6^{th} Summer School on

GEOMETRIC AND TOPOLOGICAL METHODS IN QUANTUM FIELD THEORY
Villa de Leyva, Boyacá, Colombia, 6 – 23 July, 2009

Luis J. BOYA, Departamento de Física Teórica, Universidad de Zaragoza, Spain.
luisjo@unizar.es

I.- Differential Geometry for Physicists:

1.- Manifolds

2.- Tangent space; tangent bundle; vector fields and flows

3.- p-forms; integration. de Rham cohomology. Other tensors. Metrics

4.- Connections in Tangent Bundle. Levi-Civita connection in (M, g)

5.- Curvature, and tensors: Riemann, Weyl, Ricci, Scalar curvature

6.- Homotopy groups and Spin groups.

II.- Holonomy:

1.- Parallel transport and holonomy

2.- Reduction theorem and holonomy theorem (Ambrose-Singer)

3.- The different types of holonomy: O(n) and SO(n) (for reals |R)

4.- Almost complex and Complex manifolds; symplectic; Kähler manifolds

5.- U(n) (Kähler) and SU(n) (Calabi-Yau) holonomies

6.- Quaternionic and hyperkähler manifolds (|H): holonomy q(n) and Sq(n)

7.- Exceptional holonomy and octonions O: G_2 and Spin(7) manifolds.

III.- Higher Dimensions and Strings

1.- Higher dimensions in physics: Kaluza-Klein reduction

2.- Physics in ten dimensions: The Five SuperString theories

3.-Membranes, M-Theory (11-dim) and F-Theory (12 dimensions)

4.-Standard (SM) & Minimal Supersymmetric Standard Model (MSSM).

IV.- Some issues on Compactification

1.- The general problem of Compactification

2.- Compactification from the Heterotic String H-E: Calabi-Yau 3-folds

3.- Compactification from M-Theory: G_2 holonomy manifolds

4.- Compactification from F-Theory (12-dim); K3 and del Pezzo surfaces.

Recommended Literature

General Reference on Modern Mathematics for Physicists:

M. Nakahara.- *Geometry, Topology and Physics*. Ins. of Physics (Bristol), 1st ed. 1990.- 2nd edition 2005.

Topics in String- & M-Theory: C. Johnson.- *D-Branes*. Cambridge U.P. 2003. Green-Schwarz-Witten.- *Superstring Theory*, Cambridge U.P. 1985, quoted as [GSW].
M. J. Duff (Ed.).- *The World in Eleven Dimensions*. Cambridge U.P. 1999.
For F-Theory: L.J. Boya, "Arguments for F-Theory", Mod. Phys. Lett A **21**, 1-18, 2006.
Modern Vafa´s Theory: J. Heckman and C. Vafa, "F-Theory and GUTs", arXiv: 0809.1098 (hep-th). Quoted as [Vafa].

Supersymmetry & SuperGravity.- P. West, *Introduction to Supersymmetry and Supergravity*. World Scientific 1990.

Mathematical Literature
Differential Geometry: S. Kobayashi and K. Nomizu.- *Foundations of Differential Geometry*. J.Wiley 1963 (1st Vol.) 1969 (2nd Vol.). Quoted as [KN]. Also S. Helgason.- *Differential Geometry and Symmetric Spaces"*. Academic Press, 1962 (1st edition).

Fiber Bundles, Homotopy: N. Steenrod.- *Topology of Fiber Bundles.* Princeton U. P., 1951

Id & Characteristic Classes: J. Milnor & J.D. Stasheff.- *Characteristic Classes.* Princeton U.P. 1974. Quoted as [Milnor].

For Holonomy: D. Joyce .- *Compact Manifolds of Special Holonomy.* Oxford U.P. 2002.

O(2-4-8) special: J. Conway and D. Smith.- *On quaternions and octonions.* A.K. Peters, 2003.

I.- <u>Differential Geometry for Physicists</u>

<u>1&2.</u>- <u>Manifolds and Vector Fields.</u> A *d-dimensional Manifold* M is a topological space covered by open sets U_i, each homeomorphic to an open set V_i in $|R^d$ space; if $\phi_i : U_i \to V_i$, $\phi_i(P) = \{x^1, x^2, \ldots, x^d\}$ are the *coordinates* of point $P \in M$ in the system $\{U_i, V_i\}$. If $P \in U_i \cap U_j$, the functions $\phi_i \circ \phi_j^{-1}$ are supposed to be C^∞ as maps from $|R^d$ to R^d.

We recall: M is *compact* if any open covering $\{U_i\}$ has a finite refinement. M is *connected* if one can go continuously from any point to another. M is *simply connected* if any loop (= closed path) drawn from any point is contractible (shrinks to the point continuously).

Call $\mathcal{E}(M)$ the set of C^∞ functions f: M → |R; it is a commutative, ∞-dim |R-algebra; a *vector* v in P is the operator v: $f \to \partial f/\partial n|_P$, where $\partial f/\partial n$, the *directional derivative* along v, can be expressed as $\Sigma\, a_i\, \partial/\partial x^i$. The set of vectors in P make up the <u>tangent space</u> at P, $T_P(M)$. The union of all T_P, P in M, make up the (total space of) the <u>tangent vector bundle</u> T(M):

$$\tau: \qquad R^n \to T(M) \to M \qquad \text{(I-1.1)}$$

This is a *vector bundle*, as the fibres are vector spaces; M is the *base*, TM = T(M) the *total space*, R^n (=F) is the *fibre*. If $\{e_i\}_P$ is a *frame* in P (a base of T_P), the totality of frames in P for all P make up the *frame bundle*, which is a *principal bundle*, as the group $GL_n(R)$ acts freely in the fibers:

$$\begin{array}{llll} GL_n(R) \to B \to M & & \text{Principal Bundle} & \\ \quad\downarrow & \quad \| & & \\ |R^n \to TM \to M & & \text{Associated Vector Bundle} & \text{(I-1.2)} \end{array}$$

(*Cross*) *Sections* s in a bundle π: E → M are maps: point P in M to point u in the fibre over P, $u \in \pi^{-1}(P)$. Call Γ(τ) the set of sections of TM: a section defines a vector field X; in coordinates (ξ^m are functions in $\mathcal{E}(M)$)

$$X = \xi^m\, \partial/\partial x^m \qquad \text{(I-1.3)}$$

A vector field X in a manifold M defines a *flow*, or set of curves tangent to X: if $\gamma: x^\mu(t)$ is such parameterized curve, the *system* of differential equations of the flow are

$$dx^\mu/dt = \xi^\mu, \qquad \mu: 1, 2, \ldots, n \qquad \text{(I-1.4)}$$

The set of all vector fields $\mathcal{X}(M)$ = Γ(TM) has a rich structure: on one hand is a ∞-dim <u>Lie</u> <u>algebra</u> upon commutation, [X, Y] := XY – YX (notice the second-order derivative terms cancel), and it is also a derivation algebra of the commutative algebra of functions $\mathcal{F}(M)$, as Leibniz´rule holds, $X(fh) = (Xf)h + f(Xh)$: the two structures are related, as algebras of derivations are Lie´s. $\mathcal{L}(M)$ is also, as any type of tensors, an $\mathcal{F}(M)$-module, meaning $fX=Y$ linearly.

3.- p-forms. Other tensors. Metrics. de Rham cohomology. Let $T^*_P(M)$ be the dual vector space to $T_P(M)$, and $T^*M := \cup_P T^*_P(M)$ the union for all P: it is the *cotangent bundle*, still a $|R^n$-vector bundle. Sections θ in it are called 1-forms, $\theta \in \Gamma(T^*M)$; so $\theta(X)$ is a function $f \in \mathcal{F}(M)$.

Define now a map d: $\mathcal{F}(M) \to \Gamma(T^*(M))$ by

$$df(X) = Xf, \tag{I-3.1}$$

called the *differential* of a function. Apply to $f = x^\mu$, the coordinate *functions*, to obtain

$$dx^\mu(\partial/\partial x^\nu) = \delta^\mu{}_\nu \tag{I-3.2}$$

so dx^μ is the *dual base* of $\partial/\partial x^\mu$. Hence, $\theta := p_\mu dx^\mu$ is the general dual field of a vector field, called a 1-form.

Dual to the rôle of vector fields X as derivatives, 1-forms θ serve to *integrate*: let γ be a path, a sort of 1-dim subspace of M; then

$$\int_\gamma \theta := \int_{t0}^{t_1} p_\mu(x^\nu(t))\, dx^\mu/dt \cdot dt \tag{I-3.3}$$

is the integral of 1-form θ on the curve (path) γ: it does n ot depend on parameterization γ: $x^\nu(t)$ The set of sections of the cotangent bundle $\Leftrightarrow$ set of 1-forms are written also $\Omega^1(M)$ $(=\Gamma(T^*(M)))$.

If V is a vector space, call T(V) the tensor space over it. $T^0{}_0$ are the *scalars* (field numbers), $T^1{}_0$ the vectors, $T^0{}_1$ the dual forms, $T^0{}_2$ the bilinear forms, $T^0{}_n$ include the volume forms, etc. So one can form the tensor bundle $\mathcal{T}M$ on a manifold, by union of the tensor vector spaces on each point:

$$\begin{array}{ccccc} GL_n(R) & \to & B & \to & M \\ \downarrow & & & & \| \\ T(V) & \to & \mathcal{T}M & \to & M \end{array} \tag{I-3.4}$$

whose sections define *tensor fields* on the manifold, of paramount importance in physics & maths. We shall consider two special types only, *p-forms* and *metrics*. Each type of tensor, say $T^p{}_q$, with dim n^{p+q}, where dim M = n, is an $\mathcal{F}$(M)-module.

Let us call $\wedge T^0{}_2$ the antisymmetric $\otimes$-products, formed by taking two 1-forms in a point and the wedge ($\wedge$ = antisymmetric) product: $a \wedge b := (a\otimes b - b\otimes a)/2$; generalizing, sections of $\wedge T^0{}_p$ are called *p-forms*. For example, a 2-form is expanded as

$$\omega = p_{\mu\nu}\, dx^\mu \wedge dx^\nu \text{ , etc.} \qquad \text{(I-3.5)}$$

In each point the totality of forms make up an algebra of $\Sigma_k \{n, k\} = 2^n$ dimensions. The p-forms (as tensor fields) are closed under the wedge product; the *exterior differential operator* d:

$$d: \Omega^p(M) \to \Omega^{p+1}(M) \qquad \text{(I-3.6)}$$

is defined as an *antiderivation* of the wedge product, $d(\omega_1\wedge\omega_2) = (d\omega_1) \wedge\omega_2 + \omega_1\wedge(-^{\deg 1})\, d\omega_2$, with the "correspondence" df as above, also d(cons)=0; and finally, by 2-*nilpotency*, $d^2 \equiv 0$. Notice the wedge algebra of *even* p-forms is commutative.

Functions, lying in Ω^0 , or 0-forms, "integrate" in points, $\int_P f \equiv f(P)$. 1-forms integrate on curves, $\int_\gamma \theta$; 2-form in surfaces, so $\iint_S \omega$ makes sense (e.g. in string theory), etc.: n-forms integrate in volumes. For a submanifold N with boundary ∂N, it is

$$\int_{\partial N} \omega = \int_N d\omega \qquad \text{(I-3.7)}$$

which is called *Stokes´ theorem*.

A p-form is *closed* if $d\omega=0$; it is *exact* if $\omega = d\theta$ (of course if exact is closed, as $d^2=0$); in $|R^n$ any closed form is exact (Poincaré lemma), but in general, the closed *nonexact* forms give topological information: this gives rise to the *de Rham cohomology* of the manifold M:

Call $Z^p = Z^p(M)$ the p-*cocycles* (closed forms) of M, and $B^p = B^p(M)$ the p-*coboundaries* (exact forms) of M. Also, call $H^p(M, |R) = Z^p/B^p$

$$\text{Dim } H^p(M, |R) = b_p(M) < +\infty, \quad p: 0, 1, \dots , n = \text{dim M}. \qquad \text{(I-3.8)}$$

$H^p(M, |R)$ are called the p-th (real) cohomology group of the manifold M; the dimensions b_p are the *Betti numbers*, and are *topological invariants* of the manifold: de Rham theory, suggested by É. Cartan, is an "access to the topology of manifolds via exterior forms". It is called *co*-homology, because the original definition of *homology,* H_*, makes use of cycles and boundaries as given by a *triangulation* of the manifold (topological space, in general).

For example, for R^n, b_0=1, all others = 0. b_0 measures # of components.

For S^n, we have: all b´s zero except $b_0 = b_n = 1$. For the n-Torus, $b_k = \{n, k\}$; etc. For a *compact oriented* space, $b_k = b_{n-k}$ (Poincaré duality).

Metrics. In a vector space V, a bilinear form b is a $T^0{}_2$ tensor; it is symmetric (antisymmetric) if b(v, w) = ± b(w, v); it is *regular* (or nondegenerate) if the induced map $b^*: V \rightarrow V^{dual}$ is isomorphism (equivalently, in any base b becomes a matrix (b), with det (b) ≠0).

A Riemannian manifold (B. Riemann, 1854) is a manifold endowed with a symmetric regular bilinear field, called the metric field, and definite (positive); in coordinates,

$$g;\ \ g|_U = ds^2 = g_{\mu\nu}\, dx^\mu\, dx^\nu \qquad \text{(I-3.9)}$$

and it serves to define areas: length of curves, areas o surfaces, volume of manifolds, etc. Positivity, $g_P(u, u) \geq 0$, is relaxed in the Lorenztian case.

Any manifold admits a Riemann metric:

$$\begin{array}{l} O(n) \\ \downarrow \\ GL(n) \rightarrow B(M) \rightarrow M \\ \downarrow \\ |R^N \text{ ------ } E \text{ ----- } M \end{array} \qquad \text{(I-3.10)}$$

where N = dim GL(n) – dim O(n) = n^2 – n(n-1)/2 = n(n+1)/2:

As the $|R^N$ space is *contractible*, the associated bundle ---- is trivial, hence the GL-bundle reduces to the O-bundle (this requires M *paracompact*, but this is included in the definition of M); and, of course, as O(n) is the isotropy group of a symmetric regular *positive* bilinear form, any manifold can be endowed with a Riemann metric; but *not* any manifold admits a metric with signature ≠ 0.

Riemann spaces were introduced in physics by Einstein,1915 (with signature (n-1, 1; n=4)). The idea (anticipated by Riemann and considered also by Clifford) is that the geometry of our space(time) is not fixed *a priori*, but determined by the matter/energy content of the universe. Metrics with g not definite are as easy to handle as with g positive; the crucial property is *regularity*, i.e. det (g) ≠ 0.

A *symplectic manifold* (M, ω) is a manifold M endowed with a regular 2-form ω. This requires, at least, M to be even dimension and orientable: if $A = -{}^tA$ and $\det A \neq 0$, dim A is even; on the other hand ω^n is a volume form, hence M must be orientable.

The typical domain of the symplectic geometry is *classical mechanics*: let us recall the construction of *Hamiltonian mechanics*: let H = H(q, p) the Hamilton function, and let $\omega = dp \wedge dq$ (= dp dq) the symplectic form. Construct now the row

$$H \to dH \to \omega^{-1}(dH) := X_H \to (\text{integ.})\ \tau_t^H, \quad \text{the flow or motion.} \qquad \text{(I-3.11)}$$

As ω is regular, one takes inverses, to convert 1-forms θ in vector fields X; the integration of the generator $X = X_H$ gives the curves of motion. There is a *restricted* inverse process: 1-parametric groups τ_t of symmetries generate *constants of motion*; we skip the details.

4 & 5.- Connections and curvature. Curvature as something *intrinsic* (= embedding independent) was introduced for surfaces by Gauss (1827) and generalized by Riemann (1854); it was seen as consequence of *connections*, rather than of metrics, by Levi-Civita and Weyl (ca. 1917). Understood as operation in general bundles by Ehresman (1950). We shall define connections on *vector* bundles; but more generally, they are defined in *principal* bundles [KN].

Let ξ: $V \to E \to M$ be a vector bundle. A *connection* ∇ is a *linear* map from sections in ξ to sections in the tensor product $\otimes$ of ξ with the cotangent bundle T*M, which is a *derivation*: if $f \in \mathcal{F}(M)$, $f s = s'$ makes sense, as $\Gamma(\xi)$ is $\mathcal{F}(M)$-module, and the defining properties of the connection are

$$\nabla: \Gamma(\xi) \to \Gamma(\xi \otimes T^*M); \qquad \nabla(fs) = (df)\cdot s + f\nabla s \qquad \text{(I-5.1)}$$

∇ is also called the *covariant differential* (of the connection). The covariant *derivative* with respect to a vector field X is the *contraction* with the vector field X:

$$\nabla_X s \equiv X \lrcorner \nabla s \qquad \text{(I-5.2)}$$

So covariant *differential* increases the indices by adding a covariant one, whereas covariant *derivative* conserves de index. A connection in coordinates is a matrix of 1-forms: if $\{s_i\}$ is a *frame*, $\{s_i\} = \varepsilon$ (in some coordinate patch U), $\nabla s_i = \omega_{ij}\ s_j$, where ω_{ij} is a $n \times n$ (with n=dim fibre) matrix of 1-forms in M, so for short we write $\nabla\varepsilon = \omega\varepsilon$. Connection components transform *inhomogeneously*; they are *not* tensorial: take another frame $\varepsilon' = g\cdot \varepsilon$, where g is unique, and define

$$\nabla\varepsilon' := \omega'\varepsilon'; \text{ then } \nabla\varepsilon' = \nabla(g\varepsilon) = (dg)\cdot\varepsilon + g\omega\varepsilon = \omega' g\varepsilon, \text{ or}$$

$$\omega' = dg\cdot g^{-1} + g\cdot\omega\cdot g^{-1} \qquad (1\text{-}5.3)$$

What about *curvature*? Write $\wedge^2 M$ for the (total space of) the bundle of 2-forms, and try to prolongate ∇ as *antiderivation*:
$\nabla^o\nabla := \nabla^2 : \Gamma(\xi) \to \Gamma(\xi\otimes\wedge^2 M)$

$$\nabla(\theta s) := d\theta\cdot s - \theta\wedge\nabla s \qquad (I\text{-}5.4)$$

One shows [Milnor] that $Ks := \nabla^2 s$ is *tensor*, i.e. it satisfies $K(fs) = f\,K(s)$, and it is called the *curvature* of the connection. The connection is named *flat* (not zero, which makes no sense) when K=0 (which does!). In terms of the 1-form matrix ω, it is

$$\nabla^2 s = Ks = \nabla(\omega s) = d\omega\cdot s - \omega\wedge\nabla s = d\omega\cdot s - \omega\wedge\omega\, s, \quad \text{or}$$

$$K = d\omega - \omega\wedge\omega \qquad (I\text{-}5.5)$$

formula familiar to physicists (" $F = dA + A\wedge A$" for Yang-Mills fields). Here K is clearly (in the frame s) a $n\times n$ matrix of 2-forms ("field strength" in physics); we have also

Bianchi identity: apply d (not ∇!) to (1-5.5); it is

$$dK = -(d\omega)\wedge\omega + \omega\wedge(d\omega) = -(K+\omega\wedge\omega)\wedge\omega + \omega\wedge(K+\omega\wedge\omega) = -\omega\wedge K + K\wedge\omega$$

$$\Leftrightarrow \quad \nabla K \underline{=} 0, \text{ with the natural definition } \nabla K := dK + \omega\wedge K - K\wedge\omega \qquad (I\text{-}5.6)$$

[For all this see Milnor-Stasheff, Ap. C].

Take now as bundle the *tangent bundle TM*; there is an *extra* tensor, the so-called *torsion* of the connection, $T = T^{\nabla}$

$$T(X, Y) := Z = \nabla_X Y - \nabla_Y X - [X, Y] \qquad (I\text{-}5.7)$$

It is a tensor, namely $T(fX, Y) = fT(X, Y) = T(X, fY)$ (1-5.7′)

The Levi-Civita connection. If (M, g) is a Riemannian manifold, there is a *unique* connection, characterized for being torsionless ($\Leftrightarrow$ *symmetric*, $T = T^{\nabla} = 0$) and *isometric* ($\nabla g = 0$). The theorem is *constructive,* in the sense that the explicit form of the connection is computable easily, and allows for signature; for the positive (g>0) case *or not*, the coordinates of the connection are given by the classical Christoffel formulas:

$$[\mu\nu, \lambda] := \tfrac{1}{2} (g_{\nu\lambda, \mu} - g_{\mu\nu, \lambda} + g_{\lambda\mu, \nu}) \qquad \text{(I-5.8)}$$

$$\Gamma^{\lambda}{}_{\mu\nu} = g^{\lambda\rho} [\mu\nu, \rho] \qquad \text{(I-5.9)}$$

The curvature in TM can be defined as a *2-form operator on vector fields, or a 2-form Endomorphism-valued*:

$$R(X, Y) = [\nabla_{X,} \nabla_Y] - \nabla_{[X, Y]} \qquad \text{(I-5.10)}$$

Hence $R(X, Y)\cdot Z = W$, or in coordinates $\partial_\lambda = \partial/\partial x^\mu$

$$R(\partial_\mu, \partial_\nu)\cdot\partial_\lambda := R^{\rho}{}_{\lambda\mu\nu}\, \partial_\rho \qquad \text{(I-5.11)}$$

The Riemann tensor has symmetries (consider four lower indices $R_{\sigma\lambda\mu\nu}$, called the *Riemann-Christoffel tensor* at times, $R_{\sigma\lambda\mu\nu} = g_{\sigma\rho} R^{\rho}{}_{\lambda\mu\nu}$)

i) Antisymmetric in $\mu\nu$ and in $\sigma\lambda$ independently

ii) Symmetry changing both pairs $\mu\nu \Leftrightarrow \sigma\lambda$

iii) Fully antisymmeric part =0 (from Bianchi identity*)

Hence the Young Tableau symmetry of Riemann´s is $[2^2]$, with

$$\text{Dim Riemann} = n\cdot(n+1)\cdot(n-1)\cdot n \cdot 2/4! = n^2(n^2-1)/12 \qquad \text{(I-5.12)}$$

which gives dim Riem = 0, 1,6, 20, 50 for dim = 1, 2, 3, 4, 5.

Properties i) and ii) mean the liberties are $[1^2]\vee[1^2] = [2^2]+[1^4]$; but iii) eliminates $[1^4]$.

The Riemann tensor Riem, a $T^1{}_3$ tensor, has *two* contractions: First

$$Ric_{\lambda\mu} := R^{\rho}{}_{\lambda\mu\rho} \text{ , which is } \textit{symmetric}\text{; and also} \qquad \text{(I-5.13)}$$

$$R = R_{sc} \equiv Tr\,(g^{-1}\cdot Ric) = R^m{}_m \qquad \text{(I-5.14)}$$

So in general $\quad Riem \approx Weyl + Ric(\text{trace-less}) + R_{sc} \qquad$ (I-5.14´)

* Sometimes *Bianchi identity* is reserved for a cyclic derivative property of Riem; (Cfr. [KN],I, p. 135).

where Weyl = (traceless part of Riemann´s) has dim Weyl = dim (Riem – Ric) = $n(n+1)(n+2)(n-3)/12$ (I-5.15)

Let us consider the lowest-dimension cases:

If <u>dim M = 1</u>, $g = ds^2 = f(x)\cdot dx^2$; with $y = \int \sqrt{f}\cdot dx$, it becomes $g = dy^2$: *any curve is rectifiable*: curvature requires bi-planes, $x \perp y$ and there is none in 1 dim.

If <u>dim M = 2</u>, a single curvature suffices, as only a biplane; indeed

$$K \underline{=} \mathrm{Curv(Gauss)} = R_{sc}/2 \qquad \text{(I-5.16)}$$

For example $\mathrm{Ric} = \frac{1}{2}\, g\cdot R_{sc}$. Similar for Riemann itself,

$$R_{\sigma\lambda\mu\nu} = \tfrac{1}{2}\,(g_{\sigma\nu}g_{\lambda\mu} - g_{\sigma\mu}g_{\lambda\nu})\, R_{sc} \qquad \text{(I-5.17)}$$

In this dim 2 case the choice of coordinates $(uv) \rightarrow (u'v')$ allows a general metric $g = ds^2 = E(uv)du^2 + 2F(uv)dudv + G(uv)dv^2$ to be put in <u>Geodesic</u> (I) or <u>Isothermal</u> (or *conformal*) (II) forms

(I): $ds^2 = du^2 + G(u, v)\, dv^2$ (I-5.18g)
(II): $ds^2 = \exp[2\sigma(u, v)](du^2 + dv^2)$ (I-5.18i)

In <u>dim M =3</u>, we have: dim Riem = 6 = dim Ric, hence Weyl $\underline{=}$ 0. In three dimensions, this means g can be put in orthogonal or *diagonal form*

$$G = ds^2 = E(uvw)du^2 + F(uvw)dv^2 + G(uvw)dw^2 \qquad \text{(I-5.19)}$$

It is in only in dim ≥ 4 that the Riemann tensor, Riem, exhibits all its *grandeur*; for example, in dim 4, dim Weyl = dim Ric = 10, so

dim Riem = 10 + 10 = 20. Gravitation in 3-D is "conic", with *no* propagating modes; gravitation in 4-dim has the same degrees in the *gravistatic* part as in the *radiating* part. Also, as 2 = 4/2, the curvature of 4-dim manifolds can be (anti-)*selfdual*.

One shows that Riem is the obstruction to flatness (Christoffel): any (M, g) manifold can have a metric sum of squares iff Riem = 0.

One shows likewise that the Weyl tensor W is the *obstruction to conformal flatness* (Gauss in 2-dim; Weyl in general): any (M, g) manifold, dim ≥ 4 can have a metric sum of squares but for a common factor, that is

$ds^2 = g = \exp[2\sigma(x_i)](dx_1^2 + \ldots dx_n^2)$ iff Weyl = 0 (Weyl, 1917); somehow *conformal* is associated to *traceless*:

In dim 1, Riem = 0; in dim 2 Ric(traceless) = 0; in dim 3 Riemann traceless (= Weyl) = 0. In dim 2: any metric is conformally flat; in dim 3, any metric is orthogonal (above). The obstruction to conformal flatness in 3-dim is measured by a different tensor, the *Cotton* (1899) tensor.

6.- Homotopy groups and Spin groups. The reader knows perhaps the *fundamental group* $\pi_1(X)$ or first homotopy group of a topological space X: call Ω_P the set of *loops* (= closed paths) starting and ending in $P \in X$. *Composition* of loops is immediate: the second starts when the first ends; declare two loops *equivalent* if deformable continuously one onto the other; then Ω_P/(equiv. relation) $\equiv \pi_1(X, P)$ becomes a group, the *fundamental group* of the manifold (H. Poincaré, 1896): the Id is the class of contractible loops (= deformable to a point, shrinkable to a point, nul-homotopic). If the space is *arcwise* connected (all connected manifolds are) the P-dependence is spurious, and one talks only of $\pi_1(X)$.

Some examples: $\pi_1(S^1) = Z$, $\pi_1(S^2) = 0$, $\pi_1(SU(2)=S^3) = 0$, $\pi_1(SO(3) = |RP^3) = Z_2$; other examples later. (I-6.1 & 2)

Hurewicz generalized in 1935 (in 1940 died from accident visiting the Maya ruins in Mexico) to higher homotopy group:

The $\pi_0(X)$ *set* is the set of maps: $S^0 \to X$; as S^0 is just two points, it just measures the distinct connected pieces of X:

$$X \text{ connected} \Leftrightarrow \text{card } \pi_0(X) = 1 \qquad \text{(I-6.3)}$$

If X = G is a Lie group, then $\pi_0(G)$ *is* a group; for example, =(3, 1) has four components, and π_0 (Lorentz 3, 1 group) $= Z_2 \times Z_2 = V$. (I-6.4)

The fundamental group can be seen, of course, as maps from (pointed) circles S^1 to (pointed) X, with suitable equivalence. Then the generalization is obvious: $\pi_n(X)$, n = 0, 1, 2, … are the classes of maps from $S^n \to X$ with suitable distinguished points, equivalence and quotient; we omit details (see e.g. Steenrod, Nakahara). All these are important topological invariants. J. P. Serre found (ca. 1952) that the homotopy groups of spheres were (*generally*) finite, and computable. $\pi_n(S^n) = Z$ is an old result; but $\pi_3(S^2) = Z_2$ is a surprise!

We have, summing up

0) $\pi_0(X)$ is just a discrete set; if X= G Lie group, $\pi_0(G)$ *is* a group.

1) $\pi_1(X)$ is, in general, a nonabelian group; but if X=G, a Lie group, it is abelian. For example, $\pi_1(\Sigma_g)$ is *nonabelian* for g > 1, where Σ_g is a Riemann surface of genus g: sphere, torus and "pretzel" for g= 0, 1, 2. For the tours $T^2 = (S^1)^2$, we have clearly π_1(torus)=Z+Z.

2) Any connected space with $\pi_1(X) > 0$ has a *unique* universal covering space X^, with a natural *onto* map X^→X with inverse images $\approx \pi_1(X)$:

$$\pi_1(X) \rightarrow X\hat{} \rightarrow X \qquad \text{(I-6.5)}$$

3) $\pi_{n>1}(X)$ is an *abelian* group.

4) If G is a (finite-dim!) Lie group, $\pi_2(G)=0$ (no simple proof!) (I-6.6)

5) The homotopy groups exist for any n, *regardless* the dimension of the space X; for example, $\pi_4(S^3) = Z_2$. In this it differs from co-& homology.

The homotopy groups of a space X are naturally topological *invariants*; let us include the following result of Hurewicz:

--- The *first nonnull* homology group of a manifold, $H_k(X; Z)$ coincides with the abelianized of the first nonnull homotopy group, $\pi_k(X)$ (recall: for any group G, Ab(G), the abelianized of G, is G/ΩG, with ΩG is the *commutator* or first derived group of G). For example: b_1(pretzel) = 4, and π_1(pretzel) is generated by 4 elements a, b, c, d subjected *only* to the relation $abcda^{-1}b^{-1}c^{-1}d^{-1}=1$: the abelianized H_1(pretzel; Z) is clearly Z + Z + Z + Z.

To end up this fast survey of homotopy, let us "see" the homotopy solution to the problem of the types of *principal bundles over spheres*:

$$G \rightarrow P \rightarrow S^n \qquad \text{(I-6.7)}$$

How many principal bundles P there are? The spheres are covered with two charts, skipping e.g. the poles, as $S^n\backslash\{0\} \approx R^n$. Then, as the two charts overlap at the equator, one shows the possible total spaces P´s are given by homotopy classes of maps from the equator in S^n to the group G; indeed, the precise result is (Steenrod)

$$\text{G-bundles over } S^n \approx \pi_{n-1}(G) \qquad \text{(I-6.8)}$$

<u>Spin Groups. Spin(n)</u>. Consider now the *real* orthogonal group O(n) = O(n, |R): n = 0, 1, 2, …; the n × n matrices o verify ${}^t oo=1$: hence det o =

±1: the orthogonal group has two connected components; call SO(n) the piece connected with the identity; it is called the *rotation* group. We have:

$$O(1) = Z_2 \approx S^0. \quad \text{Hence } SO(1) = \mathrm{Id}. \qquad \text{(I-6.9)}$$

$SO(2) = U(1) = S^1$, the circle. We have $\pi_1(S^1) = Z$. All higher π´s of the circle S^1 are zero, as comes from the "exact homotopy sequence", where |R is the universal covering of the circle (Steenrod):

$$Z \to |R \to SO(2) \approx S^1 \qquad \text{(I-6.10)}$$

because $|R^n$ is contractible ($\Leftrightarrow$ all homotopy groups $\neq 0$ are zero).

$SO(3) \approx RP^3$: any 3d rotation has an axis and an angle, so $SO(3) \approx$ solid 3-dim ball with radius π, but with antipodal points in the boundary 2-sphere identified; this is clearly homeomorphic to $|RP^3$, the 3d real projective space, $\approx S^3/Z_2$, where the quotient is by the antipodal map. Hence $\pi_1(SO(3)) = Z_2$, and this result holds for all higher rotation groups:

$$\pi_1(SO(n)) = Z_2,\ n \geq 3 \quad \Leftrightarrow \quad Z_2 \to \mathrm{Spin}(n) \to SO(n) \qquad \text{(I-6.11)}$$

Hence, for $n \geq 3$ there is a *unique double covering of the rotation group*, which is (clearly) also a group, called the *Spin(n)* group. The reader should see that $SO(3)\hat{} = S^3$, and also = SU(2), as he knows from quantum mechanics. There are also other *coincidences*, to wit

$\mathrm{Spin}(1) = Z_2$; $\mathrm{Spin}(2) \approx U(1)$; $\mathrm{Spin}(3) = SU(2)$; $\mathrm{Spin}(4) = [\mathrm{Spin}(3)]^2$ (I-6.12)

Spin(5) = Sq(2); Spin(6) = SU(4); Spin(7, 8, 9) relat. to octonions (I-6.13).

For n = 1, 2 the definition of Spin(1, 2) is just a *double covering*, it is *not* the universal one, but it is well defined (through Clifford algebras).

The orthogonal groups in 2, 4 and 8 dimensions are *special*. In dim 2: the rotation part is abelian and *divisible*, $SO(2)/Z_n \approx SO(2)$; the existence of the complex numbers and the regular plane polygons is related to this fact.

Dim 4: The Spin(4) group factorizes: $\mathrm{Spin}(4)=[\mathrm{Spin}(3)]^2$: the Lie algebra are 2-forms in $|R^4$, hence split in self & antiselfdual. The existence of the quaternion numbers |H and of the special regular polytopes in 4 and 3 dimensions (e.g. the 24-cell in 4d, the icosahedron in 3d) are consequences of this fact.

Dim (8): Spin(8) exhibits <u>triality</u>, an outer automorphism group S_3, of order 6: it is at the base of *SuperSymmetry* in physics! It starts with the two chiral and the vector representations of dim 8 the three: they are permuted by this S_3.

For all this see e.g. [Conway-Smith]

Finally, the *homotopy* of orthogonal groups enjoys Bott´s periodicity: for a *generic* spin (n big enough), the *stable* homotopy groups are

$$\pi_{01234567}\,(O(n), (n >>)\,) = Z_2, Z_2, 0, Z, 0, 0, 0, Z. \qquad \text{(I-6.14)}$$

whereas for the symplectic groups there is a shift by four:

$$\pi_{01234567}\,(Sq(n), (n >>)\,) = 0, 0, 0, Z, Z_2, Z_2, 0, Z. \qquad \text{(I-6.15)}$$

For completeness we add the periodicity TWO for unitary groups

$$\pi_{01234567}\,(SU(n), (n >>)\,) = 0, Z, 0, Z, 0, Z, 0, Z. \qquad \text{(I-6.16)}$$

Spin(n) groups appear often in physics through *representations*; for the rotation group SO(n) the set of irreducible (unitary) *linear* representations, *irreps*, are given, starting from the vector, dim n, by the *traceless* Young tableaux; for example, for SO(5), the first *irreps* are of dim 1 (identity); 5 (vector); $5\cdot6/2-1=14$, type [2]´; $5\cdot4/2 = 10$, type $[1^2]$. Etc

Spin(2ν+1) has a primitive irrep of dim 2^ν, call it Δ. But Spin(2ν) has *two*, of different chirality, called $\Delta_{L,R}$ and dim again 2^ν. All (true, linear) *irreps* of SO(n) are real, of course, but the character (*type*) of the (one or two) spin *irreps* varies:

$\Delta(n)$, and $\Delta_{L,R}(n)$ types: <u>real</u> for n = 0, ±1; *complex* for dim 2, 6; <u>quasireal</u> (or quaternionic, or pseudoreal) for n = 4±1 (I-6.17)

The Spin groups realize *projective* representations of the rotation group; this is why they are important in physics, as quantum mechanics seeks projective (or *ray*) representations of physical symmetries; the ubiquitous appearance of SU(2) in Quantum Mechanics is because all projective *irreps* of SO(3), the physical rotation group, come from the linear *irreps* of the covering group, SU(2) = Spin(3).

II.- Holonomy

1&2.- Parallel transport and holonomy. When there is a connection ∇ (in any bundle; in our case in the tangent bundle) vectors (or sections) are propagated along curves; in particular, *geodesics* (in the *affine* conception) are defined by the flow of auto-parallel vector fields,

$$\nabla_X X = 0 \tag{II-1.1}$$

which in coordinates becomes the conventional geodesic equation

$$d^2x^\lambda/ds^2 + \Gamma^\lambda{}_{\mu\nu}\, dx^\mu /ds\, dx^\nu /ds = 0 \tag{II-1.2}$$

with length = arc; as it is a second order equation, we have two Cauchy data: for any point and in any direction starts a geodesic (e.g. meridians in the sphere).

In a Riemannian space (M, g) there is a *metric* definition of geodesics, minimizing the distance function

$$\Delta\int ds = 0, \tag{II-1.3}$$

which yields the same equation (II-1.2).

A vector field Y is *parallel translated* along the flow of another vector field X; the first order equation is (X and Y_0 known; find Y= Y(t))

$$\nabla_X Y = 0 \tag{II-1.4}$$

with a unique solution from a particular value $Y_0=Y(P_0)=v$. We know: if the connection is *flat*, there is no curvature; how do we measure curvature from parallel transport? Through *holonomy*: consider eq. (II-1.4) for a *frame* (a base) ε in P, and make it run through a loop γ; at the end it becomes another frame ε' in the same point P: as any two frames *at the same point* are related by an isometry (when the connection, as it is our case, is the Levi Civita connection $\nabla g=0$), we have an element o of the orthogonal group O(n) depending of the loop γ: $o(\gamma) \in O(n)$. It is easy to see how all loops starting/ending from a point P_0 *compose*, and the isometries make up a subgroup of O(n), called the *holonomy group* of the connection, Hol(∇) = Hol (g) (in our case); (Cartan, 1926).

One expects Hol = Id for flat connections, but there is a *constructive* counter-result:

The Ambrose-Singer theorem, 1953: The Lie algebra of the holonomy group is generated by the curvature (see e.g. [KN], Ch. 2; Nakahara p. 343).

It is a reasonable result, as the curvature $R^{\rho}{}_{\lambda\mu\nu}$ is 2-form (μ, ν) Lie-algebra evaluated (ρ, λ): antisymmetry in the second pair implies holonomy lies inside the orthogonal group (whose Lie algebra consists of ${}^{t}A = -A$ antisymmetric matrices).

For *contractible* loops the holonomy group has to lie in SO(n), the connected part of O(n). In fact, this is the general case for orientable manifolds; let us see this in detail. A manifold is *orientable* if the transition functions between charts U_i and U_j (see Ch. I) can be chosen in GL^+, the connected part of GL (GL has det > 0 OR < – 0: two connected components). Let us see the *obstruction*:

$$\begin{array}{lllll} GL^+ \rightarrow & & & & \\ \downarrow & & & & \\ GL & \rightarrow & B(M) & \rightarrow & M \\ \downarrow & & \downarrow & & \| \\ Z_2 & \rightarrow & B'(M) & \rightarrow & M \end{array} \qquad \text{(II-1.5)}$$

The middle row generates the lower row, which *defines* the first Stiefel-Whitney class w_1 (of the tangent bundle of) the manifold M. The middle row lifts to an upper row $GL^+ \to B'' \to M$ *iff* this class is zero:

$$\text{M orientable} \Leftrightarrow w_1(M) = 0 \qquad \text{(II-1.6)}$$

This is a nice example of measuring properties by absence of obstruction: the obstruction of orientability is given by the first sw class.

By a simple extension, if one asks when an oriented manifold admits a *spin structure*; it is to ask when the tangent bundle, with group SO(n), lifts to the *spin bundle*, with group Spin(n):

$$\begin{array}{ccccc} Z_2 & & & & \\ \downarrow & & & & \\ Spin(n) & \ldots\ldots > & \hat{B}_0 & \ldots\ldots > & M \\ \downarrow & & \downarrow & & \| \\ SO(n) & \rightarrow & B_0(M) & \rightarrow & M \end{array} \qquad \text{(II-1.7)}$$

We have the result (Nakahara): the lower row lifts to the middle row, that is, the manifold admits an spin structure, *iff* the *second* Stiefel-Whitney class of the tangent bundle is zero, $w_2(TM) = 0$. For spheres it is easy to show that all are *spinable*,

$$w_2(TS^n) = 0 \qquad \text{(II-1.8)}$$

For oriented *surfaces* Σ_g, g the genus (0, 1, 2 for sphere, torus, pretzel) there is *no* obstruction to spin structures: $w_2(\Sigma_g)=0$. Let us see it just for the ordinary 2-sphere S^2: The principal bundle of the tangent bundle is

$$SO(2) \rightarrow SO(3) \rightarrow S^2 \qquad \text{(II-1-9)}$$

But this have a "square root", because SU(2) covers SO(3) universally: hence lifting: SO(2) lifts to a double covering, and this is the spin bundle. Alternatively, the Euler "class" carries to the top sw class, and as $\chi(S^2) = 2$, it is = 0 under mod 2; so $w_2(S^2)=0$; compare [Milnor, p. 99].

Once a manifold admits a spin structure, how many (inequivalent) does it admit? It is easy to see: as many as elements in $H^1(M, Z_2)$. This result bears on String Theory, as one is supposed to sum over all possible spin structures; in fact $H^1(\Sigma_g, Z_2) = 2^{2g}$, which is therefore the number of spin structures in a "worldsheet" of genus g [GWS II, p. 278].

[Any n-dim real vector bundle has $w_1, w_2, \ldots, w_{2n}$ Stiefel-Whitney classes, which take values in the Z_2 cohomology, $w_i \in H^i(M, Z_2)$; [Milnor]].

In the connection of connections, we have also the *reduction theorem* (Cfr. again [KN], Ch. 2):

Reduction Theorem: The structure group of the (vector) bundle reduces to the holonomy group.

This means: the transition functions can be taken in the holonomy (sub)group. The theorem is again reasonable, because we can arrange the transition functions from the parallel transport, so they transform among themselves with the holonomy group.

As a consequence, for an orientable manifold the holonomy group lies in SO(n); but it can be smaller, of course, when the connection conserves some particular objects (some tensor field, for example); see below.

3-6.- Classes of Holonomy groups. It is interesting to see when the holonomy group is smaller that O(n) or SO, but not Id. The problem was dealt with by M. Berger in 1955, with nearly complete results; the outcome is related to which subgroups of the orthogonal group still act *transitively* on spheres.

[Disgression: A Lie group G *acts* "differentiably" in a manifold M if g·x=y is diffeorphism (g in G; x, y in M); the action is *effective*, if g·x=x for all x => g = Id. Otherwise there is a *ineffectivity kernel* K, and G/K operates effectively. The action is *transitive (trans)*, if G·x= M, from any x.

In general, the images G·x are called "the orbit of G through x". The *isotropy* group (called *little group* (Wigner) in physics) is G_0, when G_0·x=x: points x, y in the same orbit, G·x = G·y have conjugate isotropy groups. When G acts *trans* in M, the quotient space G/H =M has a well-defined sense as manifold, and M is called a *homogeneous space*; *symmetric spaces* are an enhancement of homogeneous spaces (Helgason)], when there is a geodesic reflection symmetry.

It is interesting to see when the holonomy group is smaller that O(n), but not Id. It turns out the possible holonomy groups of any Riemann manifold are related to the four division algebras |R, C, |H and O !! They are also, as said, related to transitive action of groups on spheres, and thirdly they depend on Riemannian spaces leaving some other object (besides the metric), invariant under parallel displacement (for example, the holonomy group of orientable manifolds leaves a volume form invariant…).

Let us consider (M, g) an *irreducible nonsymmetric* Riemann manifold; the possible holonomy groups are 2×3 series for the fields (or skew-)field = |R, C and |H) plus two exceptional cases (for Octonions O). Product manifolds have product holonomy groups; and all symmetric spaces are known, so they are *excluded* from the list *by convention*.

Let us comment on the Table below. *Holonomy* reflects the "parallel" tensors $\nabla T=0$; as said, orientable manifolds maintain a volume fix, hence $Hol(\nabla) = O(n) \cap SL(n) = SO(n)$: It is the most common case, as most manifolds are required to be orientable (for integration, etc.). Recall a *complex* manifold (M, J) can acquire an hermitian metric: the "g" part is automatic (i.e., always possible) in any Riemann manifold, and ω is concocted from J: $g(J) = \omega$. The natural Levi-Civita connection leaves ONLY g invariant; but the manifold is called Kähler if *also* leaves ω fix, that is, $\nabla\omega = 0$, equivalent to ω closed, $d\omega=0$: hence the holonomy of Kähler manifolds is $O(2n) \cap Sp(2n, R) = U(n)$. If the holonomy descends to SU(n), the manifold is called Calabi-Yau (for historical reasons).

Quaternionic and hyperkähler manifolds similarly preserve some object related to the skew-field of the quaternions. As $Sq(n) \subset SU(2n) \subset SO(4n)$, hyperkähler manifolds are at least Calabi-Yau, Kähler and orientable. Quaternionic manifolds need not to be CY nor Kähler.

There also *two* singular cases related to the Octonions O.

[Recall the *complex* numbers ¢ are a 2-dim composition and division algebra over |R with the *unit* (0, 1) $\equiv$ i fulfilling $i^2 = -1$; the *quaternions* (4-dim over |R) have *two* independent units, i and j, with $i^2 = j^2 = (ij \equiv k)^2 = ijk = -1$. For the *octonions* O there are *three* independent units: i, j, k, anti-involutive and anticommuting, and the same for the products (ij, jk, ki, (ij)k), which forces *alternativity, en lieu* of associativity, namely $(ij)k = -i(jk)$. The octonions (8-dim over |R) are a division algebra; but one cannot proceed beyond: no division or composition |R-algebras except in dim 1, 2, 4, 8: (Hurwitz´ theorem, 1895); see e.g. [Conway]].

Table of HOLONOMY GROUPS

O(n) generic	SO(n) orientable, $w_1=0$	|R
U(n) Kähler, $\nabla\omega=0$	SU(n) Calabi-Yau CY, $c_1=0$	C
q(n) Quaternionic	Sq(n) Hyperkähler	|H
Spin(7) Dim M =8	G_2 Dim M = 7	O=octonions

(II-3.1)

The descent from U(n) to SU(n) is similar to the commented O(n) to SO(n): the Det map generates the bundle $U(1)=U(n)/SU(n) \to B'(M) \to M$, which defines the *first Chern class* (of the tangent bundle, $c_1(TM)$) of the complex manifold M; M Kähler becomes Calabi-Yau (CY_n, Calabi-Yau n-fold, real dim 2n) iff $c_1(M) \equiv c_1(TM) = 0$.

[For complex vector bundles η, Chern classes c_i, (i: 1 to dim η) take values in $H^{2i}(M, Z)$: there is a "trasgression" from U(1)-bundles to H^2 Z-comomology because the *resolution*, alluded to above, $Z \to |R \to S^1 = U(1)$, and similar for higher Chern classes].

The hermitian metric with quaternionic entries admits the isometry group Sq(n) (sometimes written Sp(n): it is the *real compact* form of the C_n-series of Cartan´s simple complex Lie algebras). As $Sq(n) \subset SU(2n)$, hyperkähler manifolds are also CY, with first Chern class = 0 (as they are oriented, of course, because U(n), SU(n), q(n) and Sq(n) are connected and all lie in some SO(N)).

The group q(n) [*notation not universal*!] is defined as

$$q(n) = Sq(n) \times /_2 \, Sq(1) \qquad \text{(II-3.2)}$$

and the quotient is by the common centre Z_2. We have in particular

$$Sq(1) = SU(2) = Spin(3) \approx S^3 \qquad q(1) = SU(2)^2/Z_2 = SO(4) \qquad \text{(II-3.3)}$$

7.- <u>Octonions and Octonion-related Groups</u>. Finally, as the octonions O are *not associative*, there are only FIVE groups related to then, Cartan´s exceptions G_2 (dim 14), F_4 (dim 52), E_6 (dim 78), E_7 (dim 133) and E_8 (dim 248); in some ways the spin groups Spin(7, 8 and 9) can be considered *also* as octonionic groups, and indeed G_2 as well as Spin(7) can act as exceptional holonomy groups (see Joyce´s book). It is to be remarked that both types of exceptional holonomy manifolds are Ricci flat (but neither Calabi-Yau). In particular a manifold with holonomy G_2 preserves a *generic* 3-form in 7-dim (check: $7^2 - \{7, 3\} = 49 - 35 = 14 = \dim G_2$), whereas one with Spin(7) holonomy preserves a *particular* four-form called a *Cayley four-fold*; recall the spin representation(s) of SO(7) acts in $|R^8$: $2^{(7-1)/2} = 8$, real type, so $Spin(7) \subset SO(8)$ in a natural manner.

So the G_2 & Spin(7)-manifolds have a *Ricci-flat* metric, that is, the Riemann tensor is given just by the traceless part, the Weyl tensor; this makes the construction of these manifolds difficult, as witness [Joyce]´s book.

The five exceptional groups G_2, F_4, $E_{6,7,8}$ are related to the octonions; in particular G_2 = Aut (Oct): it acts *trans* on the 6-dim sphere of unit imaginary octonions, S^6, so the defining representation has dim 7. The isotropy group is SU(3) (14–6=8), acting on the diameter $S^5 \subset R^6$ through the *real* irreducible $3 + \bar{3}$ representation; it is another suggestion that perhaps the group SU(3) appearing in physics (as color and flavour group) might be connected with the octonions, also!

F_4 acts in the 3×3 hermitian traceless Jordan octonionic matrices J(3), with real dimension (3·8+(3-1)=26, which is the defining representation of F_4; there is a famous octonionic projective plane (it should be called the Moufang plane, *not* the Cayley plane), OP^2: the isometry group is again F_4, and $OP^2 = F_4/Spin(9)$, $2\cdot 8 = 2\cdot 26 - 9\cdot 8/2 \Rightarrow 16 = 52 - 36$), and in fact there is a natural inclusion $OP^2 \subset J(3)$, as a class of idempotent elements. A noncompact form of the E_6 group also acts in O^2 as projective transformations; E_7 and E_8 have a more complicate definition (Freundenthal, 1955; Tits). We expect our understanding of these two final exceptional group will increase in the future; for example, there is a mysterious relation between the four exceptional groups (besides G_2) and spiun group, in the following sense (Adams):

F_4 can be formed from O(9) and the spin representation: 36+16=52: this is based on the OP^2 space of above (II-7.1)

E_6 can be formed from O(10) and the spin representation + U(1):

$$10\cdot 9/2 + 2\cdot(2^{10/2-1}) + 1 = 78 = \dim E_6 \qquad \text{(II-7.2)}$$

E_7 is formed starting from O(12), the spin representation and Sq(1):

$$12\cdot 11/2 + 2\cdot(2^{12/2-1}) + 3 = 66 + 64 + 3 = 133 \qquad \text{(II-7.3)}$$

E_8 is concocted from O(16) with one of the spin representation:

$$16\cdot 15/2 + 2^{16/2-1} = 248 \qquad \text{(II-7.4)}$$

The two intermediate cases need en extra ingredient, U(1) for E_6 and Sq(1) for E_7: this is well understood, as E_6 is complex-type, and E_7 is quaternionic type.

E_8 is the most spectacular of these exceptional groups: its fundamental representation is the adjoint one (dim 248) (a *unique* case among all simple Lie groups), it has no centre, neither outer automorphisms; it has 5-torsion, also unique among Lie groups. It appears in modern physics in several disguised forms: the square of it (E_8^2) is the gauge group of the heterotic exceptional strings; it acts also as gauge group in M-Theory (D. Freed); it is related to the Hodge diamond of the K3 surface (see later), etc.

III.- Strings and Higher Dimensions

1.-Higher dimensions in physics: Kaluza-Klein theories. The traditional tool for microphysics has been the *Quantum Theory of Fields*, developed since the 20´s and much improved with the renormalization program of 1947/52 (Schwinger, Feynman, Dyson…) and for the nonabelian case (`t Hooft) in 1971/73. Starting around 1975, physicists were frustrated by their unability to quantize gravitation (it is *intrinsically* non-renormalizable: the coupling constant G_N has dim $(\text{length})^2$), and one should look for new avenues; three merit our attention here:

1) Extra dimensions (of space-time)

2) Supersymmetry (mixing bosons with fermions)

3) Extended objects (like strings, membranes, etc.) (III-1.1)

In 1919 T. Kaluza wrote an letter to Einstein, showing that if one sets up general relativity in five dimensions, and somehow disregards the fifth dimension as unobservable, the fields appearing in 4-dim were three: the usual gravitation field $h = h_{\mu\nu}$, a vector field $A = A_\mu$ and a scalar ϕ. He went on to show that the usual Einstein –Hilbert (EH) lagrangian in five dimensions

$$S[h_5] = (\text{cons.}) \int \sqrt{g}\, R_{sc}(h_5)\, d^5x \qquad \text{(III-1.2)}$$

decomposed in 4-dim as the usual EH action in four, plus the lagrangian for the e.m. field $F_{\mu\nu}^2$ (plus an extra piece for the ϕ, called the *dilaton* field). Later O. Klein interpreted the unobservability of the fifth dimension as due to compactification in a very small circle S^1, which incidentally proved to quantize the electric charge in the natural quantum version of the model: the two signs of the charge were the two ways to run through the circle.

KK´s ideas were clearly premature, if very exciting; they were retaken in the 80s, sixty years later (!), when the ideas of Grand Unified Theories (GUT) appeared and also SuperSymmetry.

Let us here only observe here why graviton in 5-dim generates extra matter in four: if M, N run ≤ 5, and $\mu\nu \leq 4$; it is

$$g_{MN} \rightarrow g_{\mu\nu} + A_{\mu 5} + \Phi_{55} \qquad \text{(III-1.3)}$$

The idea of unifying e.m. with gravitation kept Einstein busy until his death, in 1955 (but not necessarily only in the KK approach).

For *Bose* fields is not difficult to generalize (III-1.3): besides the graviton, the other Bose fields which appear in modern theories are just p-forms; call them ϕ, A, B, C, D… for 0-, 1-, 2-, 3- and 4-forms respectively. Then in one-dimension descent we have

$$g_{d+1} \rightarrow g_d + A_d + \phi_d;\ \ A_{d+1} = A_d + \phi_d;$$
$$B_{d+1} \rightarrow A_d + B_d;\quad C_{d+1} \rightarrow B_d + C_d;\ \text{etc.} \qquad \text{(III-1.4)}$$

The split of Fermi fields (gravitinos and ordinary, spin ½ fermions) is not so regular, as spinors have dimension powers of two; it will be indicated in each case later.

SuperSymmetry. There are *no* compelling arguments to unify bosons and fermions, and several for *not doing* that: different statistics (in principle!) and different transformation law under the Lorentz group. Nevertheless, since 1974 (with some Russian antecedents) theorists have looked at enlarged symmetry schemes, mixing up fermions and bosons; it is an ample subject, and we just put some simple examples

1) The Wess-Zumino model (1973/74). There are a scalar field A, a pseudoscalar field B and a Majorana spinor χ living in Minkowski space. The lagrangian is free massless at first instance:

$$\mathcal{L} = - \tfrac{1}{2} (\partial_\mu A)^2 - \tfrac{1}{2} (\partial_\mu B)^2 - \tfrac{1}{2}\, \chi\, \gamma\cdot\partial\, \chi \qquad \text{(III-1.5)}$$

Although just for three massless fields unmixed, the action $\int\mathcal{L} d^4x$ has a bose-fermi symmetry: define

$$\delta A = \bar{e}\ \chi, \quad \delta B = i\ \bar{e}\,\gamma_5\, \chi, \quad \delta\, \chi = \partial\cdot\, \gamma\, (A + i\, \gamma_5\, B)\, e \qquad \text{(III-1.6)}$$

where e is a fermionic parameter ($e^2=0$); these transformations leave $\mathcal{L}$ invariant; the supersymmetry generator is called Q, so the transformations should be understood e.g. roughly as

$$\exp(-eQ)\cdot A\cdot\exp(+eQ)=(1-eQ)\cdot A\cdot(1+eQ) \;\underline{=}\; A+\delta A =A + [Q, A]e \qquad \text{(III-1.7)}$$

as {Q, e}=0. Q and e are fermionic (spinorial) objects in some precise sense. One can go on and add (equal) masses and (some) interactions preserving this (super)symmetry (see e.g. P.West´s book): supersymmetry can be maintained.

2) Super Yang-Mills model in 10 dimensions. Massless fields transform with the (compact) little group of the light cone, here O(8) (it corresponds to O(2), helicity label, in 4-dim). The three primordial representations of the Spin(8) group have dimension 8:

dimension vector, 8.- dim chiral $\Delta_{L,\,R} = 2^{8/2-1} = 8$, real type (III-1.7) which makes it easy to write down a *Susy* Yang-Mills action in 10 = (1, 1) + (8, 0) dimensions:

$$S = \int \mathcal{L}\, dx = \int dx\, (- \tfrac{1}{4} F^2 + (i/2) \quad \bar{\psi}\ \Gamma\cdot D\, \psi\ )$$
(III-1.8)

where $F\cdot F = F^2$ with the explicit form, where

$$F = F_{\mu\nu}{}^{a} = \partial_\mu A_\nu{}^{a} - \partial_\nu A_\mu{}^{a} + g\, f^{a}{}_{bc} A_\mu{}^{b} A_\nu{}^{c} \qquad \text{(III-1.9)}$$

is the field strength, and where "a" labels an index of the gauge group G, a= 1, 2, …dim G.; notice ψ in (1.8) is a chiral field, so S is parity-violating. By dimensional reduction, we get a $\mathcal{N}$=4 theory in 4-dim:

A in 8-dim gives A plus 6 ϕ´s in 4-dim; ψ in 8 gives 4ψ´s in 4; the Susy pairing is: A$\leftrightarrow\psi_1$, and 3ψ´s $\leftrightarrow$ 6 ϕ ´s. Etc. This $\mathcal{N}$=4 theory enjoys wonderful convergence properties (it *is* finite; Mandelstam).

3) <u>SuperGravity theory in eleven dimensions</u>. The particle content is

$$\underset{44}{h} - \underset{128}{\Psi} + \underset{84}{C} \quad \text{(graviton, gravitino and 3-form)} \qquad \text{(III-1.10)}$$

This is a remarkable theory: first 11 is the highest dimension one can set-up SuperGravity (Cremmer-Scherk); second, it turns out the three-form C is coupled to a *membrane* (Townsend); third, the dimensional reduction to 10 dimensions reproduces the particle content of the IIA string theory (see below). Recently has been some progress in dealing with the apparent divergences of this theory.

<u>Extended Objects</u>. As in so many things, P. A. M. Dirac was pioneer: the first study of quantum physics of extended objects was a paper of Dirac (1962), trying to describe muons as an *excited membrane* of the electron; it lead nowhere. String theory emerged around 1970, and it has an interesting story; milestones were the set-up of fermions in 1971 (P. Ramond), the ambition to cover gravitation (1975) (J. Scherk-J. Schwarz), a first hint of a theory "of everything"; SuperStrings appear in 1978, with *Susy* also on the target space, and no tachyons (GSO projection); uniqueness and claims for "Theory of Everything (T.O.E.)" (1984) (Green-Schwarz), when anomaly cancellation pointed out to select the gauge groups (besides the dimension); advent of M-Theory (1995) (below), and general dismissal by the community of physicists (~ today!).

2.- Physics in ten, eleven and twelve dimensions:
strings, membranes, M- and F-Theory.

We just shall talk briefly of the FIVE superstrings theory established around 1985 as all the five are i) viable theories, ii) include gravitation, iii) include fermions with *Susy* & *Sugra*, iv) 3 of the five include gauge groups, potentially covering the group of the standard model “3, 2, 1”, and they are v) potentially renormalizable.

The FIVE SuperSymmetric theories fill up a pentagon

```
IIA -----  Het-Exceptional
  |              |
  |              |
IIB -----    Het-Orthogonal
   \                    /
    \                  /
     \                /
         Type I                    (III-2.1)
```

A brief description follows:

IIA: Describes closed, nonchiral, $\mathcal{N}$=2 Susy strings with no gauge groups

IIB: Describes closed, chiral, $\mathcal{N}$=2 Susy strings with no gauge groups

Het-Exceptional: Describes closed, $\mathcal{N}$=1 Susy strings with E_8^2 gauge group

Het-Orthogonal: Describes closed, $\mathcal{N}$=1 Susy strings, O(32) gauge group

Type I describes open (and closed) strings, $\mathcal{N}$=1, O(32) gauge groups.

The *particle content* of SuperStrings is easy to write: The fundamental SuperSymmetry is in 8 *euclidean* dimensions, as said (triality!):

$$8_v \Leftrightarrow \Delta_L \Leftrightarrow \Delta_R \qquad \text{(III-2.2)}$$

for instance, for the IIA theory, the particle content is

$$(8_v - \Delta_L) \times (8_v - \Delta_R) = (h + B + \phi,\ A + C;\ \Psi_{1,2} + \psi_{1,2}) \qquad \text{(III-2.3)}$$

that is: graviton h, 2-form B and dilaton ϕ appear always (NS sector); in the RR sector multiply two fermions, and one gets A (a 1-form) and C (a three form); and then, two gravitinos Ψ ($\mathcal{N}$=2) and two fermions ψ.

For the IIB theory, the chiralities are the same:

$$(8_v - \Delta_L) \times (8_v - \Delta_L) = (h + B + \Phi, B' + D^+ + \Phi'; \Psi_{1,2} + \psi_{1,2}) \quad \text{(III-2.4)}$$

where $D^\pm$ is a (anti-)selfdual 4-form.

For the Type I theory, with open *and* closed strings, we get

$$(8_v - \Delta_L) \wedge (8_v - \Delta_L) = (h + \Phi, B; \Psi_1 + \psi_1) \quad \text{(III-2.5)}$$

plus the gauge group O(32). The wedge $\wedge$ just means: take the *symmetric* part of the bose product, and the *antisymmetric* one of the Fermi, etc.

The content of the heterotic strings is the same, but the gauge group is E_8^2 in the first (exceptional) heterotic string. The name “heterotic” comes from the fact that closed $\mathcal{N}$=1 Susy is performed by substituting the 24 +2 – (8+2) dimensions of the “bosonic” string with a gauge group of rank 16, either O(32) or E_8^2.

This is obviously not the place to deal with string theory *in extenso*. We mention only some of the outstanding features: one writes a lagrangian minimizing area of the *worldsheet*, and tries to quantize it. *Gravitons* follow at once from the closed sectors; the theory is potentially anomalous, (mainly because endurance of *conformal* symmetry to get quantized), that is, some of the symmetries of the naïve classical theory do not survive quantization, unless some conditions are met: this fixes *dimensions* of target space (24+2=26 for the original, bosonic string, and 8+2=10 for the superstring) and also the *gauge groups* (as stated above). This critical anomaly cancellation was the big advance in 1984/85 (Green-Schwarz).

Incidentally, we are at odds when quantizing *higher extended* objects: there is no, at the moment, accepted scheme for quantizing membranes (p = 2) or higher p-objects (it was for this reason that a long time the “pope” Ed Witten was reluctant to accept e.g. the membrane appearing naturally in 11-dim Supergravity).

The most appealing theory was the heterotic exceptional string, in part because there was only $\mathcal{N}$=1 Susy (16 supercharges, down to $\mathcal{N}$=1 or four in dim 4), and also because one hoped the E_8^2 group would naturally give rise to some of the GUT groups, like E_6, SO(10) or SU(5).

3.- M- Theory. In 1995 three fairly independent advances made the five superstrings theory merge in a so-called "M-Theory", living in eleven dimensions and including also other extended objects besides strings, in particular membranes:

a) Townsend proved (in January) that 11-dim SuGra (see above particle content) contained a *membrane* (or p=2 extended object), coupled to the three-form C (as "old" strings were coupled to the NS 2-form B), becoming the fundamental string upon circle compactification to ten dimensions.

b) Witten proved (in March) that the *strong limit* of the IIA theory developed an 11-th dimension, with particle content that of 11-dim Sugra.

c) Polchinski proved (in October) there were some "D-p Branes" as endings of open strings, a kind of "solitonic objects" he himself had discovered earlier. These p-Branes "radiate" (p+1)-forms, as charged particles (p=0) "radiate" the potential A_μ, a 1-form. One finds odd-dim Branes in the IIB theory, but even-dim Branes in the IIA. We elaborate on membranes just below.

The name "M- Theory" was coined by Witten; M stands for "Membrane", "Mother", "Mystery" etc., according to taste. To have some idea why the five theories fuse, we remark: IIA and IIB theories are the same from 9 dimensions down (no chirality in odd dimensions); same for the two heterotic strings; the responsible symmetry was called T-duality. From M-Theory in 11-dim one goes back to Het-Except by compactification *in a segment* D^1, with the two E_8 groups appearing miraculously to cancel anomalies in the boundary of the segment (we do not enter on this). Finally, Polchinski and Witten proved explicitly that in some "strong" / "weak" limit of the two (open and closed) theories with the same group, O(32), coincide: a case of the so-called S-duality.

M-Theory has not lived up the expectations; 15 years after inception has not explained anything, nor made a precise testable prediction (we do overlook some black hole entropy calculations of Strominger andd Vafa). We shall only say some words of the physical compactification problem, that is, how do we get from the fantasmal 10-dim to our mundane, open, quasi-flat Miknlowski sapace in (apparently) four dimensions!

One of the lesson we *do* have learned, however, is that particles, strings, membranes, and in general p-Branes, are related to (p+1)-forms they suppose to radiate, that is to say, membranes are charged as particles are charged, but as particles emit e.m. radiation, with 1-form potentials and 2-forms field strength, p-Branes radiate (p+1)-potentials and there are also (p+2) field intensities and even D-(p+2) duals, ending in 10 –p -4 "dual" or *magnetic* Branes; this interesting interplay is quite likely there to stay! For example, in 11-dim Supergravity, there is a dual M5 Brane (Güven), which can be seen as the magnetic dual of the better known M2 Brane.

F- Theory. In 1996 Cumrum Vafa came up with an extension of M-Theory to twelve dimensions, which he called F-Theory (F for Father? Fundamental?); the idea is that the IIB theory, known to be *self-dual* under the strong limit (S-duality) contains a complex scalar field, $z = \chi + i \exp(\varphi)$, where χ is the axion and φ the dilaton: this can be seen as a Torus fibration from 12 dimensions:

$$T^2 \rightarrow \text{F-Theory} \rightarrow \text{IIB Theory} \qquad \text{(III-3.1)}$$

as the *moduli space* of the Torus T^2 is C, the set of complex numbers.

There are several other arguments in favour of F-Theory [Boya]. The matter content of F-Theory is unclear; another confusing feature is that, at face value, F-Theory works in 12 = (10, 2), with all the associated problems related to causality, etc., for having two times. For many people (including, at the beginning, Vafa himself!) F-Theory was only a way to "track down" the complex field varying over the "surface" of the 10-dim IIB theory.

It is amusing that one can relate the particle content of 11-dim SuGra (and therefore, the low-energy limit of M-Theory) to the Moufang plane OP^2 we mentioned before: P. Ramond has shown that the three particles (h, Ψ and C) are related to the Euler number of OP^2 being 3, and in fact some representations of F_4 give rise to a triplet of representations of Spin(9): in particular, the above triplet is related to the *Id* irrep of F_4. In fact, one of the putative particle contents of F-theory is related to the "complexification" $OP_C^2 = E_6/[\text{Spin}(10)\times U(1)]$, but then there is a 27-plet, as $\chi(OP_C^2) = 27$, and it is related to the fourth power of the primordial Susy doublet,

$$|8\_v - \Delta_L|^4 \;\; \Rightarrow \text{a 27-plet of particles} \qquad \text{(III-3.2)}$$

The particles are taken as representations of O(10).

Recently (2008 on) [Vafa] has extended this theory considerably, where at the price of forgetting about gravitation one gets closer, it is hoped, to the standard model of particles and forces (see our IV Chapter); in particular GUTs, chiral matter, Yukawa couplings, not to speak of selecting gauge groups might appear possible, in principle. The development of this approach still goes on.

4.- The Standard Model and its Minimal Supersymmetric extension.

We just remind the reader that all known forces are to-day described as gauge theories, in particular, the gauge group of the Standard Model (SM) is the product SU(3) (color) × SU(2) (weak) × U(1) (≈ electromagnetism). Besides the 8+3+1=12 gauge bosons (three "swallowed" by Higg´s), the "matter" lies in the bi-fundamental representations: quarks q and $\bar{q}$ and leptons, etc. Besides, there is hope-for *God´s particle* (a left-over Higgs) supposed to be discovered next year at the LHC at CERN (?!).

We think we understand the gross features of the consequent picture: asymptotic freedom, confinement, radiative corrections in the electroweak sector, etc., but have no idea, for example, why the masses are what they are. We know (in this XXI century) neutrino have masses, but their type (Majorana?) is still unclear…In fact, the factor from the neutrino masses to the top quark mass approaches $10^{-14} \approx m_\nu/m_{top}$, beyond any reasonable calculation.

In spite of these shortcomings, people look beyond the SM. For some, Supersymmetry is irresistible, and one looks for clues as where and why. The most quoted argument (that I like, but it has still many detractors), is that the natural extrapolation of the three coupling constants g_s, g_{wk} and α_{em} by the renormalization group running (the Callan-Symanzik equation), make them cross at very high energies , $\approx 10^{16}$ GeV. It turns out that the `unique´ crossing point of the three is much improved if one completes the SM model with the so-called Minimal Supersymmetric Standard Model, MSSM.

In this model there are supersymmetric partners for each particle, with funny names (gauginos: fotino, gluino, Wino and Zino) for the fermionic partners of gauge bosons, and s-quarks and s-leptons for the bosonic partners of the matter multiplets…). Of course, in a way all this is science-fiction: there is NO the slightest experimental evidence of none of these, and the threshold for Susy partners approaches the TeV regime…

If Supersymmetry is true, where is the scale to see the Susy partners? Ee do not know for sure, but they cannot be very high in energy, otherwise no arguments for them: most of people expect them (!) in the TeV range, which probably will be observed experimentally soon. Still, we would have to understand, not only why Susy exists in first place, but also why is it broken badly, at (perhaps) the TeV scale…

IV.- **Some issues on Compactification**

1.- The general problem of compactification. We seem to live in three-plus-one dimensions (perhaps unbounded), whereas strings, M-theory, etc., live in higher dimensions, which we are not aware of. How do we cope with the "fact" of extra, unseen dimensions?

The simplest approach is through *compactification*, i.e. assuming the extra, unseen dimensions are "curled up" in a compact manifold, with dimensions well below, say, the nuclear dimensions 10^{-15} m. In principle one takes the direct product of spaces, e.g. if we take flat 4-dim space time,

$$M \approx K \times (\text{Minkowski space}, \mathsf{M} = \mathbb{R}^{3,1}) \qquad \text{(IV-1.1)}$$

where K is a compact manifold with 6, 7 or 8 dimensions for strings, M- & F-Theories.

What general properties one expects of K to satisfy, besides dimension(s) and compactness? We shall take it *oriented*, as we shall have to perform integrations, *spinable*, as we do see fermions, and whatever restrictions will make physics back in 4-dim acceptable, either the Standard Model, some GUT approximation, or just MSSM; that will depend, of course, and what physics we start from in the higher dimension, but also, as we have learned lately, the new physics generated by the compactation process: we shall see, for example, that new gauge groups might appear through potential singularities of the K space, or by wrapping on it some extended objects appearing in the total space, M. To maintain $\mathcal{N}$=1 *Supersymmetry* down to four dimensions will be a very repeated condition.

2.- Compactation from strings: Calabi-Yau 3-folds. That was historically the first realistic (but unsuccessful) attempt to rescue the SM from the (closed, $\mathcal{N}$ =1) Heterotic Exceptional ($E_8{}^2$) string (Gross-Strominger-Candelas-Witten, 1985): the two E_8 groups were very tempting as a starting point to get the GUT group: one E_8 remained hidden, the other spat a SU(3) factor to become E_6, one of the few possible GUT groups (two others were the original SU(5) (Georgi-Glashow, 1974) and SO(10) (Georgi; Fritz and Minkowski, 1976); E_6 was proposed by F. Gürsey and P. Ramond, 1976. Besides, we want to preserve $\mathcal{N}$ =1 Susy in 4-dim (explained below).

In ten dimensions the supercharge $\mathcal{N}$=1 algebra was fixed by

$$\{Q, Q\} = P_M + E^+ \qquad \text{(IV-2.1)}$$

$$\dim Q = 2^{10/2-1} = 16 \quad 16\cdot 17/2 = 136 = 10 + \{10, 2\}/2\ (=126) \qquad \text{(IV-2.2)}$$

We impose the constraint of only $\mathcal{N}$=1 Susy in 4-dim., which corresponds to 4 supercharges Q; this is *phenomenological*: we hope that there is some Susy, as we expect supersymmetric breaking to occur close to the 1 TeV scale, and $\mathcal{N} > 1$ "down to earth" means NO parity violation (+1/2 -1/2 -1/2 = -1/2 in helicities, so *both* chiralities would appear for $\mathcal{N}$= 2!); but parity violation is a conspicuous feature of nature, which we want to preserve, even with supersymmetry. That means we should break ¾ = $(16_{ini} - 4_{fin})/16$ supercharges. How do we achieve this?

It turns out that $\mathcal{N}$ =1 Susy down to 4-dim is maintained if the compactifying manifold has SU(3) holonomy, so it is, by definition, a CY_3, or Calabi-Yau 3-fold. To see this, recall the identity SU(4) = Spin(6) (stated above, Ch. II). Now under SU(3) $\subset$ SU(4), the fundamental spinor splits obviouosly as 4 = 3+1. Hence, the trivial "1" survives, and guarantees 4 Supercharges (four Q´s) in 4-dim, so $\mathcal{N}$=1, as desired, as

$$\text{Dim } Q \text{ (4-dim)} = 2^{4/2-1} = 2 \text{ complex or 1 Majorana spinor} \qquad \text{(IV-2.3)}$$

One can show directly than compactation in a CY_k preserves $1/2^{k-1}$ supersymmetries, so here we have 16/4=4: 16 means $\mathcal{N}$=1 in 10d, whereas 4 in dim 4 means also $\mathcal{N}$=1.

The search for CY_3 was an active industry in Austin, TX, during 1986/90, led by Phil Candelas (actually he has the Penrose chair in Oxford). As a by-product, his group discovered the mirror symmetry! We shall explain this.

The Hodge diamond for a CY_3 manifold is

$$\begin{array}{ccccccc} & & & 1 & & & \\ & & 0 & & 0 & & \\ & h_{2,0} & & h_{1,1} & & h_{0,2} & \\ h_{3,0} & & h_{2,1} & & h_{1,2} & & h_{0,3} \end{array} \qquad \text{(IV-2.4)}$$

plus Poincarè dual

Mirror symmetry in this CY_3 context is the exchange $h_{1,1} \leftrightarrow h_{2,1}$; it turns out the physics in mirror is the same; the two Hodge numbers interchange some complex structure with some Kähler structures, and it has been a big advance in algebraic geometry (we do not elaborate). In an arbitrary CY-k-fold, it is the exchange $h_{1,1} \leftrightarrow h_{k-1,1}$; hence e.g. the K3 manifold is self-mirror, as $h_{1,1} \equiv h_{2-1,1}$; again, this has some consequences.

3.- Compactation from M-Theory: Manifolds of G_2 Holonomy

After the 1995 revolution brought about by the discovery of (still incompletely known) M – Theory (Witten; Townsend; Polchinski) it was obvious that we have to face eleven dimensions. The supercharges and particle content (in the low-energy limit) were

$$\text{Dim Q: } 2^{(11-1)/2} = 32.- \quad \{Q, Q\} = \; P_\mu + M2 + M5 \qquad \text{(IV-3.1)}$$
$$32\cdot 33/2 = 528 = 11 + 55 + 462$$

which makes even more sense than ten dimensions: besides the linear momentum (P_μ), the theory describe the p=2 membrane M2 (already alluded to; the viewpoint of Townsend and Duff vs. Witten), and a dual, the magnetic 5-Brane (2→3→4→11-4=7, →6→5).

The seven-dim compactifying manifold has to be oriented and spinable, of course, and should also leave a spinor covariant constant, as to have, again $\mathcal{N}$=1 down to our mundane 4 dimensions; the natural spin group would be Spin(7), of course, with a real *irrep* of dim 8 = $2^{(7-1)/2}$. But happily, it has the subgroup G_2, with the natural *irrep* of dim 7 (also explained above), so G_2 acting in 8 dimensions splits as 8 = 7 + 1, just as we want! That was first emphasized (I believe) by Townsend and Papadopoulos, 1996. It reinforces the idea that octonions and related objects should play a role in our description of nature(!).

The search for manifolds with G_2 holonomy is a hard one, as the dim is odd so we have no resort no complex analysis; some examples were worked out by [Joyce].

As M-Theory is only incompletely known, there has been not much progress with this G_2 holonomy idea; in any case, in a way M-Theory was superseded, I believe, by the F-theory of Cumrum Vafa, after the original work (ca. 1996) has being reincarnated in his 2008 version; to this one we now turn.

4.- Compactation from F-Theory: Vafa´s Theory (1996, 2008).

We already talked on F-theory in (Ch. III-3). With 12 = (10, 2) signature, the type of spinor (or supercharge) is still real ($12-2 \equiv 0$ mod 8). The number of supercharges is

$$\# \, Q(12\text{-dim}) = 2^{12/2-1} = 32, \text{ real type} \qquad \text{(IV-4.1)}$$

with the superalgebra

$$\begin{array}{ccccc} \{Q, Q\} & = & J_{MN} & + & F^{\pm} \\ 528 & = & \{12, 2\}=66 & + & \{12, 6\}/2= 126 \end{array} \qquad \text{(IV-4.2)}$$

In particular, *no conventional SuperGravity possible*, as P_M does not appear in the anticommutator (recall we already said 11 was the maximum dimension for (conventional) supergravity); however, some interpretation can be given. F-Theory has other worries: for example, the two times are at odds with conventional causality; some exit avenues have been proposed (e.g. I. Bars proposes to "gauge" one of the times). We shall just address the compactification problem, following the "reincarnation" of the theory by Vafa´s group since 2008 [Vafa *et al.*, 2008].

The first idea is to forget about gravitation, and try to understand some of the feautures of the Standard Model, like GUTs, presence of chiral matter, obtaining the Yukawa couplings etc., ,and leave for the future the connection with more conventional high-dimension theories as respect gravitation and a "TOE" theory; Vafa calls this the "bottom-up approach".

The most promising compactification approach makes two setps, the first in some sense divided in another two:

First step: 12 to 8 dimensions, in a K3 manifold (IV-4.3)
Second step: 8 to 4 dimensions, in a del Pezzo surface (IV-4.4)

We explain first the new real 4-dim manifolds.

K3 complex surface.-The complex surface K3 is, topologically speaking, the only nontrivial Calabi-Yau 2-fold, i.e., with holonomy SU(2). K stands for the Karakorum High Sierra in the Himalayas, and "3" for Kummer, Kähler and Kodaira; the name is due to André Weil, 1954, at the time of the first Mount Everest climb (May, 1953). As a complex surface, it has a Hodge diamond as follows:

$$\begin{array}{ccccc} & & 1 & & \\ & 0 & & 0 & \\ 1 & & 20 & & 1 \end{array}$$

plus Poincaré dual

So it is simply connected, Euler number is $4 \cdot 1 + 20 = 24$, and signature is: $22 = (19, 3)$ [Disgression: any four-dim manifold, if compact and oriented, has another number besides Euler´s χ: it is called Hirzebruch´s signature, τ, and it is the (Sylvester) signature of the quadratic form given by the *wedge* (although commutative) product of two-forms, as $\omega_1^{(2)} \wedge \omega_2^{(2)} = (\text{real } \#) \cdot \text{volume}$].

Thus $\tau = 16 = 19 - 3$, and miraculously it happens, as $16 = 2 \cdot 8$, that two E_8 groups (or more precisely, singularities given rise to groups, by the Thom-Arnold procedure; we do not elaborate) lurk behind K3 (K3 is easily obtained from the 4-torus $T^4 \approx (S^1)^4$) by "orbifolding" and blow-up, that is, by quotienting by some fix-point group and curing the singularities; see e.g. Aspinwall, arXiv hep-th 94 04 151.

<u>del Pezzo surfaces</u>. Complex projective spaces CP^n is the set of (complex, of course!) lines in C^{n+1}. We have $CP^1 \approx S^2$, and CP^2 is the symmetric space SU(3)/U(2); it is a complex, simply-connected Kähler manifold, with Hodge diamond

$$\begin{array}{ccccc} & & 1 & & \\ & 0 & & 0 & \\ 0 & & 1 & & 0 \end{array}$$

plus Poincaré dual

hence $\chi=3$, and it does *not* admit a spin structure. It is perhaps the simplest 4-dim real manifold, after $S^{\wedge 4}$ (which is *not* complex).

del Pezzo surfaces are obtained from CP^2 after blowing up up to 8 points.

<u>Descent from 12 to 8</u>: Gauge goups appear! First of all, recall F-Theory as an elliptic fibration over IIB theory by th 2-Torus