The BV-Algebra Structure of the Hochschild Cohomology of the Group Ring of Cyclic Groups of Prime Order

Andrés Angel, Diego Duarte

September 15, 2015

1 Abstract

We construct a different Batalin-Vilkovisky algebra structure on the Hochschild cohomology of the algebra of truncated polynomials with coefficients in a field of p elements with p prime. We accomplish this by transfering the Frobenius form of the group ring of cyclic groups. We also explicitly calculate the Batalin-Vilkovisky algebra structure of the Hochschild cohomology of the group rings of cyclic groups of prime order. The algebra structure, even the Gesterhaber structure has been calculated before, but the BV structure that we calculate is a new one.

2 Introduction

The aim of these notes is to present the calculations of the Hochschild cohomology of truncated polynomials and group rings of cyclic groups. The Hochschild cohomology of truncated polynomials $HH^*(\mathbb{K}[x]/(x^n); \mathbb{K}[x]/(x^n))$ over commutative rings has been studied extensively as well as the Hochschild cohomology of group rings $HH^*(\mathbb{K}[G]; \mathbb{K}[G])$. In this article we will concentrate on calculations over $\mathbb{F}_p$, the field with p elements for p a prime.

It is possible to endow with many algebraic structures the Hochschild cohomology of an associative algebra A. It is a graded algebra given by the cup product. In [9], Gerstenhaber proves that the cup product in Hochschild cohomology is graded commutative and even more that there exists a Lie bracket that endows $HH^*(A; A)$ with a structure of Lie algebra, these two structures satisfy some compatibility conditions that are now known to define a *Gerstenhaber algebra*.

When the algebra A satisfies some sort of Poincaré duality the Hochschild cohomology has a richer algebraic structure. Tradler in [19] proves that if A is a symmetric algebra up to homotopy then $HH^*(A;A)$ is a *Batalin-Vilkovisky* algebra.

Menichi in [13] presents another proof for Tradler's result for symmetric differential graded algebras (dga A with a degree d quasi-isomorphism of A-bimodules $\Theta : A \to A^\vee$). The idea is to use an isomorphism from the algebra to its dual to transfer the Connes boundary operator on Hochschild homology to Hochschild cohomology.

We are interested in the algebra structure and the BV-structure on the Hochschild cohomology of truncated polynomial rings because of the relation with string topology.

Let M be a closed, connected, oriented manifold of dimension n. In [3], Chas and Sullivan defined a Batalin-Vilkovisky algebra structure on the re-graded homology of the loop space of M, $H_{*+n}(LM)$. These algebraic operations were defined at the level of chains using the intersection theory of the manifold and the concatenation and rotation of loops.

In [6], Cohen and Jones reinterpreted these operations at the level of stable homotopy theory and constructed an isomorphism of graded algebras between the homology of the loop space and the Hochschild cohomology of the singular cochains of the manifold

$$H_{*+n}(LM;\mathbb{K})) \cong HH^*(C^*(M;\mathbb{K});C^*(M;\mathbb{K}))$$

for M a closed, simply connected manifold.

Given that both sides admit the structure of BV-algebras it was expected that these BV-structures were isomorphic.

For spheres and complex projective spaces the algebra structure on the homology of the loop space was calculated in [5] by Cohen, Jones and Yan. In [22], C. Westerland calculates the BV-structure of $HH^*(C^*(\mathbb{S}^n,\mathbb{F}_2);C^*(\mathbb{S}^n,\mathbb{F}_2))$ and $HH^*(C^*(\mathbb{CP}^n;\mathbb{F}_2);C^*(\mathbb{CP}^n;\mathbb{F}_2))$.

In [15], Menichi calculates the BV-structures of $HH^*(C^*(\mathbb{S}^n,\mathbb{F}_2);C^*(\mathbb{S}^n,\mathbb{F}_2))$ and $H_{*+2}(LS^2;\mathbb{F}_2)$ and shows that the two BV-algebra structures are not isomorphic, even though the underlying Gesternhaber structures are isomorphic.

The calculations of Westerland and Menichi of $HH^*(C^*(M),C^*(M))$ for the cochains of spheres and complex projective spaces are based on the fact that these manifolds are formal and the cohomology rings are graded truncated polynomial rings.

According to Félix, Menichi and Thomas [7], when M is a formal manifold over a field $\mathbb{K}$, there is an isomorphism of Gerstenhaber algebras

$$HH^*(C^*(M,\mathbb{K}); C^*(M,\mathbb{K})) \cong HH^*(H^*(M;\mathbb{K}); H^*(M;\mathbb{K}))$$

And since we know that the cohomology of these spaces are truncated polynomial rings

$$H^*(\mathbb{CP}^n;\mathbb{F}_2) \cong \mathbb{F}_2[x]/(x^{n+1}) \quad \text{with} \quad |x| = 2$$
$$H^*(S^n;\mathbb{F}_2) \cong \mathbb{F}_2[x]/(x^2) \quad \text{with} \quad |x| = n$$

we can use a periodic resolution for calculations ([11], [18]). In [20], Yang uses this resolution to calculate the BV-structures of the Hochschild cohomology of truncated polynomials over $\mathbb{F}_p$. These BV-structures come from the Frobenius form induced by Poincare duality $\varepsilon\left(\sum_{i=0}^n \alpha_i x^i\right) = \alpha_n$ and $\varepsilon(a+bx) = b$ respectively.

The Gesternhaber structure only depends on the algebra structure but the BV-structure depends on the duality isomorphism, by using a different Frobenius form coming from the group ring we get a different BV-structure with the same underlying Gesternhaber algebra.

For the cyclic group of order p, $\mathbb{Z}/p\mathbb{Z}$, we have that the group ring $\mathbb{F}_p[\mathbb{Z}/p\mathbb{Z}] = \mathbb{F}_p[\sigma]/(\sigma^p - 1)$ is naturally isomorphic, as algebra, to a truncated polynomial ring $\mathbb{F}_p[x]/(x^p)$. By transferring the Frobenius form of the group ring to the truncated polynomial ring we get the following Frobenius form

$$\tilde{\varepsilon}\left(\sum_{i=0}^{p-1} \alpha_i x^i\right) = \sum_{i=0}^{p-1} (-1)^i \alpha_i$$

The new results in these notes are:

Theorem 2.1. *Let $A = \mathbb{F}_p[x]/(x^p)$ with p an odd prime and $|x| = 0$. Then $HH^*(A;A)$ has a BV-structure induced by the canonical Frobenius form of the group ring defined as follows*

$$HH^*(A;A) = \mathbb{F}_p[x,v,t]/(x^p, v^2)$$

$$\Delta(t^l x^k) = 0$$
$$\Delta(t^k v x^{2l}) = 2lt^k x^{2l-1} + \sum_{i=2l}^{p-1} (-1)^{i+1} t^k x^i$$
$$\Delta(t^k v x^{2l+1}) = (2l+1)t^k x^{2l} + \sum_{i=2l+1}^{p-1} (-1)^i t^k x^i$$

where $x \in HH^0(A;A)$, $v \in HH^1(A;A)$ and $t \in HH^2(A;A)$ with $|x| = 0$, $|v| = 1$ and $|t| = 2$.

Theorem 2.2. *Let $A = \mathbb{F}_2[x]/(x^2)$ with $|x| = 0$. Then $HH^*(A;A)$ has a BV-structure induced by the canonical Frobenius form of the group ring defined as follows*

$$HH^*(A;A) = \mathbb{F}_2[x,v,t]/(x^2, v^2 - t) \cong \Lambda(x) \otimes \mathbb{F}_2[v]$$
$$\Delta(v^k x^l) = k(1+x)v^{k-1}$$

where $x \in HH^0(A;A)$, $v \in HH^1(A;A)$ with $|x| = 0$ and $|v| = 1$.

3 Hochschild (Co)homology

Definition 3.1. Let A be a graded commutative R-algebra with R a commutative ring, suppose that $A = R\langle e\rangle \oplus \bar{A}$. Let

$$T(s\bar{A}) = \bigoplus_{k\geq 0} (s\bar{A})^{\otimes k}$$

be the universal tensor algebra of $s\bar{A}$, where we denote by sA the suspension of A, i.e., $(sA)_i = A_{i+1}$, so for $x \in A$ we have $|sx| = |x| - 1$.

The *two-sided normalized bar resolution* of A is the differential graded (A,A)-bimodule $\bar{B}(A) = A \otimes T(s\bar{A}) \otimes A$ defined as follows

- For $k \in \mathbb{N}$ let
$$\bar{B}_k(A) := A \otimes (s\bar{A})^{\otimes k} \otimes A$$
for an element $a_0\,[a_1|\cdots|a_k]\,a_{k+1} \in \bar{B}_k(A)$ of Hochschild degree k and total degree $\sum_{i=0}^{k+1} |a_i|$. We define the *internal degree* as $|a_0| + \sum_{i=1}^{k} |sa_i| + |a_{k+1}| = \sum_{i=0}^{k+1} |a_i| - k$ for $k \geq 1$ and we have
$$Internal\ degree = Total\ degree - Hochschild\ degree$$
We also define $a\,[\,]\,a' = a \otimes 1 \otimes a' \in A \otimes R \otimes A$ for $k = 0$.

- The differential $d : \bar{B}_*(A) \to \bar{B}_{*-1}(A)$ is defined by
$$\begin{aligned} d_k : \bar{B}_k(A) &\longrightarrow \bar{B}_{k-1}(A) \\ a_0\,[a_1|\cdots|a_k]\,a_{k+1} &\longmapsto (-1)^{|a_0|} a_0 a_1\,[a_2|\cdots|a_k]\,a_{k+1} \\ &\quad + \sum_{i=2}^{k} (-1)^{\epsilon_i} a_0\,[a_1|\cdots|a_{i-1}a_i|\cdots|a_k]\,a_{k+1} \\ &\quad - (-1)^{\epsilon_k} a_0\,[a_1|\cdots|a_{k-1}]\,a_k a_{k+1} \end{aligned}$$
where $\epsilon_i = |a_0| + \sum_{j<i} |sa_j|$. This differential lowers the Hochschild degree by one and preserves the total degree since A is a graded algebra.

Definition 3.2. Denote by A^{op} the opposite algebra of A and by A^e the enveloping algebra $A \otimes A^{op}$. Recall that any (A, A)-bimodule can be considered as a left (or right) A^e-module. Let M be a graded (A, A)-bimodule. The *Hochschild chain complex* is the (graded) tensor product $C_*(A; M) = M \otimes_{A^e} \bar{B}(A)$ with

$$C_n(A; M) = \bigoplus_{p+q=n} M_p \otimes_{A^e} \bar{B}_q(A)$$

and differential the (graded) tensor $D = id_M \otimes d$. The *Hochschild homology* with coefficients in M is the homology of the Hochschild chain complex

$$HH_*(A; M) := H(C_*(A; M), D)$$

The *Hochschild cochain complex* is the complex

$$C^*(A; M) = \underline{Hom}_{A^e}(\bar{B}_*(A), M) = \underline{Hom}(T(s\bar{A}), M)$$

and differential D defined by $Df = -(-1)^{|f|} f \circ d$. $\underline{Hom}$ denotes the internal Hom in the category of graded modules, i.e for U, V graded modules, the maps from U to V of homogeneous degree k are:

$$\underline{Hom}^k(U, V) = \prod_m Hom(U^m, V^{m+k})$$

So,

$$C^n(A; M) = \bigoplus_k \prod_m Hom(\left(s\bar{A}^{\otimes n}\right)^m, M^{m+k})$$

The *Hochschild cohomology of A with coefficients in M* is the homology of the Hochschild cochain complex

$$HH^*(A; M) := H(C^*(A; M), D)$$

Remark 1. For an n-cochain $\phi \in \prod_m Hom(\left(s\bar{A}^{\otimes n}\right)^m, M^{m+k})$, we say that the internal degree is k, $|\phi| = k$ and we have $|\phi(a_0 [a_1| \cdots |a_n] a_{n+1})| = |\phi| +$ internal degree of $a_0 [a_1| \cdots |a_n] a_{n+1}$, which is $|\phi| + \sum_{i=0}^{n+1} |a_i| - n$. The differential D raises the Hochschild degree by one and since

$$|D\phi(a_0 [a_1| \cdots |a_{n+1}] a_{n+2})| = |\phi(d((a_0 [a_1| \cdots |a_{n+1}] a_{n+2})|$$

this is $|\phi|$ + internal degree of $d((a_0 [a_1| \cdots |a_{n+1}] a_{n+2})$ which is $|\phi|$ + total degree $d((a_0 [a_1| \cdots |a_{n+1}] a_{n+2}) - n$. Since d preserves the total degree, we have $|\phi|$ + total degree $a_0 [a_1| \cdots |a_{n+1}] a_{n+2} - n$, so

$$|D\phi(a_0 [a_1| \cdots |a_{n+1}] a_{n+2})| = |\phi| + \sum_{i=0}^{n+2} |a_i| - n$$

which shows that $|D\phi| = |\phi| + 1$. We define the total degree of ϕ by $|\phi| - n$; therefore,

$$Total\ degree = Internal\ degree - Hochschild\ degree$$

and D preserves the total degree.

Remark 2. Hochschild (co)homology can also be defined as derived functors

$$HH_*(A; M) := Tor_*^{A^e}(M; A)$$
$$HH^*(A; M) := Ext_{A^e}^*(A; M)$$

If we take $M = A$ the Hochschild cohomology $HH^*(A; A)$ has a graded algebra structure induced from the cup product

$$(\phi \smile \psi)(a_0 [a_1| \dots |a_k] a_{k+1}) = \sum_{j=0}^{k} (-1)^{|\psi|\varepsilon_j} \phi(a_0 [a_1| \dots |a_j] 1) \psi(1 [a_{j+1}| \dots |a_k] a_{k+1})$$

It also preserves the internal degree, i.e $|\phi \smile \psi| = |\phi| + |\psi|$. For convenience we will write $\phi\psi$ instead of $\phi \smile \psi$.

In [9], Gerstenhaber proves that the cup product on Hochschild cohomology is graded commutative and that there exists a Lie bracket that endows $HH^*(A; A)$ with a structure of Lie algebra.

The Gerstenhaber bracket on $HH^*(A; A)$ is defined as follows

$$\{\phi, \psi\} = \phi \circ \psi - (-1)^{(|\phi|-1)(|\psi|-1)} \psi \circ \phi$$

where $\circ$ is defined by

$$(\phi \circ \psi)(a_0 [a_1| \dots |a_k] a_{k+1}) = \sum_{j=0}^{|\phi|-1} (-1)^{j(|\psi|-1)} \phi \left(a_0 \left[a_1| \dots |a_j| \psi(a_{j+1} \left[a_{j+2}| \dots |a_{j+|\psi|-1}\right] a_{j+|\psi|}) | \dots |a_k\right] a_{k+1}\right)$$

The cup product and the Lie bracket satisfy the following compatibility conditions.

Definition 3.3. A *Gerstenhaber algebra* is a graded commutative algebra A with a linear map $\{-, -\} : A_i \otimes A_j \to A_{i+j-1}$ of degree -1 such that

1. The bracket $\{-, -\}$ endows A with a structure of graded Lie algebra of degree 1, i.e., for all a, b and $c \in A$

$$\{a, b\} = -(-1)^{(|a|+1)(|b|+1)} \{b, a\}$$
$$\{a, \{b, c\}\} = \{\{a, b\}, c\} + (-1)^{(|a|+1)(|b|+1)} \{b, \{a, c\}\}$$

2. The product and the Lie bracket satisfy the Poisson identity, i.e., for all a, b and $c \in A$
$$\{a, bc\} = \{a, b\}\, c + (-1)^{(|a|+1)|b|} b\, \{a, c\}$$

If there is a differential of degree -1 of a Gerstenhaber algebra such that the Gerstenhaber bracket is the obstruction of the operator to be a graded derivation, then the Gerstenhaber algebra is called a Batalin-Vilkovisky algebra.

Definition 3.4. A *Batalin-Vilkovisky* algebra is a Gerstenhaber algebra A with a linear map of degree -1, $\Delta : A_i \to A_{i-1}$ such that $\Delta \circ \Delta = 0$ and

$$\{a, b\} = (-1)^{|a|}(\Delta(ab) - \Delta(a)b - (-1)^{|a|} a\Delta(b))$$

for all a and $b \in A$.

The way to construct BV-structures on Hochschild cohomology is to dualize the Connes operator on the Hochschild chains. Recall that the Hochschild homology is the homology of the complex $A \otimes_{A^e} \bar{B}(A) \cong A \otimes T(s\bar{A})$. The *Connes boundary operator*
$$B : A \otimes T(s\bar{A}) \to A \otimes T(s\bar{A})$$
is defined on the n-chains, $B : A \otimes (s\bar{A})^{\otimes n} \to A \otimes (s\bar{A})^{\otimes n+1}$, by

$$B(a_0 \otimes (a_1, \ldots, a_n)) = \sum_{i=0}^{n} (-1)^{\sum_{k=0}^{i-1} |sa_k| \sum_{k=i}^{n} |sa_k|} 1 \otimes (a_i, \ldots, a_n, a_0, a_1, \ldots, a_{i-1})$$

The dual of this operator, $B^\vee : \underline{Hom}(A \otimes T(s\bar{A}), R) \to \underline{Hom}(A \otimes T(s\bar{A}), R)$, defines by adjunction an operator on $\underline{Hom}(T(s\bar{A}), A^\vee) \cong \underline{Hom}(A \otimes T(s\bar{A}), R)$.

When the algebra A is a Poincaré duality algebra (for example a Frobenius algebra) the non-degenerate bilinear form of A induces a chain complex isomorphism
$$\underline{Hom}(T(s\bar{A}), A) \cong \underline{Hom}(T(s\bar{A}), A^\vee)$$
which defines a Δ operator on the Hochschild cochains which gives a BV-structure on $HH^*(A; A)$.

Definition 3.5. Let $\mathbb{K}$ be a field. A *Frobenius algebra* is a $\mathbb{K}$-algebra A of finite dimension with a linear function $\varepsilon : A \to \mathbb{K}$ called a *Frobenius form* (or *augmentation*), such that its kernel does not have non trivial ideals.

In a Frobenius algebra A with Frobenius form ε, a non-degenerate associative bilinear form, called *pairing*, is defined by

$$\begin{aligned} \langle , \rangle : A \otimes A &\longrightarrow \mathbb{K} \\ a \otimes b &\longmapsto \varepsilon(ab) \end{aligned}$$

which induces an isomorphism between A and its dual $A^\vee$.

Definition 3.6. We say that a Frobenius algebra is symmetric if for every $a, b \in A$

$$\langle a, b\rangle = \langle b, a\rangle$$

equivalently if for every $a, b \in A$ we have $\varepsilon(ab) = \varepsilon(ba)$.

Example 1. If G is a finite group then the group ring $\mathbb{K}[G]$, with $\mathbb{K}$ a field, is a symmetric Frobenius algebra with Frobenius form given by

$$\varepsilon : \mathbb{K}[G] \to \mathbb{K}, \quad \varepsilon\left(\sum_{g\in G} \alpha_g g\right) = \alpha_e$$

Definition 3.7. An augmented graded commutative algebra A is called a *Poincaré duality algebra* of dimension N if there exists an R-module homomorphism $\varepsilon : A^N \to R$ such that the induced bilinear forms $\langle , \rangle : A^k \otimes A^{N-k} \to R$, $a \otimes b \mapsto \varepsilon(ab)$ are non-degenerate. If $\{a_1, \ldots, a_N\}$ is a homogeneous basis of A, then the unique basis $\{a_1^\vee, \ldots, a_N^\vee\}$ of A characterized by $\langle a_i, a_j^\vee\rangle = \delta_{ij}$ is called the *Poincaré dual basis*.

Example 2. Let M be a closed, connected, oriented manifold of finite dimension n. The cohomology ring of M with coefficients in a field $\mathbb{K}$ is a Poincaré duality algebra of dimension n. The augmentation is defined by

$$\begin{aligned} \varepsilon : H^*(M;\mathbb{K}) &\to \mathbb{K} \\ \alpha &\mapsto \alpha([M]) \end{aligned}$$

where $[M]$ is the homological fundamental class of M. This augmentation induces a pairing $\langle , \rangle : H^*(M;\mathbb{K}) \otimes H^*(M;\mathbb{K}) \to \mathbb{K}$ defined by

$$\langle \alpha, \beta\rangle = \varepsilon(\alpha \smile \beta) = (\alpha \smile \beta)([M]) = \alpha([M] \frown \beta)$$

Recall that Poincaré duality theorem gives us the following isomorphism

$$\Phi : H^{n-k}(M;\mathbb{K}) \xrightarrow{h} Hom_{\mathbb{K}}(H_{n-k}(M;\mathbb{K}), \mathbb{K}) \xrightarrow{D^*} Hom_{\mathbb{K}}(H^k(M;\mathbb{K}), \mathbb{K})$$

where h is the map induced by the evaluation of cochains on chains, and D^* is the dual of Poincaré duality. Then

$$\Phi(\alpha)(\beta) = \alpha([M] \frown \beta)$$

and the pairings are non-degenerate.

In the case when A is a Poincaré duality algebra (Frobenius algebra), the BV-operator, Δ, is defined as follows

Proposition 3.1. *The operator* $\Delta : Hom(T(s\bar{A})^{\otimes n+1}, A) \to Hom(T(s\bar{A})^{\otimes n}, A)$ *is given by*

$$\Delta(f)(a_1,\dots,a_n) = \sum_{j=1}^{N} (-1)^{|f|+|a^j|\sum_{k=i}^{n}|sa_k|} \sum_{i=0}^{n} (-1)^{(|sa_0|+\sum_{k=0}^{i-1}|sa_k|)\sum_{k=i}^{n}|sa_k|} \langle 1, f(a_i,\dots,a_n,a^j,a_1,\dots,a_{i-1})\rangle {a^j}^{\vee}$$

where $\{a^1,\dots,a^N\}$ *is a homogeneous basis of* A.

The Δ operator also lowers the Hochschild degree and also the internal degree by 1, this is because $\langle , \rangle : A^k \otimes A^{N-k} \to R$ and $|1| = 0$ therefore the terms in the sum are zero except when $|f(a_i,\dots,a_n,a^j,a_1,\dots,a_{i-1})| = N$ but this is equivalent to

$$|f| + \sum_{i=1}^{n} |a_i| - n + |a^j| - 1 = N$$

since $|{a^j}^{\vee}| = N - |a^j|$, we have

$$|\Delta(f)(a_1,\dots,a_n)| = N - |a^j| = |f| + \sum_{i=1}^{n}|a_i| - n - 1$$

which proves that $|\Delta(f)| = |f| - 1$.

In [14] Menichi proves that Δ induces a BV-structure on $HH^*(A;A)$, which furthermore induces the Gerstenhaber structure of $HH^*(A;A)$.

4 Truncated Polynomials over $\mathbb{F}_p$

From example (2), the cohomology of spheres and complex projective spaces with coefficients in $\mathbb{F}_p$ are Poincaré duality algebras. Moreover, these algebras are truncated polynomial rings

$$H^*(\mathbb{S}^n;\mathbb{F}_p) = \mathbb{F}_p[x]/(x^2) \cong \Lambda(x) \quad \text{with} \quad |x| = n$$
$$\varepsilon : H^*(\mathbb{S}^n;\mathbb{F}_p) \to \mathbb{F}_p, \quad \varepsilon(a+bx) = b$$

and

$$H^*(\mathbb{C}P^n;\mathbb{F}_p) = \mathbb{F}_p[x]/(x^{n+1}) \quad \text{with} \quad |x| = 2$$
$$\varepsilon : H^*(\mathbb{C}P^n;\mathbb{F}_p) \to \mathbb{F}_p, \quad \varepsilon\left(\sum_{i=0}^{n}\alpha_i x^i\right) = \alpha_n$$

Sometimes is convenient to grade negatively the cohomology groups so we will consider the cases $|x| = \pm n$.

From now on, we assume that A is $\mathbb{F}_p[x]/(x^p)$ with $|x| = n \in \mathbb{Z}$. Let

$$P(y,z) = y^{p-1} + y^{p-2}z + \cdots + z^{p-1} \in A \otimes A = R[y,z]/(y^p, z^p)$$

with $|y| = |z| = |x|$. Note that $P(y,z)(y-z) = y^p - z^p$. Since the Hochschild (co)homology of an algebra can be calculated using projective A^e-resolutions and the bar construction is not convenient to make explicit calculations, we are going to use the following 2-periodical resolution [18], [11]. For $|x| = 0$, see [12] exercise E.4.1.8.

Proposition 4.1. *The following is a A^e-projective resolution of A*

$$P_*(A): \cdots \to A \otimes A \xrightarrow{P(y,z)} A \otimes A \xrightarrow{y-z} A \otimes A \xrightarrow{\mu} A \to 0$$

with

$$d_{2k}(a \otimes b) = y^{p-1}a \otimes b + y^{p-2}a \otimes zb + \cdots + a \otimes z^{p-1}b$$

and

$$d_{2k+1}(a \otimes b) = ya \otimes b - a \otimes zb$$

For $|x| = n$ with $n \in \mathbb{Z}$ the graded version is the following, see [20],[21].

Proposition 4.2. *The following is a A^e-projective resolution of A*

$$P_*(A): \cdots \to \Sigma^{np}(A \otimes A) \xrightarrow{P(y,z)} \Sigma^{n}(A \otimes A) \xrightarrow{y-z} A \otimes A \xrightarrow{\mu} A \to 0$$

with $P_{2k}(A) = \Sigma^{knp}(A \otimes A)$, $d_{2k}(a \otimes b) = y^{p-1}a \otimes b + y^{p-2}a \otimes zb + \cdots + a \otimes z^{p-1}b$ and $P_{2k+1}(A) = \Sigma^{nkp+n}(A \otimes A)$, $d_{2k+1}(a \otimes b) = ya \otimes b - a \otimes zb$.

Where we denote by ΣA the de-suspension of A (the inverse functor of s), i.e., $(\Sigma A)_i = A_{i-1}$ so for $x \in A$ we have $|\Sigma x| = |x| + 1$. With this regrading, the differential is of degree zero, since $d_{2k+1}: \Sigma^{nkp+n}(A \otimes A) \to \Sigma^{knp}(A \otimes A)$ and $d_{2k+1}(a \otimes b) = ya \otimes b - a \otimes zb$ with $|y| = |z| = n$. and similarly with d_{2k}.

Taking $Hom_{A^e}(\_\_, A)$ of $P_*(A)$, we get the following 2-periodical cochain complex

$$P^*(A): 0 \to A \xrightarrow{0} A \xrightarrow{px^{p-1}} A \xrightarrow{0} A \xrightarrow{px^{p-1}} \cdots \tag{1}$$

To calculate the algebra structure of $HH^*(A;A)$, we use a chain map between $P_*(A)$ and the bar construction $\bar{B}(A)$

$$\varphi_*: P_*(A) \to \bar{B}(A)$$

for $|x| = 0$, $\varphi_{2k} : A \otimes A \to A \otimes (s\bar{A})^{\otimes 2k} \otimes A$ is defined by

$$\varphi_{2k}(1 \otimes 1) = \sum 1 \left[x^{p-a_1-1}|x|x^{p-a_2-1}|x| \cdots |x^{p-a_k-1}|x\right] x^{\sum_{k=1}^{k} a_k}$$

and $\varphi_{2k+1} : A \otimes A \to A \otimes (s\bar{A})^{\otimes 2k+1} \otimes A$ is defined by

$$\varphi_{2k+1}(1 \otimes 1) = \sum 1 \left[x|x^{p-a_1-1}|x|x^{p-a_2-1}|x| \cdots |x^{p-a_k-1}|x\right] x^{\sum_{k=1}^{k} a_k}$$

where the sum is taken over $0 \leq a_1, \ldots, a_k < p-1$ and $\sum_{k=1}^{k} a_k \leq p-1$.

With the same formulas for $|x| = n$ with $n \in \mathbb{Z}$, we define

$$\varphi_* : P_*(A) \to \bar{B}(A)$$

where

$$\varphi_{2k} : \Sigma^{nkp}(A \otimes A) \to A \otimes (s\bar{A})^{\otimes 2k} \otimes A$$

and

$$\varphi_{2k+1} : \Sigma^{nkp+n}(A \otimes A) \to A \otimes (s\bar{A})^{\otimes 2k+1} \otimes A$$

By a direct computation we have,

Lemma 4.3 (Lemma 3.3 [20])**.** *$\varphi^* = Hom_{A^e}(\varphi, A) : C^*(A; A) \to P^*(A)$ is a cochain map.*

Using φ we can give the algebra structure of $HH^*(A; A)$.

Proposition 4.4 (Yang)**.** *Let $A = \mathbb{F}_p[x]/(x^p)$ with p an odd prime and $|x| = n$ with $n \in \mathbb{Z}$. Then $HH^*(A; A)$ as an algebra is*

$$HH^*(A; A) = \mathbb{F}_p[x, v, t]/(x^p, v^2)$$

where $x \in HH^0(A; A)$, $v \in HH^1(A; A)$ and $t \in HH^2(A; A)$ with $|x| = n$, $|v| = 1 - n$ and $|t| = -np + 2$.

Proof. The assertion follows from the 2-periodical resolution and the cochain map φ^* between $C^*(A; A)$ and $P^*(A)$. Recall that $P^*(A)$ is (1)

$$0 \to A \xrightarrow{0} A \xrightarrow{0} A \xrightarrow{0} A \xrightarrow{0} A \xrightarrow{0} \cdots$$

where $A \cong Hom_{A^e}(A^e, A)$. Consider the following elements

$$\begin{aligned} x \in P^0(A), &\quad x(1) = x, \\ v \in P^1(A), &\quad v(1) = 1, \\ t \in P^2(A), &\quad t(1) = 1, \end{aligned}$$

Via φ^*, these elements correspond to Hochschild cochains,

$$\begin{aligned} \bar{x} \in Hom(\mathbb{F}_p, A) = A, &\quad \bar{x}(1) = x, \\ \bar{v} \in Hom(s\bar{A}, A) = A, &\quad \bar{v}(x^i) = ix^{i-1}, \\ \bar{t} \in Hom(s\bar{A}^{\otimes 2}, A) = A, &\quad \bar{t}(x^i, x^j) = x^{i+j-p}, \end{aligned}$$

Note that the internal degree of $1 \in \mathbb{F}_p$ is 0 and the degree of $x \in A$ is n, so $|x| = n$. The internal degree of $x^i \in s\bar{A}$ is $ni - 1$ and the degree of $ix^{i-1} \in A$ is $n(i-1)$, so $|v| = n(i-1) - (ni-1) = 1 - n$, and the internal degree of $x^i \otimes x^j \in s\bar{A}^{\otimes 2}$ is $ni + nj - 2$ and the degree of $x^{i+j-p} \in A$ is $n(i+j-p)$ which gives $|t| = n(i+j-p) - (ni+nj-2) = -np + 2$.

See [20] for the proof that these elements generate and give the desired isomorphism. □

In particular for $|x| = 0$, we have

Proposition 4.5. *Let $A = \mathbb{F}_p[x]/(x^p)$ with p an odd prime and $|x| = 0$. Then $HH^*(A; A)$ as an algebra is*

$$HH^*(A; A) = \mathbb{F}_p[x, v, t]/(x^p, v^2)$$

where $x \in HH^0(A; A)$, $v \in HH^1(A; A)$ and $t \in HH^2(A; A)$ with $|x| = 0$, $|v| = 1$ and $|t| = 2$.

Proof. The proof is the same, note only that now the internal degree of $1 \in \mathbb{F}_p$ is 0 and the degree of $x \in A$ is 0, so $|x| = 0$. The internal degree of $x^i \in s\bar{A}$ is -1 and the degree of $ix^{i-1} \in A$ is 0, so $|v| = 1$, and the internal degree of $x^i \otimes x^j \in s\bar{A}^{\otimes 2}$ is -2 and the degree of $x^{i+j-p} \in A$ is 0 which gives $|t| = 2$. □

For $p = 2$ similarly we have,

Proposition 4.6. *Let $A = \mathbb{F}_2[x]/(x^2)$ with $|x| = n \in \mathbb{Z}$. Then $HH^*(A; A)$ as an algebra is*

$$HH^*(A; A) = \mathbb{F}_2[x, v]/(x^2) \cong \Lambda(x) \otimes \mathbb{F}_2[v]$$

where $x \in HH^0(A; A)$, $v \in HH^1(A; A)$, $|x| = n$ and $|v| = 1 - n$.

5 Group Ring of Prime Cyclic Groups over $\mathbb{F}_p$

It is well-known that group cohomology and Hochschild cohomology are related by Eckmann-Shapiro lemma [2]. From this lemma it can be proved directly that

$$HH^*(\mathbb{K}[G]; M) \cong H^*(G; {}_{\chi}M)$$

for G a finite group, $\mathbb{K}$ a commutative ring and M a G-module. Here ${}_{\chi}M$ denotes the left G-module with conjugation action $g \cdot x = gxg^{-1}$. In particular, for $M = \mathbb{K}[G]$ with G abelian Cibils and Solotar in [4] prove that this isomorphism is an isomorphism of algebras

$$HH^*(\mathbb{K}[G]; \mathbb{K}[G]) \cong H^*(G; {}_{\chi}\mathbb{K}[G]) \cong \mathbb{K}[G] \otimes H^*(G; \mathbb{K})$$

Also, for groups not necessary abelian, they define the product formula using conjugation classes and establish the conjecture that the isomorphism

$$HH^*(\mathbb{K}[G]; \mathbb{K}[G]) \cong H^*(G; {}_{\chi}\mathbb{K}[G])$$

is an isomorphism of algebras for any finite group. Siegel and Witherspoon in [17] prove this conjecture using the product formula defined by Cibils and Solotar. In [16] Sánchez-Flores transfers the Gerstenhaber structure of the Hochschild cohomology to the group cohomology of cyclic groups and proves that is highly non-trivial over fields of positive characteristic.

For the cyclic group of order p, $\mathbb{Z}/p\mathbb{Z}$, we have $\mathbb{F}_p[\mathbb{Z}/p\mathbb{Z}] = \mathbb{F}_p[\sigma]/(\sigma^p - 1)$ and

$$H^*(\mathbb{Z}/p\mathbb{Z}; \mathbb{F}_p) \cong \mathbb{F}_p[v,t]/(v^2) \cong \Lambda(v) \otimes \mathbb{F}_p[t] \quad \text{with} \quad |v| = 1, \ |t| = 2$$

Over $\mathbb{F}_p$ there is a natural isomorphism of algebras between the truncated polynomial ring and the group ring of a cyclic group of prime order p

$$\begin{aligned} \psi : \mathbb{F}_p[x]/(x^p) &\longrightarrow \mathbb{F}_p[\sigma]/(\sigma^p - 1) \\ x &\longmapsto \sigma - 1 \end{aligned}$$

Putting together all these, $\mathbb{F}_p[\mathbb{Z}/p\mathbb{Z}] \otimes H^*(\mathbb{Z}/p\mathbb{Z}; \mathbb{F}_p) \cong \mathbb{F}_p[x]/(x^p) \otimes \mathbb{F}_p[v,t]/(v^2)$, we have another proof of

Proposition 5.1. *Let $A = \mathbb{F}_p[x]/(x^p)$ with p an odd prime and $|x| = 0$. Then $HH^*(A; A)$ as an algebra is*

$$HH^*(A; A) = \mathbb{F}_p[x, v, t]/(x^p, v^2) \cong \mathbb{F}_p[x]/(x^p) \otimes \mathbb{F}_p[v,t]/(v^2)$$

where $x \in HH^0(A; A)$, $v \in HH^1(A; A)$ and $t \in HH^2(A; A)$ with $|x| = 0$, $|v| = 1$ and $|t| = 2$.

Notice that the BV-operator, Δ, strongly depends on the Frobenius structure of the algebra. So, it is natural to study what happens with Δ if we change the Frobenius form of the algebra.

According to Yang, when we consider the Frobenius structure given by the Frobenius form that takes the coefficient α_{p-1}

$$\varepsilon : \mathbb{F}_p[x]/(x^p) \longrightarrow \mathbb{F}_p \qquad \varepsilon\left(\sum_{i=0}^{p-1} \alpha_i x^i\right) = \alpha_{p-1}$$

we have,

Theorem 5.2. *[20] Let $A = \mathbb{F}_p[x]/(x^p)$ with p an odd prime and $|x| = n$ with $n \in \mathbb{Z}$. Then as BV-algebra*

$$\begin{aligned} HH^*(\mathbb{F}_p[x]/(x^p); \mathbb{F}_p[x]/(x^p)) &= \mathbb{F}_p[x, v, t]/(x^p, v^2) \\ \Delta(t^k x^l) &= 0 \\ \Delta(t^k v x^l) &= l t^k x^{l-1} \end{aligned} \tag{2}$$

where $x \in HH^0(A; A)$, $v \in HH^1(A; A)$ and $t \in HH^2(A; A)$ with with $|x| = n$, $|v| = 1 - n$ and $|t| = -np + 2$.

Similarly for $p = 2$,

Theorem 5.3. *Let $A = \mathbb{F}_2[x]/(x^2)$ with $|x| = n$. Then as a BV-algebra,*

$$\begin{aligned} HH^*(A; A) &= \mathbb{F}_2[x, v, t]/(x^2, v^2 - t) \cong \Lambda(x) \otimes \mathbb{F}_2[v] \\ \Delta(v^k) &= 0 \\ \Delta(x v^k) &= k v^{k-1} \end{aligned} \tag{3}$$

where $x \in HH^0(A; A)$, $v \in HH^1(A; A)$ with $|x| = n$ and $|v| = 1 - n$.

By using the formality of spheres, we have the following result

Theorem 5.4 (Westerland)**.** *As BV-algebras,*

$$\begin{aligned} HH^*(C^*(\mathbb{S}^n, \mathbb{F}_2); C^*(\mathbb{S}^n, \mathbb{F}_2)) &\cong \mathbb{F}_2[x, v]/(x^2) \\ \Delta(x^k v^l) &= k l x^{k-1} v^{l-1} \end{aligned}$$

where the topological dimensions of x and v are $-n$ and $n - 1$, respectively.

We want to define a new Frobenius form over the truncated polynomial ring by pulling-back the natural Frobenius form of the group ring

$$\varepsilon_0 : \mathbb{F}_p[\sigma]/(\sigma^p - 1) \longrightarrow \mathbb{F}_p \qquad \varepsilon_0\left(\sum_{i=0}^{p-1} \alpha_i \sigma^i\right) = \alpha_0$$

By doing this, we get a new Frobenius form on the truncated polynomial ring,

$$\tilde{\varepsilon} : \mathbb{F}_p[x]/(x^p) \longrightarrow \mathbb{F}_p \qquad \tilde{\varepsilon}\left(\sum_{i=0}^{p-1} \alpha_i x^i\right) = \sum_{i=0}^{p-1} (-1)^i \alpha_i \tag{4}$$

To calculate the new BV-structure over the Hochschild cohomology of $\mathbb{F}_p[x]/(x^p)$, we need to find the dual basis with respect to the new pairing $\langle a, b\rangle = \tilde{\varepsilon}(ab)$. By taking the set $\{1, x, \dots, x^{p-1}\}$ as a basis for $\mathbb{F}_p[x]/(x^p)$, the matrices that represent the pairing and its inverse are

$$[\langle,\rangle] = \begin{pmatrix} 1 & -1 & \cdots & -1 & 1 \\ -1 & 1 & \cdots & 1 & \\ \vdots & \vdots & \iddots & & \\ -1 & 1 & & 0 & \\ 1 & & & & \end{pmatrix} \qquad [\langle,\rangle]^{-1} = \begin{pmatrix} & & & & 1 \\ & 0 & & 1 & 1 \\ & & \iddots & 1 & \\ & 1 & \iddots & & \\ 1 & 1 & & 0 & \end{pmatrix}$$

Therefore, the dual basis is the set $\{x^{p-1}, x^{p-1} + x^{p-2}, x^{p-2} + x^{p-3}, \dots, x+1\}$.

Theorem 5.5. *Let $A = \mathbb{F}_p[x]/(x^p)$ with p an odd prime and $|x| = 0$. Then $HH^*(A; A)$ has a BV-structure induced by the canonical Frobenius form of the group ring defined as follows*

$$HH^*(A; A) = \mathbb{F}_p[x, v, t]/(x^p, v^2)$$

$$\Delta(t^l x^k) = 0$$

$$\Delta(t^k v x^{2l}) = 2lt^k x^{2l-1} + \sum_{i=2l}^{p-1} (-1)^{i+1} t^k x^i$$

$$\Delta(t^k v x^{2l+1}) = (2l+1)t^k x^{2l} + \sum_{i=2l+1}^{p-1} (-1)^i t^k x^i$$

where $x \in HH^0(A; A)$, $v \in HH^1(A; A)$ and $t \in HH^2(A; A)$ with $|x| = 0$, $|v| = 1$ and $|t| = 2$.

Proof. According to Yang [20] (with a sign correction on the last equation) $HH^*(A; A)$, as a Gerstenhaber algebra, is

$$\begin{aligned} HH^*(A; A) &= \mathbb{F}_p[x, v, t]/(x^p, v^2) \\ \{t^{k_1} x^{l_1}, t^{k_2} x^{l_2}\} &= 0 \\ \{t^{k_1} x^{l_1}, t^{k_2} v x^{l_2}\} &= l_1 t^{k_1+k_2} x^{l_1+l_2-1} \\ \{t^{k_1} v x^{l_1}, t^{k_2} v x^{l_2}\} &= (l_1 - l_2) t^{k_1+k_2} v x^{l_1+l_2-1} \end{aligned} \tag{5}$$

Since in a BV-algebra, we have the following equations

$$\Delta(ab) = \Delta(a)b + (-1)^{|a|}a\Delta(b) + (-1)^{|a|}\{a,b\} \tag{6}$$

and

$$\begin{aligned}\Delta(abc) = &\Delta(ab)c + (-1)^{|a|}a\Delta(bc) + (-1)^{(|a|-1)|b|}b\Delta(ac) \\ &- \Delta(a)bc - (-1)^{|a|}a\Delta(b)c - (-1)^{|a|+|b|}ab\Delta(c)\end{aligned} \tag{7}$$

We only need to calculate $\Delta(x)$, $\Delta(v)$, $\Delta(t)$ $\Delta(tx)$ and $\Delta(vx)$. Taking $\{1, x, \ldots, x^{p-1}\}$ as a basis for A and $\{x^{p-1}, x^{p-1}+x^{p-2}, x^{p-2}+x^{p-3}, \ldots, x+1\}$ as the dual basis induced by the new Frobenius form (4), we have

$$\boxed{\Delta(x) = 0} \text{ by degree.}$$

$$\begin{aligned}\Delta(v)(1) &= \sum_{i=0}^{p-1} -\langle 1, v(x^i)\rangle(x^{p-i}+x^{p-i-1}) = \sum_{i=0}^{p-1} -\langle 1, ix^{i-1}\rangle(x^{p-i}+x^{p-i-1}) \\ &= \sum_{i=0}^{p-1} -(-1)^{i-1}i(x^{p-i}+x^{p-i-1}) = \sum_{i=1}^{p-1}(-1)^i x^{p-i} - 1\end{aligned}$$

$$\boxed{\Delta(v) = \sum_{i=0}^{p-1}(-1)^{i+1}x^i}$$

$$\begin{aligned}\Delta(vx)(1) &= \sum_{i=0}^{p-1} -\langle 1, vx(x^i)\rangle(x^{p-i}+x^{p-i-1}) = \sum_{i=0}^{p-1} -\langle 1, ix^{i}\rangle(x^{p-i}+x^{p-i-1}) \\ &= \sum_{i=0}^{p-1} -(-1)^{i}i(x^{p-i}+x^{p-i-1})\end{aligned}$$

$$\boxed{\Delta(vx) = \sum_{i=0}^{p-1}(-1)^{i}x^i}$$

$$\Delta(t)(x^k) = \sum_{i=0}^{p-1}\langle 1, t(x^i, x^k)\rangle(x^{p-i}+x^{p-i-1}) - \sum_{i=0}^{p-1}\langle 1, t(x^k, x^i)\rangle(x^{p-i}+x^{p-i-1})$$

$$\boxed{\Delta(t) = 0}$$

$$\Delta(tx)(x^k) = \sum_{i=0}^{p-1}\langle 1, tx(x^i, x^k)\rangle(x^{p-i}+x^{p-i-1}) - \sum_{i=0}^{p-1}\langle 1, tx(x^k, x^i)\rangle(x^{p-i}+x^{p-i-1})$$

$$\boxed{\Delta(tx) = 0}$$

Note that if we calculate $\Delta(t^2)$ and $\Delta(tv)$ then Δ will be totally determined using equation (7) and induction on powers of x and t. From equation (6) and the bracket, we obtain that

$$\Delta(t^2) = \Delta(t)t + t\Delta(t) + \{t, t\}$$
$$\boxed{\Delta(t^2) = 0}$$
$$\Delta(tv) = \Delta(t)v + t\Delta(v) + \{t, v\}$$
$$\boxed{\Delta(tv) = \sum_{i=0}^{p-1}(-1)^{i+1}tx^i}$$

Therefore, the new BV-algebra structure on the Hochschild cohomology of A is

$$\Delta(t^l x^k) = 0$$
$$\Delta(t^k vx^{2l}) = 2lt^k x^{2l-1} + \sum_{i=2l}^{p-1}(-1)^{i+1}t^k x^i$$
$$\Delta(t^k vx^{2l+1}) = (2l+1)t^k x^{2l} + \sum_{i=2l+1}^{p-1}(-1)^i t^k x^i$$

Using equation (6), a straightforward calculation shows that this new BV structure gives the Gerstenhaber structure calculated by Yang. □

Similarly, for $p = 2$,

Theorem 5.6. *Let $A = \mathbb{F}_2[x]/(x^2)$ with $|x| = 0$. Then as a BV-algebra,*

$$HH^*(A; A) = \mathbb{F}_2[x, v, t]/(x^2, v^2 - t) \cong \Lambda(x) \otimes \mathbb{F}_2[v]$$
$$\Delta(v^k x^l) = k(1 + x)v^{k-1}$$

where $x \in HH^0(A; A)$, $v \in HH^1(A; A)$ with $|x| = 0$ and $|v| = 1$.

Proof. In the special case when $p = 2$, as a Gerstenhaber algebra the Hochschild cohomology of A is

$$HH^*(A; A) = \mathbb{F}_2[x, v, t]/(x^2, v^2 - t) \cong \Lambda(x) \otimes \mathbb{F}_2[v]$$
$$\{v^k, v^l\} = 0 \quad \{xv^k, v^l\} = lv^{k+l-1} \quad \{xv^k, xv^l\} = (k - l)xv^{k+l-1}$$

Following the same idea as in the case p an odd prime, we only need to calculate $\Delta(x)$, $\Delta(v)$, $\Delta(vx)$ and $\Delta(v^2)$. Taking $\{1, x\}$ as a basis for A and

$\{x, x+1\}$ as the dual basis induced by the new Frobenius form (4)

$$\begin{aligned}
\Delta(x) &= 0 \quad \text{by degree} \\
\Delta(v)(1) &= \langle 1, v(1)\rangle x + \langle 1, v(x)\rangle (x+1) = x+1 \\
\Delta(vx)(1) &= \langle 1, vx(1)\rangle x + \langle 1, vx(x)\rangle (x+1) = x+1 \\
\Delta(v^2)(x^i) &= \langle 1, v^2(1, x^i)\rangle x + \langle 1, v^2(x, x^i)\rangle (x+1) \\
&\quad + \langle 1, v^2(x^i, 1)\rangle x + \langle 1, v^2(x^i, x)\rangle (x+1) = 0
\end{aligned}$$

Now, using equation (7) and induction, the new BV-algebra structure of the Hochschild cohomology of A is

$$\begin{aligned}
\Delta(v^{2k}) = 0 \quad &\text{and} \quad \Delta(v^{2k+1}) = (1+x)v^{2k} \\
\Delta(xv^{2k}) = 0 \quad &\text{and} \quad \Delta(xv^{2k+1}) = (1+x)v^{2k}
\end{aligned}$$

Using the formula that relates the Gerstenhaber structure and the BV-structure (6), it can be checked that this new BV-structure preserves the Gerstenhaber structure

$$\begin{aligned}
\{v^{2k}, v^{2l}\} &= 0 & \{v^{2k}, v^{2l+1}\} &= 0 \\
\{xv^{2k}, v^{2l}\} &= 0 = 2lv^{2k+2l-1} & \{xv^{2k+1}, v^{2l}\} &= 0 = 2lv^{2k+2l} \\
\{xv^{2k}, v^{2l+1}\} &= v^{2k+2l} = (2l+1)v^{2k+2l} \\
\{xv^{2k+1}, v^{2l+1}\} &= v^{2k+2l+1} = (2l+1)v^{2k+2l+1} \\
\{xv^{2k}, xv^{2l}\} &= 0 = 2(k-l)xv^{2k+2l-1} \\
\{xv^{2k+1}, xv^{2l+1}\} &= 0 = 2(k-l)xv^{2k+2l+1} \\
\{xv^{2k}, xv^{2l+1}\} &= xv^{2k+2l} = (2(k-l)-1)xv^{2k+2l} \\
\{xv^{2k+1}, xv^{2l}\} &= xv^{2k+2l} = (2(k-l)+1)xv^{2k+2l}
\end{aligned}$$

□

References

[1] Brown, K.S. *Cohomology of groups*, Springer, 1982.

[2] Cartan, H.; Eilenberg, S. *Homological algebra*, Princeton Mathematical Series, **19**, Princeton University Press, 1956.

[3] Chas, M.; Sullivan, D. *String Topology*. Preprint:math.GT/9911159, 1999.

[4] Cibils, C.; Solotar, A. *Hochschild cohomology algebra of abelian groups*, Arch.Math. (Basel) **68**, 17-21, 1997.

[5] Cohen, R.L.; Jones, J.D.S.; Yan J. *The loop homology algebra of spheres and projective spaces*, arXiv preprint arXiv:math.AT/0210353 v1, 2002.

[6] Cohen, R.L; Jones, J.D.S. *A homotopy theoretic realization of string topology*, Math. Annalen, **324**, 773-798, 2002.

[7] Félix, Y.; Menichi, L.; Thomas, J.C. *Gerstenhaber duality in Hochschild cohomology*, J. Pure Appl. Algebra **199**, no. 1-3, 43-59, 2005.

[8] Félix, Y.; Thomas, J.C. *Rational BV-algebra in string topology*, arXiv preprint arXiv:0705.4194, 2007.

[9] Gerstenhaber, M. *The cohomology structure of an associative ring*, Ann. of Math. (2), **78**, 267-288, 1963.

[10] Hochschild, G. *On the cohomology groups of an associative algebra*, Annals of Mathematics. Second Series **46**, 58-67, 1945.

[11] Holm, T. *The Hochschild cohomology ring of a modular group algebra: the commutative case*, Comm. Algebra **24**, no. 6, 1957-1969, 1996.

[12] Loday, J.L. *Cyclic Homology*, Springer-Verlag , Berlin, Heidelberg, New York, 1971.

[13] Menichi, L. *Batalin-Vilkovisky algebra structures on Hochschild cohomology*, arXiv preprint arXiv:0711.1946, 2007.

[14] Menichi, L. *Batalin-Vilkovisky algebras and cyclic cohomology of Hopf algebras*, K-Theory **32**, no. 3, 231-251, 2004.

[15] Menichi, L. *String topology for spheres*, preprint: math.AT/0609304, to appear in Comment. Math. Helv.

[16] Sanchez-Flores, S. *The Lie structure on the Hochschild cohomology of a modular group algebra*, Journal of Pure and Applied Algebra (in press).

[17] Siegel, S.F.; Witherspoon, S.J. *The Hochschild cohomology ring of a group algebra*, Proceedings of the London Mathematical Society, **79**(1), 131-157, 1999.

[18] The Buenos Aires Cyclic Homology Group *Cyclic Homology of Algebras with One Generator*, K-Theory **5**, 51-69, 1991.

[19] Tradler, T. *The BV algebra on Hochschild cohomology induced by infinity inner products*, preprint: math.QA/0210150v1, 2002.

[20] Yang, T. *A Batalin-Vilkovisky Algebra Structure on the Hochschild Cohomology of Truncated Polynomials*, Topology and its Applications **160**, 1633-1651, 2013.

[21] Westerland, C. *Dyer-Lashof operations in the string topology of spheres and projective spaces*, Math.Z. **250**, 711-727, 2005.

[22] Westerland, C. *String homology of spheres and projective spaces*, Algebr. Geom. Topol. **7**, 309-325, 2007.