}_g X$ is homeomorphic to the space of paths $\gamma \in PX$ such that $\gamma(0)g = \gamma(1)$.

Define the cosimplicial space $\mathbb{P}_g X$ by dropping the last coordinate of $\bar{\mathbb{P}}_g X$ (as the last coordinate x_{n+1} on the nth level is equal to $x_0 g$), therefore having

$$\mathbb{P}_g X(\mathfrak{n}) := X^{n+1}$$

together with the induced coface and codegeneracy maps given by

$$\begin{aligned}\delta_i(x_0,\ldots,x_n) &= (x_0,\ldots,x_{i-1},x_i,x_i,x_{i+1},\ldots,x_n) \quad 0 \leqslant i \leqslant n,\\ \delta_{n+1}(x_0,\ldots,x_n) &= (x_0,\ldots,x_n,x_0 g),\\ \sigma_i(x_0,\ldots,x_n) &= (x_0,\ldots,x_i,x_{i+2},\ldots,x_n) \quad 0 \leqslant i \leqslant n-1.\end{aligned} \tag{3.1}$$

We can now consider the maps

$$\begin{aligned}\phi_k \colon \Delta_k \times P_g X &\longrightarrow X^{k+1} = \mathbb{P}_g X(\mathfrak{n}),\\ (t_1,\ldots,t_k) \times \gamma &\longmapsto (\gamma(0),\gamma(t_1),\ldots,\gamma(t_k))\end{aligned} \tag{3.2}$$

together with $\hat{\phi}_k : P_g X \to \mathrm{Map}(\Delta_k, \mathbb{P}_g X)$ their adjoints. Let

$$\phi : P_g X \longrightarrow \prod_{k \geqslant 0} \mathrm{Map}(\Delta_k, X^{k+1})$$

denote the product of the maps $\hat{\phi}_k$. Then we have the following lemma.

Lemma 3.3. *The image of the map ϕ is the total space of $\mathbb{P}_g X$ and therefore $\phi : P_g X \to |\mathbb{P}_g X|$ is a homeomorphism.*

3.3. *Cochains on loops of $[X/G]$*

Let us denote by

$$C^* := C^*(X;\mathbb{Z}),$$

the dg-ring of $\mathbb{Z}$-valued singular cochains on X. Because G acts on X, C^* is endowed with the structure of a $\mathbb{Z}G$-module-dg-ring.

For $g \in G$, let C^*_g be the submodule of $C^* \# G$ generated by the elements of the form $x \otimes g$ as defined in (2.5). Note that C^*_g inherits the structure of a C^{*e}-module when one takes $C^{*e} \subset C^* \# G^e$ and $C^*_g \subset C^* \# G$ in the following way: let $a_0 \otimes b_0 \in C^{*e}$ and $x \otimes g \in C^*_g$. Then

$$\begin{aligned}(a_0 \otimes a_1) \cdot x \otimes g &= (a_0 \otimes 1 | a_1 \otimes 1)(x \otimes g)\\ &= (a_0 \otimes 1)(x \otimes g)(a_1 \otimes 1)\\ &= a_0 x g(a_1) \otimes g.\end{aligned}$$

Theorem 3.4. *For X connected and simply connected CW-complex of finite type, there exists a homomorphism of complexes*

$$B(C^*) \otimes_{C^{*e}} C^*_g \xrightarrow{\simeq} C^*(P_g X;\mathbb{Z})$$

that moreover is a quasi-isomorphism, and therefore induces an isomorphism

$$\mathrm{Tor}^*_{C^{*e}}(C^*, C^*_g) \xrightarrow{\cong} H^*(P_g X;\mathbb{Z}).$$

Proof. Take the cosimplicial space $\mathbb{P}_g X$ and apply to it the singular cochains functor; we obtain the simplicial cochain complex $C^*(\mathbb{P}_g X)$. By Bott and Segal [**2**, Lemma 7.1], we have that the total complex (or its realization) $|C^*(\mathbb{P}_gX)|$ of the simplicial cochain complex $C^*(\mathbb{P}_gX)$ is quasi-isomorphic to $C^*(P_gX)$, as we have that P_gX is homeomorphic to the total space of $\mathbb{P}_gX$. Then we note that the total complex $|C^*(\mathbb{P}_gX)|$ is quasi-isomorphic to the complex $B(C^*)\otimes_{C^{*e}} C^*_g$ when one applies carefully the Eilenberg–Zilber theorem for product spaces [**7**]. These two facts together imply the theorem. Let us see each one in more detail.

Lemma 3.5. *There is a homomorphism of graded complexes*

$$|C^*(\mathbb{P}_gX)| \longrightarrow C^*(P_gX)$$

that when X is a connected and simply connected CW-complex of finite type, it becomes a quasi-isomorphism.

Proof. The lemma is a direct consequence of Bott and Segal [**2**, Proposition 5.3 and Lemma 7.1]. Let us see how the homomorphism is defined.

Recall from [**2**, Section 5] that the group of homogeneous elements of degree r of the total complex $|C^*(\mathbb{P}_gX)|$ is the direct sum of the groups

$$C^{n+r}(\mathbb{P}_gX(\mathfrak{n})) = C^{n+r}(X^{n+1}) = C^*(X^{n+1})[n]^r \quad \text{for } n \geqslant 0,$$

and therefore

$$|C^*(\mathbb{P}_gX)| = \bigoplus_{n\geqslant 0} C^*(X^{n+1})[n].$$

Let us consider the composition of the homomorphisms

$$C^{n+r}(X^{n+1}) \xrightarrow{\phi_n^*} C^{n+r}(\Delta_n \times P_gX) \xrightarrow{\int_{\Delta_n}} C^r(P_gX),$$

where the functions ϕ_n were defined in (3.2) and $\int_{\Delta_n}$ evaluates a cochain on the class $[\Delta_n]$, that is: the composition of the Eilenberg–Zilber map together with the evaluation on the class $[\Delta_n]$

$$C^{n+r}(\Delta_n \times P_gX) \longrightarrow C^n(\Delta_n)\otimes C^r(P_gX) \longrightarrow C^r(P_gX).$$

Because the operators $\int_{\Delta_n}$ satisfy the property

$$(-1)^n d\left(\int_{\Delta_n} \alpha\right) = \int_{\Delta_n} d\alpha - \int_{\partial\Delta_n} \alpha,$$

we have that the maps

$$C^*(X^{n+1})[n]^r = C^{n+r}(X^{n+1}) \longrightarrow C^r(P_gX)$$

assemble to define a homomorphism of complexes of degree 0

$$F : |C^*(\mathbb{P}_gX)| \longrightarrow C^*(P_gX).$$

It follows from [**2**, Proposition 5.3] that F is a quasi-isomorphism as the connectivity of X ($\geqslant 2$) is higher than the connectivity of the simplicial set that defines $\mathbb{P}_gX$ (which in this case is the circle). □

Lemma 3.6. *There is a homomorphism of graded complexes*

$$B(C^*)\otimes_{C^{*e}} C^*_g \longrightarrow |C^*(\mathbb{P}_gX)|$$

that is moreover a quasi-isomorphism.

Proof. Take the isomorphisms

$$C^{*n+1} \xrightarrow{\cong} C^{*n+2} \otimes_{C^{*e}} C^*_g$$
$$(a_0|\cdots|a_n) \longmapsto (a_0|\cdots|a_n|1) \otimes_{C^{*e}} (1)$$

together with the induced differential

$$(a_0|\cdots|a_n) \longmapsto \sum_{j=0}^{n-1}(-1)^j(a_0|\cdots|a_j a_{j+1}|\cdots|a_n) + (-1)^n(g(a_n)a_0|\cdots|a_{n-1}) \tag{3.3}$$

that comes from the Bar differential on $\bigoplus_{k\geqslant 0} C^{*k+2}$ (see Appendix A); note that the last expression in the formula (3.3) comes from the equivalences

$$\begin{aligned}(a_0|\cdots|a_n) \otimes_{C^{*e}} (1) &= (a_0|\cdots|a_{n-1}|1) \otimes_{C^{*e}} g(a_n)\\ &= (g(a_n)a_0|\cdots|a_{n-1}|1) \otimes_{C^{*e}} (1).\end{aligned}$$

Now let us consider the quasi-isomorphisms defined by the Eilenberg–Zilber map

$$C^{*n+1} \xrightarrow{\simeq} C^*(X^{n+1}) = C^*(\mathbb{P}_g X(\mathfrak{n})),$$

and note that the induced differential $C^{*n+1} \to C^{*n}$ coming from the coface maps defined in (3.1) is the same as in (3.3).

We can conclude that the compositions of the maps

$$C^{*n+2} \otimes_{C^{*e}} C^*_g \xrightarrow{\cong} C^{*n+1} \xrightarrow{\simeq} C^*(X^{n+1})$$

induce the maps

$$C^{*n+2}[n] \otimes_{C^{*e}} C^*_g \xrightarrow{\cong} C^{*n+1}[n] \xrightarrow{\simeq} C^*(X^{n+1})[n]$$

that induce the desired quasi-isomorphism

$$B(C^*) \otimes_{C^{*e}} C^*_g \xrightarrow{\simeq} |C^*(\mathbb{P}_g X)|. \qquad \square$$

Now we can complete with the proof of Theorem 3.4 by composing the quasi-isomorphisms defined in Lemmas 3.5 and 3.6

$$B(C^*) \otimes_{C^{*e}} C^*_g \xrightarrow{\simeq} |C^*(\mathbb{P}_g X)| \xrightarrow{\simeq} C^*(P_g X)$$

that moreover induce the isomorphisms

$$\mathrm{Tor}_{C^{*e}}(C^*, C^*_g) \xrightarrow{\cong} H^*(|C^*(\mathbb{P}_g X)|) \xrightarrow{\cong} H^*(P_g X). \qquad \square$$

We can assemble the C^*_g into $C^*\#G$, thus obtaining the following corollary.

Corollary 3.7. *If X is connected and simply connected and G is finite, then there is a quasi-isomorphism*

$$B(C^*(X)) \otimes_{C^*(X)^e} C^*(X)\#G \xrightarrow{\simeq} C^*(P_G X)$$

that induces the isomorphism

$$\mathrm{Tor}_{C^*(X)^e}(C^*X, C^*(X)\#G) \cong H^*(P_G X).$$

The previous corollary together with Theorem 2.2 implies the following proposition.

Proposition 3.8. *If X is connected and simply connected and G is finite, then there is an isomorphism of graded groups*

$$HH_*(C^*(X)\#G, C^*(X)\#G) \cong \mathrm{Tor}_{\mathbb{Z}G}(\mathbb{Z}, C^*(P_G X)),$$

and whenever G acts freely on X, the isomorphism becomes

$$HH_*(C^*(X)\#G, C^*(X)\#G) \cong H^*(C^*(P_GX)^G).$$

3.4. *Chains on loops of $[X/G]$*

From the previous section, we have the quasi-isomorphisms

$$B(C^*) \otimes_{C^{*e}} C_g^* \overset{\simeq}{\to} \bigoplus_{n\geqslant 0} C^{*n+1}[n] \overset{\simeq}{\to} |C^*(\mathbb{P}_gX)| \overset{\simeq}{\to} C^*(P_gX). \tag{3.4}$$

If P_gX is of finite type, that is, the cohomology is finitely generated in each degree, we can dualize the maps of (3.4) thus obtaining quasi-isomorphisms

$$C_*(P_gX) \overset{\simeq}{\to} |C^*(\mathbb{P}_gX)|^\vee \overset{\simeq}{\to} \bigoplus_{n\geqslant 0} C^{*n+1}[n]^\vee \simeq \bigoplus_{n\geqslant 0} \mathcal{H}om_{\mathbb{Z}}(C^{*n}[n], C_g^{*\vee}),$$

where $T^\vee := \mathrm{Hom}_{\mathbb{Z}}(T, \mathbb{Z})$ and the last map is induced by the isomorphisms $\mathcal{H}om_{\mathbb{Z}}(C^{*n}, C_g^{*\vee}) \to C^{*n+1\vee}$ with $\phi \mapsto \bar\phi$ and

$$\bar\phi(a_1|\cdots|a_n|b) = \phi(a_1|\cdots|a_n)(b).$$

Whenever X is a compact closed oriented manifold of dimension l, there is a homomorphism of graded groups

$$C^*(X) \longrightarrow C_{l-*}(X)$$

that induces an isomorphism in cohomology, and that moreover induces a structure of $C^*(X)^e$-module on $C_{l-*}(X)$. Therefore, we could apply the natural isomorphism

$$\mathcal{H}om_{\mathbb{Z}}(C^{*n}, C_g^{*\vee}) \cong \mathcal{H}om_{C^{*e}}(C^{*n+2}, C_g^{*\vee})$$

that yields then that there is a quasi-isomorphism

$$C_*(P_gX) \overset{\simeq}{\to} \mathcal{H}om_{C^{*e}}(B(C^*), C_g^{*\vee})$$

that induces the isomorphism

$$H_*(P_gX) \cong H^*\mathcal{H}om_{C^{*e}}(B(C^*), C_g^{*\vee}).$$

The previous proof is very sketchy and not rigorous at all; we postpone its formal proof to the following sections.

In the next section, we will show explicitly how the singular chains of P_gX are related to the complex $\mathcal{H}om_{C^{*e}}(B(C^*), C_g^*)$, and, moreover, we will show how the ring structure of $H_*(P_GX)$ defined in [**19**] is isomorphic to the ring $H^*\mathcal{H}om_{C^{*e}}(B(C^*), C^*\#G)$ whenever G is finite and X is a compact, connected and simply connected oriented manifold.

4. *String topology for orbifolds*

This section is devoted to the construction of the topological counterpart for the G-module dg-ring

$$\mathcal{H}om_{C^{*e}}(B(C^*), C^*\#G),$$

whenever we have a global quotient orbifold $[M/G]$ with M a differentiable, oriented and compact manifold, G finite group acting by orientation-preserving diffeomorphisms and $C^* = C^*(M, \mathbb{Z})$.

Subsection 4.1 generalizes the construction of Cohen [**4**] to the equivariant case to give a symmetric ring spectra over pointed G-spaces model of the Thom spectrum M^{-TM}.

In Subsection 4.2, we will start by recalling the construction of the orbifold string topology of $[M/G]$ performed in [19], which is based on the union of spectra

$$P_G M^{-TM} := \bigsqcup_{g \in G} P_g M^{-TM},$$

and the maps

$$P_g M^{-TM} \wedge P_h M^{-TM} \to P_{gh} M^{-TM}. \tag{4.1}$$

Then, in Subsection 4.3 we will construct, for each g, a cosimplicial spectrum $\mathfrak{P}_g M$ such that its total spectrum is homeomorphic to $P_g M^{-TM}$. With the cosimplicial spectrum $\mathfrak{P}_g M$ at hand, we will construct a quasi-isomorphism between $C_*(P_g M^{-TM})$ and $\mathcal{H}om_{C^{*e}}(B(C^*), C_g^*)$ that will assemble into a quasi-isomorphism

$$C_*(P_G M^{-TM}) \xrightarrow{\simeq} \mathcal{H}om_{C^{*e}}(B(C^*), C^* \# G).$$

In Subsection 4.4, we will focus on multiplicative issues. We will construct maps of cosimplicial spectra

$$\mathfrak{P}_g M \wedge \mathfrak{P}_h M \longrightarrow \mathfrak{P}_{gh} M$$

that will be compatible with the map of (4.1) and we will show that, after passing to chains, it will be compatible with the ring structure of

$$\mathcal{H}om_{C^{*e}}(B(C^*), C^* \# G).$$

We will show that the quasi-isomorphism

$$C_*(P_G M^{-TM}) \xrightarrow{\simeq} \mathcal{H}om_{C^{*e}}(B(C^*), C^* \# G)$$

will be a G-equivariant map of A_∞ rings, and in particular we will show that there is a G-equivariant isomorphism of rings

$$H_*(P_G M^{-TM}) \cong H^* \mathcal{H}om_{C^{*e}}(B(C^*), C^* \# G).$$

4.1. *Multiplicative properties of Atiyah duality*

This section is based on the construction done in [4] of a symmetric ring spectra model of M^{-TM}. Recall the classical construction of the spectrum M^{-TM}; given an embedding $e : M \hookrightarrow \mathbb{R}^k$, let $\nu(e) \to M$ be the normal bundle of the embedding and denote by $M^{\nu(e)}$ its Thom construction. Define the spectrum

$$M^{-TM} := \Sigma^{-k} M^{\nu(e)}$$

as the kth de-suspension of the Thom space $M^{\nu(e)}$.

The Spanier–Whitehead dual of M_+ is the function spectrum $F(M, S)$ and this, by Atiyah duality, can be identified with M^{-TM}. Therefore, the diagonal map $\Delta : M \to M \times M$ induces a map of spectra

$$\Delta^* : M^{-TM} \wedge M^{-TM} \longrightarrow M^{-TM},$$

which makes M^{-TM} into a ring spectrum in the stable homotopy category. Following [4], we rigidify this ring spectrum using the machinery of symmetric spectra in general model categories of Hovey [13] to take into account the G-action.

If $\mathcal{C}$ is a symmetric monoidal model category and K is a cofibrant object of $\mathcal{C}$, Hovey defined categories $\mathrm{Sp}^{\mathbb{N}}(\mathcal{C}, K)$ and $\mathrm{Sp}^{\Sigma}(\mathcal{C}, K)$ of spectra and symmetric spectra over $\mathcal{C}$. A spectrum is a sequence $\{X_n\}$ of elements of $\mathcal{C}$ together with morphisms $\epsilon_{n,m} : X_n \otimes K^{\otimes m} \to X_{n+m}$ that make $\{X_n\}$ a module over $\mathrm{Sym}(K) = (S^0, K, K^{\otimes 2}, \ldots, K^{\otimes n}, \ldots)$. Similarly, symmetric spectra are symmetric sequences $\{X_n\}$ with $\Sigma_n \times \Sigma_m$-equivariant morphisms $\epsilon_{n,m} : X_n \otimes K^{\otimes m} \to X_{n+m}$ that make it a $\mathrm{Sym}(K)$-module.

For our purposes, we take $\mathcal{C}$ the model category of based topological spaces with G-action with the (fine) equivariant model structure and $K = S^V$, the one-point compactification of the representation V.

Recall from [**14**] that every manifold with the action of a finite group admits the structure of a $G - CW$-complex, and therefore S^V is cofibrant in the (fine) equivariant model structure.

Given a manifold M with a G-action and an equivariant embedding $e : M \hookrightarrow V$ into an orthogonal representation of G, we will define symmetric ring spectra over $\mathcal{C}$, $M^{-\tau}(e)$ and $F(e)$, together with a map $\alpha : M^{-\tau}(e) \to F(e)$ that implements Atiyah duality at the level of symmetric ring spectra.

Let $\nu(e) \subseteq V$ be an equivariant tubular neighbourhood of the embedding e. Consider the space $L(V, V^n)^G$ of equivariant linear embeddings $\phi : V \to V^n$ with a trivial G-action and a Σ_n-action coming from the permutation action on V^n.

For $\phi \in L(V, V^n)^G$, let $\theta(\phi)$ be an equivariant tubular neighbourhood of $\phi \circ e$. Define the sequence of pointed G-spaces

$$\tilde{M}_n^{-\tau}(e) = \{(\phi, x) \mid \phi \in L(V, V^n)^G, x \in V^n/(V^n - \theta(\phi))\}.$$

A point in $\tilde{M}_n^{-\tau}(e)$ is an equivariant linear embedding $\phi : V \to V^n$ together with a point in the Thom space of the normal bundle to the embedding $\phi \circ e : M \hookrightarrow V^n$. We have a fibre bundle $\tilde{M}_n^{-\tau}(e) \to L(V, V^n)^G$, with fibre the Thom space $V^n/(V^n - \theta(\phi))$. The Σ_n-action on V^n induces action on $\tilde{M}_n^{-\tau}(e)$.

Consider the sequence of pointed G-spaces

$$M_n^{-\tau}(e) = \tilde{M}_n^{-\tau}(e)/\{(\phi, \infty) \mid \phi \in L(V, V^n)^G\},$$

where $\infty \in V^n/(V^n - \theta(\phi))$ is the base point. The G-spaces $M_n^{-\tau}(e)$ form a symmetric spectrum over pointed G-spaces.

Since we have a bundle $\tilde{M}_n^{-\tau}(e) \to L(V, V^n)^G$ and the connectivity of the space of equivariant embeddings from V to V^n tends to infinity, forgetting the symmetric structure, $M^{-\tau}(e)$ is equivalent to $\Sigma_V^\infty M^{\nu(e)}$, where Σ_V^∞ is the left adjoint to the functor that sends a spectrum over pointed G-spaces $\{X_n\}$ to the G-space X_1.

Similarly as in [**4**], given an equivariant embedding $e : M \hookrightarrow V$, we can define symmetric ring spectra over pointed G-spaces $F(e)$ by

$$F_n(e) = \{(\phi, \sigma) \mid \phi \in L(V, V^n)^G, \sigma : \nu(e) \longrightarrow S_x^{\oplus_n V}\}/\{(\phi, \sigma_\infty)\},$$

where, for $x \in \nu(e)$, $S_x^{\oplus_n V}$ is the compactification of a ball of radius ϵ around $\phi(e(x))$ in V^n and σ_∞ is the constant function at infinity.

The projection maps $F_n(e) \cong L(V, V^n)_+^G \wedge F(\nu(e), S^{\oplus_n V}) \to F(\nu(e), S^{\oplus_n V})$, together with the homotopy equivalence $F(\nu(e), S^{\oplus_n V}) \to F(M, S^{\oplus_n V})$, induce a π_*-equivalence of symmetric ring spectra over pointed G-spaces

$$\bar{\rho} : F(e) \longrightarrow F(M, \mathrm{Sym}(S^V)).$$

Therefore, $F(e)$ is our model for the Spanier–Whitehead dual of M_+ in the category $\mathrm{Sp}^\Sigma(\mathcal{C}, \mathrm{Sym}(S^V))$.

Define the Atiyah duality map

$$\begin{aligned} \alpha :\ M^{-\tau}(e) &\longrightarrow F(e) \\ (\phi, x) &\longrightarrow (\phi, \sigma_x), \end{aligned}$$

by $\sigma_x(y) = x - \phi(y)$ if x belongs to the ball of radius ϵ around $\phi(y)$, and to ∞ otherwise. This is a map of symmetric ring spectra.

The map α is compatible with the classical equivariant Atiyah duality map and therefore we have a commutative diagram of spectra over pointed G-spaces

$$\begin{tikzcd}
M^{-\tau}(e) \arrow[rr, "\alpha"] & & F(e) \arrow[d, "\bar{\rho}"] \\
\Sigma_V^\infty M^{\nu(e)} \arrow[u] \arrow[rr, dotted] & & F(M, \mathrm{Sym}(S^V)),
\end{tikzcd}$$

where the vertical arrows are π_*-equivalences of symmetric spectra over pointed G-spaces and the dotted arrow is the equivariant Atiyah duality equivalence [**20**], proving that α is a π_*-equivalence of symmetric ring spectra over pointed G-spaces.

By an equivariant version of the Whitney embedding theorem [**28**], the homotopy type of the spectra $\Sigma_V^\infty M^{\nu(e)}$ does not depend on the embedding and we shall denote it by M^{-TM} when no confusion arises.

The prolongation of the singular simplicial set functor gives, for every symmetric ring spectrum over pointed G-spaces, a (simplicial) $H\mathbb{Z}$-symmetric algebra spectra over pointed G-simplicial sets. In [**27**], the comparison between $H\mathbb{Z}$-symmetric modules and chain complexes is established. There is a zig-zag of weak monoidal Quillen equivalences between (simplicial) $H\mathbb{Z}$-symmetric algebra spectra and differential graded algebras, and similarly, between their module, categories. In our setting, for every symmetric spectrum over pointed G-spaces we have a chain complex with a G-action, and for every symmetric ring spectra (module) over pointed G-spaces, we have a differential graded algebra (module) with G-action.

Recall that the multiplication map

$$\mu : M^{-\tau}(e) \wedge M^{-\tau}(e) \longrightarrow M^{-\tau}(e)$$

is dual to the inclusion map

$$\begin{aligned} \Delta :\ & M \longrightarrow M \times M \\ & x \longrightarrow (x, x) \end{aligned}$$

that at the level of chains

$$\Delta^* : C_{-*}(M^{-\tau}(e)) \otimes C_{-*}(M^{-\tau}(e)) \longrightarrow C_{-*}(M^{-\tau}(e))$$

is dual to the cup product on cochains

$$\Delta^* : C^*(M) \otimes C^*(M) \longrightarrow C^*(M).$$

Now, for $g \in G$, considering the twisted diagonals

$$\begin{aligned} \Delta_g :\ & M \longrightarrow M \times M \\ & x \longrightarrow (x, xg), \end{aligned}$$

we have dual maps

$$l_g : M^{-\tau}(e) \wedge M^{-\tau}(e) \longrightarrow M^{-\tau}(e)$$

and

$$l_g : F(e) \wedge F(e) \longrightarrow F(e).$$

After applying chains (under the Atiyah equivalence), these maps correspond to the map on cochains

$$\begin{aligned} \Delta_g^* :\ & C^*(M) \otimes C^*(M) \longrightarrow C^*(M) \\ & a \otimes b \longrightarrow a \cup g(b). \end{aligned}$$

The map l_g gives a bimodule structure on $M^{-\tau}(e)$ over the symmetric ring spectrum $M^{-\tau}(e)$ that we shall denote by $M^{-\tau}(e)_g$ and a bimodule structure on $F(e)$ over $F(e)$ that we shall denote by $F(e)_g$.

Note that, for the twisted diagonals to be equivariant, we have to restrict to $C_G(g)$, the centralizer of g in G.

The G-module differential graded bimodule $C_{-*}(M^{-\tau}(e))$ over $C_{-*}(M^{-\tau}(e))$ induced by l_g and the usual multiplication will be denoted by $C_{-*}(M^{-\tau}(e))_g$. Similarly, using l_g, we can twist the bimodule structures on $C_{-*}(M^{-\tau}(e))$ over $C_{-*}(F(e))$ and using Δ_g we can twist the bimodule structures on $C^*(M)$ over $C^*(M)$. We shall denote these bimodule structures by $C_{-*}(M^{-\tau}(e))_g$ and $C^*(M)_g$, respectively.

Consider the maps on Thom spectra

$$\Delta^* : M^{-\tau}(e) \longrightarrow M^{-\tau}(e) \wedge \nu(e)_+ \quad \Delta_g^* : M^{-\tau}(e) \longrightarrow \nu(e)_+ \wedge M^{-\tau}(e)$$

induced by the map of bundles $\Delta_* : \nu(e) \to \pi_1^*(\nu(e))$ and $\Delta_{g*} : \nu(e) \to \pi_2^*(\nu(e))$ over the diagonal maps $\Delta : M \to M \times M$ and $\Delta_g : M \to M \times M$. These induce a bimodule structure on $C_{-*}(M^{-\tau}(e))$ over $C^*(\nu(e))$. We shall denote this bimodule by $C_{-*}(M^{-\tau}(e))_g$.

We have a $C_G(g)$-equivariant chain homotopy induced by the evaluation map

$$ev_* : C_{-*}(F(\nu(e), \mathrm{Sym}(S^V))) \longrightarrow C^*(\nu(e))$$

that gives equivariant chain equivalences of differential graded algebras

$$C_{-*}(F(e)) \longrightarrow C_{-*}(F(\nu(e), \mathrm{Sym}(S^V))) \longrightarrow C^*(\nu(e))$$

compatible with the bimodule structures on $C_{-*}(M^{-\tau}(e))_g$.

Since the map α is a π_*-equivalence of symmetric ring spectra, we have equivariant chain homotopy ring homomorphisms

$$C_{-*}(M^{-\tau}(e)) \longrightarrow C^*(F(M, \mathrm{Sym}(S^V))) \longrightarrow C^*(M),$$

which induces the usual Poincare duality map.

Theorem 4.1. *Given a manifold M with an action of a finite group G embedded equivariantly in a representation V, there are $C_G(g)$-equivariant chain homotopy equivalences of differential graded algebras*

$$C_{-*}(F(e)) \longrightarrow C^*(\nu(e))$$

and

$$C_{-*}(M^{-\tau}(e)) \longrightarrow C_{-*}(F(e)),$$

which are compatible with the bimodule structures on $C_{-*}(M^{-\tau}(e))_g$.

4.2. *Orbifold string topology*

This section is based on the construction done in [**19**] of the orbifold loop spectra $P_G M^{-TM}$.

Now, for $t \in \mathbb{R}$ let $e_t : P_g M \to M$ be the evaluation of a map at the time t: $e_t(f) := f(t)$. Consider the space of composable maps

$$P_g M \ {}_1\!\times_0 P_h M = \{(\phi, \psi) \in P_g M \times P_h M | \phi(1) = \psi(0)\},$$

and note that they fit into the pullback square

$$\begin{tikzcd}
P_gM \,{}_1\times_0\, P_hM \arrow[r] \arrow[d, "e_0\circ\pi_1"] & P_gM \times P_hM \arrow[d, "e_0\times e_0"] \\
M \arrow[r, "\Delta_g"] & M \times M,
\end{tikzcd}$$

where the downward arrow $e_0 \circ \pi_1$ is the evaluation at zero of the first map.

The normal bundle of the upper horizontal map becomes isomorphic to the pullback under $e_0 \circ \pi_1$ of the normal bundle of the map Δ_g and therefore we can construct the Thom–Pontrjagin collapse maps that make the diagram commutative

$$\begin{tikzcd}
P_gM \times P_hM \arrow[r] \arrow[d, "e_0\times e_0"] & P_gM \,{}_1\times_0\, P_hM^{\pi_1^* e_0^* N_g} \arrow[d, "e_0\circ\pi_1"] \\
M \times M \arrow[r] & M^{N_g},
\end{tikzcd}$$

and by inverting the tangent bundle of $M \times M$ on the lower left hand, we obtain the commutative square

$$\begin{tikzcd}
P_gM^{-e_0^*TM} \wedge P_hM^{-e_0^*TM} \arrow[r] \arrow[d, "e_0\times e_0"] & P_gM \,{}_1\times_0\, P_hM^{-\pi_1^* e_0^* TM} \arrow[d, "e_0\circ\pi_1"] \\
M^{-TM} \wedge M^{-TM} \arrow[r, "\Delta_g^*"] & M^{-TM}.
\end{tikzcd}$$

The concatenation of paths in $P_gM \,{}_1\times_0\, P_hM$ produces a map

$$P_gM \,{}_1\times_0\, P_hM \longrightarrow P_{gh}M$$

that induces the map of spectra

$$P_gM \,{}_1\times_0\, P_hM^{-\pi_1^* e_0^* TM} \longrightarrow P_{gh}M^{-e_0^*TM},$$

which, composed with the upper horizontal map of diagram defines the map

$$\mu_{g,h} : P_gM^{-e_0^*TM} \wedge P_hM^{-e_0^*TM} \longrightarrow P_{gh}M^{-e_0^*TM},$$

which fits into the commutative diagram of spectra

$$\begin{tikzcd}
P_gM^{-e_0^*TM} \wedge P_hM^{-e_0^*TM} \arrow[r, "\mu_{g,h}"] \arrow[d, "e_0\times e_0"] & P_{gh}M^{-e_0^*TM} \arrow[d, "e_0"] \\
M^{-TM} \wedge M^{-TM} \arrow[r, "\Delta_g^*"] & M^{-TM}.
\end{tikzcd}$$

If we denote the spectrum P_gM^{-TM} as

$$P_gM^{-TM} := P_gM^{-e_0^*TM},$$

we define

$$P_GM^{-TM} := \bigsqcup_{g\in G} P_gM^{-TM}.$$

Assembling the maps $\mu_{g,h}$ we define a map

$$\mu : P_GM^{-TM} \wedge P_GM^{-TM} \longrightarrow P_GM^{-TM}, \tag{4.2}$$

where we have denoted

$$P_GM^{-TM} \wedge P_GM^{-TM} := \bigsqcup_{(g,h)\in G\times G} P_gM^{-TM} \wedge P_hM^{-TM}.$$

Note that the compositions $\mu \circ (\mu \times 1)$ and $\mu \circ (1 \times \mu)$ do not agree, but they are homotopically equivalent.

Recall from Definition 3.1 that $P_G M$ is a G-space where, for $k \in G$ and $f \in P_g M$, we have that $fk \in P_{k^{-1}gk} M$. Because the embedding of M to the representation $e : M \hookrightarrow V$ is G-equivariant, the spectra M^{-TM} acquire an action of G, and moreover, for $k \in G$ there is an induced map $P_g M^{-TM} \to P_{k^{-1}gk} M^{-TM}$ of spectra, where the based point in $P_g M^{-TM}$ is mapped to the based point in $P_{k^{-1}gk} M^{-TM}$.

It follows that, for $k \in G$, we have the commutative diagram

$$\begin{tikzcd}
P_g M^{-TM} \wedge P_h M^{-TM} \arrow[r, "\mu_{g,h}"] \arrow[d, "k \times k"] & P_{gh} M^{-TM} \arrow[d, "k"] \\
P_{k^{-1}gk} M^{-e_0^* TM} \wedge P_{k^{-1}hk} M^{-TM} \arrow[r, "\mu_{k^{-1}gk,k^{-1}hk}"] & P_{k^{-1}ghk} M^{-TM},
\end{tikzcd}$$

which implies that the map $\mu : P_G M^{TM} \wedge P_G M^{-TM} \to P_G M^{-TM}$ is G-equivariant.

Therefore, we could think of $P_G M^{-TM}$ as a ring spectrum with a G-action in the homotopy category; taking homology, we get a G-module ring $H_*(P_G M^{-TM}; \mathbb{Z})$ whose product structure was called the G-string product in [**19**], and the induced ring structure on the invariant set

$$H_*(P_G M^{-TM}; \mathbb{Q})^G$$

was called in [**19**] the *orbifold string topology ring*.

4.3. *Cosimplicial spectra*

In this section, we shall describe a cosimplicial model for the spectra $P_g M^{-TM}$ that will allow us to give a natural way of relating the singular chains $C_*(P_g M^{-TM})$ to the complex $\mathcal{H}om_{C^{*e}}(B(C^*), C_g^*)$. This section is a generalization of Cohen and Jones [**5**, Section 3] and we shall mimic their construction.

Recall from (3.2) the functions ϕ_k

$$\begin{aligned}
\phi_k \colon \Delta_k \times P_g X &\longrightarrow M^{k+1} = \mathbb{P}_g M(\mathfrak{n}), \\
(t_1, \ldots, t_k) \times \gamma &\longmapsto (\gamma(0), \gamma(t_1), \ldots, \gamma(t_k)),
\end{aligned}$$

and consider the commutative diagram

$$\begin{tikzcd}
\Delta_k \times P_g M \arrow[r, "\phi_k"] \arrow[d, "e"] & M^{k+1} \arrow[d, "\pi_1"] \\
M \arrow[r, "="] & M
\end{tikzcd}$$

where $e((t_0, t_1, \ldots, t_k), \gamma) \mapsto \gamma(0)$ and the right-hand vertical map is the projection on the first coordinate. Pulling back the virtual bundle $-TM$ under e and π_1, we get a map of Thom spectra that, by abuse of notation, we still call ϕ_k:

$$\phi_k : (\Delta_k)_+ \wedge P_g M^{-TM} \longrightarrow M^{-TM} \wedge (M^k)_+.$$

Taking adjoints, we get a map of spectra

$$\phi : P_g M^{-TM} \longrightarrow \prod_k \mathrm{Map}((\Delta_k)_+, M^{-TM} \wedge (M^k)_+)$$

that is just the induced map of Thom spectra of the map

$$\phi : P_gM \longrightarrow \prod_{k\geqslant 0} \mathrm{Map}(\Delta_k, M^{k+1})$$

described in Lemma 3.3.

We are now ready to define the cosimplicial spectrum $\mathfrak{P}_gM$; it will be the cosimplicial Thom spectrum of the cosimplicial virtual bundle $-TM$ on $\mathbb{P}_gM$. More explicitly, let $\mathfrak{P}_gM$ be the cosimplicial spectrum whose k-simplices are the spectrum

$$\mathfrak{P}_gM_k := M^{-TM} \wedge (M^k)_+,$$

and whose coface and codegeneracy maps are

$$\begin{aligned}
\delta_0(u; x_1, \dots, x_{k-1}) &= (u; y, x_1, \dots, x_{k-1}),\\
\delta_i(u; x_1, \dots, x_{k-1}) &= (u; x_1, \dots, x_{i-1}, x_i, x_i, x_{i+1}, \dots, x_{k-1}), \quad 1 \leqslant i \leqslant k-1,\\
\delta_k(u; x_1, \dots, x_{k-1}) &= (v; x_1, \dots, x_{k-1}, z),\\
\sigma_i(u; x_1, \dots, x_{k+1}) &= (u; x_1, \dots, x_i, x_{i+2}, \dots, x_{k+1}), \quad 0 \leqslant i \leqslant k,
\end{aligned}$$

with $\mu(u) = (u, y)$ and $\nu(u) = (z, v)$, where

$$\mu : M^{-TM} \longrightarrow M^{-TM} \wedge M_+ \quad \nu : M^{-TM} \longrightarrow M_+ \wedge M^{-TM}$$

are the maps of Thom spectra induced over the diagonal maps $\Delta : M \to M \times M$ and $\Delta_g : M \to M \times M$ by the maps of virtual bundles $\Delta_* : -TM \to -\pi_1^*TM$ and $\Delta_{g*} : -TM \to -\pi_2^*TM$.

Let $|\mathfrak{P}_gM|$ be the total spectrum of the cosimplicial spectrum $\mathfrak{P}_gM$, that is, it consists of sequences of maps $\{\gamma_k\}$ in

$$\prod_k \mathrm{Map}((\Delta_k)_+; M^{-TM} \wedge (M^k)_+)$$

that commute with the coface and codegeneracy maps. Therefore, we can conclude that, by applying the Thom spectrum functor for the virtual bundle of Lemma 3.3, we obtain the following proposition.

Proposition 4.2. *The map*

$$\phi : P_gM^{-TM} \longrightarrow \prod_k Map((\Delta_k)_+, M^{-TM} \wedge (M^k)_+)$$

induces a homeomorphism between the spectrum P_gM^{-TM} and the total spectrum of $\mathfrak{P}_gM$:

$$\phi : P_gM^{-TM} \xrightarrow{\cong} |\mathfrak{P}_gM|.$$

As a consequence of Lemma 3.5 after passing to Thom spectra, we get that the maps $\phi_k : (\Delta_k)_+ \wedge P_gM^{-TM} \to M^{-TM} \wedge (M^k)_+$ define maps of cochains

$$C^*(M^{-TM} \wedge (M^k)_+)[k] \longrightarrow C^*(P_gM^{-TM}),$$

which assemble to give a map from the total complex of the simplicial cochain complex $|C^*(\mathfrak{P}_gM)|$ to the cochain complex of the total spectrum $|\mathfrak{P}_gM|$; therefore, we have the following lemma.

Lemma 4.3. *There is a homomorphism of graded complexes*

$$|C^*(\mathfrak{P}_gM)| \longrightarrow C^*(P_gM^{-TM})$$

such that when M is a connected and simply connected manifold, it becomes a quasi-isomorphism.

If we define

$$|C^*(\mathfrak{P}_g M)|^\vee := \mathcal{H}om_{\mathbb{Z}}(|C^*(\mathfrak{P}_g M)|, \mathbb{Z}),$$

then let us show the following lemma.

Lemma 4.4. *There is a map of graded complexes*

$$|C^*(\mathfrak{P}_g M)|^\vee \longrightarrow \mathcal{H}om_{C^{*e}}(B(C^*), C^*_g)$$

that moreover is a quasi-isomorphism.

Proof. The cochains of the k simplices of $\mathfrak{P}_g M$ become chain homotopy equivalent to

$$C^*(M^{-TM} \wedge (M^k)_+) \cong C^*(M)^k \otimes C^*(M^{-TM}),$$

and, after dualizing, we get that

$$\begin{aligned} C^*(M^{-TM} \wedge (M^k)_+)[k]^\vee &\cong \mathcal{H}om_{\mathbb{Z}}(C^*(M)[1]^{\otimes k} \otimes C^*(M^{-TM}), \mathbb{Z}) \\ &\cong \mathcal{H}om_{\mathbb{Z}}(C^*(M)[1]^{\otimes k}, C_{-*}(M^{-TM})). \end{aligned}$$

Now we want to use the fact that Atiyah duality produces an equivalence of symmetric spectra that induces a chain homotopy equivalence

$$\alpha_* : C_{-*}(M^{-TM}) \longrightarrow C^*(M)$$

compatible with the $C^*(M)^e$-module structure of $C_{-*}(M^{-\tau}(e))$; this is the content of Theorem 4.1. For this, let us modify (by a homotopy equivalent model) the cosimplicial spectrum $\mathfrak{P}_g M$. Its k-cosimplices are

$$M^{-\tau}(e) \wedge (\nu(e)_+)^k$$

with the corresponding codegeneracy and coface maps defined using μ and ν_g. As before, the chains of the k-cosimplices are given by

$$\begin{aligned} C^*(M^{-\tau}(e) \wedge (\nu(e)^k)_+)[k]^\vee &\cong \mathrm{Hom}(C^*(\nu(e))[1]^{\otimes k}, C_{-*}(M^{-\tau}(e))_g) \\ &\cong \mathcal{H}om_{C^{*e}}(C^* \otimes C^*[1]^{\otimes k} \otimes C^*, C_{-*}(M^{-\tau}(e))_g) \\ &\cong CH^*(C^*(\nu(e)), C_{-*}(M^{-\tau}(e))_g)[k]. \end{aligned}$$

Using the first part of Theorem 4.1, we can replace $C^*(\nu(e))$ by $C_{-*}(F(e))$ to obtain a $C_G(g)$-equivariant chain equivalence

$$|C^*(\mathfrak{P}_g M)|^\vee \longrightarrow CH^*(C_{-*}(F(e)), C_{-*}(M^{-\tau}(e))_g).$$

Now, by the second part of Theorem 4.1, we have a $C_G(g)$-equivariant chain homotopy equivalence

$$CH^*(C_{-*}(F(e)), C_{-*}(M^{-\tau}(e))_g) \longrightarrow CH^*(C_{-*}(M^{-\tau}(e)), C_{-*}(M^{-\tau}(e))_g),$$

which all together gives a $C_G(g)$-equivariant chain homotopy equivalence

$$|C^*(\mathfrak{P}_g M)|^\vee \longrightarrow CH^*(C_{-*}(M^{-\tau}(e)), C_{-*}(M^{-\tau}(e))_g). \tag{4.3}$$

Since we have a π_*-equivalence of symmetric ring spectra over pointed G-spaces $\alpha : M^{-\tau}(e) \to F(e)$ that is compatible with the twisted bimodule structure on $M^{-\tau}(e)_g$ and $F(e)_g$ (both bimodule structures come as duals to the twisted diagonals), it follows that it induces an equivalence of their topological Hochschild cohomologies

$$THH^*(M^{-\tau}(e), M^{-\tau}(e)_g) \cong THH^*(F(e), F(e)_g).$$

By applying the chain functor, we obtain $C_G(g)$-equivariant chain homotopies

$$CH^*(C_{-*}(M^{-\tau}(e)), C_{-*}(M^{-\tau}(e))_g) \cong CH^*(C^*(M), C^*(M)_g). \tag{4.4}$$

Finally, putting together the isomorphism equations (4.3) and (4.4), we obtain the desired result, a $C_G(g)$-equivariant chain homotopy equivalence

$$|C^*(\mathfrak{P}_g M)|^\vee \longrightarrow CH^*(C^*(M), C^*(M)_g) = \mathcal{H}om_{C^{*e}}(B(C^*), C^*_g). \qquad \square$$

Lemma 4.3 together with Lemma 4.4 allows us to conclude the following theorem.

Theorem 4.5. *For M a connected and simply connected compact manifold, there exists a homomorphism of graded complexes*

$$C_*(P_g M^{-TM}) \xrightarrow{\simeq} \mathcal{H}om_{C^{*e}}(B(C^*), C^*_g)$$

that moreover is a quasi-isomorphism, and therefore induces an isomorphism of graded groups

$$H_*(P_g M^{-TM}) \xrightarrow{\cong} H^*\mathcal{H}om_{C^{*e}}(B(C^*), C^*_g).$$

From the previous theorem, assembling the maps for all g in G, we can deduce the following corollary.

Corollary 4.6. *For M a connected and simply connected compact manifold, there exists a homomorphism of graded complexes*

$$C_*(P_G M^{-TM}) \xrightarrow{\simeq} \mathcal{H}om_{C^{*e}}(B(C^*), C^*\#G)$$

that moreover is a quasi-isomorphism and G-equivariant, and therefore induces a G-equivariant isomorphism of graded groups

$$H_*(P_G M^{-TM}) \xrightarrow{\cong} H^*\mathcal{H}om_{C^{*e}}(B(C^*), C^*\#G).$$

Proof. The only thing left to prove is that the maps $P_g M^{-TM} \to |\mathfrak{P}_g M|$ induce an equivariant map

$$P_G M^{-TM} \longrightarrow \coprod_{g \in G} |\mathfrak{P}_g M|,$$

but this follows from the fact that the evaluation maps $e_t : P_G M \to M$ are G-equivariant. $\square$

4.4. *Multiplicative structures*

In this section, we will show how the map of Corollary 4.6 is compatible with the multiplicative structure of $C_*(P_G M^{-TM})$ coming from the map μ of (4.2), and the natural ring structure on $\mathcal{H}om_{C^{*e}}(B(C^*), C^*\#G)$ that was explained in Paragraph 2.3.1. To achieve this, we shall endow the cosimplicial spectra $\mathfrak{P}_g M$ with multiplicative maps

$$\tilde{\mu}_{g,h} : \mathfrak{P}_g M \wedge \mathfrak{P}_h M \longrightarrow \mathfrak{P}_{gh} M$$

that, once realized, will be compatible with the maps $\mu_{g,h}$ and that, after passing to chains, will realize the ring structure of $\mathcal{H}om_{C^{*e}}(B(C^*), C^*\#G)$.

Let us define the maps

$$\begin{aligned} \tilde{\mu}^{k,l}_{g,h} : (M^{-TM} \wedge (M^k)_+) \wedge (M^{-TM} \wedge (M^l)_+) &\longrightarrow M^{-TM} \wedge (M^{k+l})_+ \\ (u; x_1, \ldots, x_k) \wedge (v; y_1, \ldots, y_l) &\longmapsto (\Delta^*_g(u, v); x_1, \ldots, x_k, y_1, \ldots, y_l) \end{aligned}$$

that define maps of the simplices

$$\tilde{\mu}^{k,l}_{g,h} : (\mathfrak{P}_g M)_k \wedge (\mathfrak{P}_h M)_l \longrightarrow (\mathfrak{P}_{gh} M)_{k+l},$$

which commute with the coface and codegeneracy operators, and therefore induce maps of cosimplicial spectra

$$\tilde{\mu}_{g,h} : \mathfrak{P}_g M \wedge \mathfrak{P}_h M \longrightarrow \mathfrak{P}_{gh} M.$$

Taking the total spectrum, we get maps of spectra

$$|\tilde{\mu}_{g,h}| : |\mathfrak{P}_g M| \wedge |\mathfrak{P}_h M| \longrightarrow |\mathfrak{P}_{gh} M|$$

that induce pairings that are A_∞-associative.

Applying the chains functor to the map $\tilde{\mu}^{k,l}_{g,h}$, we get the map

$$\begin{aligned}(\tilde{\mu}^{k,l}_{g,h})_* : &(C_{-*}(M)^k \otimes C_{-*}(M^{-TM})) \otimes (C_{-*}(M)^l \otimes C_{-*}(M^{-TM}))\\ &\longrightarrow C_{-*}(M)^{k+l} \otimes C_{-*}(M^{-TM})\end{aligned}$$

that, by Theorem 4.1, induces the map

$$\begin{aligned}&(\tilde{\mu}^{k,l}_{g,h})_* : (C_{-*}(M)^k \otimes C^*(M)) \otimes (C_{-*}(M)^l \otimes C^*(M))\\ &\quad \longrightarrow C_{-*}(M)^{k+l} \otimes C^*(M)\\ &(a_1 \otimes \cdots \otimes a_k \otimes \alpha) \otimes (b_1 \otimes \cdots \otimes b_l \otimes \beta)\\ &\quad \longmapsto a_1 \otimes \cdots \otimes a_k \otimes b_1 \otimes \cdots \otimes b_l \otimes \Delta_g^*(\alpha \otimes \beta)\\ &\quad = a_1 \otimes \cdots \otimes a_k \otimes b_1 \otimes \cdots \otimes b_l \otimes \alpha \cdot g(\beta).\end{aligned}$$

Composing with the natural isomorphisms

$$C_{-*}(M)^k \otimes C^*(M) \cong \mathcal{H}om_{C^*(M)^e}(C^*(M)^{k+2}, C^*(M)),$$

we see that the map $(\tilde{\mu}^{k,l}_{g,h})_*$ induces the multiplicative structure that, for

$$\phi_g \in \mathcal{H}om_{C^*(M)^e}(C^*(M)^{k+2}, C^*(M)), \quad \psi_h \in \mathcal{H}om_{C^*(M)^e}(C^*(M)^{l+2}, C^*(M)),$$

defines the function

$$(a_0|\cdots|a_{k+l+1}) \longmapsto (-1)^{|\psi_h|\varepsilon_k} \phi_g(a_0|\cdots|a_k|1) g(\psi_h(1|a_{k+1}|\cdots|a_{k+l+1})),$$

which agrees with the definition of the product structure of $\mathcal{H}om_{C^{*e}}(B(C^*), C^*\#G)$ done in Paragraph 2.3.1.

Then we have the following lemma.

LEMMA 4.7. *The maps of Lemma* 4.4

$$|C^*(\mathfrak{P}_g M)|^\vee \longrightarrow \mathcal{H}om_{C^{*e}}(B(C^*), C^*_g)$$

are compatible with the multiplicative structures on $\bigoplus_{g\in G} |C^*(\mathfrak{P}_g M)|^\vee$ *and* $\mathcal{H}om_{C^{*e}}(B(C^*), C^*\#G)$*, where the multiplicative structure on the left hand side is the one induced by the maps* $(\tilde{\mu}^{k,l}_{g,h})_*$*. In particular, we have an isomorphism of rings*

$$\bigoplus_{g\in G} H^*(|C^*(\mathfrak{P}_g M)|^\vee) \xrightarrow{\cong} H^*\mathcal{H}om_{C^{*e}}(B(C^*), C^*\#G).$$

We are left with providing the relationship between the maps $\mu_{g,h}$ and the maps $\tilde{\mu}_{g,h}$; this will be achieved with the following theorem.

Theorem 4.8. *The multiplicative structures that the maps $\mu_{g,h}$ and $\tilde{\mu}_{g,h}$ define are compatible in the sense that the following diagram homotopy commutes*:

$$\begin{tikzcd}
P_gM^{-TM} \wedge P_hM^{-TM} \arrow[r, "\mu_{g,h}"] \arrow[d, "\phi\wedge\phi"] & P_{gh}M^{-TM} \arrow[d, "\phi"] \\
|\mathfrak{P}_gM| \wedge |\mathfrak{P}_hM| \arrow[r, "\tilde{\mu}_{g,h}"] & |\mathfrak{P}_{gh}M|
\end{tikzcd}$$

where ϕ is the map of Proposition 4.2.

Proof. The proof is almost identical to [**5**, Proof of Theorem 13]. The only difference is that, whenever it is used the diagonal map Δ on [**5**, Proof of Theorem 13] it needs to be replaced by the twisted diagonal Δ_g. We shall not reproduce the proof here. □

Lemma 4.7 together with Theorem 4.8 implies that the G-equivariant quasi-isomorphism of Corollary 4.6 is compatible with the multiplicative structures previously defined, and therefore we can complete this section with the following theorem.

Theorem 4.9. *For M a connected and simply connected compact manifold, there exists a quasi-isomorphism of graded complexes*

$$\Phi : C_*(P_GM^{-TM}) \xrightarrow{\simeq} \mathcal{H}om_{C^{*e}}(B(C^*), C^*\#G)$$

that is G-equivariant, and that furthermore makes the following diagram commute:

$$\begin{tikzcd}
C_*(P_GM^{-TM}) \otimes C_*(P_GM^{-TM}) \arrow[r, "\mu_*"] \arrow[d, "\phi_*\otimes\phi_*"] & C_*(P_GM^{-TM}) \arrow[d, "\phi_*"] \\
\mathcal{H}om_{C^{*e}}(B(C^*), C^*\#G) \otimes \mathcal{H}om_{C^{*e}}(B(C^*), C^*\#G) \arrow[r, "\cdot"] & \mathcal{H}om_{C^{*e}}(B(C^*), C^*\#G).
\end{tikzcd}$$

Hence, Φ induces a G-equivariant isomorphism of rings

$$\Phi : H_*(P_GM^{-TM}) \xrightarrow{\cong} H^*\mathcal{H}om_{C^{*e}}(B(C^*), C^*\#G).$$

The previous Theorem 4.9, together with Theorem 2.2 and the results of the Appendix B, imply the following theorem.

Theorem 4.10. *For M connected and simply connected and G a finite group acting on M, there exists an isomorphism of graded rings*

$$HH^*(C^*(M)\#G, C^*(M)\#G) \cong \mathrm{Ext}^*_{\mathbb{Z}G}(\mathbb{Z}, C_*(P_GM^{-TM})).$$

5. *Homotopical realization for $HH^*(C^*(M)\#G, C^*(M)\#G)$*

From Theorem 4.10, we know that we have an isomorphism of rings

$$HH^*(C^*(M)\#G, C^*(M)\#G) \cong \mathrm{Ext}^*_{\mathbb{Z}G}(\mathbb{Z}, C_*(P_GM^{-TM})),$$

which can be further extended to an isomorphism of rings

$$HH^*(C^*(M)\#G, C^*(M)\#G) \cong H^*(C^*(EG) \otimes C_*(P_GM^{-TM}))^G \tag{5.1}$$

as we know that

$$\begin{aligned}\mathrm{Ext}^*_{\mathbb{Z}G}(\mathbb{Z}, C_*(P_GM^{-TM})) &\cong H^*\mathcal{H}om_{\mathbb{Z}G}(C_*(EG), C_*(P_GM^{-TM}))\\ &\cong H^*\mathcal{H}om_{\mathbb{Z}G}(\mathbb{Z}, C^*(EG)\otimes C_*(P_GM^{-TM}))\\ &= H^*(C^*(EG)\otimes C_*(P_GM^{-TM}))^G. \end{aligned}\tag{5.2}$$

We would like to have a topological construction associated to the Hochschild cohomology of $C^*(M)\#G$. One deterrent for its existence comes from the fact that on the right-hand side of (5.1) we have a mixture of chains with cochain complexes.

To overcome this problem, we will construct a pro-ring spectrum EG^{-TEG} associated to EG, which will allow us to change the cochain complex $C^*(EG)$ by the chain complex of EG^{-TEG}. With this idea in mind, let us generalize the construction of the string topology of BG that can be found in [**10**, **11**] to the orbifold case.

Let $EG_1 \subset \cdots \subset EG_n \subset EG_{n+1} \subset \cdots \subset EG$ be a finite-dimensional manifold approximation of the universal G-principal bundle $EG \to BG$. Consider the maps

$$P_GM \times_G EG_n \xrightarrow{e_0} M \times_G EG_n,$$

where, by abuse of notation, we denote the evaluation at the time t of a pair $(f,\lambda) \in P_GM \times_G EG_n$ also by e_t. For convenience, let us call $M_n = M \times_G EG_n$.

Take the Thom spectra

$$(P_GM \times_G EG_n)^{-e_0^*TM_n},$$

and let us show that indeed it is a ring spectra. Consider the diagram

$$\begin{tikzcd}
(P_GM\,{}_1\!\times_0 P_GM) \times_G EG_n \arrow[r, "\tilde{\Delta}"] \arrow[d, "e_0"] & (P_GM \times_G EG_n) \times (P_GM \times_G EG_n) \arrow[d, "e_0\times e_0"] \\
M \times_G EG_n \arrow[r, "\Delta"] & (M \times_G EG_n) \times (M \times_G EG_n)
\end{tikzcd}$$

where Δ is the diagonal inclusion and $P_GM\,{}_1\!\times_0 P_GM$ fits in the pullback square

$$\begin{tikzcd}
P_GM\,{}_1\!\times_0 P_GM \arrow[r] \arrow[d, "e_\infty"] & P_GM \times P_GM \arrow[d, "e_1\times e_0"] \\
M \arrow[r, "\mathrm{diag}"] & M \times M.
\end{tikzcd}$$

As the normal bundle of the inclusion $P_GM\,{}_1\!\times_0 P_GM \to P_GM \times P_GM$ is isomorphic to e_∞^*TM, the normal bundle of the inclusion

$$\tilde{\Delta} : (P_GM\,{}_1\!\times_0 P_GM) \times_G EG_n \longrightarrow (P_GM \times_G EG_n) \times (P_GM \times_G EG_n)$$

is isomorphic to $e_\infty^*TM_n$. Then, by the Thom–Pontryagin construction, we have a map

$$(P_GM \times_G EG_n) \times (P_GM \times_G EG_n) \longrightarrow ((P_GM\,{}_1\!\times_0 P_GM) \times_G EG_n)^{e_\infty^*TM_n},$$

which induces a map of spectra

$$\begin{aligned}\nu : (P_GM \times_G EG_n)^{-e_1^*TM_n} &\wedge (P_GM \times_G EG_n)^{-e_0^*TM_n}\\ &\longrightarrow ((P_GM\,{}_1\!\times_0 P_GM) \times_G EG_n)^{-e_\infty^*TM_n}.\end{aligned}$$

Let us recall the concatenation map

$$\begin{aligned}\mu :\ P_GM\,{}_0\!\times_1 P_GM &\longrightarrow P_GM\\ ((\phi, g), (\psi, h)) &\longmapsto (\phi \circ \psi, gh),\end{aligned}$$

where

$$\phi \circ \psi(t) := \begin{cases} \phi(2t) & \text{for } 0 \leqslant t < \frac{1}{2}, \\ \psi(2t-1) & \text{for } \frac{1}{2} \leqslant t \leqslant 1. \end{cases}$$

We have the commutative square

$$\begin{tikzcd}
(P_GM \, {}_0{\times}_1 \, P_GM) \times_G EG_n \arrow[r, "\mu"] \arrow[d, "e_\infty"] & P_GM \times_G EG_n \arrow[d, "e_{\frac{1}{2}}"] \\
M \times_G EG_n \arrow[r, "="] & M \times_G EG_n
\end{tikzcd}$$

that induces a map on spectra

$$((P_GM \, {}_0{\times}_1 \, P_GM) \times_G EG_n)^{-e_\infty^* T(M \times_G EG_n)} \xrightarrow{\bar{\mu}} (P_GM \times_G EG_n)^{-e_{1/2}{}^* T(M \times_G EG_n)},$$

which, composed with the map ν, gives us a map of spectra

$$\begin{aligned} &(P_GM \times_G EG_n)^{-e_1^* T(M \times_G EG_n)} \wedge (P_GM \times_G EG_n)^{-e_0^* T(M \times_G EG_n)} \\ &\quad \longrightarrow (P_GM \times_G EG_n)^{-e_{1/2}{}^* T(M \times_G EG_n)}. \end{aligned}$$

Because all the maps $e_0, e_{1/2}, e_1$ are homotopy equivalent, the bundles $e_0^* TM_n \cong e_{1/2}^* TM_n \cong e_1^* TM_n$ are all isomorphic. Therefore, we can construct a map of spectra

$$\begin{aligned} &(P_GM \times_G EG_n)^{-e_0^* T(M \times_G EG_n)} \wedge (P_GM \times_G EG_n)^{-e_0^* T(M \times_G EG_n)} \\ &\quad \longrightarrow (P_GM \times_G EG_n)^{-e_0^* T(M \times_G EG_n)} \end{aligned}$$

that makes $(P_GM \times_G EG_n)^{-e_0^* TM_n}$ into a ring spectra.

The inclusions

$$\begin{tikzcd}
\cdots \arrow[r] & P_GM \times_G EG_n \arrow[r] \arrow[d] & P_GM \times_G EG_{n+1} \arrow[r] \arrow[d] & \cdots \\
\cdots \arrow[r] & M \times_G EG_n \arrow[r] & M \times_G EG_{n+1} \arrow[r] & \cdots
\end{tikzcd}$$

induce maps of ring spectra

$$(P_GM \times_G EG_n)^{-e_0^* T(M \times_G EG_n)} \overset{\rho_n^{n+1}}{\longleftarrow} (P_GM \times_G EG_{n+1})^{-e_0^* T(M \times_G EG_{n+1})}. \tag{5.3}$$

Definition 5.1. The previous system of ring spectra will be denoted by

$$\mathcal{L}(M \times_G EG)^{-T(M \times_G EG)},$$

and we will call it *the free loop space pro-ring spectrum associated to the orbifold.*

Note that the maps of ring spectra in (5.3) do not assemble into an inverse system of ring spectra. The reason for this is that each of the maps ρ_n^{n+1} of ring spectra in (5.3) depends explicitly on an equivariant embedding $EG_{n+1} \to V_{n+1}$ of EG_{n+1} into a representation of G. Therefore, the composition $\rho_{n-1}^n \circ \rho_n^{n+1}$ does not necessarily agree with the map of spectra

$$(P_GM \times_G EG_{n-1})^{-e_0^* T(M \times_G EG_{n-1})} \overset{\rho_{n-1}^{n+1}}{\longleftarrow} (P_GM \times_G EG_{n+1})^{-e_0^* T(M \times_G EG_{n+1})}$$

induced by the inclusion $EG_{n-1} \subset EG_{n+1}$.

Nevertheless, the maps ρ_{n-1}^{n+1} and $\rho_{n-1}^n \circ \rho_n^{n+1}$ are homotopically equivalent, and therefore, after applying the homology functor, we indeed get an inverse system of graded rings. Therefore, we give the following definition.

Definition 5.2. The homology of the free loop space pro-ring spectrum associated to the orbifold $[M/G]$ is

$$H_*^{\mathrm{pro}}(\mathcal{L}(M \times_G EG)^{-T(M\times_G EG)}) := \lim_{\longleftarrow n} H_*((P_G M \times_G EG_n)^{-e_0^* T(M\times_G EG_n)}).$$

We obtain the following theorem.

Theorem 5.3. *For M connected and simply connected, and G a finite group acting on M, there is an isomorphism of graded rings*

$$HH^*(C^*(M)\#G, C^*(M)\#G) \cong H_*^{\mathrm{pro}}(\mathcal{L}(M \times_G EG)^{-T(M\times_G EG)})$$

between the Hochschild cohomology of $C^*(M)\#G$ and the homology of the free loop space pro-ring spectrum associated to the orbifold $[M/G]$.

Proof. Consider the following sequence of graded ring isomorphisms:

$$\begin{aligned}
&H_*^{\mathrm{pro}}(\mathcal{L}(M \times_G EG)^{-T(M\times_G EG)})\\
&\quad := \lim_{\longleftarrow n} H_*((P_G M \times_G EG_n)^{-e_0^* T(M\times_G EG_n)}) &(5.4)\\
&\quad \cong \lim_{\longleftarrow n} H_*(C_*((P_G M \times EG_n)^{-e_0^* T(M\times EG_n)})^G) &(5.5)\\
&\quad \cong \lim_{\longleftarrow n} H_*(C_*(P_G M^{-TM} \wedge EG_n^{-TEG_n})^G) &(5.6)\\
&\quad \cong \lim_{\longleftarrow n} H_*((C_*(EG_n^{-TEG_n}) \otimes C_*(P_G M^{-TM}))^G) &(5.7)\\
&\quad \cong \lim_{\longleftarrow n} H^*((C^*(EG_n) \otimes C_*(P_G M^{-TM}))^G) &(5.8)\\
&\quad \cong H^*(\lim_{\longleftarrow n}(C^*(EG_n) \otimes C_*(P_G M^{-TM}))^G) &(5.9)\\
&\quad \cong H^*((C^*EG \otimes C_*(P_G M^{-TM}))^G) &(5.10)\\
&\quad \cong \mathrm{Ext}^*_{\mathbb{Z}G}(\mathbb{Z}, C_*(P_G M^{-TM})) &(5.11)\\
&\quad \cong HH^*(C^*(M)\#G, C^*(M)\#G),
\end{aligned}$$

where the isomorphism between (5.4) and (5.5) is due to the fact that G acts freely on EG_n; the isomorphism between (5.5) and (5.6) is due to the fact that as ring spectra $X^{-TX} \wedge Y^{-TY}$ and $(X \times Y)^{-T(X\times Y)}$ are homotopic; the isomorphism between (5.6) and (5.7) follows by the Eilenberg–Zilber theorem; the isomorphism between (5.7) and (5.8) follows by S-duality between $C_*(EG_n^{-TEG_n})$ and $C^*(EG_n)$; the isomorphism between (5.8) and (5.9) follows from the fact that the inverse system of graded rings

$$H^*((C^*(EG_n) \otimes C_*(P_G M^{-TM}))^G)$$

satisfies the Mittag–Leffler condition; the isomorphism between (5.9) and (5.10) follows from the fact that the inverse system $H^*(EG_n)$ of graded rings satisfies the Mittag–Leffler condition; the isomorphism between (5.10) and (5.11) follows from (5.2); and the last isomorphism is proved in Theorem 4.10.

Then the theorem follows from the previous isomorphisms. □

With Theorem 5.3 at hand, we make the following definition.

Definition 5.4. The string topology ring associated to a global quotient orbifold $[M/G]$ is the graded ring

$$H_*^{\mathrm{pro}}(\mathcal{L}(M \times_G EG)^{-T(M\times_G EG)}; \mathbb{Z}).$$

Note that the previous definition is indeed a homotopy invariant of the orbifold, and hence, it is well defined in the Morita equivalence class of the orbifold $[M/G]$. The isomorphism with the Hochschild cohomology ring of $C^*(M)\#G$ depended explicitly on the fact that $H^1(M) = 0$, and hence it is needed that M be simply connected.

6. *Applications*

6.1. *Rational coefficients*

When we use rational coefficients, the G-invariants functor is exact, and therefore there is no need to derive it. Hence, the string topology ring of the orbifold $[M/G]$ becomes isomorphic as graded rings to

$$H_*^{\mathrm{pro}}(\mathcal{L}(M \times_G EG)^{-T(M\times_G EG)}; \mathbb{Q}) \cong H_*(P_G M^{-TM}; \mathbb{Q})^G,$$

which is precisely what was defined in [**19**] as the orbifold string topology ring.

So, we have that in the case where M is connected and simply connected, the orbifold string topology of $[M/G]$ with coefficients in $\mathbb{Q}$ is isomorphic to the ring

$$H_*(P_G M^{-TM}; \mathbb{Q})^G \cong HH^*(C^*(M;\mathbb{Q})\#G, C^*(M;\mathbb{Q})\#G).$$

6.2. $M = point$

The string topology for $[*/G]$ turns out to be a ring that can be reproduced with the pull–push formalism (see [**30**]). Let us see first that as a graded $\mathbb{Z}$-module, we have an isomorphism

$$H_*^{\mathrm{pro}}(\mathcal{L}BG^{-TBG}) \cong \bigoplus_{(g)} H^*(BC(g)),$$

where (g) runs over the conjugacy classes of elements in G and $C(g)$ denotes the centralizer of g in G.

The groupoid $[P_G M/G]$ becomes simply $[G/G]$ where G acts by conjugation on G, thus we obtain

$$(G \times_G EG_n)^{-TBG_n} = \bigsqcup_{(g)} (EG_n/C(g))^{-\pi^* TBG_n} \cong \bigsqcup_{(g)} (BC(g)_n)^{-TBC(g)_n}$$

with $\pi : BC(g)_n := EG_n/C(g) \to EG_n/G = BG_n$; the second equality follows from the fact that π is a cover map and therefore $TBC(g)_n \cong \pi^* TBG_n$.

Hence,

$$\begin{aligned} H_*^{\mathrm{pro}}(\mathcal{L}BG^{-TBG}) &= \bigoplus_{(g)} \lim_{\leftarrow n} H_*((BC(g)_n)^{-TBC(g)_n}) \\ &= \bigoplus_{(g)} \lim_{\leftarrow n} H^*(BC(g)_n) \\ &= \bigoplus_{(g)} H^*(BC(g)). \end{aligned}$$

Now let us see what is the induced ring structure: we have the maps

$$\begin{tikzcd} EG_n/C(g) \times EG_n/C(h) & EG_n/(C(g)\cap C(h)) \arrow[l, "\Delta"'] \arrow[r] & EG_n/C(gh), \end{tikzcd}$$

and all the pullbacks of the bundle TBG_n are isomorphic to the corresponding tangent bundles. Note that the map Δ is injective and therefore we can perform the Thom–Pontrjagin construction giving us the map in homology

$$H_*(EG_n/C(g) \times EG_n/C(h)) \longrightarrow H_{*-k_n}(EG_n/(C(g)\cap C(h)))$$

with $k_n = \dim(EG_n)$, which is Poincaré dual to the pullback map in cohomology

$$\Delta^* : H^*(EG_n/C(g) \times EG_n/C(h)) \longrightarrow H^*(EG_n/(C(g) \cap C(h))).$$

The natural map in homology

$$H_*(EG_n/(C(g) \cap C(h))) \longrightarrow H_*(EG_n/C(gh))$$

is Poincaré dual to the push-forward map in cohomology

$$H^*(EG_n/(C(g) \cap C(h))) \longrightarrow H^*(EG_n/C(gh))$$

that defines the induction map

$$H^*(B(C(g) \cap C(h))) \longrightarrow H^*(BC(gh)).$$

We therefore see that the ring structure in $H_*^{\text{pro}}(\mathcal{L}BG^{-TBG})$ is obtained by taking classes in $H^*(BC(g))$ and $H^*(BC(h))$, pulling them back to $H^*B(C(g) \cap C(h))$ and then pushing them forward to $H^*(BC(gh))$. This procedure is what is known as the pull–push formalism, and it is a well-known fact among algebraists that the ring structure $HH^*(\mathbb{Z}G, \mathbb{Z}G)$ could be recovered with this formalism (see [**30**, Example 2.7] and the references therein).

Note that in the case where G is abelian, $BG = K(G, 1) = \Omega K(G, 2)$ is a topological group and therefore, $LBG \cong G \times BG$, and we have

$$HH^*(\mathbb{Z}G, \mathbb{Z}G) \cong \mathbb{Z}G \otimes_{\mathbb{Z}} \operatorname{Ext}^*_{\mathbb{Z}G}(\mathbb{Z}, \mathbb{Z}) \cong \mathbb{Z}G \otimes_{\mathbb{Z}} H^*(BG),$$

and therefore the string topology ring for $[*/G]$ is the ring $\mathbb{Z}G \otimes_{\mathbb{Z}} H^*(BG)$.

6.3. *Manifold with finite fundamental group*

Consider a connected compact manifold N with finite fundamental group $G = \pi_1 N$. Then, by Theorem 5.3, we have the following theorem.

THEOREM 6.1. *The string topology ring $H_*(\mathcal{L}N^{-TN}; \mathbb{Z})$ as defined in [**5**] is isomorphic to the ring*

$$HH^*(C^*(\tilde{N})\#G, C^*(\tilde{N})\#G),$$

where $\tilde{N}$ is the universal cover of N.

Proof. Because G acts freely on $\tilde{N}$, we have that

$$H_*^{\text{pro}}(\mathcal{L}(\tilde{N} \times_G EG)^{-T(\tilde{N} \times_G EG)}) \cong H_*(P_G\tilde{N}^{-T\tilde{N}})^G,$$

and moreover we have that as ring spectra $P_G\tilde{N}^{-T\tilde{N}}/G \cong \mathcal{L}N^{-TN}$. Together with Theorem 5.3, we have the desired isomorphism of graded rings

$$H_*(\mathcal{L}N^{-TN}; \mathbb{Z}) \cong HH^*(C^*(\tilde{N})\#G, C^*(\tilde{N})\#G). \qquad \square$$

7. *Failure of Hochschild cohomology invariance under orbifold equivalence*

Taking a closer look at Theorem 5.3, we see that we can only relate the string topology of an orbifold to the Hochschild cohomology ring of the group dg-ring, whenever we can write the orbifold as the groupoid $[M/G]$ with M simply connected and connected. The reason for this restriction lies in the use of the Eilenberg–Moore spectral sequence to relate the complex

$$C^*(M) \overset{L}{\otimes}_{C^*(M)^e} C^*(M) \quad \text{with } C^*(\mathcal{L}M),$$

which, in the case where M is not simply connected, does not converge (see [**6**]).

But it is natural to ask whether the Hochschild cohomology ring

$$HH^*(C^*(M)\#G, C^*(M)\#G)$$

is independent of the presentation of the orbifold; in other words, for two Morita equivalent orbifold groupoids $[M/G]$ and $[N/H]$ in the sense of Moerdijk [**22**], are the graded rings

$$HH^*(C^*(M)\#G, C^*(M)\#G) \quad \text{and} \quad HH^*(C^*(N)\#H, C^*(N)\#H)$$

isomorphic?

The following result tells us that, in general, the answer to the previous question is negative.

Proposition 7.1. *Let $M = S^1$ be the circle equipped with the antipodal action of $G = \mathbb{Z}/2$. Let us replace the cochains by differential forms ΩM and let us work with complex coefficients. Consider the Morita equivalent groupoids $[M/G]$ and the topological quotient M/G, together with the associated dg-rings $\Omega M\#G$ and $\Omega(M/G)$. Then $HH^*(\Omega M\#G, \Omega M\#G)$ is not isomorphic to $HH^*(\Omega(M/G), \Omega(M/G))$ as rings.*

Proof. Let $\mathbb{C}[x]/x^2$ be the dg-algebra with trivial differential and $\deg x = 1$. Consider the trivial G-action on $\mathbb{C}[x]/x^2$. Note that $H^*(\Omega M)$ is spanned by the classes of 1 and $d\Theta$. Since the elements 1 and $d\Theta$ are G-invariant, we get a G-equivariant quasi-isomorphism

$$\mathbb{C}[x]/x^2 \longrightarrow \Omega M, \quad x \longmapsto d\Theta. \tag{7.1}$$

Since M/G is diffeomorphic to M, a straightforward computation now gives

$$HH^*(\Omega(M/G)) \cong HH^*(\Omega M) \cong HH^*(\mathbb{C}[x]/x^2) \cong \mathbb{C}[y,z]/z^2,$$

where $\deg y = 0$ and $\deg z = 1$. On the other hand, the map in (7.1) gives a quasi-isomorphism

$$\mathbb{C}[x]/x^2\#G \cong \Omega M\#G.$$

Since $\mathbb{C}[x]/x^2\#G$ is the usual algebra tensor product of $\mathbb{C}[x]/x^2$ and $\mathbb{C}G$ and since $\mathbb{C}G$ is commutative and semi-simple, this gives

$$HH^*(\Omega M\#G) \cong HH^*(\mathbb{C}[x]/x^2\#G) = HH^*(\mathbb{C}[x]/x^2) \otimes \mathbb{C}G \cong \mathbb{C}[y,z]/z^2 \otimes \mathbb{C}G.$$

This proves the proposition since $\mathbb{C}[y,z]/z^2$ and $\mathbb{C}[y,z]/z^2 \otimes \mathbb{C}G$ are non-isomorphic rings. □

Since Hochschild cohomology commutes with the extension of scalars and ΩM is Morita equivalent to $C^*(M) \otimes_{\mathbb{Z}} \mathbb{C}$, we conclude that $HH^*(C^*(M)\#G, C^*(M)\#G)$ and $HH^*(C^*(M/G), C^*(M/G))$ are also non-isomorphic rings.

Because the Hochschild cohomology is invariant under derived equivalence (see [**17**]), we can conclude the following corollary.

Corollary 7.2. *Let $M = S^1$ and endow it with the antipodal action of $G = \mathbb{Z}/2$. Then the derived categories of dg-modules*

$$\mathcal{D}(C^*(M)\#G) \quad \text{and} \quad \mathcal{D}(C^*(M/G))$$

*(as in [**25**]) are not equivalent.*

(Note that, by Proposition 7.1, these categories are non-equivalent also with coefficients in the field of complex numbers.) Therefore, one cannot expect that there is an isomorphism of Hochschild cohomology rings for the dg-rings of Morita equivalent groupoids, and in some sense, one can only produce an isomorphism with the string topology ring whenever one can find a description of the orbifold given by the quotient of a simply connected manifold by the action of a finite group. When this is not the case, as, for example, the case of manifolds with a non-finite fundamental group, we do not know how to recover the string topology ring via the Hochschild cohomology of some dg-ring associated to the orbifold. We leave this question open.

Appendix A

A.1. *Derived category of dg-modules over a dg-ring*

In this section, we will give the preliminaries on differential graded modules over differential graded rings and we will set up the notation. We give a rather detailed exposition partly because we felt there was a need in the literature for an elementary introduction to Hochschild (co)homology in the dg-setting that relates the derived category of dg-modules and the derived functor approach to the down-to-earth formulas used by the working topologists and algebraists.

This summary is based on the papers [**16**, **25**] and the book [**1**]. In what follows, all complexes will be cohomological, that is, the differentials will raise the degree by 1.

A *differential graded ring* (dg-ring) is a pair $\mathcal{A} = (A, d)$ consisting of a $\mathbb{Z}$-graded ring A together with a differential d of degree 1 that satisfies the Leibniz rules

$$d(ab) = d(a)b + (-1)^{|a|}ad(b)$$

for all homogeneous elements $a, b \in A$. In this paper, we will assume that the dg-rings come endowed with a unit element. The dg-ring A is called (graded) commutative if $ab = (-1)^{|a||b|}ba$. For the rest of this section, $\mathcal{A}$ denotes a dg-ring.

A *differential graded left $\mathcal{A}$-module* (left $\mathcal{A}$-module) consists of a graded left A-module M together with a differential d_M of degree 1 that satisfies the Leibniz rule

$$d_M(ab) = d(a)b + (-1)^{|a|}ad_M(b)$$

for all homogeneous elements $a \in A$ and $b \in M$. A morphism $f : M \to N$ of $\mathcal{A}$-modules is an A-linear map that is homogeneous of degree 0 and commutes with the differentials.

Denote by $\mathcal{A} - \mathrm{mod}$ the category of left $\mathcal{A}$-modules. We denote homomorphisms in this category by

$$\mathrm{Hom}_{\mathcal{A}}(\_\,,\_\,) := \mathrm{Hom}_{\mathcal{A}-\mathrm{mod}}(\_\,,\_\,).$$

The category of graded left A-modules is denoted by $A - \mathrm{mod}$. Morphisms in this category are by definition A-linear maps homogeneous of degree 0. We put

$$\mathrm{Hom}_{A}(\_\,,\_\,) := \mathrm{Hom}_{A-\mathrm{mod}}(\_\,,\_\,).$$

Thus,

$$\mathrm{Hom}_{\mathcal{A}}(M, N) = \{f \in \mathrm{Hom}_A(M, N) | d_N \circ f - f \circ d_M = 0\}.$$

Similarly, there are the categories $\mathrm{mod} - \mathcal{A}$ of right $\mathcal{A}$-modules and $\mathcal{A} - \mathrm{mod} - \mathcal{A}$ of $\mathcal{A}$-bimodules.

The *opposite* dg-ring $\mathcal{A}^o$ of $\mathcal{A}$ is defined to be $\mathcal{A}^o = (A^o, d)$ where its elements are the same ones as in $\mathcal{A}$ and with the same differential, but the multiplication $a \circ b$ is the opposite of the one in A, that is,

$$a \circ b := (-1)^{|a||b|}ba.$$

Note that if $\mathcal{A}'$ is any dg-ring, then $\mathcal{A} \otimes_{\mathbb{Z}} \mathcal{A}'$ is a dg-ring with multiplication $a \otimes a' \cdot b \otimes b' = (-1)^{|a'||b|} ab \otimes a'b'$ and differential $d(a \otimes a') = da \otimes a' + (-1)^{|a'|} a \otimes da'$ for $a, b \in \mathcal{A}$ and $a', b' \in \mathcal{A}'$.

The *shift functor* [1] on $\mathcal{A} - \mathrm{mod}$ is given by shifting the degree of a complex by 1

$$M[1]^k = M^{1+k},$$

and with differential $d_{M[1]} = -d_M$. We therefore have a canonical isomorphism of degree 1

$$s : M \longrightarrow M[1] \quad \text{such that } sd_M x = -d_{M[1]} sx. \tag{A.1}$$

The *tensor product* of a right $\mathcal{A}$-module M and a left $\mathcal{A}$-module N consists of the graded complex of abelian groups $M \otimes_A N$ together with the differential

$$d(m \otimes n) = d_M m \otimes n + (-1)^{|m|} m \otimes d_N n.$$

If A is graded commutative, then M is automatically an $\mathcal{A}$-bimodule by $amb = (-1)^{|m||b|} mab$ for $a, b \in A$ and $m \in M$. Hence, in this case, $M \otimes_{\mathcal{A}} N$ is a left $\mathcal{A}$-module by $a \cdot m \otimes n = (am) \otimes n$.

A homomorphism $f : M \to N$ in $\mathcal{A} - \mathrm{mod}$ is a *quasi-isomorphism* if f induces an isomorphism in cohomology $H^*f : H^*M \cong H^*N$.

A *chain homotopy* between homomorphisms $f_0, f_1 \in \mathrm{Hom}_{\mathcal{A}}(M, N)$ of $\mathcal{A}$-modules is a homomorphism of graded A-modules $k \in \mathrm{Hom}_A(M, N[-1])$ of degree -1 that satisfies

$$f_1 - f_0 = d_M \circ k + k \circ d_N.$$

The *homotopy category* $\mathcal{K}(\mathcal{A})$ of the dg-ring $\mathcal{A}$ has the same objects as $\mathcal{A} - \mathrm{mod}$ and for morphisms has the chain homotopy equivalence classes of morphisms in equivalence classes of morphisms in $\mathcal{A} - \mathrm{mod}$.

The *derived category* $\mathcal{D}(\mathcal{A})$ is the localization of the homotopy category $\mathcal{K}(\mathcal{A})$ with respect to quasi-isomorphisms.

In more sophisticated presentations than ours, one can define a structure of the Quillen model category on $\mathcal{A} - \mathrm{mod}$; see [**12**, **23**]. Although this topic is not discussed here, we use below the model category term 'cofibrant' for objects that essentially do the same work in the dg-world (where objects, for example, the Bar construction, automatically behave like unbounded complexes) as bounded above projective resolutions do in classical homological algebra.

A (left $\mathcal{A}$-)module M is *cofibrant with respect to* $\mathcal{A}$ if there exists an exhaustive increasing filtration by submodules

$$0 = M^0 \subset M^1 \subset M^2 \subset \cdots \subset M^k \subset \cdots \subset M$$

such that each subquotient M^k/M^{k-1} is a direct summand of a direct sum of shifted copies of $\mathcal{A}$. An $\mathcal{A}$-module M is *cofibrant* if it is cofibrant with respect to both $\mathcal{A}$ and the base ring $\mathbb{Z}$. In the case that $\mathcal{A}$ is free over $\mathbb{Z}$, the cofibrant modules are the same as the cofibrant modules with respect to $\mathcal{A}$.

Up to chain homotopy equivalence, the cofibrant modules are the ones that possess Keller's property (P) [**16**]. Every quasi-isomorphism between cofibrant modules is a chain homotopy equivalence and every module can be approximated up to quasi-isomorphism by a cofibrant module.

The derived category $\mathcal{D}(\mathcal{A})$ is equivalent to the full subcategory of $\mathcal{K}(\mathcal{A})$ whose objects are cofibrant $\mathcal{A}$-modules; see [**16**].

If $f : \mathcal{A} \to \mathcal{B}$ is a quasi-isomorphism of dg-rings, then the derived functors of restriction and extension of scalars induce equivalences between the derived categories $\mathcal{D}(\mathcal{A})$ and $\mathcal{D}(\mathcal{B})$.

Let M and N be left $\mathcal{A}$-modules. The *homomorphism complex*

$$\mathcal{H}om_{\mathcal{A}}(M, N)$$

between M and N is the complex defined as follows: in dimension $k \in \mathbb{Z}$, the chain group $\mathcal{H}om_{\mathcal{A}}(M,N)^k$ is the group of graded A-module homomorphisms of degree k, that is,

$$\mathcal{H}om_{\mathcal{A}}(M,N)^k := \mathrm{Hom}_A(M, N[k]),$$

and the differential

$$d : \mathcal{H}om_{\mathcal{A}}(M,N)^k \longrightarrow \mathcal{H}om_{\mathcal{A}}(M,N)^{k+1}$$

is defined by

$$d(f) := d_N \circ f - (-1)^k f \circ d_M.$$

With this definition in mind, the 0-cycles of the homomorphism complex $\mathcal{H}om_{\mathcal{A}}(M,N)$ are precisely the $\mathcal{A}$-module homomorphisms between M and N:

$$\mathcal{Z}^0\mathcal{H}om_{\mathcal{A}}(M,N) = \mathrm{Hom}_{\mathcal{A}}(M,N),$$

and the zeroth cohomology of the complex $\mathcal{H}om_{\mathcal{A}}(M,N)$ is precisely the set of equivalence classes of chain homotopic maps:

$$H^0\mathcal{H}om_{\mathcal{A}}(M,N) = \mathrm{Hom}_{\mathcal{K}(\mathcal{A})}(M,N).$$

For M, N and L left A-modules, the composition of homomorphisms induces a bilinear pairing

$$\mathcal{H}om_{\mathcal{A}}(N,L)^k \times \mathcal{H}om_{\mathcal{A}}(M,N)^l \to \mathcal{H}om_{\mathcal{A}}(M,L)^{k+l} \tag{A.2}$$

that moreover satisfies the Leibniz rule: for $f \in \mathcal{H}om_{\mathcal{A}}(M,N)^l$ and $g \in \mathcal{H}om_{\mathcal{A}}(N,L)^k$ one can check that

$$d(g \circ f) = dg \circ f + (-1)^k g \circ df.$$

From this it follows, in particular, that the endomorphism complex $\mathcal{H}om_{\mathcal{A}}(M,M)$ is a differential graded ring with multiplication the composition of homomorphisms.

A.2. *Derived functors*

For M a left $\mathcal{A}$-module and N a right $\mathcal{A}$-module, the derived tensor $\overset{L}{\otimes}_{\mathcal{A}}$ between N and M is defined as the complex

$$N \overset{L}{\otimes}_{\mathcal{A}} M := N \otimes_{\mathcal{A}} M',$$

where $M' \to M$ is a cofibrant replacement for M. The derived tensor product defines a functor $\overset{L}{\bigotimes}_{\mathcal{A}} \colon \mathcal{D}(\mathcal{A}^o) \times \mathcal{D}(\mathcal{A}) \longrightarrow \mathcal{D}(\mathbb{Z})$.

The Tor-groups between N and M are defined as the cohomology of the derived tensor between N and M,

$$\mathrm{Tor}^*_{\mathcal{A}}(N,M) = H^*(N \overset{L}{\otimes}_{\mathcal{A}} M).$$

Let M and N be two $\mathcal{A}$-modules. The derived functor of $\mathcal{H}om_{\mathcal{A}}$ is the functor $\mathcal{RH}om_{\mathcal{A}}$ which is defined as the complex

$$\mathcal{RH}om_{\mathcal{A}}(M,N) := \mathcal{H}om_{\mathcal{A}}(M',N),$$

where $M' \to M$ is a cofibrant replacement of M. We see that $\mathcal{RH}om$ is well defined up to chain homotopies and it defines a functor $\mathcal{RH}om_{\mathcal{A}}(\ ,\ ) : \mathcal{D}(\mathcal{A}) \times \mathcal{D}(\mathcal{A}) \longrightarrow \mathcal{D}(\mathbb{Z})$.

By definition, we have

$$\mathrm{Ext}^k_{\mathcal{A}}(M,N) := \mathrm{Hom}_{\mathcal{D}(\mathcal{A})}(M, N[k]) = H^k\mathcal{RH}om_{\mathcal{A}}(M,N) = H^k\mathcal{H}om_{\mathcal{A}}(M',N).$$

We have seen that the endomorphism complex

$$\mathcal{H}om_{\mathcal{A}}(M', M')$$

becomes a differential graded ring by the composition of homomorphisms. This in particular implies that the graded Ext-group

$$\mathrm{Ext}^*_{\mathcal{A}}(M, M) = H^*\mathcal{H}om_{\mathcal{A}}(M', M')$$

becomes a graded ring.

A.3. *Hochschild (co) homology for dg-rings*

In this section, we define the Hochschild homology and cohomology for a dg-ring $\mathcal{A}$ and we will list some of its properties.

Consider the dg-ring

$$\mathcal{A}^e := \mathcal{A} \otimes_{\mathbb{Z}} \mathcal{A}^o.$$

Then an $\mathcal{A}$-bimodule is the same thing as a left $\mathcal{A}^e$-module. Let us consider $\mathcal{A}$ as a left $\mathcal{A}^e$-module in the natural way.

In order to define the Hochschild (co)homology, we need to define a replacement of $\mathcal{A}$ in $\mathcal{A}^e$ − mod that is known as the Bar construction.

A.3.1. *Bar construction.* The Bar construction is based on the Bar resolution for modules over rings. We shall use the sign conventions defined in [**8**, **9**] and, in Appendix B, we will show how these sign conventions arise.

For $k \geqslant 0$, let

$$P^{-k} := (\mathcal{A} \otimes \mathcal{A}[1]^{\otimes k} \otimes \mathcal{A}) = \mathcal{A}^{\otimes k+2}[k]$$

be the $\mathcal{A}^e$-module defined in the natural way by

$$(a \otimes b)(x_0|x_1|\cdots|x_{k+1}) = (ax_0|x_1|\cdots|x_{k+1}b),$$

where $a \otimes b \in \mathcal{A}^e$, and $(x_0|x_1|\cdots|x_{k+1})$ denotes the element

$$x_0 \otimes sx_1 \otimes \cdots \otimes sx_n \otimes x_{j+1}$$

in P^{-k} where sx denotes the image in $\mathcal{A}[1]$ of $x \in \mathcal{A}$ under the isomorphism $s : \mathcal{A} \to \mathcal{A}[1]$ induced by the shift functor; see (A.1).

We have that the degree of an element in P^{-k} is

$$|(a_0|\cdots|a_{k+1})| := |a_0| + \cdots + |a_{k+1}| - k,$$

and the differential in P^{-k} becomes

$$\begin{aligned} d(x_0|x_1|\cdots|x_{k+1}) := (dx_0|x_1|\cdots|x_{k+1}) &- \sum_{j=1}^{k}(-1)^{\varepsilon_{j-1}}(x_0|\cdots|dx_j|\cdots|x_{k+1}) \\ &+ (-1)^{\varepsilon_k}(x_0|\cdots|x_k|dx_{k+1}), \end{aligned} \tag{A.3}$$

where

$$\varepsilon_j := |x_0| + |x_1| + \cdots + |x_j| - j$$

denotes the degree of the first $j+1$ elements of $(a_0|\cdots|a_{k+1})$ as an element in $\mathcal{A} \otimes \mathcal{A}[1]^k \otimes \mathcal{A}$.

Define the homomorphism of $\mathcal{A}^e$-modules as

$$\begin{aligned} \delta^{-k} : P^{-k} &\longrightarrow P^{-k+1} \\ (x_0|x_1|\cdots|x_{k+1}) &\longmapsto \sum_{j=0}^{k-1}(-1)^{\varepsilon_j}(x_0|\cdots|x_{j-1}|x_jx_{j+1}|x_{j+2}|\cdots|x_{k+1}) \\ &\qquad - (-1)^{\varepsilon_k}(x_0|\cdots|x_{k-1}|x_kx_{k+1}), \end{aligned}$$

which together with the module homomorphism

$$\begin{aligned}\epsilon : P^0 &\longrightarrow \mathcal{A}\\ (x_0|x_1) &\longmapsto x_0x_1\end{aligned}$$

determines a complex of $\mathcal{A}^e$-modules over

$$\cdots P^{-3} \xrightarrow{\delta^{-3}} P^{-2} \xrightarrow{\delta^{-2}} P^{-1} \xrightarrow{\delta^{-1}} P^0 \xrightarrow{\epsilon} \mathcal{A} \longrightarrow 0,$$

which turns out to be acyclic if we consider it as a complex of modules over A^e.

The Bar construction is the $\mathcal{A}^e$-module

$$B(\mathcal{A}) := \bigoplus_{k=0}^{\infty} P^{-k} = \bigoplus_{k=0}^{\infty} (\mathcal{A} \otimes \mathcal{A}[1]^{\otimes k} \otimes \mathcal{A})$$

with differential

$$\begin{aligned}D : P^{-k} &\longrightarrow P^{-k} \oplus P^{-k+1}\\ D(p) &= d_{P^{-k}}p + \delta_{-k}p\end{aligned}$$

that we shall simply define by $D = d + \delta$.

It is straightforward to check that the differentials d and δ commute as operators, that is, $[d, \delta] = d\delta - (-1)^{|d||\delta|}\delta d = 0$, and therefore we have that $D^2 = 0$.

The sign conventions for the differentials d and δ are obtained by transporting the structure of the usual differentials d and the Bar differential δ from $\bigoplus_k \mathcal{A}^{k+2}$ to $\bigoplus_k (\mathcal{A} \otimes \mathcal{A}[1]^k \otimes \mathcal{A})$; see Appendix B for a proof of this fact.

We extend the map ϵ of (A.4) to a morphism with $\epsilon : B(\mathcal{A}) \to \mathcal{A}$ (the augmentation morphism) by requiring $\epsilon|_{P^{-k}} = 0$ for $k > 0$.

Lemma A.1. *Assume that the dg-ring $\mathcal{A}$ is free over $\mathbb{Z}$. Then $B(\mathcal{A}) \xrightarrow{\epsilon} \mathcal{A}$ is a cofibrant replacement of $\mathcal{A}$.*

A.3.2. *Definition of Hochschild (co) homology.* With the Bar construction at hand, we can define the Hochschild (co)homology groups.

Definition A.2. The Hochschild cohomology ring of a dg-ring $\mathcal{A}$ is the ring

$$HH^*(\mathcal{A}, \mathcal{A}) := H^*\mathcal{H}om_{\mathcal{A}^e}(B(\mathcal{A}), B(\mathcal{A})),$$

where the ring structure is given by the composition of maps, and the Hochschild homology group is given by the graded group

$$HH_*(\mathcal{A}, \mathcal{A}) := H^*(B(\mathcal{A}) \otimes_{\mathcal{A}^e} \mathcal{A}).$$

In the case where $\mathcal{A}$ is free over $\mathbb{Z}$, we know from Lemma A.1 that $B(\mathcal{A})$ is a cofibrant replacement for $\mathcal{A}$ in $\mathcal{A}^e$-mod, and therefore we could alternatively define the Hochschild cohomology as the graded ring

$$HH^*(\mathcal{A}, \mathcal{A}) := \mathrm{Ext}^*_{\mathcal{A}^e}(\mathcal{A}, \mathcal{A}),$$

and the Hochschild homology as the graded group

$$HH_*(\mathcal{A}, \mathcal{A}) := \mathrm{Tor}^*_{\mathcal{A}^e}(\mathcal{A}, \mathcal{A}).$$

A.3.3. *Properties of the Hochschild cohomology.* The Hochschild cohomology can also be calculated using the complex

$$\mathcal{H}om_{\mathcal{A}^e}(B(\mathcal{A}),\mathcal{A}).$$

This complex is better suited for the topological constructions that are done in the rest of the paper in order to relate the homology of the free loops on a manifold and the Hochschild cohomology of the dg-ring of cochains.

We define the product $\phi\cdot\psi$ of $\phi,\psi\in\mathcal{H}om_{\mathcal{A}^e}(B(\mathcal{A}),\mathcal{A})$ by

$$(\phi\cdot\psi)(a_0|\cdots|a_{k+1})=\sum_{j=0}^{k}(-1)^{|\psi|\varepsilon_j}\phi(a_0|\cdots|a_j|1)\psi(1|a_{j+1}|\cdots|a_{k+1}).$$

We will show that this ring structure makes $\mathcal{H}om_{\mathcal{A}^e}(B(\mathcal{A}),\mathcal{A})$ into a dg-ring, which is moreover quasi-isomorphic to the dg-ring $\mathcal{H}om_{\mathcal{A}^e}(B(\mathcal{A}),B(\mathcal{A}))$, where the dg-ring structure on the latter complex is a composition of maps; see (A.2).

Proposition A.3. *The complex $\mathcal{H}om_{\mathcal{A}^e}(B(\mathcal{A}),\mathcal{A})$ is a dg-ring.*

Proof. The product is clearly associative, and moreover it defines the structure of a unitary ring on $\mathcal{H}om_{\mathcal{A}^e}(B(\mathcal{A}),\mathcal{A})$; the unit element of this ring is the augmentation map ϵ.

The proof of the Leibniz rule is a calculation based on the fact that the ring structure can also be obtained from the diagonal map $\Delta: B(\mathcal{A})\to B(\mathcal{A})\otimes_{\mathcal{A}} B(\mathcal{A})$ of $\mathcal{A}^e$-modules via a pullback.

The diagonal map $\Delta: B(\mathcal{A})\to B(\mathcal{A})\otimes_{\mathcal{A}} B(\mathcal{A})$ is defined by

$$\Delta(a_0|\cdots|a_{k+1})=\sum_{j=0}^{k}(a_0|\cdots|a_j|1)\otimes(1|a_{j+1}|\cdots|a_{k+1}). \tag{A.4}$$

For $\phi,\psi\in\mathcal{H}om_{\mathcal{A}^e}(B(\mathcal{A}),\mathcal{A})$, we see that

$$\phi\cdot\psi=\phi\otimes\psi\circ\Delta=\Delta^*(\phi\otimes\psi),$$

where $\phi\otimes\psi: B(\mathcal{A})\otimes_{\mathcal{A}} B(\mathcal{A})\to\mathcal{A}\otimes_{\mathcal{A}}\mathcal{A}=\mathcal{A}$ is given by $a\otimes a'\mapsto(-1)^{|a||\psi|}\phi(a)\psi(a')$. ☐

Let us state some facts whose proofs are not difficult.

Lemma A.4. *Assume that $\mathcal{P}\xrightarrow{\mu}\mathcal{A}$ is a cofibrant replacement of $\mathcal{A}$ in $\mathcal{A}^e-\mathrm{mod}$. Then $\mathcal{P}\otimes_{\mathcal{A}}\mathcal{P}\xrightarrow{\mu\otimes\mu}\mathcal{A}\otimes_{\mathcal{A}}\mathcal{A}=\mathcal{A}$ is also a cofibrant replacement of $\mathcal{A}$ in $\mathcal{A}^e-\mathrm{mod}$.*

Lemma A.5. *The map $\Delta: B(\mathcal{A})\to B(\mathcal{A})\otimes_{\mathcal{A}} B(\mathcal{A})$ is a map of $\mathcal{A}^e$-modules.*

Note that the compositions

$$B(\mathcal{A})\xrightarrow{\Delta}B(\mathcal{A})\otimes_{\mathcal{A}}B(\mathcal{A})\xrightarrow{1\otimes\epsilon}B(\mathcal{A})\otimes_{\mathcal{A}}\mathcal{A}=B(\mathcal{A}),$$
$$B(\mathcal{A})\xrightarrow{\Delta}B(\mathcal{A})\otimes_{\mathcal{A}}B(\mathcal{A})\xrightarrow{\epsilon\otimes 1}\mathcal{A}\otimes_{\mathcal{A}}B(\mathcal{A})=B(\mathcal{A})$$

are the identity and that, by Lemma A.4, $B(\mathcal{A})\otimes_{\mathcal{A}}B(\mathcal{A})\xrightarrow{\epsilon\otimes\epsilon}\mathcal{A}\otimes_{\mathcal{A}}\mathcal{A}=\mathcal{A}$ is a cofibrant replacement for $\mathcal{A}$ (in the category of left $\mathcal{A}^e$-modules). Since the maps $1\otimes\epsilon$ and $(1\otimes\epsilon)\circ\Delta$ are quasi-isomorphisms, also Δ is a quasi-isomorphism.

Now note that since $\epsilon: B(\mathcal{A})\to\mathcal{A}$ is a cofibrant replacement, it induces a quasi-isomorphism of complexes

$$\epsilon_*:\mathcal{H}om_{\mathcal{A}^e}(B(\mathcal{A}),B(\mathcal{A}))\longrightarrow\mathcal{H}om_{\mathcal{A}^e}(B(\mathcal{A}),\mathcal{A}),$$

but note that its kernel is not a right ideal and, consequently, ϵ_* cannot be a ring homomorphism; nevertheless, the properties of the diagonal map Δ give us the following proposition.

PROPOSITION A.6. *The map on cohomology induced by ϵ_* is multiplicative, and hence gives a canonical ring isomorphism between*

$$H^*\mathcal{H}om_{\mathcal{A}^e}(B(\mathcal{A}),\mathcal{A}) \quad \text{and} \quad H^*\mathcal{H}om_{\mathcal{A}^e}(B(\mathcal{A}),B(\mathcal{A})).$$

To complete this section, let us state that the construction performed with the diagonal map for $B(\mathcal{A})$ can be generalized to other cofibrant replacements of $\mathcal{A}$.

PROPOSITION A.7. *Let $\mathcal{P} \xrightarrow{\mu} \mathcal{A}$ be a cofibrant replacement of $\mathcal{A}$ and, furthermore, assume that there exists a homomorphism of $\mathcal{A}^e$-modules*

$$\Delta_{\mathcal{P}} : \mathcal{P} \longrightarrow \mathcal{P} \otimes_{\mathcal{A}} \mathcal{P}$$

such that the compositions

$$\begin{aligned} &\mathcal{P} \xrightarrow{\Delta_{\mathcal{P}}} \mathcal{P} \otimes_{\mathcal{A}} \mathcal{P} \xrightarrow{1\otimes\mu} \mathcal{A} \otimes_{\mathcal{A}} \mathcal{P} = \mathcal{P}, \\ &\mathcal{P} \xrightarrow{\Delta_{\mathcal{P}}} \mathcal{P} \otimes_{\mathcal{A}} \mathcal{P} \xrightarrow{\mu\otimes 1} \mathcal{P} \otimes_{\mathcal{A}} \mathcal{A} = \mathcal{P} \end{aligned} \tag{A.5}$$

are both the identity, and that the map $\Delta_{\mathcal{P}}$ is coassociative. Then the product structure defined by the map

$$\begin{aligned} \mathcal{H}om_{\mathcal{A}^e}(\mathcal{P},\mathcal{A}) \times \mathcal{H}om_{\mathcal{A}^e}(\mathcal{P},\mathcal{A}) &\longrightarrow \mathcal{H}om_{\mathcal{A}^e}(\mathcal{P},\mathcal{A}) \\ \phi \times \psi &\longmapsto \Delta_{\mathcal{P}}^*(\phi \otimes \psi) \end{aligned}$$

makes $\mathcal{H}om_{\mathcal{A}^e}(\mathcal{P},\mathcal{A})$ into an associative dg-ring that induces an associative ring structure on $H^*\mathcal{H}om_{\mathcal{A}^e}(\mathcal{P},\mathcal{A})$. This ring is canonically isomorphic to $HH^*(\mathcal{A},\mathcal{A})$.

Proof. The same argument as in Lemma A.3 shows that $\mathcal{H}om_{\mathcal{A}^e}(\mathcal{P},\mathcal{A})$ is a dg-ring; its associativity follows from the coassociativity of $\Delta_{\mathcal{P}}$. The fact that μ is a unit follows from the compositions of (A.5).

Now, since $B(\mathcal{A})$ and $\mathcal{P}$ are cofibrant, there exists a quasi-isomorphism of $\mathcal{A}^e$-modules $\alpha : B(\mathcal{A}) \to \mathcal{P}$, unique up to homotopy such that $\mu \circ \alpha = \epsilon$. This gives a map $\phi \mapsto \alpha^*(\phi)$ from $\mathcal{H}om_{\mathcal{A}^e}(\mathcal{P},\mathcal{A})$ to $\mathcal{H}om_{\mathcal{A}^e}(B(\mathcal{A}),\mathcal{A})$. Taking cohomology, we get a graded group isomorphism

$$H^*\mathcal{H}om_{\mathcal{A}^e}(\mathcal{P},\mathcal{A}) \longrightarrow H^*\mathcal{H}om_{\mathcal{A}^e}(B(\mathcal{A}),\mathcal{A}), \phi \longmapsto \alpha^*(\phi), \tag{A.6}$$

which is independent of the choice of α.

Now consider the maps

$$\Delta_P \circ \alpha, \alpha \otimes \alpha \circ \Delta : B(\mathcal{A}) \longrightarrow \mathcal{P} \otimes_{\mathcal{A}} \mathcal{P}.$$

We have that $\mu \otimes \mu : \mathcal{P} \otimes_{\mathcal{A}} \mathcal{P} \to \mathcal{A}$ is a cofibrant replacement and that

$$\mu \otimes \mu \circ \Delta_P \circ \alpha = \mu \otimes \mu \circ \alpha \otimes \alpha \circ \Delta = \epsilon \otimes \epsilon.$$

We conclude that $\Delta_P \circ \alpha$ and $\alpha \circ \Delta$ are homotopic. Hence, they induce the same map on cohomology which implies that (A.6) is a ring homomorphism. As $H^*\mathcal{H}om_{\mathcal{A}^e}(B(\mathcal{A}),\mathcal{A})$ is canonically isomorphic to $HH^*(\mathcal{A},\mathcal{A})$, the proposition follows. ☐

Appendix B

In this section, we will explain how the sign conventions arise for the Bar construction of Paragraph A.3.1. The idea consists in transporting the standard differentials of the complex $\bigoplus_k \mathcal{A}^{k+2}$ to the complex $\bigoplus_k (\mathcal{A} \otimes \mathcal{A}[1]^k \otimes \mathcal{A})$ via some chosen isomorphisms.

Let us recall from (A.1) that there is an isomorphism $s : \mathcal{A} \to \mathcal{A}[1]$; $a \mapsto sa$ such that $dsa = -sda$. Define the isomorphism

$$\Phi_k : \mathcal{A}^{k+2} \longrightarrow \mathcal{A} \otimes \mathcal{A}[1]^k \otimes \mathcal{A}$$

as the composition of the maps

$$(\mathrm{Id} \otimes s \otimes \mathrm{Id}^{\otimes k}) \circ \cdots \circ (\mathrm{Id}^{\otimes k-1} \otimes s \otimes \mathrm{Id}^{\otimes 2}) \circ (\mathrm{Id}^{\otimes k} \otimes s \otimes \mathrm{Id}).$$

Because s is an odd map, we get

$$\begin{aligned}&(\mathrm{Id}^{\otimes i} \otimes s \otimes \mathrm{Id}^{\otimes k-i+1})(a_0|\cdots|a_{k+1})\\ &\quad = (-1)^{|a_0|+\cdots+|a_i|}(a_0|\cdots|a_i|sa_{i+1}|a_{i+2}|\cdots|a_{k+1}),\end{aligned}$$

and therefore

$$\Phi_k(a_0|\cdots|a_{k+1}) = (-1)^{\sum_{i=0}^{k-1}(k-i)|a_i|}(a_0|sa_1|\cdots|sa_k|a_{k+1}).$$

The standard differential for the Bar resolution is defined as the sum $\delta^0 + \cdots + \delta^k$ with

$$\begin{aligned}\delta^j &: \mathcal{A}^{k+2} \longrightarrow \mathcal{A}^{k+1}\\ (a_0|\cdots|a_{k+1}) &\longmapsto (-1)^j (a_0|\cdots|a_{j-1}|a_j a_{j+1}|a_{j+2}|\cdots|a_{k+1}).\end{aligned}$$

We will transport these maps δ^j as maps $\mathcal{A} \otimes \mathcal{A}[1]^k \otimes \mathcal{A} \to \mathcal{A} \otimes \mathcal{A}[1]^{k-1} \otimes \mathcal{A}$ to define the differential defined in (A.4).

We have that

$$\begin{aligned}\Phi_k(a_0|\cdots|a_j a_{j+1}|\cdots|a_{k+1}) &= (-1)^{\sum_{i=0}^{j}(k-i-1)|a_i| + \sum_{i=j+1}^{k-1}(k-i)|a_i|}\\ &\quad \times (a_0|sa_1|\cdots|s(a_j a_{j+1})|\cdots|sa_k|a_{k+1}),\end{aligned}$$

which implies that the induced map becomes

$$\begin{aligned}\bar{\delta}^j &: \mathcal{A} \otimes \mathcal{A}[1]^k \otimes \mathcal{A} \longrightarrow \mathcal{A} \otimes \mathcal{A}[1]^{k-1} \otimes \mathcal{A}\\ (a_0|sa_1|\cdots|sa_k|a_{k+1}) &\longmapsto (-1)^{\varepsilon_j}(a_0|sa_1|\cdots|s(a_j a_{j+1})|\cdots|sa_k|a_{k+1}),\end{aligned}$$

which satisfies $\bar{\delta}^j \circ \Phi_k = \Phi_{k-1} \circ \delta^j$, with

$$\varepsilon_j := |a_0| + |a_1| + \cdots + |a_j| - j.$$

Note that, when $j = k$, we get that

$$\bar{\delta}^k(a_0|sa_1|\cdots|sa_k|a_{k+1}) = -(-1)^{\varepsilon_k}(a_0|sa_1|\cdots|sa_{k-1}|a_k a_{k+1}),$$

and therefore the differential δ in the Bar construction $B(\mathcal{A})$ becomes

$$\delta := \bar{\delta}^0 + \cdots + \bar{\delta}^k,$$

which is the one defined in (A.4).

To finalize, let us explain how the differential defined in (A.3) comes from the internal differential of $\mathcal{A} \otimes \mathcal{A}^k \otimes \mathcal{A}$ and the graded commutation of s and d:

$$\begin{aligned}
&d(a_0|sa_1|\cdots|sa_k|a_{k+1})\\
&\quad= (da_0|sa_1|\cdots|sa_k|a_{k+1}) + \sum_{j=1}^{k}(-1)^{\varepsilon_{j-1}}(a_0|sa_1|\cdots|dsa_j|\cdots|sa_k|a_{k+1})\\
&\qquad+ (-1)^{\varepsilon_k}(a_0|sa_1|\cdots|sa_k|da_{k+1})\\
&\quad= (da_0|sa_1|\cdots|sa_k|a_{k+1}) - \sum_{j=1}^{k}(-1)^{\varepsilon_{j-1}}(a_0|sa_1|\cdots|sda_j|\cdots|sa_k|a_{k+1})\\
&\qquad+ (-1)^{\varepsilon_k}(a_0|sa_1|\cdots|sa_k|da_{k+1}).
\end{aligned}$$

Appendix C

Let M be a compact, oriented, differentiable, connected and simply connected manifold. In this section, we resolve the technical problems about the Hochschild cohomology of the singular cochains $C^* := C^*(M)$ that arise from the fact that C^* is not a free $\mathbb{Z}$-module (unless M is a point). This is so because an infinite product of copies of $\mathbb{Z}$ is not free over $\mathbb{Z}$. This problem would disappear if one used coefficients over a field, but we prefer to develop the theory over $\mathbb{Z}$ as it will turn out that, with some effort, it is possible to do so. On the algebraic side, we could solve this problem by considering simplicial cochains S^* instead of singular cochains (having an explicit simplicial decomposition of M). Note, however, that the algebraic object that naturally corresponds to the topological side of this paper is the Hochschild cohomology of C^* and not that of S^*, so we are forced to deal with C^*.

Recall that one has

$$HH^*(C^*, C^*) = H^*\mathcal{H}om_{C^{*e}}(B(C^*), C^*),$$

and that

$$HH^*(S^*, S^*) = H^*\mathcal{H}om_{S^{*e}}(B(S^*), S^*) \cong \mathrm{Ext}_{S^{*e}}(S^*, S^*).$$

The fact that $C^*(M)$ is not free over $\mathbb{Z}$ implies that $B(C^*)$ cannot be cofibrant in the sense of Appendix A. Nevertheless, this quantity can be interpreted as a *relative* Ext-group (see [**29**, Lemma 9.1.3]) and it is even possible to define a model category structure on the category of dg-modules over C^{*e} in which $B(C^*)$ is cofibrant. In principle, all the homological algebra of this paper could be translated to this setup, but it turns out that this is not necessary since we will prove the following theorem.

THEOREM C.1. *There is a canonical isomorphism*

$$HH^*(C^*, C^*) \cong HH^*(S^*, S^*)$$

as graded $\mathbb{Z}$-algebras.

The reason for this to be possible is that C^* is the dual of C_* which is free over $\mathbb{Z}$. We have the following lemma.

LEMMA C.2. *Let (X, d) and (Y, ∂) be (unbounded) complexes of free $\mathbb{Z}$-modules such that $H^*(X) \cong H^*(Y)$. Then X and Y are homotopic and any quasi-isomorphism $q : X \to Y$ is a homotopy equivalence.*

Proof. For every n, we have that $X^n \to \operatorname{Im} d^n$ splits, since $\operatorname{Im} d^n$ is a free $\mathbb{Z}$-module, being a submodule of the free $\mathbb{Z}$-module X^{n+1}. Therefore, $X^n \cong \operatorname{Ker} d^n \oplus \operatorname{Im} d^n$ and the differential $d^n : X^n \to X^{n+1}$ corresponds to $(a, b) \mapsto (b, 0)$.

Let A_n denote the subcomplex $\operatorname{Im} d^{n-1} \hookrightarrow \operatorname{Ker} d^n$ of X. It follows that

$$X \cong \bigoplus_{n\in\mathbb{Z}} A_n.$$

Similarly, $Y \cong \bigoplus_{n\in\mathbb{Z}} B_n$, where B_n is the subcomplex $\operatorname{Im} \partial^n \hookrightarrow \operatorname{Ker} \partial^{n+1}$ of Y.

Since $H^n(A_n) \cong H^n(X) \cong H^n(Y) \cong H^n(B_n)$ and A_n is a projective resolution of $H^n(A_n)[-n]$ and B_n is a projective resolution of $H^n(B_n)[-n]$, we conclude that A_n is homotopic to B_n. Thus, X is homotopic to Y.

For the last assertion, note that each A_n and B_n are cofibrant objects of $C(\mathbb{Z})$. Thus, X and Y are cofibrant as well and hence any quasi-isomorphism between them is a homotopy. □

We fix a finite simplicial decomposition of M on which G acts simplicially and we consider the simplicial decomposition that the barycentric subdivision defines (this is in order to get a simplicial decomposition that induces a G-CW decomposition). If we think of a k-simplex in M as a certain map from Δ_k to M, it is clear how to get an embedding $i_* : S_* \to C_*$ that is a G-equivariant quasi-isomorphism. Moreover, if we denote by $f = i^* : C^* \to S^*$ the corresponding map on cochains, then f is a morphism of dg-algebras. We have the following proposition.

Proposition C.3. *The homomorphism $f : C^* \to S^*$ is a homotopy equivalence.*

Proof. Since S_* and C_* are free $\mathbb{Z}$-modules, Lemma C.2 shows that i_* is a homotopy equivalence. Hence, also $f = i^*$ is a homotopy equivalence. □

We can now give the following proof.

Proof of Theorem C.1. The dg-ring map $f : C^* \to S^*$ induces a C^{*e}-module structure on S^* and a $B(C^*)^e$-linear map

$$\bar{f} : B(C^*) \longrightarrow B(S^*), (c_0|\cdots|c_{k+1}) \longmapsto (f(c_0)|\cdots|f(c_{k+1})).$$

Note that f will not have a multiplicative homotopy inverse and hence that $\bar{f}$ is not a homotopy equivalence.

Nevertheless, we have canonical maps

$$\begin{aligned} \mathcal{H}om_{C^{*e}}(B(C^*), C^*) &\longrightarrow \mathcal{H}om_{C^{e*}}(B(C^*), S^*), \quad \phi \longmapsto f \circ \phi, \\ \mathcal{H}om_{S^{*e}}(B(S^*), S^*) &\longrightarrow \mathcal{H}om_{C^{*e}}(B(C^*), S^*), \quad \phi \longmapsto \phi \circ \bar{f} \end{aligned}$$

that we claim are both quasi-isomorphisms. This would prove Theorem C.1.

To show that each of the previous maps are quasi-isomorphisms, we will use specific spectral sequences on both sides of each map, whose zeroth differentials will avoid the part of the total differential that reflects the multiplication, and that will therefore become isomorphic at the first page. Let us be more explicit.

For $\mathcal{A}$ a dg-ring define $T(\mathcal{A}) = \bigoplus_{i\geqslant 0} \mathcal{A}^{\otimes i}[i]$. Consider the restriction isomorphisms

$$\begin{aligned} \mathcal{H}om_{\mathcal{A}^e}(\mathcal{A}^{\otimes i+2}[i], \mathcal{A}) &\longrightarrow \mathcal{H}om_{\mathbb{Z}}(\mathcal{A}^{\otimes i}[i], \mathcal{A}) \\ f &\longmapsto \{\mathbf{a} \longmapsto f(1 \otimes \mathbf{a} \otimes 1)\}, \end{aligned}$$

whose inverse isomorphism is

$$g \longmapsto \{(a_0|\ldots|a_{i+1}) \longmapsto a_0 g(a_1|\ldots|a_i)a_{i+1}\}.$$

If we use these isomorphisms to transport the differential in $\mathcal{H}om_{\mathcal{A}^e}(B(\mathcal{A}),\mathcal{A})$ to $\mathcal{H}om_{\mathbb{Z}}(T(\mathcal{A}),\mathcal{A})$, we get an isomorphism of these two homomorphism complexes. Explicitly, the differential d on $\mathcal{H}om_{\mathbb{Z}}(T(\mathcal{A}),\mathcal{A})$ is given by

$$df(\mathbf{a}) = d_t(f(\mathbf{a})) - (-1)^{|f|}(\delta f(\mathbf{a}) + f(d_s(\mathbf{a}))),$$

where d_t is the differential of $\mathcal{A}$, the target of the homomorphism, d_s is the differential on the source, which becomes

$$d_s(a_1|\cdots|a_k) = -\sum_{i=1}^{k}(-1)^{\varepsilon_{i-1}}(a_1|\cdots|da_i|\cdots|a_k),$$

and δf is the homomorphism defined as

$$\begin{aligned}\delta f(a_1|\cdots|a_k) = a_1 f(a_2|\cdots|a_k) &+ \sum_{i=2}^{k-1}(-1)^{\varepsilon_i} f(a_1|\cdots|a_i a_{i+1}|\cdots|a_k)\\ &- (-1)^{\varepsilon_k} f(a_1|\cdots|a_{k-1})a_k.\end{aligned}$$

Applying the previous discussion to $\mathcal{A} = C^*$ and $\mathcal{A} = S^*$, we see that it suffices to show that the following two dg-maps are quasi-isomorphisms:

$$\begin{aligned}\mathcal{H}om_{\mathbb{Z}}(T(C^*),C^*) &\longrightarrow \mathcal{H}om_{\mathbb{Z}}(T(C^*),S^*), \quad \phi \longmapsto f\circ\phi,\\ \mathcal{H}om_{\mathbb{Z}}(T(S^*),S^*) &\longrightarrow \mathcal{H}om_{\mathbb{Z}}(T(C^*),S^*), \quad \phi \longmapsto \phi\circ\bar{f}.\end{aligned}$$

Let us prove that the first map is a quasi-isomorphism.

Lemma C.4. *The map*

$$\Phi : \mathcal{H}om_{\mathbb{Z}}(T(C^*),C^*) \longrightarrow \mathcal{H}om_{\mathbb{Z}}(T(C^*),S^*), \quad \phi \longmapsto f\circ\phi$$

is a quasi-isomorphism.

Proof. Consider the filtrations

$$F_k = \mathcal{H}om_{\mathbb{Z}}\left(\bigoplus_{i=k}^{\infty} C^{*\otimes i}[i], C^*\right) \quad \bar{F}_k = \mathcal{H}om_{\mathbb{Z}}\left(\bigoplus_{i=k}^{\infty} C^{*\otimes i}[i], S^*\right)$$

of $\mathcal{H}om_{\mathbb{Z}}(T(C^*),C^*)$ and $\mathcal{H}om_{\mathbb{Z}}(T(C^*),S^*)$, respectively, which are compatible with the homomorphism Φ.

The zeroth page of the spectral sequences associated to F_* and $\bar{F}_*$ are the complexes

$$\begin{aligned}E_0 &= \bigoplus_{k=0}^{\infty} F_k/F_{k+1} = \bigoplus_{k=0}^{\infty} \mathcal{H}om_{\mathbb{Z}}(C^{*\otimes k}[k], C^*),\\ \bar{E}_0 &= \bigoplus_{k=0}^{\infty} \bar{F}_k/\bar{F}_{k+1} = \bigoplus_{k=0}^{\infty} \mathcal{H}om_{\mathbb{Z}}(C^{*\otimes k}[k], S^*),\end{aligned}$$

whose zeroth differentials are

$$\begin{aligned}d^0\phi(\mathbf{a}) &= d_{C^*}(\phi(\mathbf{a})) - (-1)^{|\phi|}\phi(d_s(\mathbf{a})),\\ \bar{d}^0\bar{\phi}(\mathbf{a}) &= d_{S^*}(\bar{\phi}(\mathbf{a})) - (-1)^{|\bar{\phi}|}\bar{\phi}(d_s(\mathbf{a})),\end{aligned}$$

where d_s is the internal differential of the complex $C^{*\otimes k}[k]$.

Applying the canonical isomorphisms

$$\begin{aligned}\mathcal{H}om_{\mathbb{Z}}(C^{*\otimes k}[k], C^*) &\cong \mathcal{H}om_{\mathbb{Z}}(C^{*\otimes k}[k]\otimes C_*, \mathbb{Z}),\\ \mathcal{H}om_{\mathbb{Z}}(C^{*\otimes k}[k], S^*) &\cong \mathcal{H}om_{\mathbb{Z}}(C^{*\otimes k}[k]\otimes S_*, \mathbb{Z}),\end{aligned}$$

we see that the induced differentials on the right-hand side are just the canonical differentials given by the dualization of the tensor product of d_s with ∂_{C_*} and of d_s with ∂_{S_*}, respectively, where ∂ denotes the differential at the chains level.

Because the map $i : S_* \to C_*$ induces a homotopy equivalence, we see that the induced map on the zeroth pages $\Phi : E_0 \to \bar{E}_0$ is a quasi-isomorphism, and therefore the induced homomorphism on the first pages becomes an isomorphism $\Phi : E_1 \overset{\cong}{\to} \bar{E}_1$. We have therefore that the spectral sequences are isomorphic after the first page and moreover that the filtrations are both Hausdorff and weakly convergent.

The convergence of the spectral sequences tells us that Φ induces an isomorphism between the inverse limits (see [**21**, Theorem 3.9])

$$\begin{aligned}\Phi : &\lim_{\leftarrow k}(H^*\mathcal{H}om_{\mathbb{Z}}(T(C^*), C^*)/\mathrm{Im}(H^*F_k \longrightarrow H^*\mathcal{H}om_{\mathbb{Z}}(T(C^*), C^*)))\\ &\overset{\cong}{\to} \lim_{\leftarrow k}(H^*\mathcal{H}om_{\mathbb{Z}}(T(C^*), S^*)/\mathrm{Im}(H^*\bar{F}_k \longrightarrow H^*\mathcal{H}om_{\mathbb{Z}}(T(C^*), S^*))),\end{aligned} \tag{C.1}$$

but as the cohomology groups $H^*\mathcal{H}om_{\mathbb{Z}}(T(C^*), C^*)$ and $H^*\mathcal{H}om_{\mathbb{Z}}(T(C^*), S^*)$ are both graded groups, and in each degree they are finitely generated (as we know that $H^*\mathcal{H}om_{\mathbb{Z}}(T(C^*), C^*)$ is isomorphic to the homology of $\mathcal{L}M^{-TM}$), we have that (C.1) implies that there is a Φ that induces an isomorphism at the level of the cohomologies

$$\Phi : H^*\mathcal{H}om_{\mathbb{Z}}(T(C^*), C^*) \overset{\cong}{\to} H^*\mathcal{H}om_{\mathbb{Z}}(T(C^*), S^*).$$

This completes the proof of Lemma C.4. □

Let us prove that the second map is a quasi-isomorphism.

Lemma C.5. *The map*

$$F : \mathcal{H}om_{\mathbb{Z}}(T(S^*), S^*) \longrightarrow \mathcal{H}om_{\mathbb{Z}}(T(C^*), S^*), \quad \phi \longmapsto \phi \circ \bar{f}$$

is a quasi-isomorphism.

Proof. Consider the following double filtrations:

$$P^{j.k} = \mathcal{H}om_{\mathbb{Z}}\left(\bigoplus_{i\geqslant k} S^{*\otimes i}, S^{\geqslant j}\right),$$

$$\bar{P}^{j.k} = \mathcal{H}om_{\mathbb{Z}}\left(\bigoplus_{i\geqslant k} C^{*\otimes i}, S^{\geqslant j}\right),$$

and note that the filtrations are compatible with the map F, and that the differentials δ and $d_t = d_{S^*}$ raise the degree, namely, we have that

$$\begin{aligned}&\delta : P^{j,k} \longrightarrow P^{j,k+1}, \quad d_t : P^{j,k} \longrightarrow P^{j+1,k}, \quad \delta : \bar{P}^{j,k} \longrightarrow \bar{P}^{j,k+1}\\ &\text{and } d_t : \bar{P}^{j,k} \longrightarrow \bar{P}^{j+1,k}.\end{aligned}$$

Let us take the filtrations defined by these double filtrations

$$Q^r = \sum_{k+j=r} P^{j,k} \quad \bar{Q}^r = \sum_{k+j=r} \bar{P}^{j,k},$$

noting that both δ and d_t raise the degree $\delta, d_t : Q^r \to Q^{r+1}$. Therefore, the associated spectral sequences E_* and $\bar{E}_*$ are compatible via the map F defined by

$$F : E_* \longrightarrow \bar{E}_*,$$

and on the zeroth pages of both spectral sequences we get the associated graded complexes

$$E_0 \cong \bigoplus_{r=0}^{\infty} \bigoplus_{k+j=r} \mathcal{H}om_{\mathbb{Z}}(S^{*\otimes k}, S^j),$$

$$\bar{E}_0 \cong \bigoplus_{r=0}^{\infty} \bigoplus_{k+j=r} \mathcal{H}om_{\mathbb{Z}}(C^{*\otimes k}, S^j).$$

The zeroth differential d^0 on the group $\mathcal{H}om_{\mathbb{Z}}(S^{*\otimes k}, S^j)$ becomes the differential obtained by pre-composing with the internal differential of the source $S^{*\otimes k}$:

$$(d^0\phi)(a_1|\cdots|a_k) = (-1)^{|\phi|}\phi(d(a_0|\cdots|a_k))$$

(the same happens with the zeroth differential $\bar{d}^0$ of $\bar{E}_0$), and therefore we have that the first levels of the spectral sequences become the sum of the cohomologies of the duals of $S^{\otimes k}$ and $C^{\otimes k}$, respectively, tensored with S^*

$$E_1 \cong \bigoplus_{r=0}^{\infty} H^*((S^{*\otimes r})^\vee) \otimes S^*,$$
$$\bar{E}_1 \cong \bigoplus_{r=0}^{\infty} H^*((C^{*\otimes r})^\vee) \otimes S^*,$$

as we know that S^* is a finitely generated free $\mathbb{Z}$-module.

The map $F : E_1 \to \bar{E}_1$ at the first level is clearly an isomorphism as the map $f : C^* \to S^*$ induces a quasi-isomorphism at the level of the tensor products $C^{*\otimes k} \xrightarrow{\simeq} S^{*\otimes k}$ and their duals $(S^{*\otimes k})^\vee \xrightarrow{\simeq} (C^{*\otimes k})^\vee$.

We have now that F becomes an isomorphism at the first level of the spectral sequences. We also have that the filtration given by Q^r is Hausdorff; $\bigcap_r Q^r = \{0\}$, therefore weakly convergent, and applying the same argument as in the previous lemma, namely that the cohomologies

$$H^*\mathcal{H}om_{\mathbb{Z}}(T(S^*), S^*) \quad \text{and} \quad H^*\mathcal{H}om_{\mathbb{Z}}(T(C^*), S^*)$$

are both graded rings, we have that the isomorphism of the spectral sequences implies that the cohomologies

$$H^*\mathcal{H}om_{\mathbb{Z}}(T(S^*), S^*) \quad \text{and} \quad H^*\mathcal{H}om_{\mathbb{Z}}(T(C^*), S^*)$$

are isomorphic. Therefore, F induces a q.i. on the complexes

$$\mathcal{H}om_{\mathbb{Z}}(T(S^*), S^*) \xrightarrow{\simeq} \mathcal{H}om_{\mathbb{Z}}(T(C^*), S^*). \qquad \square$$

This completes the proof of Theorem C.1. □

Using the notation of (2.6), we see that Theorem C.1 implies that there are canonical isomorphisms

$$H^*\mathcal{H}om_{C^{*e}}(B(C^*), C^*_g) \cong H^*\mathcal{H}om_{S^{*e}}(B(S^*), S^*_g) = \mathrm{Ext}_{S^{*e}}(S^*, S^*_g),$$
$$H^*\mathcal{H}om_{C^{*e}}(B(C^*), C^*\#G) \cong H^*\mathcal{H}om_{S^{*e}}(B(S^*), S^*\#G) = \mathrm{Ext}_{S^{*e}}(S^*, S^*\#G)$$

that induce a canonical isomorphism between the cohomologies

$$H^*\mathcal{H}om_{\mathbb{Z}G}(\bar{B}(\mathbb{Z}G), \mathcal{H}om_{C^{*e}}(B(C^*), C^*\#G))$$
$$\cong H^*\mathcal{H}om_{\mathbb{Z}G}(\bar{B}(\mathbb{Z}G), \mathcal{H}om_{S^{*e}}(B(S^*), S^*\#G)).$$

Therefore, we can conclude the following corollary.

Corollary C.6. *There is a canonical ring isomorphism*

$$HH^*(C^*\#G, C^*\#G) \cong HH^*(S^*\#G, S^*\#G)$$

that can also be seen as a ring isomorphism

$$HH^*(C^*\#G, C^*\#G) \cong \mathrm{Ext}_{S^*\#G^e}(S^*\#G, S^*\#G).$$

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References

1. J. Bernstein and V. Lunts, *Equivariant sheaves and functors*, Lecture Notes in Mathematics 1578 (Springer, Berlin, 1994).

2. R. Bott and G. Segal, 'The cohomology of the vector fields on a manifold', *Topology* 16 (1977) 285–298.

3. M. Chas and D. Sullivan, 'String topology', Preprint, 1999, arXiv:math/9911159v1.

4. R. L. Cohen, 'Multiplicative properties of Atiyah duality', *Homology Homotopy Appl.* 6 (2004) 269–281.

5. R. L. Cohen and J. D. S. Jones, 'A homotopy theoretic realization of string topology', *Math. Ann.* 324 (2002) 773–798.

6. W. G. Dwyer, 'Strong convergence of the Eilenberg–Moore spectral sequence', *Topology* 13 (1974) 255–265.

7. S. Eilenberg and J. A. Zilber, 'On products of complexes', *Amer. J. Math.* 75 (1953) 200–204.

8. Y. Félix, S. Halperin and J.-C. Thomas, *Rational homotopy theory*, Graduate Texts in Mathematics 205 (Springer, New York, 2001).

9. Y. Félix, J.-C. Thomas and M. Vigué-Poirrier, 'Rational string topology', *J. Eur. Math. Soc. (JEMS)* 9 (2007) 123–156.

10. K. Gruher and P. Salvatore, 'Generalized string topology operations', *Proc. London Math. Soc.* (3) 96 (2008) 78–106.

11. K. Gruher and C. Westerland, 'String topology prospectra and Hochschild cohomology', *J. Topol.* 1 (2008) 837–856.

12. M. Hovey, 'Model categories', Mathematical Surveys and Monographs 63 (American Mathematical Society, Providence, RI, 1999).

13. M. Hovey, 'Spectra and symmetric spectra in general model categories', *J. Pure Appl. Algebra* 165 (2001) 63–127.

14. S. Illman, 'Smooth equivariant triangulations of G-manifolds for G a finite group', *Math. Ann.* 233 (1978) 199–220.

15. J. D. S. Jones, 'Cyclic homology and equivariant homology', *Invent. Math.* 87 (1987) 403–423.

16. B. Keller, 'Deriving DG categories', *Ann. Sci. École Norm. Sup.* (4) 27 (1994) 63–102.

17. B. Keller, 'Hochschild cohomology and derived Picard groups', *J. Pure Appl. Algebra* 190 (2004) 177–196.

18. E. Lupercio and B. Uribe, 'Loop groupoids, gerbes, and twisted sectors on orbifolds', *Orbifolds in mathematics and physics (Madison, WI, 2001)*, Contemporary Mathematics 310 (American Mathematical Society Providence, RI, 2002) 163–184.

19. E. Lupercio, B. Uribe and M. A. Xicoténcatl, 'Orbifold string topology', *Geom. Topol.* 12 (2008) 2203–2248.

20. J. P. May, *Equivariant homotopy and cohomology theory*, CBMS Regional Conference Series in Mathematics 91 (Published for the Conference Board of the Mathematical Sciences, Washington, DC, 1996). With contributions by M. Cole, G. Comezaña, S. Costenoble, A. D. Elmendorf, J. P. C. Greenlees, L. G. Lewis, Jr., R. J. Piacenza, G. Triantafillou and S. Waner.

21. J. McCleary, *A user's guide to spectral sequences*, 2nd edn, Cambridge Studies in Advanced Mathematics 58 (Cambridge University Press, Cambridge, 2001).

22. I. Moerdijk, 'Orbifolds as groupoids: an introduction', *Orbifolds in mathematics and physics (Madison, WI, 2001)*, Contemporary Mathematics 310 (American Mathematical Society, Providence, RI, 2002) 205–222.

23. D. G. Quillen, *Homotopical algebra*, Lecture Notes in Mathematics 43 (Springer, Berlin, 1967).

24. K. Sanada, 'On the Hochschild cohomology of crossed products', *Comm. Algebra* 21 (1993) 2727–2748.

25. S. Schwede, 'Morita theory in abelian, derived and stable model categories', *Structured ring spectra*, London Mathematical Society Lecture Note Series 315 (Cambridge University Press, Cambridge, 2004) 33–86.

26. G. Segal, 'Categories and cohomology theories', *Topology* 13 (1974) 293–312.

27. B. Shipley, '$H\mathbb{Z}$-algebra spectra are differential graded algebras', *Amer. J. Math.* 129 (2007) 351–379.

28. A. G. Wasserman, ‘Equivariant differential topology’, *Topology* 8 (1969) 127–150.
29. C. A. Weibel, *An introduction to homological algebra*, Cambridge Studies in Advanced Mathematics 38 (Cambridge University Press, Cambridge, 1994).
30. S. J. Witherspoon, ‘Products in Hochschild cohomology and Grothendieck rings of group crossed products’, *Adv. Math.* 185 (2004) 136–158.

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