Anomalies and Noncommutative Index Theory

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Abstract

These lectures are devoted to a description of anomalies in quantum field theory from the point of view of noncommutative geometry and topology. We will in particular introduce the basic methods of cyclic cohomology and explain the noncommutative counterparts of the Atiyah-Singer index theorem.

Contents

1 Introduction

The aim of these lectures is to provide a modest insight into the interplay between Quantum Field Theory and Noncommutative Geometry. We choose to focus on the very specialized problem of chiral anomalies in gauge theories, from the viewpoint of noncommutative index theorems. In fact both subjects can be considered as equivalent, in a certain sense. On one side, chiral anomalies arise as the lack of gauge invariance for a quantum field theory after renormalization. This confers a fundamental *local* nature to anomalies, in a geometric sense. One the other hand, a particular attention has been drawn in the last years to local index formulas in noncommutative geometry, as formulated by Connes and Moscovici [9]. We propose to explain in these notes how local index formulas can be extracted from quantum anomalies.

These lectures are organized as follows. In the first section, we recall some (very) basic material on quantum field theory, and explain how chiral anomalies appear. The second section deals with noncommutative geometry, in particular K-theory and cyclic cohomology. In the third section we review the Chern-Connes character construction and establish the link between the noncommutative index theorem and chiral anomalies.

2 Quantum Field Theory and Anomalies

2.1 Classical field theory

The simplest field theory over a smooth manifold M is provided by a (real) scalar field ϕ, that is, a smooth function over M with values in the real numbers

$$\phi : M \to \mathbb{R} \ . \tag{1}$$

It should be understood that M plays the rôle of "space-time", although it is not necessarily endowed with a Minkowski or a pseudoriemannian metric. In general M is simply a smooth manifold without boundary. Important examples arising from physics include the case of the euclidian space $M = \mathbb{R}^n$, or the n-sphere $M = S^n$. The dynamics of the field ϕ is entirely described by an action functional $S(\phi)$. The *locality* principle states that the action is the integral over M of a lagrangian density $\mathcal{L}(x, \phi)$:

$$S(\phi) = \int_M dx\, \mathcal{L}(x, \phi) \ , \tag{2}$$

where dx is the volume element associated, for example, to a riemannian (or pseudoriemannian) metric. The lagrangian $\mathcal{L}(x, \phi)$ must be local in the sense that its value at a point $x \in M$ only depends on x and a finite number of the derivatives of the field ϕ at x. A classical example is the Klein-Gordon theory in Minkowski space: here $M = \mathbb{R}^4$ is endowed with the (pseudoeuclidian) Minkowski metric g of signature $(+1, -1, -1, -1)$ and the action reads

$$S(\phi) = \int_{\mathbb{R}^4} d^4x\, \frac{1}{2} g^{\mu\nu} \partial_\mu \phi(x) \partial_\nu \phi(x) \ . \tag{3}$$

One can also add a potential term $V(x, \phi)$ to the lagrangian, with $V(x, \phi)$ any function depending polynomially on ϕ, describing a self-interaction of the field. Then the full action is

$$S(\phi) = \int_{\mathbb{R}^4} d^4x\, (\frac{1}{2} g^{\mu\nu} \partial_\mu \phi(x) \partial_\nu \phi(x) + V(x, \phi)) \ . \tag{4}$$

The Klein-Gordon lagrangian thus depends only on ϕ and its first derivatives. Returning to the general case, the equations of motion for the field ϕ are obtained from a variational principle: ϕ verifies the equations of motion if and only if the action is stationary at point ϕ:

$$\frac{\delta S(\phi)}{\delta \phi} = 0 \ . \tag{5}$$

To be precise, $\delta/\delta\phi$ is the Fréchet derivative with respect to ϕ, hence $\delta S(\phi)/\delta\phi$ is a (tempered) distribution over M depending on ϕ. For the Klein-Gordon theory this distribution is actually a smooth function whose value at a point $x \in M$ yields the Euler-Lagrange equations

$$\frac{\delta S(\phi)}{\delta \phi(x)} = 0 = -\Delta\phi(x) + V'(x,\phi) \qquad \forall x \in M \ , \tag{6}$$

where $\Delta = g^{\mu\nu}\partial_\mu\partial_\nu$ is the laplacian in Minkowski space (or d'alembertian), and V' is the derivative of $V(x,\phi)$ with respect to the variable ϕ. For a generic potential, the differential equation (6) is nonlinear and cannot be solved explicitly. However, it is always possible to find a solution ϕ in terms of formal power series with respect to some set of parameters. Let us illustrate this in the case of a massive field ϕ with cubic interaction, in the presence of sources. This corresponds to the following potential:

$$V(x,\phi) = \frac{m^2}{2}\phi(x)^2 + \frac{\lambda}{3!}\phi(x)^3 - j(x)\phi(x) \ , \tag{7}$$

so that the lagrangian density is given by

$$\mathcal{L}(x,\phi) = \frac{1}{2}(g^{\mu\nu}\partial_\mu\phi(x)\partial_\nu\phi(x) + m^2\phi(x)^2) + \frac{\lambda}{3!}\phi(x)^3 - j(x)\phi(x) \ . \tag{8}$$

Here $m \in \mathbb{R}$ is a mass term, $\lambda \in \mathbb{R}$ is the coupling constant of the cubic interacion, and j is a function over M which plays the rôle of a source for the field ϕ. The Euler-Lagrange equation is

$$0 = (-\Delta + m^2)\phi(x) + \frac{\lambda}{2}\phi(x)^2 - j(x) \qquad \forall x \in M \ . \tag{9}$$

We will construct a solution ϕ as a power series in the coupling constant λ and the source function j. Rewrite the equation of motion as

$$(-\Delta + m^2)\phi(x) = j(x) - \frac{\lambda}{2}\phi(x)^2 \ . \tag{10}$$

The wave operator $(-\Delta + m^2)$ is a differential operator acting on the space of smooth functions over M with rapid decay at infinity. It has an inverse G, the so-called propagator, which may be represented by its distributional kernel $G(x,y)$, $x, y \in M$. The action of G on a smooth function ϕ reads

$$(G\phi)(x) = \int_{\mathbb{R}^4} d^4y\, G(x,y)\phi(y) \ , \tag{11}$$

so that the condition "G is the inverse operator of $(-\Delta + m^2)$" is equivalent to

$$(-\Delta_x + m^2)G(x,y) = \delta^4(x-y) \ , \tag{12}$$

where $\delta^4(x-y)$ is the Dirac distribution in Minkowski space (i.e. the identity operator on functions). It is often more convenient to represent $G(x,y)$ by its Fourier transform

$$G(x,y) = \int_{\mathbb{R}^4} \frac{d^4p}{(2\pi)^4}\, G(p) e^{-ip(x-y)} \ , \tag{13}$$

Now the above condition in Fourier space is simply

$$(p^2+m^2)G(p) = 1 \ , \tag{14}$$

with $p^2 = g^{\mu\nu}p_\mu p_\nu$, whence

$$G(p) = \frac{1}{p^2+m^2} \ . \tag{15}$$

$G(x,y)$ is a distribution in the variables x, y, whose singular part is restricted to the "diagonal" $x = y$. Applying the propagator on both sides of (10), the equation of motion is transformed into the integral equation

$$\phi(x) = \int_M d^4y\, G(x,y)(j(y) - \frac{\lambda}{2}\phi(y)^2) \ . \tag{16}$$

The latter can be solved perturbatively, i.e. as a power series in j and λ, by reinjecting the expression of ϕ in the right-hand side. This is better achieved by using a diagrammatic representation of the equation: the propagator $G(x,y)$ is represented by a line joining the points x and y, and the interaction term is represented by a vertex connecting three lines. One obtains the resolution of the integral equation as an expansion in tree graphs (the Feynman graphs for the classical theory).

The same methods can be applied without change for a scalar field theory over any riemannian or pseudoriemannian manifold M, provided the lagrangian density is the sum of a kinetic term (quadratic in ϕ) and of a polynomial potential. Typically

$$\mathcal{L}(x,\phi) = \frac{1}{2}(g^{\mu\nu}\partial_\mu\phi(x)\partial_\nu\phi(x) + m^2\phi(x)^2) + V(x,\phi) \ , \tag{17}$$

with $g^{\mu\nu} = g^{\mu\nu}(x)$ the metric tensor of the manifold. In the remainder of these lectures we will mostly be concerned with compact riemannian manifolds (without boundary), for which the metric is definite and positive. Up to integration by parts, the kinetic term of the action may be written as

$$S(\phi)_{\mathrm{kin}} = \int_M dx\, \frac{1}{2}\phi(x)(-\Delta + m^2)\phi(x) \ , \tag{18}$$

where Δ is the laplacian in curved space. For $m \neq 0$ the operator $(-\Delta + m^2)$ has an inverse G as in the flat case, and the same diagrammatic expansion in terms of the propagator solves the equation of motion.

2.2 Quantum field theory

Consider a scalar field theory ϕ over a riemannian manifold M. Its classical dynamics is described by a local action functional

$$S(\phi) = \int_M dx\, \mathcal{L}(x, \phi) \ , \tag{19}$$

from which are deduced the Euler-Lagrange equations. In quantum field theory, ϕ does no longer satisfies equations of motion and is replaced by a fluctuating field. The physical quantities are not extracted from the exact values of ϕ, but rather from the correlation functions (in the statistical sense) of the field evaluated at different points. If $x_1, \dots x_n$ denote n arbitrary points in M, the associated correlation function is given by the functional integral

$$\langle \phi(x_1) \dots \phi(x_n) \rangle = \int \mathcal{D}\phi\ \phi(x_1) \dots \phi(x_n) e^{-S(\phi)/\hbar} \ . \tag{20}$$

Here $\mathcal{D}\phi$ is the formal euclidian integration measure over the infinite-dimensional space of classical fields over M, and $\hbar$ is the Planck constant. The statistical weight $\exp(-S(\phi)/\hbar)$ is computed from the classical action. The integration measure is supposed to be normalized so that

$$\int \mathcal{D}\phi\ e^{-S(\phi)/\hbar} = 1 \ . \tag{21}$$

When the Planck constant $\hbar$ becomes "small" in appropriate units, one sees that the statistical weight selects essentially the fields ϕ which are close to realize the minimal values of the action, or in other words, the fields satisfying the classical equation of motion

$$\frac{\delta S(\phi)}{\delta \phi(x)} = 0 \ . \tag{22}$$

This result is known as the classical limit, when the mean value of the quantum field tends to its classical value:

$$\langle \phi(x) \rangle \underset{\hbar \to 0}{\sim} \phi(x)_{\mathrm{class}} \ . \tag{23}$$

Hence in quantum field theory ($\hbar \neq 0$), the correlation functions include the contribution of quantum fluctuations around the classical field configurations. The main problem, however, is that the integration measure $\mathcal{D}\phi$ does not rigorously exist in the infinite-dimensional space of classical fields. In order to give a sense to the formal integral (20), we will show that it satisfies some functional equation known as the *Dyson-Schwinger equation*. We first introduce a classical source $j(x)$ for the field ϕ (cf. section 2.1). This is obtained by replacing the action $S(\phi)$ by

$$S(\phi) - \int_M dx\, j(x)\phi(x) \ . \tag{24}$$

The partition function of the quantum field ϕ is by definition the following functional of the source j

$$Z(j) = \int \mathcal{D}\phi\ e^{(-S(\phi) + \int dx\, j(x)\phi(x))/\hbar} \ , \tag{25}$$

and the normalization of the measure is such that

$$Z(0) = 1 \ . \tag{26}$$

An easy computation shows the correlation functions coincide with the coefficients of the Taylor expansion of the functional $Z(j)$ around zero:

$$\langle \phi(x_1) \dots \phi(x_n) \rangle = \hbar^n \frac{\delta}{\delta j(x_1)} \dots \frac{\delta}{\delta j(x_n)} Z(j)|_{j=0} \ . \tag{27}$$

Hence all the information we need about the quantum theory is contained in the partition function. More generally if $\mathcal{O}(\phi)$ is any reasonable functional of ϕ (typically, a polynomial), its expectation value is given by

$$\langle \mathcal{O}(\phi) \rangle = \int \mathcal{D}\phi \ \mathcal{O}(\phi) e^{-S(\phi)/\hbar} = \mathcal{O}(\hbar \frac{\delta}{\delta j}) Z(j)|_{j=0} \ , \tag{28}$$

since each insertion of a factor ϕ is implemented by the derivation $\hbar \frac{\delta}{\delta j}$ on $Z(j)$. It is clear that, $\mathcal{D}\phi$ being a formal euclidian integration measure, it should be invariant under the translation of ϕ by a smooth function ξ:

$$\phi(x) \to \phi(x) + \xi(x) \ , \qquad \mathcal{D}\phi \to \mathcal{D}(\phi + \xi) = \mathcal{D}\phi \ . \tag{29}$$

Hence the corresponding change of variable in the partition function yields

$$Z(j) = \int \mathcal{D}\phi \ e^{(-S(\phi+\xi) + \int dx \, j(x)(\phi(x) + \xi(x)))/\hbar} \tag{30}$$

for any smooth function ξ over M. It follows that the functional (or Fréchet) derivative of $Z(j)$ with respect to ξ vanishes:

$$\hbar \frac{\delta Z(j)}{\delta \xi(x)} = 0 = \int \mathcal{D}\phi \ (-\frac{\delta S(\phi)}{\delta \phi(x)} + j(x)) e^{(-S(\phi) + \int dy \, j(y)\phi(y))/\hbar} \tag{31}$$

for any $x \in M$. This is equivalent to the Dyson-Schwinger equation

$$0 = \big(- \frac{\delta S}{\delta \phi(x)} (\hbar \frac{\delta}{\delta j}) + j(x) \big) Z(j) \qquad \forall x \in M \ . \tag{32}$$

Here $\frac{\delta S}{\delta \phi(x)}(\hbar \frac{\delta}{\delta j})$ is a polynomial in the functional derivative $\frac{\delta}{\delta j}$. This differential equation together with the initial condition $Z(0) = 1$ allows to determine the functional $Z(j)$ as a power series in the source field j and other parameters. Let us show this in the example of the massive scalar field with cubic interaction, as in section 2.1. The classical action is

$$S(\phi) = \int_M dx \, \frac{1}{2} (g^{\mu\nu} \partial_\mu \phi(x) \partial_\nu \phi(x) + m^2 \phi(x)^2) + \frac{\lambda}{3!} \phi(x)^3 \ , \tag{33}$$

with $g^{\mu\nu}$ the metric tensor of the riemannian manifold M and dx the volume element. One has

$$\frac{\delta S(\phi)}{\delta \phi(x)} = (-\Delta + m^2)\phi(x) + \frac{\lambda}{2} \phi(x)^2 \ , \tag{34}$$

where Δ is the laplacian in curved space. Hence

$$\frac{\delta S}{\delta\phi(x)}(\hbar\frac{\delta}{\delta j}) = (-\Delta + m^2)\hbar\frac{\delta}{\delta j(x)} + \frac{\lambda}{2}\left(\hbar\frac{\delta}{\delta j(x)}\right)^2 , \tag{35}$$

and the Dyson-Schwinger equation reads (compare with the classical equation of motion (9))

$$0 = \left((-\Delta + m^2)\hbar\frac{\delta}{\delta j(x)} + \frac{\lambda}{2}\left(\hbar\frac{\delta}{\delta j(x)}\right)^2 - j(x)\right) Z(j) \qquad \forall x \in M . \tag{36}$$

It is more convenient to determine the logarithm of $Z(j)$

$$W(j) := \hbar \ln Z(j) , \qquad W(0) = 0 . \tag{37}$$

The functional $W(j)$ is called the *free energy* of the quantum field ϕ, by analogy with statistical thermodynamics [12]. One has

$$\begin{aligned} Z(j) = e^{W(j)/\hbar} , \qquad \hbar\frac{\delta}{\delta j(x)} Z(j) &= \frac{\delta W(j)}{\delta j(x)} Z(j) , \\ \left(\hbar\frac{\delta}{\delta j(x)}\right)^2 Z(j) &= \left(\left(\frac{\delta W(j)}{\delta j(x)}\right)^2 + \hbar\frac{\delta^2 W(j)}{\delta j(x)^2}\right) Z(j) , \end{aligned} \tag{38}$$

and the Dyson-Schwinger equation yields for the free energy

$$(-\Delta + m^2)\frac{\delta W(j)}{\delta j(x)} + \frac{\lambda}{2}\left(\left(\frac{\delta W(j)}{\delta j(x)}\right)^2 + \hbar\frac{\delta^2 W(j)}{\delta j(x)^2}\right) = j(x) . \tag{39}$$

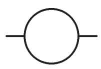
As in the classical case, we will solve this equation in terms of formal power series in the source j and the coupling constant λ. Introduce the propagator G of the field as the inverse of the wave operator $(-\Delta + m^2)$:

$$(-\Delta_x + m^2) G(x, y) = \delta(x - y) . \tag{40}$$

Then (39) is equivalent to the integral equation

$$\frac{\delta W(j)}{\delta j(x)} = \int_M dy\, G(x, y)\left(j(y) - \frac{\lambda}{2}\left(\left(\frac{\delta W(j)}{\delta j(y)}\right)^2 + \hbar\frac{\delta^2 W(j)}{\delta j(y)^2}\right)\right) \tag{41}$$

The iterative resolution of this equation may be depicted by an expansion in terms of Feynman graphs. One recovers the same tree graphs as the resolution of the classical field equations, plus new graphs involving loops. Each loop comes with a factor $\hbar$: it corresponds to the quantum fluctuations of the theory. All the terms of this formal expansion are not well-defined however. It happens that in many cases the loops are given by divergent integrals. This comes from the ambiguity in the definition of the infinite-dimensional integration measure $\mathcal{D}\phi$, which was used purely formally in our derivation of the Dyson-Schwinger equations. Let us for example examine the following simple loop contributing to the j^2 term of $W(j)$, in the case of the four-dimensional euclidian manifold $M = \mathbb{R}^4$:

$L =$  (42)

Its contribution to $W(j)$ is given by integrating all the vertices over the manifold $\mathbb{R}^4$:

$$L = \int_{\mathbb{R}^8} d^4x_1\, d^4x_2\, j(x_1)j(x_2) \int_{\mathbb{R}^8} d^4x\, d^4y\, G(x_1, x)(G(x, y))^2 G(y, x_2) \; . \tag{43}$$

The ambiguity comes from the integral over the vertices x, y. Indeed the propagator $G(x, y)$ being a singular distribution at $x = y$, its square is not determined as a distribution, and consequently the integral diverges when x tends to y. This can be seen by Fourier transform,

$$G(x, y) = \int_{\mathbb{R}^4} \frac{d^4p}{(2\pi)^4}\, G(p) e^{-ip(x-y)} \tag{44}$$

with $G(p) = 1/(p^2 + m^2)$, so that after a naive change of variables one gets

$$(G(x, y))^2 = \int \frac{d^4p}{(2\pi)^4} \frac{d^4k}{(2\pi)^4}\, G(k)G(p - k)e^{-ip(x-y)} \; . \tag{45}$$

The latter is not a distribution, because its Fourier transform is given by an integral over k diverging logarithmically as $|k| \to \infty$:

$$I(p) = \int \frac{d^4k}{(2\pi)^4} \frac{1}{(k^2 + m^2)((p - k)^2 + m^2)} = \infty \; . \tag{46}$$

This divergence reveals we have not defined correctly the product of distributions. $(G(x, y))^2$ is actually a well-defined function of x and y provided $x \neq y$, but the problem appears on the diagonal $x = y$. Hence the solution is to find a good distribution which extends the function $(G(x, y))^2$ to the entire space, including the diagonal. This program is called *renormalization*. We can renormalize the product $(G(x, y))^2$ as follows. Derivating (46) under the integral with respect to $p \in \mathbb{R}^4$ gives a convergent integral:

$$\frac{\partial I}{\partial p^\mu} = \int \frac{d^4k}{(2\pi)^4} \frac{-2(p - k)_\mu}{(k^2 + m^2)((p - k)^2 + m^2)^2} \; . \tag{47}$$

Hence, it is possible to *define* the Fourier transform of $(G(x, y))^2$ as a primitive of (47), which is of course determined up to a constant. Hence the renormalized integral is

$$I_R(p) = \int \frac{d^4k}{(2\pi)^4} \frac{k^2 - (p - k)^2}{(k^2 + m^2)^2((p - k)^2 + m^2)} + C \; , \tag{48}$$

where the integrand behaves as $|k|^{-5}$ when $|k| \to \infty$ and is therefore absolutely integrable. Hence by Fourier transform the renormalized loop is a distribution defined modulo a term proportional to a Dirac distribution centered on the diagonal $x = y$,

$$(G(x, y))^2_R = \int \frac{d^4p}{(2\pi)^4} \frac{k^2 - (p - k)^2}{(k^2 + m^2)^2((p - k)^2 + m^2)}\, e^{-ip(x-y)} + C\delta^4(x - y) \; , \tag{49}$$

with C an arbitrary constant. Hence, we see that it is always possible to give a sense to the loop amplitude L, but the renormalized result is *not unique* since it involves the introduction of an arbitrary parameter C. This is a general feature of renormalization theory. Given any Feynman diagram with an arbitrary number of loops, we can always find a renormalized value which is defined up to addition of one or several local terms with parameters $C_1, \ldots C_n$. Renormalizable quantum field theories are those for which all the renormalization parameters can be fixed by imposing a finite number of constraints on the physical quantities such a masses, coupling constants, etc... This is precisely the case for the ϕ^3 scalar theory on the euclidian manifold $\mathbb{R}^p$ up to $p = 6$. In space-time dimension grater than 6 however, the ϕ^3 theory becomes non-renormalizable, in the sense that the number of renormalization constants in each Feynman diagram increases with the number of loops, hence it becomes impossible to fix all of these by physical constraints. Another example of renormalizable theory is the ϕ^4 theory in 4 space-time dimensions.

2.3 Gauge theories

Gauge theories form one of the most important class of quantum field theories. They are relevant both in physics, especially in the description of fundamental interactions between elementary particles, and in mathematics due to their deep interplay with geometry and topology. The quantization of gauge fields requires some care because we must take into account the presence of symmetries.

Let us start with a riemannian manifold M of dimension n. We suppose M is provided with a spin structure, which means that we can speak about spinor fields over M. It is sufficient to formulate all that in a local coordinate system $\{x^\mu\}$, $\mu = 1, \ldots n$. It provides a local basis of one-forms (sections of the cotangent bundle) $\{dx^\mu\}$ in the following sense: any smooth one-form $\alpha \in C^\infty(T^*M)$ is written uniquely as a linear combination

$$\alpha = \sum_\mu \alpha_\mu(x) dx^\mu \ , \tag{50}$$

the coefficients α_μ being smooth functions. We introduce also a system of orthonormal frames $\{e^a\}$, $a = 1, \ldots n$, which provides locally another basis of one-forms. One thus has

$$e^a = \sum_\mu e^a_\mu(x) dx^\mu \ , \tag{51}$$

where the coefficients functions $e^a_\mu(x)$ are the components of an invertible $n \times n$ matrix at any point x. The Levi-Civita connection is the unique torsion-free affine connection

$$\nabla^{LC} : C^\infty(T^*M) \to C^\infty(T^*M \otimes T^*M) \tag{52}$$

preserving the riemannian metric. Its action on the orthonormal frame $\{e^a\}$ may be written in terms of its component functions $\omega_{\mu ab}$

$$\nabla^{LC} e^a = -\sum_{\mu,b} \omega_{\mu ab}(x)\, dx^\mu \otimes e^b \ . \tag{53}$$

Because the Levi-Civita connection preserves the metric, the antisymmetry condition $\omega_{\mu ab} = -\omega_{\mu ba}$ holds. Now any smooth one-form $\alpha \in C^\infty(T^*M)$ may be decomposed in the orthonormal basis

$$\alpha = \sum_a \alpha_a(x) e^a \ , \tag{54}$$

so that we have

$$\nabla^{LC}\alpha = \sum_{\mu,a} \nabla_\mu \alpha_a(x)\, dx^\mu \otimes e^a \ , \quad \nabla_\mu \alpha_a = \partial_\mu \alpha_a + \sum_b \omega_{\mu ab}\alpha_b \ . \tag{55}$$

The components of the Riemann curvature tensor are deduced from the commutator of the covariant derivatives

$$[\nabla_\mu, \nabla_\nu] e^a = -\sum_b R_{\mu\nu ab} e^b \ , \tag{56}$$

which yields

$$R_{\mu\nu ab} = \partial_\mu \omega_{\nu ab} - \partial_\nu \omega_{\mu ab} + \sum_c (\omega_{\mu ac}\omega_{\nu cb} - \omega_{\nu ac}\omega_{\mu cb}) \ . \tag{57}$$

The Clifford bundle $\mathrm{Cl}(M)$ is the vector bundle over M whose fiber at any point $x \in M$ is the Clifford algebra of the cotangent space at x endowed with the riemannian metric. Hence the sections of the Clifford bundle form a unital algebra generated by a set of covectors $\{\gamma^\mu\}$ and the algebraic rule

$$\gamma^\mu \gamma^\nu + \gamma^\nu \gamma^\mu = -2 g^{\mu\nu}(x) \ , \tag{58}$$

where the functions $g^{\mu\nu}$ are the components of the metric tensor in the coordinate basis $\{x^\mu\}$. Hence any section of the Clifford bundle may be written uniquely as a linear combination

$$\sum_{i=0}^{n} \sum_{\mu_1 < \ldots < \mu_i} \alpha_{\mu_1 \ldots \mu_i}(x) \gamma^{\mu_1} \ldots \gamma^{\mu_i} \ , \tag{59}$$

with smooth functions coefficients $\alpha_{\mu_1 \ldots \mu_i}$. By changing to the orthonormal base

$$\gamma^a = \sum_\mu e^a_\mu(x) \gamma^\mu \ , \tag{60}$$

the new generators of the Clifford sections verify the algebraic rule

$$\gamma^a \gamma^b + \gamma^b \gamma^a = -2\delta^{ab} \ , \tag{61}$$

and any section of the Clifford bundle may be as well decomposed linearly on the products of the γ^a.
The spinor bundle $S(M)$ is the vector bundle over M whose fiber is the spinor representation of the fiber of the Clifford bundle. $S(M)$ exists globally exactly when M has a spin structure. In the local orthonormal frame $\{e^a\}$, a smooth section $\psi \in C^\infty(S)$ is given by a set of smooth functions with values in $\mathbb{C}$

$$\begin{aligned} \psi = \{\psi^j\} \ , \quad & j = 1, \ldots, 2^{n/2} && \text{if } n = \dim M \text{ is even} \ , \\ & j = 1, \ldots, 2^{(n-1)/2} && \text{if } n = \dim M \text{ is odd} \ . \end{aligned} \tag{62}$$

The sections γ^a generating the Clifford algebra are then represented by constant matrices on the spinor space (e.g. the Pauli matrices times i when dim $M = 3$ or the Dirac matrices of the euclidian space $\mathbb{R}^4$ when dim $M = 4$). The space of smooth sections of the spinor bundle (i.e. spinor fields) is endowed with the spin connection

$$\nabla^S : C^\infty(S) \to C^\infty(T^*M \otimes S) \tag{63}$$

locally expressed by

$$\nabla^S_\mu = \partial_\mu - \frac{1}{4} \sum_{a,b} \omega_{\mu ab}(x) \gamma^a \gamma^b \ . \tag{64}$$

We are ready now to introduce the Dirac operator as the differential operator on spinors

$$D : C^\infty(S) \to C^\infty(S) \tag{65}$$

defined by Clifford multiplication

$$D = \sum_\mu \gamma^\mu \nabla^S_\mu \ . \tag{66}$$

From now on we suppose that the manifold M is compact (without boundary) and has *even* dimension. Then the Clifford bundle is provided with a *chirality section*

$$\gamma = i^{n/2} \prod_{a=1}^{n} \gamma^a \ , \tag{67}$$

with the important property that it is an involution for the Clifford product

$$(\gamma)^2 = 1 \ . \tag{68}$$

Consequently the bundle of spinors splits into the direct sum of two subbundles of opposite chiralities $S = S_+ \oplus S_-$,

$$\psi_+ = \frac{1+\gamma}{2} \psi \ , \qquad \psi_- = \frac{1-\gamma}{2} \psi \ , \tag{69}$$

where $\frac{1 \pm \gamma}{2}$ is the projector onto the eigenspace of γ corresponding to the eigenvalue ± 1. One checks that γ anticommutes with all the generators γ^a,

$$\gamma \gamma^a + \gamma^a \gamma = 0 \ , \tag{70}$$

hence it also anticommutes with the Dirac operator:

$$D\gamma + \gamma D = 0 \ . \tag{71}$$

An operator anticommuting with γ is called *odd*, which means it changes the chirality of spinors:

$$D : C^\infty(S_\pm) \to C^\infty(S_\mp) \ . \tag{72}$$

We will often use the following matricial representation

$$\psi = \begin{pmatrix} \psi_+ \\ \psi_- \end{pmatrix} \ , \quad D = \begin{pmatrix} 0 & D_- \\ D_+ & 0 \end{pmatrix} \ , \quad \gamma = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \ . \tag{73}$$

The classical dynamics of a spinor field $\psi \in C^\infty(S)$ over the riemannian manifold M is governed by the action

$$S(\psi,\overline{\psi}) = \int_M dx\ \overline{\psi}(x)(D+m\gamma)\psi(x)\ , \tag{74}$$

where $\overline{\psi} \in C^\infty(S^*)$ is a dual spinor, and the parameter $m \in \mathbb{R}$ defines a pseudoscalar mass term. The classical equations of motion come from a variational principle

$$\frac{\delta S}{\delta\overline{\psi}(x)} = (D+m\gamma)\psi(x) = 0\ , \qquad \frac{\delta S}{\delta\psi(x)} = \overline{\psi}(D+m\gamma) = 0\ . \tag{75}$$

The pseudoscalar mass term is chosen so that the operator

$$Q = D + m\gamma \tag{76}$$

is invertible. Indeed, using the anticommutation relation (71), its square $Q^2 = D^2 + m^2$ is clearly a strictly positive operator, and the inverse of Q is given by

$$Q^{-1} = Q/Q^2\ . \tag{77}$$

This is the propagator of the spinor field theory over the riemannian (spin) manifold M. As in the case of the scalar field (section 2.1), we introduce a source function $\eta(x)$ for the field ψ by adding a term in the action:

$$S(\psi,\overline{\psi},\eta) = \int_M dx\ (\overline{\psi}(x)(D+m\gamma)\psi(x) - \overline{\psi}(x)\eta(x))\ . \tag{78}$$

The equations of motion for ψ are modified by insertion of the source

$$(D+m\gamma)\psi(x) = \eta(x)\ , \tag{79}$$

which can be solved explicitly with the use of the propagator:

$$\psi(x) = (Q^{-1}\eta)(x) = \int_M dy\, Q^{-1}(x,y)\eta(y)\ . \tag{80}$$

We obtain a more interesting theory by coupling the spinors to a Yang-Mills field. Consider a hermitian complex vector bundle E of rank N over M, provided with a hermitian connection

$$\nabla^E : C^\infty(E) \to C^\infty(T^*M \otimes E)\ . \tag{81}$$

In the local coordinate system $\{x^\mu\}$, the connection is described by a *gauge field* whose components $A_\mu(x)$ are smooth functions with values in the Lie algebra of the unitary matrix group $U_N(\mathbb{C})$:

$$\nabla^E_\mu = \partial_\mu + A_\mu(x)\ . \tag{82}$$

The coupling of the spinors with the gauge field is obtained by considering the tensor product of vector bundles $S \otimes E$, endowed with the connection

$$\nabla_A = \nabla^S \otimes 1 + 1 \otimes \nabla^E : C^\infty(S\otimes E) \to C^\infty(T^*M \otimes S \otimes E)\ . \tag{83}$$

We use the subscript A in order to remind that the connection depends on the gauge field. Hence in the local coordinate system,

$$(\nabla_A)_\mu = \partial_\mu - \frac{1}{4}\sum_{a,b} \omega_{\mu ab}(x)\gamma^a\gamma^b + A_\mu(x) \ . \tag{84}$$

The associated Dirac operator is

$$D_A : C^\infty(S\otimes E) \to C^\infty(S\otimes E) \ , \qquad D_A = \sum_\mu \gamma^\mu (\nabla_A)_\mu \ . \tag{85}$$

Now a smooth section ψ of the tensor product $S\otimes E$ is locally given by a collection of N spinor fields (this is not true globally on M if the vector bundle E is not topologically trivial). The $\pm$ decomposition via the chirality operator γ still holds in this more general setting, as well as the matricial decomposition (73). The new action is a functional of $\psi \in C^\infty(S\otimes E)$, $\overline{\psi} \in C^\infty(S\otimes E)^*$ and the gauge field A:

$$S_f(\psi, \overline{\psi}, A) = \int_M dx \ \overline{\psi}(x)(D_A + m\gamma)\psi(x) \ . \tag{86}$$

The fundamental property of this action is its local gauge invariance. Let g be a unitary endomorphism of E. This means that g can be locally represented by a smooth function with values in the unitary group $U_N(\mathbb{C})$. The corresponding gauge transformation acts on the fields as follows,

$$\psi \to g^{-1}\psi \ , \quad \overline{\psi} \to \overline{\psi}g \ , \quad A \to A^g = g^{-1}dg + g^{-1}Ag \ , \tag{87}$$

where the connection $A = A_\mu(x)dx^\mu$ is considered locally as a matrix-valued one-form. These transformation laws imply that the Dirac operator transforms equivariantly,

$$D_{A^g} = g^{-1}(D_A)g \ , \tag{88}$$

and therefore the action is invariant:

$$S_f(g^{-1}\psi, \overline{\psi}g, A^g) = S_f(\psi, \overline{\psi}, A) \tag{89}$$

for any unitary endomorphism g. One can add a kinetic term for the gauge field A constructed from the curvature

$$(\nabla_A)^2 = \Theta(A) \ . \tag{90}$$

The curvature is a two-form over M with values in the endomorphisms of E. One has locally

$$\Theta(A) = dA + A^2 \ , \tag{91}$$

or in components

$$\Theta_{\mu\nu}(A) = \partial_\mu A_\nu - \partial_\nu A_\mu - [A_\mu, A_\nu] \ . \tag{92}$$

Under a gauge transformation, the curvature changes according to the adjoint representation

$$\Theta(A^g) = g^{-1}\Theta(A)g \ . \tag{93}$$

Let tr be the trace on the algebra of $n \times n$ matrices. The functional

$$S_b(A) = \int_M dx \, \frac{1}{4e^2} \sum_{\mu,\nu} \mathrm{tr}(\Theta_{\mu\nu}(A)\Theta^{\mu\nu}(A)) \tag{94}$$

is gauge-invariant. Here $\Theta^{\mu\nu}(A) = g^{\mu\sigma} g^{\nu\tau} \Theta_{\sigma\tau}(A)$, and $e \in \mathbb{R}$ is a coupling constant (the "charge" of the spinor fields). The full action of our gauge theory is therefore the sum $S_b + S_f$

$$S(\psi, \overline{\psi}, A) = \int_M dx \, (\frac{1}{4e^2} \sum_{\mu,\nu} \mathrm{tr}(\Theta_{\mu\nu}(A)\Theta^{\mu\nu}(A)) + \overline{\psi}(D_A + m\gamma)\psi) \ . \tag{95}$$

As in the case of the scalar field, we would like to quantize both spinors $\psi, \overline{\psi}$ and gauge field A. It amounts to compute the expectation value of some gauge-invariant functionals $\mathcal{O}$ via a formal integral

$$\langle \mathcal{O} \rangle = \int \mathcal{D}A \, \mathcal{D}\psi \, \mathcal{D}\overline{\psi} \; \mathcal{O}(\psi, \overline{\psi}, A) \; e^{-S(\psi,\overline{\psi},A)/\hbar} \ . \tag{96}$$

Such gauge-invariant functionals $\mathcal{O}$ are the physical "observables" of the theory. For example, the action S itself is an observable. Also, in physically relevant theories the spin-statistics theorem [12] dictates that the one-form A must be quantized as a bosonic field, whereas the spinors $\psi, \overline{\psi}$ must be quantized as fermionic fields. Hence the functional integration "measure" $\mathcal{D}\mathcal{A}$ is, as for the scalar field, the formal euclidian measure in the infinite-dimensional space of gauge fields A, whereas $\mathcal{D}\psi \, \mathcal{D}\overline{\psi}$ denotes the Berezin integrals over the anticommuting variables $\psi, \overline{\psi}$, see [12]. Since the action S_f depends quadratically on the fermions, it is easy to show that the Berezin integral of $e^{-S_f/\hbar}$ is proportional to the determinant of the operator $Q_A = D_A + m\gamma$. We must choose the normalization factor in such a way that the Berezin integral equals one when A is equal to a fixed background connection, say A_0. With $Q = Q_{A_0}$, we thus have

$$\int \mathcal{D}\psi \, \mathcal{D}\overline{\psi} \; e^{-S_f(\psi,\overline{\psi},A)/\hbar} = \det(Q^{-1} Q_A) \ . \tag{97}$$

As usual, we can insert an observable $\mathcal{O}$ by introducing anticommuting sources $\eta(x), \overline{\eta}(x)$ for the fermionic fields:

$$\begin{aligned}
&\int \mathcal{D}\psi \, \mathcal{D}\overline{\psi} \; \mathcal{O}(\psi, \overline{\psi}, A) \; e^{(-S_f(\psi,\overline{\psi},A) + \int dx \, (\overline{\eta}\psi + \overline{\psi}\eta)(x))/\hbar} \\
&= \; \mathcal{O}(\hbar \frac{\delta}{\delta \overline{\eta}}, -\hbar \frac{\delta}{\delta \eta}, A) \int \mathcal{D}\psi \, \mathcal{D}\overline{\psi} \; e^{(-S_f(\psi,\overline{\psi},A) + \int dx \, (\overline{\eta}\psi + \overline{\psi}\eta)(x))/\hbar} \\
&= \; \det(Q^{-1} Q_A) \mathcal{O}(\hbar \frac{\delta}{\delta \overline{\eta}}, -\hbar \frac{\delta}{\delta \eta}, A) e^{\int dx \, dy \, \overline{\eta}(x) Q_A^{-1}(x,y) \eta(y)/\hbar} \ .
\end{aligned} \tag{98}$$

Therefore the expectation value of $\mathcal{O}$ reads

$$\langle \mathcal{O} \rangle = \int \mathcal{D}A \; \mathcal{O}(\hbar \frac{\delta}{\delta \overline{\eta}}, -\hbar \frac{\delta}{\delta \eta}, A) \; e^{(-S_b(A) + W(\eta, \overline{\eta}, A))/\hbar}|_{\eta, \overline{\eta} = 0} \ , \tag{99}$$

where the free energy of the fermions in the presence of sources $W(\eta, \overline{\eta}, A)$ is by definition

$$W(\eta, \overline{\eta}, A) = \int dx \, dy \, \overline{\eta}(x) Q_A^{-1}(x,y) \eta(y) + \hbar \ln \det(Q^{-1} Q_A) \ . \tag{100}$$

The first term of the free energy involves the inverse of the operator $Q_A = D_A + m\gamma$, which may be obtained as a power series in the gauge field parameter $A = Q_A - Q$:

$$Q_A^{-1} = Q^{-1} - Q^{-1}AQ^{-1} + Q^{-1}AQ^{-1}AQ^{-1} - \ldots \tag{101}$$

Consequently the source term of W is well-defined as a power series in A, each order being represented by a tree diagram. The second term, proportional to the Planck constant $\hbar$, is a quantum fluctuation. It is not well-defined because the operator $Q^{-1}Q_A$ acts on the infinite-dimensional space of fields ψ; the determinant of such an operator may not exist. This is a renormalization problem. We first remark that formally the logarithm of a determinant is the trace of the logarithm (this is true in finite dimensions):

$$\ln \det(Q^{-1}Q_A) = \mathrm{Tr} \ln(Q^{-1}Q_A) \ . \tag{102}$$

Then expand the logarithm in power series of A:

$$\mathrm{Tr} \ln(Q^{-1}Q_A) = \mathrm{Tr} \ln(1 + Q^{-1}A) = \sum_{n=1}^{\infty} \frac{(-)^{n+1}}{n} \mathrm{Tr}\,(Q^{-1}A)^n \ . \tag{103}$$

Note that this expansion may not converge for "large" values of A. Each term of the series is a multiple integral over M:

$$\begin{aligned} \mathrm{Tr}\,(Q^{-1}A)^n \quad = \quad & \int_{M^n} dx_1 \ldots dx_n \ A(x_1)Q^{-1}(x_1,x_2)A(x_2)Q^{-1}(x_2,x_3) \\ & \ldots A(x_n)Q^{-1}(x_n,x_1) \ . \end{aligned} \tag{104}$$

This integral may also diverge because the product of distributions

$$Q^{-1}(x_1,x_2)Q^{-1}(x_2,x_3)\ldots Q^{-1}(x_n,x_1) \tag{105}$$

is not well-defined in general. The quantum part of the free energy is therefore represented by an expansion of one-loop Feynman diagrams (recall we have a factor $\hbar$). Now observe the following. The operators A and Q^{-1} actually extend to bounded operators on the Hilbert space of square-integrable spinors $\mathcal{H} = L^2(S)$ over M. Moreover, one can show that Q^{-1} is a compact operator which actually lives in the Schatten class $\ell^p(\mathcal{H})$ for any $p > \dim M$; this means that the trace of the operator $(Q^2)^{-p/2}$ is finite. The Hölder inequality implies that for any integer $n > \dim M$, the trace

$$\mathrm{Tr}\,(Q^{-1}A)^n < \infty \tag{106}$$

is well-defined. This is not the case, however, if $n \leq \dim M$: the trace diverges, because the kernel (105) is not a distribution near the diagonal $x_1 = x_2 \ldots = x_n$. Consequently only *finitely many loops* need to be renormalized, namely those corresponding to $n \leq \dim M$. This is achieved, following the general principles of perturbative renormalization, by choosing a distributional extention of the kernel near the diagonal. Once this is done, the free energy W is determined modulo addition of a finite number of local counterterms, i.e. the integral over M of a polynomial in the gauge field A and its derivatives.

2.4 Chiral anomalies

We have seen that in the quantum theory of spinors $\psi, \overline{\psi}$ coupled to a gauge field A, the expectation value of an observable $\mathcal{O}$ follows from the evaluation of the functional integral (99). Two difficulties then appear. First, the gauge invariance of the bosonic action $S_b(A)$ implies that the gauge field A has no propagator (the quadratic term in $S_b(A)$ is not an invertible operator). This problem can be successfully removed within a perturbative scheme using the Faddeev-Popov procedure, which amounts to introduce ghost fields. A second difficulty is that after renormalization of the fermionic loops, the gauge symmetry may be broken; therefore the expectation values of observables cannot be defined consistently. This phenomenon is called an *anomaly*. It arises at the limit of vanishing mass $m \to 0$, when only *chiral* fermions are present. Let us describe chiral gauge theories.

We know that on an even-dimensional spin manifold M, a spinor field $\psi \in C^\infty(S \otimes E)$ splits into semi-spinors of opposite chiralities:

$$\psi = \begin{pmatrix} \psi_+ \\ \psi_- \end{pmatrix} , \qquad \overline{\psi} = \begin{pmatrix} \overline{\psi}_+ & \overline{\psi}_- \end{pmatrix} , \tag{107}$$

and the Dirac is odd according to the $\mathbb{Z}_2$-grading of operators

$$D_A = \begin{pmatrix} 0 & (D_A)_- \\ (D_A)_+ & 0 \end{pmatrix} . \tag{108}$$

The fermionic action with mass term may thus be written as

$$\begin{aligned} S_f(\psi, \overline{\psi}, A) &= \int_M dx\ \overline{\psi}(x)(D_A + m\gamma)\psi(x) \qquad (109) \\ &= \int_M dx\ (\overline{\psi}_-(D_A)_+\psi_+ + \overline{\psi}_+(D_A)_-\psi_- \\ &\qquad + m(\overline{\psi}_+\psi_+ - \overline{\psi}_-\psi_-)) . \end{aligned}$$

We know perform a gauge transformation by a unitary endomorphism $g \in \mathrm{End}(E)$. Its action on spinors is an even operator, according to the $\mathbb{Z}_2$-graduation. We may write

$$g = \begin{pmatrix} g_+ & 0 \\ 0 & g_- \end{pmatrix} , \tag{110}$$

where $g_\pm$ acts on the semi-spinor $\psi_\pm$. The gauge transformation

$$\psi \to g^{-1}\psi , \quad \overline{\psi} \to \overline{\psi} g , \quad D_A \to D_{A^g} = g^{-1} D_A g \tag{111}$$

leaves the fermionic action S_f invariant, but the quantum part of the free energy

$$W(A) = \hbar \ln \det(Q^{-1}Q_A) , \qquad Q_A = D_A + m\gamma \tag{112}$$

transforms as

$$\ln \det(Q^{-1}Q_A) \to \ln \det(Q^{-1}g^{-1}Q_A g) . \tag{113}$$

If we naively assume that the determinant is multiplicative, as in finite dimensions, then we expect $W(A^g) = W(A)$. However, the determinant is not well-defined and the renormalization procedure may a priori break this gauge invariance. Fortunately, one can show there always exists renormalized determinants (i.e. choices of counterterms) which preserve gauge-invariance. Hence there is no anomaly for these theories. The situation is different if we consider chiral gauge transformations. The latter affect only half of the spinors and are defined as

$$\begin{aligned} \begin{pmatrix} \psi_+ \\ \psi_- \end{pmatrix} &\rightarrow \begin{pmatrix} g_+^{-1} & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} \psi_+ \\ \psi_- \end{pmatrix} , \\ \begin{pmatrix} \overline{\psi}_+ & \overline{\psi}_- \end{pmatrix} &\rightarrow \begin{pmatrix} \overline{\psi}_+ & \overline{\psi}_- \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & g_- \end{pmatrix} , \end{aligned} \tag{114}$$

so that S_f will remain invariant if we transform the operator $Q_A = D_A + m\gamma$ according to

$$Q_A \rightarrow g_-^{-1} Q_A g_+ \ . \tag{115}$$

The free energy then changes as

$$\ln\det(Q^{-1}Q_A) \rightarrow \ln\det(Q^{-1}g_-^{-1}Q_A g_+) \ . \tag{116}$$

It should be noted that the transformation (115) cannot be absorbed by a change of potential A, unless we enlarge the space of potentials itself. Chiral transformations are of physical interest, however: at the limit $m \rightarrow 0$, the semi-spinors of opposite chirality become independent and the action is the sum of two terms

$$\begin{aligned} S_f(\psi, \overline{\psi}, A) &= \int_M dx\, \overline{\psi}_-(D_A)_+\psi_+ + \int_M dx\, \overline{\psi}_+(D_A)_-\psi_- \\ &= S_f^+(\psi_+, \overline{\psi}_-, A) + S_f^-(\psi_-, \overline{\psi}_+, A) \end{aligned} \tag{117}$$

Hence we can consider a gauge theory involving only half of the fermions via the chiral action S_f^+ for example. S_f^+ is invariant under the chiral gauge transformations

$$\psi_+ \rightarrow g_+^{-1}\psi_+ \ , \quad \overline{\psi}_- \rightarrow \overline{\psi}_- g_- \ , \quad A \rightarrow g^{-1}dg + g^{-1}Ag \ . \tag{118}$$

In this chiral gauge theory, we are thus led to calculate functional integrals of the type

$$\langle \mathcal{O} \rangle = \int \mathcal{D}A\, \mathcal{D}\psi_+\, \mathcal{D}\overline{\psi}_-\ \mathcal{O}(\psi_+, \overline{\psi}_-, A)\ e^{-(S_b(A) + S_f^+(\psi_+, \overline{\psi}_-, A))/\hbar} \ . \tag{119}$$

This is performed in exactly the same way as before. The Berezin integral over chiral fermions yields a determinant

$$\int \mathcal{D}\psi_+\, \mathcal{D}\overline{\psi}_-\ e^{-S_f^+(\psi_+, \overline{\psi}_-, A)/\hbar} = \det(D_+^{-1}(D_A)_+) \ , \tag{120}$$

where D is the chiral Dirac operator associated to a background connection. For $m = 0$ one has $Q_A = D_A$, $Q = D$ so that

$$\ln\det(Q^{-1}Q_A) = \ln\det(D_+^{-1}(D_A)_+) + \ln\det(D_-^{-1}(D_A)_-) \ . \tag{121}$$

Applying a chiral gauge transformation

$$D_A \to g_-^{-1} D_A g_+ \ , \quad (D_A)_+ \to g_-^{-1}(D_A)_+ g_+ \ , \quad (D_A)_- \to (D_A)_- \tag{122}$$

shows that the variation of the free energy of the chiral gauge theory $W_+(A) = \hbar \ln \det(D_+^{-1}(D_A)_+)$ is the limit $m \to 0$ of

$$\hbar \ln \det(Q^{-1} g_-^{-1} Q_A g_+) - \hbar \ln \det(Q^{-1} Q_A) \ . \tag{123}$$

There are topological reasons for which this variation cannot vanish. It is related to the first homotopy of the group of gauge transformations $\mathcal{G} \subset \mathrm{End}(E)$. Let us consider a smooth family of gauge transformations parametrized by the circle:

$$g : S^1 \to \mathcal{G} \ , \qquad t \mapsto g(t) \ . \tag{124}$$

This yields a smooth family of free energies

$$W(g) = \hbar \ln \det(Q^{-1} g_-^{-1} Q_A g_+) \ . \tag{125}$$

According to the preceding section, $W(g)$ will be defined as a formal power series of the operator $\widetilde{A} = g_-^{-1} Q_A g_+ - Q$:

$$\begin{aligned} \ln \det(Q^{-1} g_-^{-1} Q_A g_+) &= \mathrm{Tr} \ln(Q^{-1} g_-^{-1} Q_A g_+) \ = \ \mathrm{Tr} \ln(1 + Q^{-1} \widetilde{A}) \\ &= \sum_{n=1}^{\infty} \frac{(-)^{n+1}}{n} \mathrm{Tr}\, (Q^{-1} \widetilde{A})^n \ . \end{aligned} \tag{126}$$

Only the first few terms need renormalization. This formal power series is not convergent for "large" values of $\widetilde{A}$, since it is the expansion of a logarithm around 1. Denote by $s : C^\infty(S^1) \to \Omega^1(S^1)$ the de Rham differential on the circle (we choose the notation s in relation with the corresponding BRS operator [10, 14]). The anomaly $\Delta(g)$ is the differential of the free energy function along the circle

$$\Delta(g) = sW(g) \ . \tag{127}$$

It is a closed one-form in $\Omega^1(S^1)$. Its cohomology class does *not* vanish because the logarithm of the determinant is not uniquely determined over the circle in general (a fact expressed by the divergence of the formal power series in $\widetilde{\mathcal{A}}$). In other words the integral

$$\frac{1}{2\pi i} \oint \Delta(g) = \frac{1}{2\pi i} \oint sW(g) \ \in \mathbb{Z} \tag{128}$$

is an integer which counts the winding number of the $\mathbb{C}$-valued determinant function $e^{W(g)/\hbar}$ over the circle. We call this winding number the *topological* anomaly of the loop g. It is clearly defined at the level of homotopy classes of loops in the gauge group $\mathcal{G}$. The thus obtained map

$$\pi_1(\mathcal{G}) \to \mathbb{Z} \ , \qquad g \mapsto \frac{1}{2\pi i} \oint \Delta(g) \tag{129}$$

will be eventually related to an index computation, using K-theoretic tools. We will do this in the completely general framework of non-commutative geometry.

3 Noncommutative Geometry

3.1 Noncommutative spaces

The aim of noncommutative geometry is to extend the tools of classical geometry to more general "spaces" represented by associative algebras. The basic observation comes from the fact that an ordirary space X, viewed as a set of points with an additional structure (for example a measure, or a topology, or a differentiable structure), is entirely characterized by a suitable algebra of functions $f : X \to \mathbb{C}$. For example

1) A space X endowed with a measure Ω is described by the algebra $L^\infty(X, \Omega)$ of essentially bounded measurable functions over X.

2) A locally compact topological space X is described by the algebra $C_0(X)$ of continuous functions vanishing at infinity.

3) A smooth manifold X is described by the algebra $C_c^\infty(X)$ of smooth functions with compact support on it.

In all the above cases, the algebra of functions is commutative, the product being given by pointwise multiplication:

$$(f_1 f_2)(x) = f_1(x) f_2(x) \qquad \forall x \in X \; . \tag{130}$$

The idea of noncommutative geometry is that any algebra, not necessarily commutative, should be thought of as a kind of noncommutative "space" X. Of course such an X is no longer given by a set of points. The examples above motivate the following definitions of different types of noncommutative spaces:

1) Noncommutative measurable spaces = von Neumann algebras.

2) Noncommutative topological spaces = C^*-algebras.

3) Noncommutative differentiable manifolds = "smooth" subalgebras of C^*-algebras.

It is worth mentioning that these algebras are always realized as subalgebras of bounded operators on a separable Hilbert space. It allows to use the powerful tools of functional analysis. The force of noncommutative geometry is that singular spaces, generally badly behaved as set of points, become more tractable when described by noncommutative algebras. We list below some well-known examples of noncommutative spaces.

Example 3.1 The dual of a discrete group. Let G be a discrete (countable) group. The convolution algebra $\mathcal{A} = C_c(G)$ of G is the space of functions $f : G \to \mathbb{C}$ with finite support, endowed with the convolution product

$$(f_1 f_2)(g) = \sum_{h \in G} f_1(h) f_2(h^{-1} g) \qquad \forall g \in G \; . \tag{131}$$

The convolution algebra is always associative. It is commutative exactly when G is abelian. $\mathcal{A}$ is represented by bounded operators on the Hilbert space $\mathcal{H} = \ell^2(G)$ of square-integrable functions on G, via the left regular representation

$$(f\xi)(g) = \sum_{h \in G} f(h)\xi(h^{-1}g) \qquad \forall\, f \in \mathcal{A}\ ,\ \xi \in \mathcal{H}\ . \tag{132}$$

The completion of $\mathcal{A}$ in the operator norm is the reduced C^*-algebra of the group $C^*_r(G)$. When G is commutative, $C^*_r(G)$ is isomorphic, by Fourier transform, to the commutative algebra of continuous functions over the Pontrjagin dual $\widehat{G}$ (the compact space of characters of G):

$$C^*_r(G) \cong C(\widehat{G})\ . \tag{133}$$

the product on $C(\widehat{G})$ is given by pointwise multiplication. This shows that when G is not abelian, the noncommutative topological space described by the C^*-algebra $C^*_r(G)$ may be considered as the "Pontrjagin dual" of G. It bears essential information about the representation theory of G, and is therefore of great importance.

Example 3.2 Groupoids. Groupoids provide an extremely rich class of noncommutative spaces, ranging from groups at one end, to ordinary (commutative) spaces at the other end. A groupoid G is a small category in which all the arrows are invertible. In other words, there is a distinguished subset $X \subset G$ of the set G, and two maps $r, s : G \rightrightarrows X$, the range and source maps. r and s restricted to the subset X are the identity map. An element $\gamma \in G$ is represented by an arrow from $s(\gamma)$ to $r(\gamma)$:

$$\begin{tikzcd} r(\gamma) & s(\gamma) \arrow[l, bend right, "\gamma"'] \end{tikzcd} \tag{134}$$

The structure of the groupoid is given by an associative composition law of compatible arrows: if $\gamma_1, \gamma_2 \in G$ are such that $s(\gamma_1) = r(\gamma_2)$, there is a composite arrow $\gamma_1\gamma_2 \in G$ such that $r(\gamma_1\gamma_2) = r(\gamma_1)$ and $s(\gamma_1\gamma_2) = s(\gamma_2)$:

$$\begin{tikzcd} r(\gamma_1) & s(\gamma_1) = r(\gamma_2) \arrow[l, bend right, "\gamma_1"'] & s(\gamma_2) \arrow[l, bend right, "\gamma_2"'] \arrow[ll, bend left, "\gamma_1\gamma_2"] \end{tikzcd} \tag{135}$$

Moreover, the elements of X are units for the composition law, and any arrow $\gamma \in G$ has an inverse $\gamma^{-1} \in G$ such that $\gamma\gamma^{-1} = r(\gamma)$ and $\gamma^{-1}\gamma = s(\gamma)$. The inverse is necessarily unique. Therefore, one can think of G as the space of units X whose points are connected by invertible arrows.
The convolution algebra of the groupoid is the associative algebra $\mathcal{A} = C_c(G)$ of functions $f : G \to \mathbb{C}$ with finite support, endowed with the product

$$(f_1 f_2)(\gamma) = \sum_{\gamma = \gamma_1\gamma_2} f_1(\gamma_1) f_2(\gamma_2)\ . \tag{136}$$

There are two extremal cases. When the space of units X is reduced to only one element $e \in G$, then all the arrows of G can be composed. Hence there is an associative composition law $G \times G \to G$, a unit element e and all the arrows are invertible: it means that G is a group. The algebra $\mathcal{A}$ coincides with the convolution algebra of the group. Another extremal case is when $X = G$. Then each arrow is a unit and can be composed only with itself: the structure of the groupoid G reduces to an ordinary set of points X, and the algebra $\mathcal{A}$ becomes the commutative algebra of functions over X, with pointwise multiplication.
In general, the noncommutative space described by the convolution algebra $\mathcal{A}$ corresponds to the space of units X, whose points are identified in a complicated way via the arrows. $\mathcal{A}$ encodes these relations. More generally X and G can be topological spaces or manifolds, in which case the convolution algebra will be obtained from a suitable space of continuous, or differentiable, functions $f : G \to \mathbb{C}$. Groupoids are extremely useful in a wide range of geometric situations, including for example group actions on manifolds, quotient spaces by an equivalence relation, or foliations.

Example 3.3 Almost commutative geometries. If X is a compact manifold, we can take the tensor product of the commutative algebra of smooth functions over M with a matrix algebra:

$$\mathcal{A} = C^\infty(M) \otimes M_N(\mathbb{C}) \ . \tag{137}$$

Then $\mathcal{A}$ is noncommutative, but does not differs very much from $C^\infty(M)$. This kind of noncommutative spaces are called almost commutative geometries. They appeared as an attempt to describe the standard model of elementary particles from first principles of noncommutative geometry [7].

As we have seen, a smooth algebra $\mathcal{A}$ describes a noncommutative "manifold", whereas its C^*-completion detects only the underlying noncommutative "topological space". If one wants to refine further the geometric description of a noncommutative space, including for example the notion of distance or riemannian metric, it is necessary to bring new elements into play. The correct definition of a noncommutative riemannian manifold is introduced by Connes as a *spectral triple*

$$(\mathcal{A}, \mathcal{H}, D) \tag{138}$$

where $\mathcal{A}$ is an involutive "smooth" algebra represented in the algebra of bounded operators on a Hilbert space $\mathcal{H}$, and D is a selfadjoint, unbounded operator on $\mathcal{H}$ such that

i) $(1 + D^2)^{-1/2}$ is a compact operator,
ii) For any $a \in \mathcal{A}$, the commutator $[D, a]$ extends to a bounded operator on $\mathcal{H}$.

In addition, the spectral triple is said to be of even degree if there is an involutive operator γ inducing a $\mathbb{Z}_2$-grading on $\mathcal{H}$, such that D is odd and $\mathcal{A}$ is represented by even operators. If there is no such γ the spectral triple is said to be of odd degree.

The classical commutative example is provided by a riemannian, closed, spin manifold M. Here $\mathcal{A} = C^\infty(M)$ is the algebra of smooth functions acting by pointwise multiplication on the Hilbert space $\mathcal{H} = L^2(S)$ of square-integrable sections of the spinor bundle, and $D : C^\infty(S) \to C^\infty(S)$ is the usual Dirac operator. The degree of the spectral triple corresponds to the parity of the dimension of M. It is remarkable that the geodesic distance on M and the riemannian metric can be extracted from the Dirac operator and the Hilbert space representation of $\mathcal{A}$ [7]. Spectral triples thus offer the possibility to generalize all the tools of ordinary differential and riemannian geometry to noncommutative spaces, using functional analysis methods.

3.2 K-theory and index theory

We know that an associative algebra $\mathcal{A}$ is the noncommutative generalization of a manifold. It would be convenient to extend the classical notions of algebraic topology, K-theory, characteristic classes, etc ... to the realm of noncommutative geometry. This is indeed possible. In this section we will discuss the noncommutative version of K-theory, whose classical counterpart provides topological invariants of manifolds, and the corresponding index theory.

We first review the classical K-theory of manifolds (this works more generally for locally compact spaces). In order to avoid unnecessary complications, we suppose that M is a compact manifold. Consider the set $\mathrm{Vect}(M)$ of isomorphism classes of complex vector bundles $E \to M$ with finite rank. There is a natural semigroup structure on $\mathrm{Vect}(M)$ given by direct sum

$$[E] + [F] = [E \oplus F] \ , \tag{139}$$

and the vector bundle of rank zero $[0]$ is obviously a neutral element. The Grothendieck group $K(M)$ is the abelian group generated by $\mathrm{Vect}(M)$. Its elements are obtained by considering pairs of isomorphism classes of vector bundles $([E], [F])$ subject to the addition law

$$([E], [F]) + ([E'], [F']) = ([E \oplus E'], [F \oplus F']) \tag{140}$$

and the equivalence relation

$$([E], [F]) \sim ([E \oplus G], [F \oplus G]) \tag{141}$$

for any $[G] \in \mathrm{Vect}(M)$. The equivalence class of the pair $([E], [F])$ for this relation is by definition the difference $[E] - [F] \in K(M)$. The opposite of the difference $[E] - [F]$ is given by $[F] - [E]$, since one has

$$([E], [F]) + ([F], [E]) = ([E \oplus F], [F \oplus E]) \sim ([0], [0]) \ . \tag{142}$$

Hence $K(M)$ is indeed an abelian group. We would like to obtain the same group directly from the commutative algebra of smooth functions

$$\mathcal{A} = C^\infty(M) \ . \tag{143}$$

Hence we must know how to describe vector bundles over M in terms of $\mathcal{A}$. It turns out that vector bundles exactly correspond to finitely generated projective modules over $\mathcal{A}$. Indeed, the space of smooth sections $C^\infty(E)$ is a (say, right) module over $\mathcal{A}$, for the action given by pointwise multiplication:

$$C^\infty(E) \times \mathcal{A} \to C^\infty(E) \ , \qquad (\xi a)(x) = \xi(x) a(x) \quad \forall \xi \in C^\infty(E), a \in \mathcal{A} \ . \tag{144}$$

This right $\mathcal{A}$-module completely describes the vector bundle E. Equivalently, finitely generated projective modules correspond to idempotents in the matrix algebra over $\mathcal{A}$. On obtains an idempotent from a vector bundle E of rank n as follows. Since M is compact, there exists a finite open covering $\{U_i\}$, $i = 1, \ldots, m$ of M such that E is trivial over each U_i:

$$E|_{U_i} \cong U_i \times \mathbb{C}^n \ . \tag{145}$$

For any pair i, j, let $g_{ij} : U_i \cap U_j \to Gl_n(\mathbb{C})$ be the transition functions defining E. They fulfill the cocycle condition $g_{ij} g_{jk} = g_{ik}$ for any i, j, k. Choose a partition of unity $\{\rho_i\}$ relative to the open covering. Each ρ_i is a positive real-valued function with compact support on U_i such that:

$$\sum_{i=1}^{m} \rho_i(x) = 1 \qquad \forall x \in M \ . \tag{146}$$

Then the following matrix $e \in M_m(M_n(\mathcal{A}))$ defines an idempotent:

$$e_{ij} = \sqrt{\rho_i}\, g_{ij} \sqrt{\rho_j} \qquad \forall i, j = 1, \ldots, m \ . \tag{147}$$

Indeed, each component e_{ij} is an element of the matrix algebra $M_n(\mathcal{A})$ and the cocycle condition implies

$$\sum_j e_{ij} e_{jk} = \sum_j \sqrt{\rho_i} g_{ij} \rho_j g_{jk} \sqrt{\rho_k} = (\sum_j \rho_j) \sqrt{\rho_i} g_{ik} \sqrt{\rho_k} = e_{ik} \ . \tag{148}$$

Hence e is an idempotent of the matrix algebra $M_N(\mathcal{A})$ with $N = mn$. We have shown that any vector bundle E gives rise to an idempotent of the matrix algebra $M_N(\mathcal{A})$, for a given size N. Conversely, any idempotent $e \in M_N(\mathcal{A})$ always determines a vector bundle over M: let 1_N denote the trivial complex line bundle of rank N over M. Then e is an idempotent endomorphism of 1_N. Its range yields a vector bundle E as a direct summand:

$$1_N = E \oplus F \ , \qquad E = \mathrm{Im}(e) \ , \ F = \mathrm{Im}(1 - e). \tag{149}$$

Two idempotents e, f in the matrix algebra $M_\infty(\mathcal{A}) = \cup_{N \in \mathbb{N}} M_N(\mathcal{A})$ are said to be isomorphic if there exists two matrices u and v such that

$$e = uv \quad \text{and} \quad f = vu \ . \tag{150}$$

It is not hard to show that the isomorphism classes of vector bundles over M are in one-to-one correspondence with the isomorphism classes of idempotents in the matrix algebra $M_\infty(\mathcal{A})$. The set of isomorphism classes of idempotents is naturally a semigroup under the direct sum

$$e \oplus f = \begin{pmatrix} e & 0 \\ 0 & f \end{pmatrix} \ . \tag{151}$$

Hence the K-theory of M may be alternatively defined as the Grothendieck group of the semigroup of equivalence classes of idempotents in the matrix algebra $M_\infty(\mathcal{A})$. This motivates the definition of the K-theory of any associative (not necessarily commutative) algebra.

Definition 3.4 *Let $\mathcal{A}$ be an associative algebra. The K-theory group $K_0(\mathcal{A})$ is the Grothendieck group associated to the semigroup of isomorphism classes of idempotents in the matrix algebra $M_\infty(\mathcal{A})$.*

$K_0(\mathcal{A})$ is the first of a hierarchy of abelian groups [15]. For our purposes we only need to introduce the next algebraic K-group $K_1(\mathcal{A})$. Let $M_N(\widetilde{\mathcal{A}})$ be the algebra of $N \times N$ matrices over the unitalized algebra $\widetilde{\mathcal{A}} = \mathcal{A} \oplus \mathbb{C}$. The elements of $\widetilde{\mathcal{A}}$ are thus pairs (a, λ) with $a \in \mathcal{A}$, $\lambda \in \mathbb{C}$ and the multiplication rule is

$$(a_1, \lambda_1)(a_2, \lambda_2) = (a_1 a_2 + \lambda_1 a_2 + \lambda_2 a_1, \lambda_1 \lambda_2) \ . \tag{152}$$

The linear group $Gl_N(\mathcal{A})$ is the group of invertible elements of the matrix algebra $M_N(\widetilde{\mathcal{A}})$ of the form 1 modulo $M_N(\mathcal{A})$:

$$Gl_N(\mathcal{A}) = \{x \in Gl_N(\widetilde{\mathcal{A}}) \mid x \equiv 1_N \mod M_N(\mathcal{A})\} \ . \tag{153}$$

Let $Gl_\infty(\mathcal{A})$ denote the inductive limit of the groups $Gl_N(\mathcal{A})$ under the inclusions

$$Gl_N(\mathcal{A}) \hookrightarrow Gl_{N+1}(\mathcal{A}) \ , \qquad x \hookrightarrow \begin{pmatrix} x & 0 \\ 0 & 1 \end{pmatrix} \ . \tag{154}$$

By definition the K-theory group $K_1(\mathcal{A})$ is the abelianization of $Gl_\infty(\mathcal{A})$:

$$K_1(\mathcal{A}) = Gl_\infty(\mathcal{A}) / [Gl_\infty(\mathcal{A}), Gl_\infty(\mathcal{A})] \ , \tag{155}$$

where $[Gl_\infty(\mathcal{A}), Gl_\infty(\mathcal{A})]$ is the normal subgroup generated by the "commutators" $xyx^{-1}y^{-1}$, for $x, y \in Gl_\infty(\mathcal{A})$.

Remark 3.5 When $\mathcal{A}$ is a banach algebra, the algebraic K-group $K_1(\mathcal{A})$ should not be confused with the topological group $K_1^{\rm top}(\mathcal{A})$, defined as the quotient of $Gl_\infty(\mathcal{A})$ by homotopy. There always exists a surjective map $K_1(\mathcal{A}) \to K_1^{\rm top}(\mathcal{A})$ induced by the universal property of $K_1(\mathcal{A})$, but these two abelian groups differ in general.

In the classical case of a manifold M, elliptic pseudodifferential operators provide invariants of the K-theory group $K(M)$. This is the content of the celebrated Atiyah-Singer index theorem [2]. In the noncommutative context, the Dirac operator of a spectral triple $(\mathcal{A}, \mathcal{H}, D)$ plays the rôle of an elliptic operator for the algebra $\mathcal{A}$. We will restrict our attention to spectral triples of even degree only. Hence, $\mathcal{H} = \mathcal{H}_+ \oplus \mathcal{H}_-$ is a $\mathbb{Z}_2$-graded Hilbert space on which $\mathcal{A}$ acts as bounded operators of even degree, and D is a selfadjoint unbounded operator of odd degree verifying the conditions listed in section 3.1. An index pairing gives rise to an additive map

$$[D] : K_0(\mathcal{A}) \to \mathbb{Z} \ . \tag{156}$$

The latter is constructed as follows. Let $e \in M_N(\mathcal{A})$ be an idempotent. At the expense of replacing $\mathcal{A}$ by $M_N(\mathcal{A})$, $\mathcal{H}$ by $\mathcal{H} \otimes \mathbb{C}^N$ and D by $D \otimes 1$, we may assume that e belongs to $\mathcal{A}$. Hence e defines an idempotent operator on the Hilbert space $\mathcal{H}$. Define the bounded selfadjoint operator

$$F = D/(1 + D^2)^{1/2} \ . \tag{157}$$

The commutator $[F,e]$ is a compact, as well as the difference F^2-1. We regard the compression eFe as a bounded operator of odd degree on the Hilbert space $\mathcal{H}'=e\mathcal{H}=\mathcal{H}'_+\oplus\mathcal{H}'_-$. It is written in 2×2 matrix notation

$$eFe=\begin{pmatrix} 0 & P \\ Q & 0 \end{pmatrix}\ , \tag{158}$$

with $Q:\mathcal{H}'_+\to\mathcal{H}'_-$ and $P:\mathcal{H}'_-\to\mathcal{H}'_+$. One shows that $PQ-1$ and $QP-1$ are compact operators, hence Q is a Fredholm operator [11]: it has finite-dimensional kernel and cokernel. We define the index of D against the idempotent e as the integer

$$\mathrm{Ind}(D,e)=\dim\mathrm{Ker}\,Q-\dim\mathrm{Coker}\,Q\ \in\mathbb{Z}\ . \tag{159}$$

One can show that this number only depends on the K-theory class of e. This gives rise to the index map (156). The aim of cyclic cohomology is to provide tools for the computation of this map, and to obtain explicit formulas generalizing the Atiyah-Singer index theorem in the noncommutative setting.

3.3 Cyclic cohomology

Cyclic homology and cohomology is the noncommutative generalization of de Rham cohomology. It was introduced by Connes in the context of index theory for noncommutative spaces, and independently by Tsygan in an additive approach to algebraic K-theory.

The Loday-Quillen bicomplex

We introduce here cyclic homology in the simplest way, using the so-called Loday-Quillen bicomplex [13]. Let $\mathcal{A}$ be an associative algebra over $\mathbb{C}$. The cyclic bicomplex of $\mathcal{A}$ is a collection of vector spaces indexed by two integers p,n constructed from the tensor powers of $\mathcal{A}$

$$CC_{p,n}(\mathcal{A})=\mathcal{A}^{\otimes(n+1)}\ ,\quad p,n\in\mathbb{N} \tag{160}$$

and linear maps b,b',λ,N (arrows) organized as follows:

$$\begin{tikzcd}
{} \arrow[d, "b"] & {} \arrow[d, "-b'"] & {} \arrow[d, "b"] & \\
CC_{0,2}(\mathcal{A}) & CC_{1,2}(\mathcal{A}) \arrow[l, "1-\lambda"'] \arrow[d, "-b'"] & CC_{2,2}(\mathcal{A}) \arrow[l, "N"'] \arrow[d, "b"] & \cdots \arrow[l, "1-\lambda"'] \\
CC_{0,1}(\mathcal{A}) \arrow[u, phantom] & CC_{1,1}(\mathcal{A}) \arrow[l, "1-\lambda"'] \arrow[d, "-b'"] & CC_{2,1}(\mathcal{A}) \arrow[l, "N"'] \arrow[d, "b"] & \cdots \arrow[l, "1-\lambda"'] \\
CC_{0,0}(\mathcal{A}) & CC_{1,0}(\mathcal{A}) \arrow[l, "1-\lambda"'] & CC_{2,0}(\mathcal{A}) \arrow[l, "N"'] & \cdots \arrow[l, "1-\lambda"']
\end{tikzcd} \tag{161}$$

with $CC_{0,2}(\mathcal{A})\xrightarrow{b}CC_{0,1}(\mathcal{A})\xrightarrow{b}CC_{0,0}(\mathcal{A})$ in the first column.

The linear map $b':\mathcal{A}^{\otimes(n+1)}\to\mathcal{A}^{\otimes n}$ is given by its action on a $(n+1)$-tensor product of elements $a_0,\ldots,a_n\in\mathcal{A}$ by the formula

$$b'(a_0\otimes a_1\otimes\ldots\otimes a_n)=\sum_{i=0}^{n-1}(-)^i a_0\otimes\ldots\otimes a_ia_{i+1}\otimes\ldots\otimes a_n\ , \tag{162}$$

and $b : \mathcal{A}^{\otimes(n+1)} \to \mathcal{A}^{\otimes n}$ is the Hochschild operator

$$b(a_0 \otimes \ldots \otimes a_n) = b'(a_0 \otimes \ldots \otimes a_n) + (-)^n a_n a_0 \otimes a_1 \otimes \ldots \otimes a_{n-1} \ . \tag{163}$$

These are coboundary maps, in the sense that $b^2 = 0 = b'^2$. The next operator $\lambda : \mathcal{A}^{\otimes(n+1)} \to \mathcal{A}^{\otimes(n+1)}$ is the signed cyclic permutation

$$\lambda(a_0 \otimes \ldots \otimes a_n) = (-)^n a_n \otimes a_0 \otimes \ldots \otimes a_{n-1} \ , \tag{164}$$

and N is the cyclic antisymmetrization

$$N = \sum_{i=0}^{n} \lambda^i \quad \text{on } \mathcal{A}^{\otimes(n+1)} \ . \tag{165}$$

One has $(1-\lambda)N = 0 = N(1-\lambda)$ and the following equalities show that the vertical and horizontal maps commute:

$$\begin{aligned} Nb &= b'N \ , \\ (1-\lambda)b' &= b(1-\lambda) \ . \end{aligned} \tag{166}$$

Hence $CC_{p,n}(\mathcal{A})$ is indeed a bicomplex. One can show that each row

$$CC_{0,n}(\mathcal{A}) \xleftarrow{1-\lambda} CC_{1,n}(\mathcal{A}) \xleftarrow{N} CC_{2,n}(\mathcal{A}) \xleftarrow{1-\lambda} \cdots \tag{167}$$

is exact except at the first term, that is, the kernel of $(1-\lambda)$ coincides with the image of N and conversely. We can now give the definition of the cyclic homology of $\mathcal{A}$.

Definition 3.6 *The cyclic homology $HC_*(\mathcal{A})$ is the homology of the total complex $CC_{p,n}(\mathcal{A})$. More precisely for any $n \in \mathbb{N}$, $HC_n(\mathcal{A})$ is the homology in degree n of the complex*

$$\ldots \to D_n \to D_{n-1} \to \ldots \to D_0 \to 0 \ , \tag{168}$$

formed by the diagonals of $CC_{p,n}(\mathcal{A})$:

$$\begin{aligned} D_n &= CC_{0,n} \oplus CC_{1,n-1} \oplus \ldots \oplus CC_{n,0} \\ &= \mathcal{A}^{\otimes(n+1)} \oplus \mathcal{A}^{\otimes n} \oplus \ldots \oplus \mathcal{A} \ . \end{aligned} \tag{169}$$

Cyclic homology can be defined in an alternative way, using the complex of cyclic chains. Introduce first the following sub-bicomplex $CC^{triv}_{*,*}(\mathcal{A})$:

$$\begin{array}{ccccccc}
\downarrow & & \downarrow & & \downarrow & & \\
(1-\lambda)\mathcal{A}^{\otimes 3} & \xleftarrow{1-\lambda} & \mathcal{A}^{\otimes 3} & \xleftarrow{N} & \mathcal{A}^{\otimes 3} & \xleftarrow{1-\lambda} & \cdots \\
\downarrow{\scriptstyle b} & & \downarrow{\scriptstyle -b'} & & \downarrow{\scriptstyle b} & & \\
(1-\lambda)\mathcal{A}^{\otimes 2} & \xleftarrow{1-\lambda} & \mathcal{A}^{\otimes 2} & \xleftarrow{N} & \mathcal{A}^{\otimes 2} & \xleftarrow{1-\lambda} & \cdots \\
\downarrow{\scriptstyle b} & & \downarrow{\scriptstyle -b'} & & \downarrow{\scriptstyle b} & & \\
(1-\lambda)\mathcal{A} & \xleftarrow{1-\lambda} & \mathcal{A} & \xleftarrow{N} & \mathcal{A} & \xleftarrow{1-\lambda} & \cdots
\end{array} \tag{170}$$

Hence only the first column differs from the original cyclic bicomplex, since $(1-\lambda)\mathcal{A}^{\otimes n} \subset \mathcal{A}^{\otimes n}$. Now the rows of $CC^{triv}(\mathcal{A})$ are exact even at the first term. Therefore, a classical diagram-chasing argument implies the vanishing of the cohomology of the total complex $CC^{triv}(\mathcal{A})$. Let $C^{\lambda}_{*}(\mathcal{A})$ be the complex of *cyclic chains*

$$C^{\lambda}_{n}(\mathcal{A}) = D_n(\mathcal{A})/D^{triv}_{n}(\mathcal{A}) = \mathcal{A}^{\otimes(n+1)}/(1-\lambda)\mathcal{A}^{\otimes(n+1)} \ , \tag{171}$$

which takes place in a short exact sequence of total complexes

$$0 \to D^{triv} \to D \to C^{\lambda}(\mathcal{A}) \to 0 \ . \tag{172}$$

Then the associated homology long exact sequence implies the following proposition:

Proposition 3.7 *For any $n \in \mathbb{N}$, the cyclic homology $HC_n(\mathcal{A})$ is isomorphic to the homology in degree n of the complex of cyclic chains*

$$\ldots \longrightarrow C^{\lambda}_{n}(\mathcal{A}) \xrightarrow{b} C^{\lambda}_{n-1}(\mathcal{A}) \xrightarrow{b} \ldots \longrightarrow C^{\lambda}_{0}(\mathcal{A}) \longrightarrow 0 \ . \tag{173}$$

Connes' long exact sequence

Let us isolate the first two columns of the cyclic bicomplex

$$\begin{array}{ccc}
\downarrow & & \downarrow \\
\mathcal{A}^{\otimes 3} & \xleftarrow{1-\lambda} & \mathcal{A}^{\otimes 3} \\
\downarrow b & & \downarrow -b' \\
\mathcal{A}^{\otimes 2} & \xleftarrow{1-\lambda} & \mathcal{A}^{\otimes 2} \\
\downarrow b & & \downarrow -b' \\
\mathcal{A} & \xleftarrow{1-\lambda} & \mathcal{A}
\end{array} \tag{174}$$

and consider the total complex obtained by taking the diagonals $D^{H}_{n} = \mathcal{A}^{\otimes(n+1)} \oplus \mathcal{A}^{n}$, $D^{H}_{0} = \mathcal{A}$:

$$\ldots \longrightarrow D^{H}_{n} \longrightarrow D^{H}_{n-1} \longrightarrow \ldots \longrightarrow D^{H}_{0} \longrightarrow 0 \ . \tag{175}$$

Definition 3.8 *The nth degree homology of the complex D^H is the Hochschild homology $HH_n(\mathcal{A})$.*

For any $n \in \mathbb{N}$, D^{H}_{n} is the kernel of the projection $D_n \to D_{n-2}$ of the corresponding diagonals of the cyclic bicomplex, whence a short exact sequence of complexes

$$0 \to D^{H} \to D \to D[2] \to 0 \ , \tag{176}$$

where $D[2]$ denotes the cyclic bicomplex with degrees shifted by two. The associated homology long exact sequence leads to the following theorem relating Hochschild and cyclic homologies:

Theorem 3.9 (Connes' exact sequence) *The cyclic homology of $\mathcal{A}$ is related to the Hochschild homology through the long exact sequence*

$$\ldots \longrightarrow HC_n(\mathcal{A}) \xrightarrow{S} HC_{n-2}(\mathcal{A}) \xrightarrow{B} HH_{n-1}(\mathcal{A}) \xrightarrow{I} HC_{n-1}(\mathcal{A}) \xrightarrow{S} HC_{n-3}(\mathcal{A}) \xrightarrow{B}$$

$$\ldots \xrightarrow{I} HC_1(\mathcal{A}) \longrightarrow 0 \longrightarrow HH_0(\mathcal{A}) \xrightarrow{I} HC_0(\mathcal{A}) \longrightarrow 0 \ ,$$

where the map I is induced by the inclusion $D_n^H \hookrightarrow D_n$, S by the projection $D_n \to D_{n-2}$ and B is the boundary of the exact sequence (176).

Cyclic cohomology

The preceding constructions can be straightforwardly dualized by introducing the bicomplex of cyclic cochains $CC^{p,n}(\mathcal{A}) = \mathrm{Hom}(\mathcal{A}^{\otimes(n+1)}, \mathbb{C})$, $\forall p, n \in \mathbb{N}$:

$$\begin{array}{ccccccc}
\uparrow & & \uparrow & & \uparrow & & \\
CC^{0,2} & \xrightarrow{1-\lambda} & CC^{1,2} & \xrightarrow{N} & CC^{2,2} & \xrightarrow{1-\lambda} & \cdots \\
\uparrow b & & \uparrow -b' & & \uparrow b & & \\
CC^{0,1} & \xrightarrow{1-\lambda} & CC^{1,1} & \xrightarrow{N} & CC^{2,1} & \xrightarrow{1-\lambda} & \cdots \\
\uparrow b & & \uparrow -b' & & \uparrow b & & \\
CC^{0,0} & \xrightarrow{1-\lambda} & CC^{1,0} & \xrightarrow{N} & CC^{2,0} & \xrightarrow{1-\lambda} & \cdots
\end{array} \tag{177}$$

where b, b', λ, N are transposed of the preceding operators. The cyclic cohomology $HC^n(\mathcal{A})$ is the cohomology of the associated total complex in degree n. As before, cyclic cohomology may be obtained via a complex of cyclic cochains as follows. Define the complex of cyclic cochains $C_\lambda(\mathcal{A})$ by

$$C^n_\lambda(\mathcal{A}) = \mathrm{Ker}(1-\lambda) \cap \mathrm{Hom}(\mathcal{A}^{\otimes(n+1)}, \mathbb{C}) \ . \tag{178}$$

$\mathrm{Ker}(1-\lambda)$ is stable by b, hence $C_\lambda(\mathcal{A})$ is also stable. If one considers the cyclic bicomplex (177) augmented on the left by the column C_λ, one obtains a bicomplex with trivial total cohomology, because its rows are exact:

$$\begin{array}{ccccccccc}
 & & \uparrow & & \uparrow & & \uparrow & & \\
0 & \longrightarrow & C^2_\lambda(\mathcal{A}) & \longrightarrow & CC^{0,2} & \xrightarrow{1-\lambda} & CC^{1,2} & \xrightarrow{N} & \cdots \\
 & & \uparrow b & & \uparrow b & & \uparrow -b' & & \\
0 & \longrightarrow & C^1_\lambda(\mathcal{A}) & \longrightarrow & CC^{0,1} & \xrightarrow{1-\lambda} & CC^{1,1} & \xrightarrow{N} & \cdots \\
 & & \uparrow b & & \uparrow b & & \uparrow -b' & & \\
0 & \longrightarrow & C^0_\lambda(\mathcal{A}) & \longrightarrow & CC^{0,0} & \xrightarrow{1-\lambda} & CC^{1,0} & \xrightarrow{N} & \cdots
\end{array} \tag{179}$$

Since $C_\lambda(\mathcal{A})$ is the quotient complex of (179) by (177), one is led to the following proposition:

Proposition 3.10 *The cyclic cohomology $HC^n(\mathcal{A})$ corresponds to the nth degree cohomology of the complex of cyclic cochains*

$$0 \longrightarrow C^0_\lambda(\mathcal{A}) \xrightarrow{b} \ldots \longrightarrow C^n_\lambda(\mathcal{A}) \xrightarrow{b} C^{n+1}_\lambda(\mathcal{A}) \xrightarrow{b} \ldots \tag{180}$$

Therefore, a cyclic cocycle $\varphi \in Z(C^n_\lambda(\mathcal{A}))$ is a $(n+1)$-linear form over $\mathcal{A}$ verifying the cyclicity and closedness conditions

$$\begin{aligned}&\varphi(a_0,\ldots,a_n) = (-)^n\varphi(a_n,a_0,\ldots,a_{n-1}) \ ,\\ &0 = \varphi(a_0a_1,a_2,\ldots,a_{n+1}) - \varphi(a_0,a_1a_2,\ldots,a_{n+1}) + \ldots\\ &\ldots + (-)^{n+1}\varphi(a_{n+1}a_0,a_1,\ldots,a_n) \ .\end{aligned} \tag{181}$$

It remains to describe Connes' long exact sequence for cyclic cohomology. By definition, the Hochschild cohomology $HH^n(\mathcal{A})$ is the cohomology of the total complex obtained by isolating the first two columns of (177). Hence, the short exact sequence dual of (176) implies the following theorem:

Theorem 3.11 (Connes) *There is a long exact sequence relating the cyclic and Hochschild homologies of $\mathcal{A}$:*

$$0 \longrightarrow HC^0(\mathcal{A}) \xrightarrow{I} HH^0(\mathcal{A}) \longrightarrow 0 \longrightarrow HC^1(\mathcal{A}) \xrightarrow{I} \ldots$$

$$\longrightarrow HC^{n-3}(\mathcal{A}) \xrightarrow{S} HC^{n-1}(\mathcal{A}) \xrightarrow{I} HH^{n-1}(\mathcal{A}) \xrightarrow{B} HC^{n-2}(\mathcal{A}) \xrightarrow{S} HC^n(\mathcal{A}) \xrightarrow{I} \ldots$$

Noncommutative differential forms

A completely equivalent, but sometimes more convenient way to describe the cyclic bicomplex is via noncommutative differential forms. It also makes the link with de Rham theory more transparent. Let $\mathcal{A}$ be an associative algebra, and $\widetilde{\mathcal{A}} = \mathcal{A} \oplus \mathbb{C}$ denote the algebra obtained by adding a unit 1 (even if $\mathcal{A}$ is already unital). The space of noncommutative differential forms is the direct sum

$$\Omega\mathcal{A} = \bigoplus_{n\geq 0} \Omega^n\mathcal{A} \tag{182}$$

with $\Omega^n\mathcal{A} = \widetilde{\mathcal{A}}\otimes\mathcal{A}^{\otimes n}$ for $n \geq 1$ and $\Omega^0\mathcal{A} = \mathcal{A}$. It is customary to denote a string $a_0\otimes\ldots\otimes a_n \in \Omega^n\mathcal{A}$ (resp. $1\otimes a_1\ldots\otimes a_n$) by the differential form $a_0da_1\ldots da_n$ (resp. $da_1\ldots da_n$). A differential $d : \Omega\mathcal{A} \to \Omega^{n+1}\mathcal{A}$ is uniquely specified by setting

$$d(a_0da_1\ldots da_n) = da_0da_1\ldots da_n \ , \qquad d(da_1\ldots da_n) = 0 \ , \tag{183}$$

which automatically implies $d^2 = 0$. We endow the space $\Omega\mathcal{A}$ with the natural product coming from the usual multiplication of differential forms:

$$\begin{aligned}&(a_0da_1\ldots da_n)(a_{n+1}da_{n+2}\ldots da_{n+p}) =\\ &\quad a_0da_1\ldots d(a_na_{n+1})da_{n+2}\ldots da_{n+p} - a_0da_1\ldots d(a_{n-1}a_n)da_{n+1}\ldots da_{n+p} +\\ &\quad \ldots + (-)^n a_0a_1da_2\ldots da_nda_{n+1}\ldots da_{n+p} \ .\end{aligned} \tag{184}$$

Then $\Omega\mathcal{A}$ becomes a differential algebra graded over $\mathbb{N}$, that is, the product satisfies the Leibniz rule

$$d(\omega_0\omega_1) = d\omega_0\,\omega_1 + (-)^{|\omega_0|}\omega_0 d\omega_1 \ , \tag{185}$$

for any differential forms $\omega_0, \omega_1 \in \Omega\mathcal{A}$, where $|\omega_0| \in \mathbb{N}$ is the degree of ω_0. It turns out that $\Omega\mathcal{A}$ is a universal differential graded (DG) algebra over $\mathcal{A}$ in the following sense. Let Ω be any DG algebra and $\rho : \mathcal{A} \to \Omega^0$ an algebra homomorphism. Then there exists a unique DG algebra homomorphism $\phi : \Omega\mathcal{A} \to \Omega$ which extends ρ in a commutative diagram

$$\begin{tikzcd} \mathcal{A} \arrow[rr, "\rho"] \arrow[dr] & & \Omega \\ & \Omega\mathcal{A} \arrow[ur, "\phi"'] & \end{tikzcd} \tag{186}$$

Here the arrow $\mathcal{A} \to \Omega\mathcal{A}$ is the canonical inclusion coming from the identification $\mathcal{A} = \Omega^0\mathcal{A}$.

Let us now describe Hochschild homology by means of noncommutative differential forms. The Hochschild operator $b : \Omega^{n+1}\mathcal{A} \to \Omega^n\mathcal{A}$ is defined by

$$b(\omega da) = (-)^n[\omega, a] \ , \qquad \forall \omega \in \Omega^n\mathcal{A} \ , \ a \in \mathcal{A} \ , \tag{187}$$

and $b = 0$ on $\Omega^0\mathcal{A}$. It is easy to check that $b^2 = 0$ and the complex

$$\ldots \longrightarrow \Omega^n\mathcal{A} \xrightarrow{b} \Omega^{n-1}\mathcal{A} \xrightarrow{b} \ldots \longrightarrow \Omega^1\mathcal{A} \xrightarrow{b} \mathcal{A} \longrightarrow 0 \tag{188}$$

is actually *isomorphic* to the Hochschild complex (175), under the identification $\Omega^n\mathcal{A} \cong D^H_n$. Hence the complex (188) calculates the Hochschild homology of $\mathcal{A}$.

Next we introduce the Karoubi operator $k : \Omega^n\mathcal{A} \to \Omega^n\mathcal{A}$ and Connes' boundary $B : \Omega^n\mathcal{A} \to \Omega^{n+1}\mathcal{A}$:

$$\begin{aligned} k &= 1 - (bd + db) \\ B &= (1 + k + ... + k^n)d \end{aligned} \tag{189}$$

One has

$$B^2 = 0 \ , \quad b^2 = 0 \ , \quad bB + Bb = 0 \ , \tag{190}$$

whence the existence of the so-called (b, B)-bicomplex

$$\begin{tikzcd} \arrow[d] & \arrow[d] & \arrow[d] & \arrow[d] \\ \Omega^3\mathcal{A} \arrow[d, "b"'] & \Omega^2\mathcal{A} \arrow[l, "B"'] \arrow[d, "b"'] & \Omega^1\mathcal{A} \arrow[l, "B"'] \arrow[d, "b"'] & \mathcal{A} \arrow[l, "B"'] \\ \Omega^2\mathcal{A} \arrow[d, "b"'] & \Omega^1\mathcal{A} \arrow[l, "B"'] \arrow[d, "b"'] & \mathcal{A} \arrow[l, "B"'] & \\ \Omega^1\mathcal{A} \arrow[d, "b"'] & \mathcal{A} \arrow[l, "B"'] & & \\ \mathcal{A} & & & \end{tikzcd} \tag{191}$$

We already know that each column is isomorphic to the Hochschild complex (174). On the other hand, an easy computation shows the boundary B acting on $\Omega^n\mathcal{A} = \mathcal{A}^{\otimes(n+1)} \oplus \mathcal{A}^{\otimes n}$ coincides with the map N defined on the first factor $\mathcal{A}^{\otimes(n+1)}$, see equation (165). Consequently, the (b, B)-bicomplex of differential forms is exactly isomorphic to the Loday-Quillen bicomplex computing cyclic homology.

Proposition 3.12 *The cyclic homology $HC_*(\mathcal{A})$ is the homology of the total complex associated to the (b, B)-bicomplex of differential forms (191).*

Remark also that the boundary of the exact sequence (176) is explicitly given, at the level of differential forms, by the operator B. This justifies the use of the notation $B : HC_n(\mathcal{A}) \to HH_{n+1}(\mathcal{A})$ in the SBI sequence of Connes (theorem (3.9)).

Passing to the dual complexes, one gets of course an analogous description of cyclic cohomology in terms of linear maps on the space of differential forms. In particular, a cyclic n-cochain $\varphi \in C^n_\lambda(\mathcal{A})$ defines a linear map $\widehat{\varphi}$ on $\Omega^n\mathcal{A}$ as follows:

$$\begin{aligned} \widehat{\varphi}(a_0 da_1 \dots da_n) &:= (-)^{[n/2]} n!\, \varphi(a_0, a_1, \dots, a_n)\ , \\ \widehat{\varphi}(da_1 \dots da_n) &:= 0\ , \end{aligned} \tag{192}$$

where $[n/2]$ is the largest integer $\leq n/2$. It is not hard to show that this correspondence induces an *isomorphism* between the space of cyclic cocycles $Z(C^*_\lambda(\mathcal{A}))$ and the space of *closed graded traces* on the differential algebra $(\Omega\mathcal{A}, d)$. Hence a cyclic n-cocycle $\widehat{\varphi}$ is a linear map on $\Omega^n\mathcal{A}$ verifying the two conditions

$$\begin{aligned} \text{closedness:} &\quad \widehat{\varphi}(d\omega) = 0\ , \quad \forall \omega \in \Omega^{n-1}\mathcal{A} \qquad (193) \\ \text{graded trace:} &\quad \widehat{\varphi}(\omega_p\omega_q) = (-)^{pq}\widehat{\varphi}(\omega_q\omega_p)\ , \quad \forall \omega_p \in \Omega^p\ ,\ \omega_q \in \Omega^q\ . \end{aligned}$$

This correspondence between cyclic cocycles over $\mathcal{A}$ and closed graded traces on $\Omega\mathcal{A}$ allows to construct *cup-products*. Indeed, given two algebras $\mathcal{A}$ and $\mathcal{B}$, one can form the graded tensor product of their universal DG algebras $\Omega\mathcal{A}\widehat{\otimes}\Omega\mathcal{B}$. It is a DG algebra, with differential given by the sum $d \otimes 1 + 1 \otimes d$. The universal property of $\Omega(\mathcal{A} \otimes \mathcal{B})$ implies the existence of a unique DG algebra morphism

$$\pi : \Omega(\mathcal{A} \otimes \mathcal{B}) \to \Omega\mathcal{A}\widehat{\otimes}\Omega\mathcal{B}\ . \tag{194}$$

Now let $\varphi \in Z(C^n_\lambda(\mathcal{A}))$ and $\psi \in Z(C^m_\lambda(\mathcal{B}))$ be cyclic cocycles over $\mathcal{A}$ and $\mathcal{B}$ respectively, and denote by $\widehat{\varphi}$ and $\widehat{\psi}$ the associated closed graded traces on $\Omega\mathcal{A}$ and $\Omega\mathcal{B}$. The graded tensor product

$$\widehat{\varphi} \otimes \widehat{\psi} : \Omega\mathcal{A}\widehat{\otimes}\Omega\mathcal{B} \to \mathbb{C} \tag{195}$$

is a closed graded trace on $\Omega\mathcal{A}\widehat{\otimes}\Omega\mathcal{B}$. The cup-product $\varphi\#\psi$ is then the cyclic cocycle over $\mathcal{A} \otimes \mathcal{B}$ defined by the closed graded trace

$$\widehat{(\varphi\#\psi)} = \pi^*(\widehat{\varphi} \otimes \widehat{\psi})\ . \tag{196}$$

One can show that the cup-product is well-defined at the level of cyclic cohomology classes, thus induces a bilinear map

$$\# : HC^n(\mathcal{A}) \otimes HC^m(\mathcal{B}) \to HC^{n+m}(\mathcal{A} \otimes \mathcal{B})\ . \tag{197}$$

Periodic theory

The cyclic (co)homology we have introduced so far is called the *non-periodic* theory. There is however another version of cyclic (co)homology, called the *periodic* theory, defined via a slight modification of the cyclic bicomplexes. The resulting theory is gifted with good properties, like homotopy invariance, and is really the generalization of de Rham theory from manifolds to noncommutative spaces. The periodic theory will be used as a receptacle for the Chern-Connes character in the next section.

Consider an associative algebra $\mathcal{A}$ and its DG algebra of noncommutative differential forms $\Omega\mathcal{A}$. Let us modify the (b, B)-bicomplex (191) by adding infinitely many columns to the left, connected by the boundary B:

$$\begin{tikzcd}
 & {} \arrow[d] & {} \arrow[d] & {} \arrow[d] & {} \arrow[d] \\
\dots & \Omega^3\mathcal{A} \arrow[l, "B"'] \arrow[d, "b"'] & \Omega^2\mathcal{A} \arrow[l, "B"'] \arrow[d, "b"'] & \Omega^1\mathcal{A} \arrow[l, "B"'] \arrow[d, "b"'] & \mathcal{A} \arrow[l, "B"'] \\
\dots & \Omega^2\mathcal{A} \arrow[l, "B"'] \arrow[d, "b"'] & \Omega^1\mathcal{A} \arrow[l, "B"'] \arrow[d, "b"'] & \mathcal{A} \arrow[l, "B"'] & \\
\dots & \Omega^1\mathcal{A} \arrow[l, "B"'] \arrow[d, "b"'] & \mathcal{A} \arrow[l, "B"'] & & \\
\dots & \mathcal{A} \arrow[l, "B"'] & & &
\end{tikzcd} \tag{198}$$

Now the diagonals of this bicomplex have infinite length. They are no longer indexed by natural integers, but are only distinguished by their parity. A diagonal is either a sum of the forms of even degree:

$$D_{even} = \mathcal{A} \oplus \Omega^2\mathcal{A} \oplus \Omega^4\mathcal{A} \oplus \dots\ , \tag{199}$$

or its is a sum of forms of odd degree:

$$D_{odd} = \Omega^1\mathcal{A} \oplus \Omega^3\mathcal{A} \oplus \Omega^5\mathcal{A} \oplus \dots\ . \tag{200}$$

Now we want to take the total complex with boundary $b + B$ and consider its homology. However, we must take care of the following subtlety. Since the diagonals have infinite length, there are actually two different ways of forming a total complex. The first one is to consider that the diagonals are indeed direct *sums*, that is, their elements are *finite* linear combinations of differential forms:

$$\omega \in D_{even} \Leftrightarrow \omega = \sum_{k=0}^{n} \omega_{2k} \tag{201}$$

for some n and $\omega_{2k} \in \Omega^{2k}\mathcal{A}$. Unfortunately, it turns out that the homology of this total complex with boundary $b + B$ vanishes. It can be shown using the exactness of the rows in the Loday-Quillen description of the cyclic bicomplex.

Hence the bicomplex of finite chains is not interesting. The second possibility is to complete the diagonals by taking the direct *products*

$$\widehat{D}_{even} = \prod_{k=0}^{\infty} \Omega^{2k}\mathcal{A} \ , \qquad \widehat{D}_{odd} = \prod_{k=0}^{\infty} \Omega^{2k+1}\mathcal{A} \ . \tag{202}$$

It means that an element of, say, $\widehat{D}_{even}$ is an *infinite sequence* of forms of even degree, all of which may be non-zero:

$$\omega \in \widehat{D}_{even} \Leftrightarrow \omega = \sum_{k=0}^{\infty} \omega_{2k} = (\omega_0, \omega_2, \omega_4, \ldots) \ . \tag{203}$$

Now, the total boundary $b + B$ is still defined on these infinite diagonals. We get this way a $\mathbb{Z}_2$-graded complex

$$\widehat{\Omega}\mathcal{A} = \widehat{D}_{even} \oplus \widehat{D}_{odd} = \prod_{n=0}^{\infty} \Omega^n \mathcal{A} \ , \tag{204}$$

with boundary $b + B$ mapping the even subspace $\widehat{D}_{even}$ to the odd subspace $\widehat{D}_{odd}$ and conversely. The homology of this complex is nontrivial in general.

Definition 3.13 *The periodic cyclic homology $HP_*(\mathcal{A})$ is the homology of the $\mathbb{Z}_2$-graded complex $(\widehat{\Omega}\mathcal{A}, b + B)$ of infinite chains over $\mathcal{A}$:*

$$HP_i(\mathcal{A}) = H_i(\widehat{\Omega}\mathcal{A}) \ , \quad i = 0, 1 \ . \tag{205}$$

Concretely, a periodic cycle of even degree is an infinite sequence annihilated by the total boundary $b + B$,

$$\omega = \sum_{k=0}^{\infty} \omega_{2k} \ , \qquad (b + B)\omega = 0 \ , \tag{206}$$

which may be reexpressed in terms of its components of various degrees:

$$\begin{array}{rcl} B\omega_0 + b\omega_2 & = & 0 \\ B\omega_2 + b\omega_4 & = & 0 \\ & \vdots & \\ B\omega_{2k} + b\omega_{2k+2} & = & 0 \\ & \vdots & \end{array} \tag{207}$$

Analogously, a periodic cycle of odd degree is an infinite sequence of forms of odd degrees

$$\omega = \sum_{k=0}^{\infty} \omega_{2k+1} \ , \qquad (b + B)\omega = 0 \ , \tag{208}$$

verifying the sequence of identities

$$\begin{aligned} b\omega_1 &= 0 \\ B\omega_1 + b\omega_3 &= 0 \\ B\omega_3 + b\omega_5 &= 0 \\ &\vdots \\ B\omega_{2k+1} + b\omega_{2k+3} &= 0 \\ &\vdots \end{aligned} \tag{209}$$

Passing to the dual theory, we can define periodic cyclic cohomology by considering the complex of *finite* cochains over $\mathcal{A}$ with total boundary $b+B$. It is given by the direct *sum* of the spaces of linear maps $\Omega^n\mathcal{A} \to \mathbb{C}$

$$(\widehat{\Omega}\mathcal{A})' = \bigoplus_{n=0}^{\infty} \mathrm{Hom}(\Omega^n\mathcal{A}, \mathbb{C}) \ , \tag{210}$$

on which act the transposed of the boundaries b and B. It is crucial here to define a periodic cochain $\varphi \in (\widehat{\Omega}\mathcal{A})'$ as a *finite* linear combination of maps $\varphi_n : \Omega^n\mathcal{A} \to \mathbb{C}$, so that its pairing with a periodic chain $\omega = \sum_{n=0}^{\infty} \omega_n$ is given by the finite sum

$$\langle \varphi, \omega \rangle = \sum_{n \geq 0} \varphi_n(\omega_n) \ . \tag{211}$$

Definition 3.14 *The periodic cyclic cohomology $HP^*(\mathcal{A})$ is the cohomology of the $\mathbb{Z}_2$-graded complex $((\widehat{\Omega}\mathcal{A})', b+B)$ of finite cochains over $\mathcal{A}$:*

$$HP^i(\mathcal{A}) = H^i((\widehat{\Omega}\mathcal{A})') \ , \quad i = 0, 1 \ . \tag{212}$$

It is isomorphic to the inductive limit of the non-periodic cyclic cohomology groups $HC^*(\mathcal{A})$ under the suspension operation $S : HC^n(\mathcal{A}) \to HC^{n+2}(\mathcal{A})$:

$$HP^i(\mathcal{A}) = \varinjlim_{k} HC^{2k+i}(\mathcal{A}) \ , \quad i = 0, 1 \ . \tag{213}$$

Concretely, a periodic cocycle of even degree is a finite sum annihilated by the total boundary $b+B$,

$$\varphi = \sum_{k=0}^{n} \varphi_{2k} \ , \quad \varphi_{2k} : \Omega^{2k}\mathcal{A} \to \mathbb{C} \ , \quad (b+B)\varphi = 0 \ , \tag{214}$$

which means in terms of components

$$\begin{aligned} b\varphi_0 + B\varphi_2 &= 0 \\ b\varphi_2 + B\varphi_4 &= 0 \\ &\vdots \\ b\varphi_{2n-2} + B\varphi_{2n} &= 0 \\ b\varphi_{2n} &= 0 \ . \end{aligned} \tag{215}$$

Analogously for a periodic cocycle of odd degree:

$$\varphi = \sum_{k=0}^{n} \varphi_{2k+1} \; , \quad \varphi_{2k+1} : \Omega^{2k+1}\mathcal{A} \to \mathbb{C} \; , \quad (b+B)\varphi = 0 \; , \tag{216}$$

$$\begin{aligned} B\varphi_1 &= 0 \\ b\varphi_1 + B\varphi_3 &= 0 \\ b\varphi_3 + B\varphi_5 &= 0 \\ &\vdots \\ b\varphi_{2n-1} + B\varphi_{2n+1} &= 0 \\ b\varphi_{2n+1} &= 0 \; . \end{aligned} \tag{217}$$

Since the complex of periodic cochains is dual to the complex of periodic chains, the pairing $\langle \varphi, \omega \rangle$ induces a bilinear product between periodic cyclic cohomology and periodic cyclic homology, with values in $\mathbb{C}$:

$$\langle \; , \; \rangle : HP^i(\mathcal{A}) \times HP_i(\mathcal{A}) \to \mathbb{C} \; , \qquad i = 0, 1 \; . \tag{218}$$

Example 3.15 In the classical case of a smooth compact manifold M, one can take the commutative algebra $\mathcal{A} = C^\infty(M)$. It is naturally endowed with a locally convex topology coming from the family of seminorms $p_n(a) = \sup_M |\partial^n a(x)|$ for any $a \in \mathcal{A}$. A theorem of Connes [6] shows that the periodic cyclic cohomology of $\mathcal{A}$ is isomorphic to the de Rham homology of M,

$$HP^i(\mathcal{A}) \cong \bigoplus_{k \geq 0} H_{2k+i}(M) \; , \tag{219}$$

provided one takes the cohomology of *continuous* periodic cochains for the topology of $\mathcal{A}$.

4 Index Theorems

4.1 The Chern-Connes character

Chern character in K-theory

Cyclic homology was introduced as a noncommutative analogue of de Rham theory. As in the classical (commutative) case, it is the natural receptacle for a Chern character defined on K-theory. Here we will describe how to construct such Chern characters for the first two algebraic K-groups.

Let $\mathcal{A}$ be an associative algebra over $\mathbb{C}$. We have seen in section 3.2 that the K-theory group $K_0(\mathcal{A})$ is the Grothendieck group associated to the semigroup of isomorphism classes of idempotents in the matrix algebra $M_\infty(\mathcal{A}) = \cup_k M_k(\mathcal{A})$. Recall that two such idempotents $e \in M_k(\mathcal{A})$ and $f \in M_n(\mathcal{A})$ are isomorphic if and only if there exist rectangular matrices u and v of appropriate size over $\mathcal{A}$, such that $e = uv$ and $f = vu$. Moreover, the sum of idempotents is given by

$$e \oplus f = \begin{pmatrix} e & 0 \\ 0 & f \end{pmatrix} \; . \tag{220}$$

The Grothendieck group is then the abelian group generated by isomorphism classes of idempotents and constrained by the equivalence relation

$$[e] + [f] \sim [e \oplus f] \ . \tag{221}$$

Finally, recall that any element of $K_0(\mathcal{A})$ can be written as a difference of two classes $[e]-[f]$. We wish to construct a Chern character on $K_0(\mathcal{A})$ with values in the periodic cyclic homology of even degree $HP_0(\mathcal{A})$. According to section 3.3, a class in $HP_0(\mathcal{A})$ is represented by a sequence of noncommutative differential forms of even degree

$$\omega = \sum_{n=0}^{\infty} \omega_{2n} \ , \qquad \omega_{2n} \in \Omega^{2n}\mathcal{A} \ , \tag{222}$$

subject to the closedness condition $(b+B)\omega = 0$.

Proposition-Definition 4.1 *Let $e \in M_\infty(\mathcal{A})$ be an idempotent. The following differential form defines a periodic cycle of even degree over $\mathcal{A}$:*

$$\mathrm{ch}(e) = \mathrm{tr}(e) + \sum_{n=0}^{\infty} (-)^n \frac{(2n)!}{n!} \mathrm{tr}((e - \frac{1}{2})(dede)^n) \ , \tag{223}$$

where the map $\mathrm{tr} : \Omega(M_\infty(\mathcal{A})) \to \Omega\mathcal{A}$ is induced by the trace of matrices. The periodic cyclic homology class of $\mathrm{ch}(e)$ in $HP_0(\mathcal{A})$ is the Chern character of the idempotent e.

One can check that this definition is compatible with the isomorphism relation of idempotents and yields a Chern character at the level of K-theory classes:

Proposition 4.2 *The Chern character defines an additive map*

$$\mathrm{ch} : K_0(\mathcal{A}) \to HP_0(\mathcal{A}) \ . \tag{224}$$

Recall that there exists a bilinear pairing between periodic cyclic cohomology and periodic cyclic homology, induced by the evaluation of a cycle $\omega = \sum_{k=0}^{\infty} \omega_{2k}$ on a cocycle $\varphi = \sum_{k=0}^{n} \varphi_{2k}$:

$$\langle \varphi, \omega \rangle = \sum_{k=0}^{n} \varphi_{2k}(\omega_{2k}) \ . \tag{225}$$

Therefore, the Chern character on K-theory allows one to define a bilinear pairing between periodic cyclic cohomology and K-theory:

Corollary 4.3 *Let φ be a periodic cocycle of even degree over $\mathcal{A}$ and $e \in M_\infty(\mathcal{A})$ an idempotent. The finite sum*

$$\langle \varphi, e \rangle = \varphi_0 \mathrm{tr}(e) + \sum_{n \geq 0} (-)^n \frac{(2n)!}{n!} \varphi_{2n} \mathrm{tr}((e - \frac{1}{2})(dede)^n) \tag{226}$$

induces a bilinear pairing

$$\langle \ , \ \rangle : HP^0(\mathcal{A}) \times K_0(\mathcal{A}) \to \mathbb{C} \ . \tag{227}$$

There is a somewhat simpler formula, when cyclic cocycles are described in terms of closed graded traces over $\Omega\mathcal{A}$. Recall from section 3.3 that to any closed graded trace $\widehat{\varphi} : \Omega^{2n}\mathcal{A} \to \mathbb{C}$ of degree $2n$ corresponds a periodic cocycle

$$\varphi = (-)^n \frac{1}{(2n)!}\, \widehat{\varphi} \ , \tag{228}$$

and all the periodic cyclic cohomology classes can be represented this way. Hence, the pairing of φ with an idempotent e is simply given by

$$\langle \varphi, e\rangle = \frac{1}{n!}\, (\widehat{\varphi}\#\mathrm{tr})(e(dede)^n) \ . \tag{229}$$

We now turn to the Chern character on the first algebraic K-group $K_1(\mathcal{A})$. Denote by $M_k(\widetilde{\mathcal{A}})$ the algebra of $k\times k$ matrices over the unitalized algebra $\widetilde{\mathcal{A}} = \mathcal{A}\oplus\mathbb{C}$ (we add a unit to $\mathcal{A}$ even when the latter is already unital). We have introduced the linear group $Gl_k(\mathcal{A})$ as the set of invertible elements of $M_k(\widetilde{\mathcal{A}})$ equal to 1 modulo $M_k(\mathcal{A})$:

$$Gl_k(\mathcal{A}) = \{x \in Gl_k(\widetilde{\mathcal{A}}) \mid x \equiv 1_k \mod M_k(\mathcal{A})\} \ . \tag{230}$$

$Gl_\infty(\mathcal{A})$ denotes the inductive limit of the groups $Gl_k(\mathcal{A})$ under the inclusions

$$Gl_k(\mathcal{A}) \hookrightarrow Gl_{k+1}(\mathcal{A}) \ , \qquad x \hookrightarrow \begin{pmatrix} x & 0 \\ 0 & 1 \end{pmatrix} \ . \tag{231}$$

By definition $K_1(\mathcal{A})$ is the abelianized group

$$K_1(\mathcal{A}) = Gl_\infty(\mathcal{A})/[Gl_\infty(\mathcal{A}), Gl_\infty(\mathcal{A})] \ . \tag{232}$$

As in the even case, we shall construct a Chern character with values in the odd periodic cyclic homology $HP_1(\mathcal{A})$.

Proposition-Definition 4.4 *Let $g \in Gl_\infty(\mathcal{A})$. The following odd differential form over $\mathcal{A}$*

$$\mathrm{ch}(g) = \sum_{n=0}^{\infty} (-)^n n!\, \mathrm{tr}(g^{-1}dg(dg^{-1}dg)^n) \tag{233}$$

is closed for the boundary $b+B$ and thus defines a periodic cycle of odd degree over $\mathcal{A}$. Its periodic cyclic homology class in $HP_1(\mathcal{A})$ is the Chern character of g.

One can check the Chern character vanishes on the commutator subgroup $[Gl_\infty(\mathcal{A}), Gl_\infty(\mathcal{A})]$ and hences passes to the quotient $K_1(\mathcal{A})$:

Proposition 4.5 *The Chern character induces an additive map*

$$\mathrm{ch} : K_1(\mathcal{A}) \to HP_1(\mathcal{A}) \ . \tag{234}$$

Corollary 4.6 *Let φ be a periodic cocycle of odd degree over $\mathcal{A}$, and $g \in Gl_\infty(\mathcal{A})$ be an invertible element. The finite sum*

$$\langle \varphi, g\rangle = \sum_{n\geq 0} (-)^n n!\, \varphi_{2n+1}\mathrm{tr}(g^{-1}dg(dg^{-1}dg)^n) \tag{235}$$

induces a bilinear pairing

$$\langle\ ,\ \rangle : HP^1(\mathcal{A}) \times K_1(\mathcal{A}) \to \mathbb{C} \ . \tag{236}$$

Again, the pairing is simpler once periodic cyclic cohomology classes are represented in terms of closed graded traces of odd degre over $\Omega\mathcal{A}$. If $\widehat{\varphi} : \Omega^{2n+1}\mathcal{A} \to \mathbb{C}$ is a closed graded trace of degree $2n+1$, the periodic cochain

$$\varphi = (-)^n \frac{1}{(2n+1)!}\, \widehat{\varphi} \ , \tag{237}$$

is $(b+B)$-closed and its pairing with an invertible g is given by

$$\langle \varphi, g \rangle = \frac{n!}{(2n+1)!}\, (\varphi \# \mathrm{tr})(g^{-1}dg(dg^{-1}dg)^n) \ . \tag{238}$$

We shall end this section by mentioning that the pairings between K_* and HP^* are compatible with the cup-products in K-theory and cyclic cohomology. Given two algebras $\mathcal{A}$ and $\mathcal{B}$, there exists a cup-product

$$\boxtimes : K_i(\mathcal{A}) \times K_j(\mathcal{B}) \to K_{i+j}(\mathcal{A} \otimes \mathcal{B}) \quad i, j \in \mathbb{Z}_2 \tag{239}$$

defined as follows. Let $e \in M_\infty(\mathcal{A})$ and $f \in M_\infty(\mathcal{B})$ be two idempotents. The cup-product of their K-theory classes is represented by the tensor product

$$[e] \boxtimes [f] = [e \otimes f] \ \in K_0(\mathcal{A} \otimes \mathcal{B}) \ . \tag{240}$$

If $u \in Gl_\infty(\mathcal{B})$ is an invertible element, the cup-product between $K_0(\mathcal{A})$ and $K_1(\mathcal{B})$ reads

$$[e] \boxtimes [u] = [1 \otimes 1 + e \otimes (u-1)] \ \in K_1(\mathcal{A} \otimes \mathcal{B}) \ , \tag{241}$$

where $1 \otimes 1 + e \otimes (u-1)$ lies in $Gl_\infty(\mathcal{A} \otimes \mathcal{B})$.

Proposition 4.7 *i) Let $[e] \in K_0(\mathcal{A})$ and $[f] \in K_0(\mathcal{B})$ be two K-theory classes, and $\varphi \in HC^{even}(\mathcal{A})$, $\psi \in HC^{even}(\mathcal{B})$ be two cyclic cohomology classes of even degree respectively over $\mathcal{A}$ and $\mathcal{B}$. then*

$$\langle \varphi \# \psi, [e] \boxtimes [f] \rangle = \langle \varphi, [e] \rangle \langle \psi, [f] \rangle \ . \tag{242}$$

ii) Let $[e] \in K_0(\mathcal{A})$, $[u] \in K_1(\mathcal{B})$ be K-theory classes, and $\varphi \in HC^{even}(\mathcal{A})$ (resp. $\psi \in HC^{odd}(\mathcal{B})$) an even cyclic cohomology class over $\mathcal{A}$ (resp. odd cyclic cohomology class over $\mathcal{B}$). Then

$$\langle \varphi \# \psi, [e] \boxtimes [u] \rangle = \langle \varphi, [e] \rangle \langle \psi, [u] \rangle \ . \tag{243}$$

Chern character for spectral triples

We have seen in section 3.2 that spectral triples over an algebra $\mathcal{A}$ provide a theory "dual" to K-theory in the sense that there are pairings

$$\text{spectral triples} \times K_*(\mathcal{A}) \to \mathbb{Z} \tag{244}$$

given by the index of abstract Dirac-type operators. When a spectral triple satisfies suitable "finite-dimensionality" conditions (p-summability), one can associate to it a periodic cyclic cocycle over $\mathcal{A}$. This will be detailed in the construction below. One is left with the so-called Chern-Connes character

$$\mathrm{ch}: \begin{array}{c} p\text{-summable} \\ \text{spectral triples} \end{array} \to HP^*(\mathcal{A}) \tag{245}$$

with values in the periodic cyclic cohomology of $\mathcal{A}$. This Chern character is "dual" to the Chern character on K-theory, because it is constructed so that the following diagram

$$\begin{array}{ccccc} \begin{array}{c} p\text{-summable} \\ \text{spectral triples} \end{array} & \times\, K_*(\mathcal{A}) & \xrightarrow{\mathrm{Ind}} & \mathbb{Z} \\ \downarrow{\scriptstyle \mathrm{ch}} & \downarrow{\scriptstyle \mathrm{ch}} & & \downarrow \\ HP^*(\mathcal{A}) & \times HP_*(\mathcal{A}) & \xrightarrow{\langle\ ,\ \rangle} & \mathbb{C} \end{array} \tag{246}$$

commutes. In particular, this means that the index of an abstract Dirac operator D against an idempotent e, representing respectively a spectral triple and a K-theory class, can be computed by means of the duality bracket between their respective Chern characters:

$$\mathrm{Ind}(D, e) = \langle \mathrm{ch}(D), \mathrm{ch}(e) \rangle \ . \tag{247}$$

Recall that a spectral triple $(\mathcal{A}, \mathcal{H}, D)$ corresponds to the following data:
i) $\mathcal{A}$ is a (unital) algebra represented in the algebra $\mathcal{L}(\mathcal{H})$ of bounded linear operators on a separable Hilbert space $\mathcal{H}$.
ii) D is a densely defined unbounded self-adjoint operator on $\mathcal{H}$ with compact resolvent. For any $a \in \mathcal{A}$, the commutator $[D, a]$ is densely defined and extends to a bounded operator on $\mathcal{H}$.
iii) If the spectral triple is of even parity, there is a self-adjoint involution $\gamma \in \mathcal{L}(\mathcal{H})$, $\gamma^2 = 1$, such that $\gamma D = -D\gamma$, $\gamma a = a\gamma \quad \forall a \in \mathcal{A}$. That is, $\mathcal{H}$ is a $\mathbb{Z}_2$-graded Hilbert space and D is an operator of odd degree.

The spectral triple $(\mathcal{A}, \mathcal{H}, D)$ is called p-summable, for $p \geq 1$ a real number, if is it satisfies the following additional property [6]:
iv) $(1 + D^2)^{-1/2} \in \ell^p(\mathcal{H})$,
where $\ell^p(\mathcal{H})$ is the Schatten p-class.

The Chern character of a p-summable spectral triple $(\mathcal{A}, \mathcal{H}, D)$ in the periodic cyclic cohomology of $\mathcal{A}$ is given by explicit formulas involving the operator D and the Hilbert space representation of $\mathcal{A}$. Let us start with the case of a spectral triple of even parity. We can choose the following matricial representation of operators acting on the $\mathbb{Z}_2$-graded space $\mathcal{H} = \mathcal{H}_+ \oplus \mathcal{H}_-$:

$$a = \begin{pmatrix} a_+ & 0 \\ 0 & a_- \end{pmatrix} , \quad D = \begin{pmatrix} 0 & D_- \\ D_+ & 0 \end{pmatrix} , \quad \gamma = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} , \tag{248}$$

for any $a \in \mathcal{A}$. Hence a_+ (resp. a_-) denotes the representation of a on the subspace $\mathcal{H}_+$ (resp. $\mathcal{H}_-$), $D_+ : \mathcal{H}_+ \to \mathcal{H}_-$ is the chiral Dirac operator and $D_- : \mathcal{H}_- \to \mathcal{H}_+$ is its adjoint. D may not be invertible, so we need to modify slightly the spectral triple in order to deal with an invertible Dirac operator.

This is done as follows. Let $m \in \mathbb{R}$ be an arbitrary "mass", and consider the selfadjoint unbounded operator

$$Q = D + \gamma m \ . \tag{249}$$

Since γ anticommutes with D, the square $Q^2 = D^2 + m^2$ is a strictly positive selfadjoint operator when $m \neq 0$, hence is invertible. Replace $(\mathcal{A}, \mathcal{H}, D)$ by the new spectral triple of even degree $(\mathcal{A}, \mathcal{H}', D')$, where the Hilbert space $\mathcal{H}'$ and the new Dirac and chirality operators D' and γ' read

$$\mathcal{H}' = \begin{pmatrix} \mathcal{H} \\ \mathcal{H} \end{pmatrix} \ , \quad D' = \begin{pmatrix} 0 & Q \\ Q & 0 \end{pmatrix} \ , \quad \gamma' = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \ , \tag{250}$$

hence the 2×2 matrix representation is replaced by 4×4 matrices, and the representation of $\mathcal{A}$ into the algebra of bounded operators $\mathcal{L}(\mathcal{H}')$ is defined to be

$$a = \begin{pmatrix} a'_+ & 0 \\ 0 & a'_- \end{pmatrix} \ , \text{ with } \ a'_+ = \begin{pmatrix} a_+ & 0 \\ 0 & 0 \end{pmatrix} \ , \quad a'_- = \begin{pmatrix} 0 & 0 \\ 0 & a_- \end{pmatrix} \ . \tag{251}$$

What we have done so far is just a dual analogue of stabilization by matrices in K-theory, and the two spectral triples $(\mathcal{A}, \mathcal{H}, D)$ and $(\mathcal{A}, \mathcal{H}', D')$ are equivalent in the sense of K-homology (the theory dual to K-theory). Now for $m \neq 0$, the spectral triple $(\mathcal{A}, \mathcal{H}', D')$ has an invertible Dirac operator D'. We will construct periodic cyclic cocycles over $\mathcal{A}$ from the operator

$$F = \begin{pmatrix} 0 & Q^{-1} \\ Q & 0 \end{pmatrix} \ . \tag{252}$$

One obviously has $F^2 = 1$. This property enables us to consider the graded commutator by F

$$a \mapsto [F, a] \tag{253}$$

as a kind of noncommutative differential. More precisely, consider the unital algebra Ω of operators on the $\mathbb{Z}_2$-graded Hilbert space $\mathcal{H}'$ generated by the elements $a \in \mathcal{A}$ (which are of even degree) and the commutators $[F, a]$ of odd degree. The graded commutator is a map of odd degree

$$[F, \ ] : \Omega \to \Omega \ , \qquad \omega \mapsto [F, \omega] = F\omega - (-)^{|\omega|}\omega F \ , \tag{254}$$

where $|\omega|$ denotes the degree of $\omega \in \Omega$. This is a graded derivation because $[F, \omega_1\omega_2] = [F, \omega_1]\omega_2 + (-)^{|\omega_1|}\omega_1[F, \omega_2]$ for any $\omega_1, \omega_2 \in \Omega$. Moreover, its square vanishes by virtue of the graded Jacobi identity

$$[F, [F, \omega]] = [F^2, \omega] = [1, \omega] = 0 \ . \tag{255}$$

Hence Ω is a $\mathbb{Z}_2$-graded differential algebra. By the universal property of the DG algebra of noncommutative differential forms $\Omega\mathcal{A}$ (section 3.3), there is a unique DG algebra homomorphism

$$\Omega\mathcal{A} \to \Omega \ , \quad a_0 da_1 \dots da_n \mapsto a_0[F, a_1] \dots [F, a_n] \tag{256}$$

which transforms the universal differential d on $\Omega\mathcal{A}$ into the differential $[F,\ ]$ on Ω. Let us calculate explicitly the commutator $[F,a]$ in matricial form. One has

$$\begin{aligned}[F,a] &= \begin{pmatrix} 0 & Q^{-1} \\ Q & 0 \end{pmatrix}\begin{pmatrix} a'_+ & 0 \\ 0 & a'_- \end{pmatrix} - \begin{pmatrix} a'_+ & 0 \\ 0 & a'_- \end{pmatrix}\begin{pmatrix} 0 & Q^{-1} \\ Q & 0 \end{pmatrix} \\ &= \begin{pmatrix} 0 & Q^{-1}a'_- - a'_+Q^{-1} \\ Qa'_+ - a'_-Q & 0 \end{pmatrix} . \end{aligned} \tag{257}$$

The bottom left corner reads

$$\begin{aligned} Qa'_+ - a'_-Q &= \begin{pmatrix} m & D_- \\ D_+ & -m \end{pmatrix}\begin{pmatrix} a_+ & 0 \\ 0 & 0 \end{pmatrix} - \begin{pmatrix} 0 & 0 \\ 0 & a_- \end{pmatrix}\begin{pmatrix} m & D_- \\ D_+ & -m \end{pmatrix} \\ &= \begin{pmatrix} ma_+ & 0 \\ D_+a_+ - a_-D_+ & ma_- \end{pmatrix} , \end{aligned} \tag{258}$$

whereas the top right corner may be rewritten as

$$Q^{-1}a'_- - a'_+Q^{-1} = -Q^{-1}(Qa'_+ - a'_-Q)Q^{-1} . \tag{259}$$

Now by definition, the commutator $[D,a]$ is bounded on $\mathcal{H}$, which implies $D_+a_+ - a_-D_+$ is bounded and also $Qa'_+ - a'_-Q$. Moreover, by p-summability the operator Q^{-1} lies in the Schatten class $\ell^p(\mathcal{H})$ hence

$$Q^{-1}a'_- - a'_+Q^{-1} \in \ell^{p/2}(\mathcal{H}) . \tag{260}$$

It follows that for any $a \in \mathcal{A}$, the operator $F[F,a] \in \ell^p(\mathcal{H}')$, and more generally for any even integer $n \in 2\mathbb{N}$, the expression

$$F[F,a_0][F,a_1]\dots[F,a_n] \tag{261}$$

lies in the Schatten class $\ell^{\frac{p}{n+1}}(\mathcal{H}')$, and is a trace-class operator when $n+1>p$. Let $\mathrm{Tr}_s : \ell^1(\mathcal{H}') \to \mathbb{C}$ be the supertrace of operators on $\mathcal{H}'$, defined by

$$\mathrm{Tr}_s(x) = \mathrm{Tr}(\gamma' x) \tag{262}$$

for any $x \in \ell^1(\mathcal{H}')$. By construction, Tr_s vanishes on graded commutators.

Proposition 4.8 *Let $(\mathcal{A},\mathcal{H},D)$ be a p-summable spectral triple of even degree, and $Q = D + \gamma m$ the massive amplification of D by a mass term $m \neq 0$. Then for any even integer $n > p-1$, the linear map $\mathrm{ch}_n(Q) : \Omega^n\mathcal{A} \to \mathbb{C}$ defined by*

$$\mathrm{ch}_n(Q)(a_0da_1\dots da_n) = \frac{1}{2}\frac{(n/2)!}{n!}\mathrm{Tr}_s(F[F,a_0][F,a_1]\dots[F,a_n]) \tag{263}$$

is a cyclic cocycle of degree n over $\mathcal{A}$.

Proof: First observe that since $[F,1]=0$, one has

$$\mathrm{ch}_n(Q)(da_1\dots da_n) = 0 .$$

Hence, we only need to show that the $(n+1)$-linear map

$$\varphi(a_0,a_1,\dots,a_n) = \mathrm{ch}_n(Q)(a_0da_1\dots da_n)$$

is a cyclic cocycle, that is, fulfills the cyclicity and closedness conditions (181). By the tracial property of Tr_s one has

$$\mathrm{Tr}_s(F[F,a_0][F,a_1]\dots[F,a_n]) = -\mathrm{Tr}_s([F,a_n]F[F,a_0]\dots[F,a_{n-1}])$$

and since $[F,a_n]F = -F[F,a_n]$, one deduces the cyclicity property

$$\varphi(a_0,a_1,\dots,a_n) = \varphi(a_n,a_0,\dots,a_{n-1}) \ .$$

It remains to show that φ is closed with respect to the Hochschild operator b. The commutator $[F,\ ]$ being a derivation, one has

$$\begin{aligned}
&\mathrm{Tr}_s(F[F,a_0a_1][F,a_2]\dots[F,a_{n+1}]) - \mathrm{Tr}_s(F[F,a_0][F,a_1a_2]\dots[F,a_{n+1}]) + \dots \\
&\qquad \dots - \mathrm{Tr}_s(F[F,a_{n+1}a_0][F,a_1]\dots[F,a_n]) \\
&\quad = \mathrm{Tr}_s(Fa_0[F,a_1][F,a_2]\dots[F,a_{n+1}]) + \mathrm{Tr}_s(F[F,a_0][F,a_1]\dots[F,a_n]a_{n+1}) \\
&\qquad -\mathrm{Tr}_s(F[F,a_{n+1}a_0][F,a_1]\dots[F,a_n]) \\
&\quad = 0
\end{aligned}$$

whence

$$\begin{aligned}
b\varphi(a_0,\dots,a_{n+1}) &= \varphi(a_0a_1,a_2,\dots,a_{n+1}) - \varphi(a_0,a_1a_2,\dots,a_{n+1}) + \dots \\
&\quad \dots + (-)^{n+1}\varphi(a_{n+1}a_0,a_1,\dots,a_n) \\
&= 0 \ .
\end{aligned}$$

It follows that φ is a cyclic cocycle. ■

Since any cyclic cocycle on $\mathcal{A}$ is annihilated by both the Hochschild operator b and Connes' boundary B, one may consider $\mathrm{ch}_n(D)$ as a periodic cocycle over $\mathcal{A}$:

$$b\mathrm{ch}_n(Q) = 0 \ , \ B\mathrm{ch}_n(D) = 0 \Rightarrow (b+B)\mathrm{ch}_n(D) = 0 \ . \tag{264}$$

Remark that the cocycle $\mathrm{ch}_n(Q)$ depends on the choice of the mass term $m \neq 0$. However, the following proposition shows that the periodic cyclic cohomology class of $\mathrm{ch}_n(Q)$ is independent of m, and even better, is independent of the degree $n > p-1$.

Proposition 4.9 *Let $(\mathcal{A},\mathcal{H},D)$ be a p-summable spectral triple of even degree and $Q = D+\gamma m$. The periodic cyclic cohomology class of $\mathrm{ch}_n(Q)$ is independent of the choice of mass m and of the degree $n > p-1$. Hence there exists a unique periodic cyclic cohomology class of even degree*

$$\mathrm{ch}(D) \in HP_0(\mathcal{A}) \tag{265}$$

called the Chern-Connes character of the spectral triple, represented by any of the above cyclic cocycles.

Proof: a) Independence with respect to the degree n. We will show that two cocycles $\mathrm{ch}_n(Q)$ and $\mathrm{ch}_{n+2}(Q)$ corresponding to consecutive values of n are cohomologous as periodic cocycles. We introduce a certain transgressed cochain of odd degree $n+1$:

$$\mathrm{cs}_{n+1}(a_0da_1\dots da_{n+1}) = \frac{1}{2}\frac{(n/2)!}{(n+1)!}\mathrm{Tr}_s(Fa_0[F,a_1]\dots[F,a_{n+1}]) \ .$$

Elementary algebraic manipulations allow to calculate the cochain $b\mathrm{cs}_{n+1}$ of degree $n+2$,

$$\begin{aligned} b\mathrm{cs}_{n+1}(a_0da_1\dots da_{n+2}) &= -\frac{1}{2}\frac{(n/2)!}{(n+1)!}\mathrm{Tr}_s(a_0[F,a_1]\dots[F,a_{n+2}]) \\ &= -\mathrm{ch}_{n+2}(Q)(a_0da_1\dots da_{n+2}) \end{aligned}$$

and the cochain $B\mathrm{cs}_{n+1}$ of degree n:

$$\begin{aligned} B\mathrm{cs}_{n+1}(a_0da_1\dots da_n) &= \frac{1}{2}\frac{(n/2)!}{n!}\mathrm{Tr}_s(F[F,a_0][F,a_1]\dots[F,a_n]) \\ &= \mathrm{ch}_n(Q)(a_0da_1\dots da_n) \end{aligned}$$

Therefore, the periodic coboundary of the cochain cs_{n+1} is

$$(b+B)\mathrm{cs}_{n+1} = \mathrm{ch}_n(Q) - \mathrm{ch}_{n+2}(Q) \ ,$$

which shows the the cocycles $\mathrm{ch}_n(Q)$ and $\mathrm{ch}_{n+2}(Q)$ are cohomologous in periodic cyclic cohomology.
b) Independence with respect to the mass m. First remark that m only appears in the diagonal of the matrix Q. Since $\mathrm{ch}_n(Q)$ involves a polynomial of even degree in the variables Q and $Q^{-1} = Q/(D^2+m^2)$, it is not hard to see that actually $\mathrm{ch}_n(Q)$ depends on m only through its square m^2. Hence it is sufficient to consider m, say, stricly positive. We will show that the periodic cyclic cohomology class of $\mathrm{ch}_n(Q)$ is invariant under an infinitesimal variation of m. Let $\dot{Q} = \gamma$ be the derivative of Q with respect to m, one has

$$\dot{F} = \begin{pmatrix} 0 & \dot{Q^{-1}} \\ \dot{Q} & 0 \end{pmatrix} = \begin{pmatrix} 0 & -Q^{-1}\gamma Q^{-1} \\ \gamma & 0 \end{pmatrix} \ .$$

Deriving the relation $F^2 = 0$ implies $\dot{F}F + F\dot{F} = 0 = [F,\dot{F}]$. Take $n \in 2\mathbb{N}$ sufficiently large, for example $n > p+1$, and introduce the following odd cochain of degree $n-1$,

$$\begin{aligned} \psi_{n-1}(a_0da_1\dots da_{n-1}) &= \frac{1}{2}\frac{(n/2)!}{n!}(-\mathrm{Tr}_s(F[F,a_0]\dot{F}[F,a_1]\dots[F,a_{n-1}]) \\ &\quad +\mathrm{Tr}_s(F[F,a_0][F,a_1]\dot{F}\dots[F,a_{n-1}]) - \dots \\ &\quad \dots + \mathrm{Tr}_s(F[F,a_0][F,a_1]\dots[F,a_{n-1}]\dot{F}) \end{aligned}$$

by inserting the operator $\dot{F}$ between the commutators, with alternate signs. We compute the Hochschild coboundary of ψ_{n-1} using the algebraic rule $[F,\dot{F}] = 0$ and the fact that $[F,\ ]$ is a derivation. One finds

$$\begin{aligned} b\psi_{n-1}(a_0da_1\dots da_n) &= \frac{1}{2}\frac{(n/2)!}{n!}\frac{\partial}{\partial m}\mathrm{Tr}_s(F[F,a_0][F,a_1]\dots[F,a_n]) \\ &= \frac{\partial}{\partial m}(\mathrm{ch}_n(Q))(a_0da_1\dots da_n) \ . \end{aligned}$$

Moreover, the boundary B vanishes on ψ_{n-1}, which shows the derivative of $\mathrm{ch}_n(Q)$ with respect to m is a periodic coboundary:

$$\frac{\partial}{\partial m}\mathrm{ch}_n(Q) = (b+B)\psi_{n-1} \ .$$

Integrating this equation between two strictly positive values m_1 and m_2 implies that the difference $\mathrm{ch}_n(Q_{m_2}) - \mathrm{ch}_n(Q_{m_1})$ is a periodic coboundary, whence the invariance of the periodic cohomology class $\mathrm{ch}_n(Q_m)$. ■

Now given an even spectral triple $(\mathcal{A}, \mathcal{H}, D)$ and an idempotent $e \in M_\infty(\mathcal{A})$ representing a class in $K_0(\mathcal{A})$, one can calculate the pairing between their Chern characters lying respectively in $HP^0(\mathcal{A})$ and $HP_0(\mathcal{A})$. It turns out that the result is an integer and coincides with the index of the Dirac operator D against e. This is the content of the following index theorem due to Connes [6].

Theorem 4.10 (Connes) *Let $(\mathcal{A}, \mathcal{H}, D)$ be a p-summable spectral triple of even degree over $\mathcal{A}$ and $[e] \in K_0(\mathcal{A})$ be a K-theory class represented by the idempotent $e \in M_N(\mathcal{A})$. Then the index of D against e is computed by the duality pairing of their Chern characters in periodic cyclic (co)homology*

$$\mathrm{Ind}(D, e) = \langle \mathrm{ch}(D), \mathrm{ch}(e) \rangle \ . \tag{266}$$

Proof: Replacing the Hilbert space $\mathcal{H}$ by $\mathcal{H} \otimes \mathbb{C}^N$, the Dirac operator D by $D \otimes \mathrm{id}_N$ and the representation of $a \in \mathcal{A}$ by $a \otimes \mathrm{id}_N$ if necessary, we may assume the idempotent e belongs to $\mathcal{A}$. Let $Q = D + \gamma m$ with $m \neq 0$ be the massive amplification of the Dirac operator, then

$$F = \begin{pmatrix} 0 & Q^{-1} \\ Q & 0 \end{pmatrix} .$$

acts on the Hilbert space $\mathcal{H}' = \mathcal{H} \oplus \mathcal{H}$, and the idempotent e reads

$$e = \begin{pmatrix} e'_+ & 0 \\ 0 & e'_- \end{pmatrix} \ , \text{ with } \quad e'_+ = \begin{pmatrix} e_+ & 0 \\ 0 & 0 \end{pmatrix} \ , \quad e'_- = \begin{pmatrix} 0 & 0 \\ 0 & e_- \end{pmatrix} \ .$$

Set $\mathcal{H}_1 = e'_+\mathcal{H}$, $\mathcal{H}_2 = e'_-\mathcal{H}$, and denote by $Q' : \mathcal{H}_1 \to \mathcal{H}_2$ the restriction of Q to these subspaces, and by $P' : \mathcal{H}_2 \to \mathcal{H}_1$ the restriction of Q^{-1}. Then Q' and P' are no longer invertible operators, but an easy computation shows that $\mathrm{id}_{\mathcal{H}_1} - P'Q'$ (resp. $\mathrm{id}_{\mathcal{H}_2} - Q'P'$) is the restriction to $\mathcal{H}_1$ (resp. $\mathcal{H}_2$) of the operator $e - eFeFe$. Since e is an idempotent, one has

$$e - eFeFe = e[F, e]^2 e$$

with $[F, e]^2 \in \ell^{p/2}(\mathcal{H}')$. It is not too difficult to show that

$$\mathrm{Ind}(P') = \mathrm{Ind}(D, e) = \mathrm{Tr}_s((e - eFeFe)^{n/2+1})$$

for any even integer $n > p - 1$. It remains to identify the above supertrace with the pairing $\langle \mathrm{ch}(D), \mathrm{ch}(e) \rangle$. Recall the Chern character of $(\mathcal{A}, \mathcal{H}, D)$ is represented by the cyclic n-cocycle

$$\mathrm{ch}_n(Q)(a_0 da_1 \dots da_n) = \frac{(n/2)!}{n!} \mathrm{Tr}_s(F[F, a_0][F, a_1] \dots [F, a_n])$$

and the Chern character of e is the periodic cycle

$$\mathrm{ch}(e) = e + \sum_{k=0}^{\infty} (-)^k \frac{(2k)!}{k!} (e - \frac{1}{2})(dede)^k \ .$$

Therefore the pairing selects only the term $k = n/2$ of the above series, so that

$$\langle \mathrm{ch}(D), \mathrm{ch}(e) \rangle = (-)^{n/2} \mathrm{Tr}_s(F[F,e]^{n+1}) \ .$$

It is now just a matter of algebraic manipulations, using $e[F,e]^2 = [F,e]^2 e$, to show the equality

$$(-)^{n/2} \mathrm{Tr}_s(F[F,e]^{n+1}) = \mathrm{Tr}_s((e - eFeFe)^{n/2+1}) \ ,$$

hence the index of the Dirac operator against e is obtained by the pairing of the Chern characters as wanted. ∎

4.2 Local formulas and residues

Although the n-dimensional Chern character of a p-summable spectral triple $(\mathcal{A}, \mathcal{H}, D)$ is given by an apparently simple formula

$$\mathrm{ch}_n(Q)(a_0 da_1 \dots da_n) = \frac{1}{2} \frac{(n/2)!}{n!} \mathrm{Tr}_s(F[F,a_0][F,a_1]\dots[F,a_n]) \ , \quad n > p-1 \ , \tag{267}$$

it is not always easily computable in concrete situations. Even in the classical example, when $\mathcal{A} = C^\infty(M)$ is the algebra of smooth functions on a closed riemannian spin manifold M, and D is the usual Dirac operator acting on a dense subspace of the Hilbert space $\mathcal{H} = L^2(S)$ of square-integrable sections of the spinor bundle, the computation of the cocycle $\mathrm{ch}_n(Q)$ is tedious. The reason is that the operator

$$F = \begin{pmatrix} 0 & Q^{-1} \\ Q & 0 \end{pmatrix} \tag{268}$$

involves the inverse of the massive amplification $Q = D + \gamma m$. Therefore Q^{-1} is a non-local operator, in the sense that its action on a smooth section ξ of the spinor bundle is given in terms of its distributional kernel $Q^{-1}(x,y)$ which is non-zero outside the diagonal $x = y$:

$$(Q^{-1}\xi)(x) = \int_M dy \, Q^{-1}(x,y)\xi(y) \ . \tag{269}$$

We recognize here that $Q^{-1}(x,y)$ is the propagator of the fermionic spinor field with mass m, see section 2.3. As a consequence of this non-locality, taking the operator supertrace in the Chern character formula (267) yields a multi-integral expression

$$\int_M dy_0 \int_M dy_1 \dots \int_M dy_n \, k(y_0, y_1, \dots, y_n) \ , \tag{270}$$

with k some distributional kernel in the variables y_i. This is in contrast with the idea that Chern characters and consequently index formulas should be given by local expressions, of the form

$$\int_M dy \, k(y) \tag{271}$$

with $k(y)$ some polynomial involving curvatures of connections, like in the Atiyah-Singer index theorem. This difficulty is solved by Connes and Moscovici in a series of articles [8, 9] by constructing a periodic cocycle over $\mathcal{A}$ cohomologous to $\mathrm{ch}_n(Q)$, whose components are given by residues of zeta-functions and are therefore automatically local. It is worth mentioning that this notion of “locality” is quite general and extends the classical notion from manifold to noncommutative spaces.

In order to write this local representative of the Chern character, it is necessary to give more informations about the noncommutative space described by the spectral triple $(\mathcal{A}, \mathcal{H}, D)$. The correct notion of “dimension” for this space is not entirely given by p-summability, which is a real number, but is rather encoded in a subset of points in the complex plane

$$Sd \subset \mathbb{C} \tag{272}$$

called the dimension spectrum of the spectral triple. These points correspond to the poles of some zeta-functions obtained as follows. Denote by δ the derivation $\delta(T) = [|D|, T]$ for any operator T on $\mathcal{H}$. We assume that the representation of $\mathcal{A}$ in $\mathcal{H}$ is contained in the domain of all the iterations δ^n of this derivation. Let $\mathcal{B}$ be the algebra generated by $\mathcal{A}$, $[D, \mathcal{A}]$ and all the derivatives $\delta^n(a)$ and $\delta^n([D, a])$, $a \in \mathcal{A}$. We make the following analytical assumptions about the spectral triple:

i) The dimension spectrum $Sd \subset \mathbb{C}$ is discrete.

ii) For any element b of the algebra $\mathcal{B}$, the zeta-function

$$\zeta_b(z) = \mathrm{Tr}(b|D|^{-z}) \tag{273}$$

extends holomorphically to the complementary $\mathbb{C}\backslash Sd$, and the order of the poles of ζ_b is bounded by a fixed integer.

iii) The function $\Gamma(z)\zeta_b(z)$ is of rapid decay along the vertical lines $z = r + it$, for any fixed $r > 0$.

Under these assumptions, Connes and Moscovici give a representative of the Chern character of the spectral triple as a periocic cocycle over $\mathcal{A}$, that is, a finite collection of linear maps $\varphi_n : \Omega^n\mathcal{A} \to \mathbb{C}$ verifying the cocycle condition

$$b\varphi_n + B\varphi_{n+2} = 0 \quad \forall n \ . \tag{274}$$

We state the result below for a triple of even degree.

Theorem 4.11 (Connes-Moscovici) *Let $(\mathcal{A}, \mathcal{H}, D)$ be a p-summable spectral triple with discrete dimension spectrum verifying the assumptions above. Then its Chern character is represented by the following periodic cocycle φ of even degree over $\mathcal{A}$:*

$$\varphi_0(a) = \operatorname*{Res}_{z=0} (\Gamma(z)\mathrm{Tr}_s(a|D|^{-2z})) \ , \tag{275}$$

and for any even integer $n > 0$,

$$\varphi_n(a_0da_1 \dots da_n) = \sum_{q\geq 0, k_i\geq 0} c_{n,k,q} \times \tag{276}$$
$$\times \operatorname*{Res}_{z=0} (z^q \mathrm{Tr}_s(a_0[D, a_1]^{(k_1)} \dots [D, a_n]^{(k_n)}|D|^{-(2z+n+2\sum k_i)})) \ ,$$

where $[D,a]^{(k)}$ *denotes the k-th iterated commutator* $[D^2,\ ]$ *on* $[D,a]$*, and the coefficients* $c_{n,k,q}$ *are given by*

$$c_{n,k,q} = (-)^{k_1+\ldots+k_n} \frac{\Gamma^{(q)}(k_1+\ldots+k_n+n/2)}{q!k_1!\ldots k_n!} \times \\ \times((k_1+1)(k_1+k_2+2)\ldots(k_1+\ldots+k_n+n))^{-1} \ . \tag{277}$$

The finite summability of the spectral triple implies that only a finite number of components φ_n are non-zero, and the presence of residues necessarily gives local expressions. In the classical example $\mathcal{A} = C^\infty(M)$ and D the Dirac operator on spinors, one can show that the dimension spectrum consists only of simple poles and is contained in the set

$$Sd \subset \{k \in \mathbb{Z} \,|\, k \leq \dim M\} \ . \tag{278}$$

Hence we retain only the residues of the form ($q = 0$)

$$\operatorname*{Res}_{z=0} \mathrm{Tr}_s(a_0[D,a_1]^{(k_1)} \ldots [D,a_n]^{(k_n)}|D|^{-(2z+n+2\sum k_i)}) \tag{279}$$

which vanish unless all the k_i's are zero, in which case

$$\operatorname*{Res}_{z=0} \mathrm{Tr}_s(a_0[D,a_1]\ldots[D,a_n]|D|^{-(2z+n)}) = \lambda_n \int_M \hat{A}(M) \wedge a_0 da_1 \wedge \ldots \wedge da_n \tag{280}$$

for some coefficient λ_n. Here we recognize the Atiyah-Hirzebruch $\hat{A}$-genus of the spin manifold. By pairing the Chern character of the triple with an idempotent $e \in M_N(\mathcal{A})$ representing a K-theory class of M, one thus recovers the Atiyah-Singer index theorem for spin manifolds:

$$\mathrm{Ind}(D,e) = \int_M \hat{A}(M)\mathrm{ch}(e) \ . \tag{281}$$

In the general case the dimension spectrum may have poles of higher order and the terms involving the derivatives of gamma-functions ($q \neq 0$) in the coefficients $c_{n,k,q}$ contribute to the Chern character. This happens for example in the case of manifolds with singularities. One might be surprised by the appearance of transcendantal coefficients $c_{n,k,q}$ in an index theorem, however these are largely irrelevant because, as shown in [9], one can modify the cocycle φ by adding a coboundary such that the result involves only rational coefficients.

4.3 Anomalies revisited

We see from the preceding discussion about the noncommutative index theorem that the formalism used there is very close to the description of chiral anomalies in quantum field theory. To make this link more precise, we will state here a theorem relating the index of an abstract Dirac operator and a chiral anomaly, in the most general framework of noncommutative geometry. Since anomalies are always local, we will find an alternative way of computing the index pairing through local formulas, in the spirit of Connes and Moscovici.

So let us start with a spectral triple $(\mathcal{A}, \mathcal{H}, D)$ over an associative algebra $\mathcal{A}$. We assume the triple has even parity and is p-summable for a given real number $p \geq 1$. Hence the Dirac operator is such that

$$(1 + D^2)^{-1/2} \in \ell^p(\mathcal{H}) \ . \tag{282}$$

There is an obvious way of considering a kind of noncommutative quantum field theory associated to the spectral triple, by generalizing the fermionic action encountered in section 2.3 in the case of closed manifolds. For any vector $\psi \in \mathcal{H}$ belonging to the domain of D, and any dual vector $\overline{\psi} \in \mathcal{H}^* \cong \mathcal{H}$, we introduce the action

$$S(\psi, \overline{\psi}) = \langle \overline{\psi}, Q\psi \rangle \ , \quad Q = D + \gamma m \tag{283}$$

where $m \in \mathbb{R}$ is a mass term. This action therefore describes the classical dynamics of a free massive fermionic field ψ and its adjoint $\overline{\psi}$ on the noncommutative "manifold" $(\mathcal{A}, \mathcal{H}, D)$. Since a free theory is rather trivial, we would like to improve it a little bit by coupling the fermions to a gauge potential A, such that the total action is invariant under *chiral* gauge transformations, described as follows. Let $\widetilde{\mathcal{A}}$ be the unitalization of $\mathcal{A}$ (we add a unit to $\mathcal{A}$ even though $\mathcal{A}$ is already supposed to be unital) and let $g \in \widetilde{\mathcal{A}}$ be an invertible element of the form

$$g = 1 + u \ , \qquad u \in \mathcal{A} \ , \tag{284}$$

where 1 is the unit of $\widetilde{\mathcal{A}}$. The representation of g as a bounded operator on the $\mathbb{Z}_2$-graded Hilbert space $\mathcal{H} = \mathcal{H}_+ \oplus \mathcal{H}_-$ reads, using 2×2 matrices,

$$g = 1 + \begin{pmatrix} u_+ & 0 \\ 0 & u_- \end{pmatrix} , \tag{285}$$

with $u_\pm$ the action of u on the subspaces $\mathcal{H}_\pm$. Let us split g as the product $g_+ g_-$ of partial transformations

$$g_+ = 1 + \begin{pmatrix} u_+ & 0 \\ 0 & 0 \end{pmatrix} , \quad g_- = 1 + \begin{pmatrix} 0 & 0 \\ 0 & u_- \end{pmatrix} . \tag{286}$$

Then the chiral transformation of parameter g is defined on the fields $\psi \in \mathcal{H}$, $\overline{\psi} \in \mathcal{H}^*$ as

$$\psi \mapsto g_+^{-1} \psi \ , \qquad \overline{\psi} \mapsto \overline{\psi} g_- \ , \tag{287}$$

so that the free action transforms as

$$\langle \overline{\psi}, Q\psi \rangle \mapsto \langle \overline{\psi}, g_- Q g_+^{-1} \psi \rangle \ . \tag{288}$$

We shall absorb the lack of chiral invariance of the free action by introducing a gauge potential A as a bounded operator on $\mathcal{H}$. In practice, A will be an element of the unital algebra of bounded operators generated by $\mathcal{A}$, $[D, \mathcal{A}]$ and the involution γ. Then we perturb the massive operator Q by adding

$$Q_A = Q + A \ . \tag{289}$$

Note that the unbounded operator Q_A is not selfadjoint in general, but differs from the selfadjoint Q only by a bounded operator. The chiral transformation

A^g of the potential is specified so that Q_A transforms adequately under the action of g:

$$Q_A \mapsto g_-^{-1} Q_A g_+ = Q_{A^g} \ . \tag{290}$$

One finds

$$A^g = g_-^{-1} Q_A g_+ - Q = (g_-^{-1} Q g_+ - Q) + g_-^{-1} A g_+ \ , \tag{291}$$

which is reminiscent of the chiral gauge transformations obtained in the case of manifolds. Remark that A^g is bounded. Indeed, the first term of the right hand side is

$$g_-^{-1} Q g_+ - Q = (g_-^{-1} D g_+ - D) + m(g_-^{-1} \gamma g_+ - \gamma) \ . \tag{292}$$

The mass term is clearly bounded, whereas the term involving the Dirac operator reads

$$\begin{aligned} g_-^{-1} D g_+ - D &= g_-^{-1}(D g_+ - g_- D) \\ &= g_-^{-1} \left(\begin{pmatrix} 0 & D_- \\ D_+ & 0 \end{pmatrix} \begin{pmatrix} 1+u_+ & 0 \\ 0 & 1 \end{pmatrix} - \begin{pmatrix} 1 & 0 \\ 0 & 1+u_- \end{pmatrix} \begin{pmatrix} 0 & D_- \\ D_+ & 0 \end{pmatrix} \right) \\ &= g_-^{-1} \begin{pmatrix} 0 & 0 \\ (D_+ u_+ - u_- D_+) & 0 \end{pmatrix} \ . \end{aligned} \tag{293}$$

By hypothesis on the spectral triple, the commutator $[D, u]$ is bounded on $\mathcal{H}$, hence is also $D_+ u_+ - u_- D_+$ on $\mathcal{H}_+$. It follows that the chiral transformation maps A to another bounded operator A^g of the same kind. Now, we add an interaction term to the free action, in order to get the full action

$$S(\psi, \overline{\psi}, A) = \langle \overline{\psi}, (Q + A)\psi \rangle \tag{294}$$

describing massive fermions $\psi, \overline{\psi}$ coupled to an external potential A. By construction, this action is invariant under chiral transformations:

$$S(g_+^{-1}\psi, \overline{\psi} g_-, A^g) = S(\psi, \overline{\psi}, A) \tag{295}$$

for any invertible element $g = 1 + u \in \widetilde{\mathcal{A}}$. We would like now to quantize the fermionic fields, retaining A as a fixed (classical) external potential. This is achieved by considering the partition function (we set $\hbar = 1$)

$$Z(A) = \int \mathcal{D}\psi \mathcal{D}\overline{\psi} \, e^{-S(\psi, \overline{\psi}, A)} \tag{296}$$

where $\mathcal{D}\psi\mathcal{D}\overline{\psi}$ is the formal Berezin integration measure over the infinite dimensional space of fermionic fields. In order to give a sense to the functional integral, we just remark that the action is quadratic in $\psi, \overline{\psi}$ so that by a usual argument, $Z(A)$ is just defined to be a regularized determinant of the unbounded operator Q_A:

$$Z(A) = {}'' \det(Q^{-1} Q_A)'' \ , \tag{297}$$

where the normalization condition $Z(0) = 1$ is taken into account. Following the basic principles of quantum field theory, it is easier to deal with the free energy

$$W(A) = \ln Z(A) \tag{298}$$

as a formal power series in A, with a finite number of terms to be renormalized. Hence we are trying to define

$$W(A) = \ln\det(Q^{-1}Q_A) = \mathrm{Tr}\ln(1+Q^{-1}A)\ . \tag{299}$$

The trace stands for the ordinary operator trace on the Hilbert space $\mathcal{H}$. Developing the logarithm as a formal power series in A, we find

$$W(A) = \mathrm{Tr}(Q^{-1}A) - \frac{1}{2}\mathrm{Tr}(Q^{-1}AQ^{-1}A) + \ldots + \frac{(-1)^{n+1}}{n}\,\mathrm{Tr}\,(Q^{-1}A)^n + \ldots \tag{300}$$

Now the p-summability assumption $Q^{-1} \in \ell^p(\mathcal{H})$ tells us that the operator $(Q^{-1}A)^n$ is trace-class whenever $n \geq p$, hence the corresponding term in the series $W(A)$ is well-defined. Therefore only the first $p-1$ terms need to be renormalized. This can be done, in a very general manner, as follows. Denote by $\Psi(\mathcal{A})$ the algebra of (possibly unbounded) operators generated by $\mathcal{A}$, Q, Q^{-1} and the chirality operator γ. This plays the role of an algebra of abstract pseudodifferential operators. The trace Tr is well-defined on the subalgebra of trace-class operators $\Psi(\mathcal{A}) \cap \ell^1(\mathcal{H})$. Choose an arbitrary linear extension

$$\tau : \Psi(\mathcal{A}) \to \mathbb{C} \tag{301}$$

which coincides with Tr on trace-class operators. In general, it is impossible to find an extension τ with the additional property of being a trace on the whole algebra $\Psi(\mathcal{A})$: rather τ is simply a linear map and one has

$$\tau(T_1T_2) \neq \tau(T_2T_1) \tag{302}$$

for arbitrary elements $T_1, T_2 \in \Psi(\mathcal{A})$. On the other hand, there are many ways of constructing such a linear extension. For example, suppose as in the preceding section that the spectral triple has a discrete dimension spectrum verifying the conditions of Connes-Moscovici. Then for any element $T \in \Psi(\mathcal{A})$, the zeta-function

$$\zeta_T(z) = \mathrm{Tr}(T|D|^{-z}) \tag{303}$$

has a meromorphic extension to the complex plane. One could define $\tau(T)$ as the finite part of the Laurent expansion of $\zeta_T(z)$ around zero, or equivalently, as the residue

$$\tau(T) = \operatorname*{Res}_{z=0} \frac{1}{z}\mathrm{Tr}(T|Q|^{-z})\ . \tag{304}$$

If T is trace-class, then $\zeta_T(z)$ has no pole at $z=0$ and clearly $\tau(T)$ reduces to the trace $\mathrm{Tr}(T)$. Hence τ is indeed a linear extension of Tr, but it is not a trace on $\Psi(\mathcal{A})$. Of course such an extension is not unique. One could as well define another extension τ' with the help of the gamma-function

$$\tau'(T) = \operatorname*{Res}_{z=0} \Gamma(z)\mathrm{Tr}(T|Q|^{-z})\ , \tag{305}$$

which in fact coincides with the finite part of the trace regularised by the heat kernel operator $\exp(-tQ^2)$, $t>0$:

$$\tau'(T) = \operatorname*{Pf}_{t\to 0} \mathrm{Tr}(Te^{-tQ^2})\ . \tag{306}$$

It is important, however, to observe the following fact. In the vicinity of zero the function $\Gamma(z)$ is the sum of a holomorphic function $h(z)$ and a simple pole $1/z$, so that the difference of the two extensions $\tau' - \tau$

$$\tau'(T) - \tau(T) = \operatorname*{Res}_{z=0} h(z)\mathrm{Tr}(T|D|^{-z}) \tag{307}$$

only involves the *poles* of the zeta-function $\zeta_T(z)$. By a general principle, the poles are always "local" terms, in the noncommutative geometry sense. This is exactly what happens in renormalization theory: one is looking for an extension of some divergent quantity, the result is defined only modulo local counterterms. Turning back to the free energy $W(A)$, its renormalized form is obtained by replacing the ill-defined trace Tr by any choice of extension τ:

$$W_\tau(A) = \tau(Q^{-1}A) - \frac{1}{2}\tau(Q^{-1}AQ^{-1}A) + \ldots + \frac{(-1)^{n+1}}{n}\tau\,(Q^{-1}A)^n + \ldots \tag{308}$$

For n sufficiently large ($n \geq p$), τ coincides with the operator trace, which shows that different renormalizations associated to different choices of extensions differ only by a *finite number of local counterterms*

$$W_{\tau'}(A) - W_\tau(A) = \sum_{n=1}^{m} \frac{(-1)^{n+1}}{n}(\tau' - \tau)\,(Q^{-1}A)^n\ , \tag{309}$$

where m is the largest integer $< p$.

Once a renormalization $W_\tau(A)$ is chosen, we must keep in mind that the formal power series (308) is still divergent for large values of the potential A, since it is the expansion of a logarithm. Let us now examine how $W_\tau(A)$ behaves under chiral gauge transformations. Its non-invariance would imply a chiral anomaly, in our noncommutative context. Since we are interested mainly in infinitesimal chiral transformations, we formulate the situation as follows. Consider a family of gauge transformations parametrized by the circle,

$$g : S^1 \to Gl_1(\widetilde{\mathcal{A}})\ , \tag{310}$$

so that for any $t \in S^1$, $g(t)$ is an invertible element of the unitalization $\widetilde{\mathcal{A}}$ of the form $1 + u(t)$, $u(t) \in \mathcal{A}$. Some "smoothness" conditions must be imposed on the function g. Since the algebra $\mathcal{A}$ is not endowed with any topology, we will say that the function g is smooth if it is an invertible element of the algebraic tensor product $C^\infty(S^1) \otimes \widetilde{\mathcal{A}}$. Hence g may be written as a finite sum

$$g = \sum_i f_i \otimes \widetilde{a}_i\ , \quad f_i \in C^\infty(S^1)\ ,\ \widetilde{a}_i = 1 + a_i\ ,\ a_i \in \mathcal{A}\ . \tag{311}$$

A very important example of such invertible functions is provided by idempotent loops [5]. Let $e \in \mathcal{A}$ be an idempotent, and $\beta \in C^\infty(S^1)$ be the Bott generator of the circle:

$$\beta(t) = e^{2\pi i t}\ . \tag{312}$$

The idempotent loop g associated to e is then given by

$$g = 1 + (\beta - 1)e\ \in C^\infty(S^1) \otimes \widetilde{\mathcal{A}}\ , \tag{313}$$

with inverse

$$g^{-1} = 1 + (\beta^{-1} - 1)e \ . \tag{314}$$

Actually, in our formulation of the index theorem from chiral anomalies, all the interesting gauge transformations we will be concerned with arise from idempotent loops (this is connected with Bott periodicity [5] for some Banach completion of the algebra $\mathcal{A}$).
Now we fix such an invertible element $g \in C^\infty(S^1) \otimes \widetilde{\mathcal{A}}$. Denote by

$$s : C^\infty(S^1) \to \Omega^1(S^1) \tag{315}$$

the de Rham differential mapping smooth functions to smooth one-forms over the circle. It extends to a differential on the tensor product

$$s : C^\infty(S^1) \otimes \widetilde{\mathcal{A}} \to \Omega^1(S^1) \otimes \widetilde{\mathcal{A}} \ . \tag{316}$$

The Maurer-Cartan one-form $\omega \in \Omega^1(S^1) \otimes \widetilde{\mathcal{A}}$ is defined as

$$\omega = g^{-1} sg \ . \tag{317}$$

We now use the representation of $\mathcal{A}$ in $\mathcal{L}(\mathcal{H})$ and consider g and ω as a function (resp. one-form) over the circle, with values in the algebra of bounded operators on $\mathcal{H}$:

$$\begin{aligned} g &= g_+ g_- = \begin{pmatrix} 1+u_+ & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 1+u_- \end{pmatrix} , \\ \omega &= g_+^{-1} sg_+ + g_-^{-1} sg_- = \omega_+ + \omega_- \\ \omega_+ &= \begin{pmatrix} (1+u_+)^{-1} su_+ & 0 \\ 0 & 0 \end{pmatrix} , \quad \omega_- = \begin{pmatrix} 0 & 0 \\ 0 & (1+u_-)^{-1} su_- \end{pmatrix} . \end{aligned} \tag{318}$$

Take $A_0 = 0$ as a fixed background potential. Its chiral gauge transform with respect to the loop g

$$A = A_0^g = g_-^{-1} Q g_+ - Q \ \in C^\infty(S^1) \otimes \mathcal{L}(\mathcal{H}) \tag{319}$$

is an operator-valued function on the circle, whose differential is the operator-valued one-form (observe the analogy with BRS transformations [10, 14])

$$sA = (Q + A)\omega_+ - \omega_-(Q + A) \ \in \Omega^1(S^1) \otimes \mathcal{L}(\mathcal{H}) \ . \tag{320}$$

Now, the free energy $W_\tau(A)$ is a formal series of smooth complex-valued functions

$$\frac{(-1)^{n+1}}{n} \tau\,(Q^{-1}A)^n \ \in C^\infty(S^1) \ . \tag{321}$$

Therefore, the differential of each term of the series may be computed from

$$\begin{aligned} s\,\tau\,(Q^{-1}A)^n &= \tau(Q^{-1}sA(Q^{-1}A)^{n-1} + Q^{-1}AQ^{-1}sA(Q^{-1}A)^{n-2} + \ldots \\ &\quad \ldots + (Q^{-1}A)^{n-1}Q^{-1}sA) \ . \end{aligned} \tag{322}$$

Suppose n is greater than the summabibity degree p of the spectral triple. Then τ is simply the operator trace Tr and the cyclicity of the trace implies

$$s\,\tau\,(Q^{-1}A)^n = n\,\mathrm{Tr}(Q^{-1}sA(Q^{-1}A)^{n-1}) \ , \quad n \geq p \ . \tag{323}$$

We know think in terms of formal power series in A. Replacing sA by its expression (320), we separate a term of degree $n-1$ and another term of degree n w.r.t. A:

$$
\begin{aligned}
s\,\tau\,(Q^{-1}A)^n &= n\,\mathrm{Tr}(Q^{-1}(Q\omega_+ - \omega_- Q + A\omega_+ - \omega_- A)(Q^{-1}A)^{n-1}) \qquad (324)\\
&= n\,\mathrm{Tr}((\omega_+ - Q^{-1}\omega_- Q)(Q^{-1}A)^{n-1}) + n\,\mathrm{Tr}((\omega_+ - Q^{-1}\omega_- Q)(Q^{-1}A)^{n})\ .
\end{aligned}
$$

Consequently, we see that in the formal series

$$
s\sum_{n\geq p}\frac{(-1)^{n+1}}{n}\,\tau\,(Q^{-1}A)^n \qquad (325)
$$

the terms cancel each other except for the first one, namely

$$
(-1)^m\mathrm{Tr}((\omega_+ - Q^{-1}\omega_- Q)(Q^{-1}A)^m)\ , \qquad (326)
$$

where m is the biggest integer $< p$. This cancellation does not hold true for the terms of lower degree, because in general τ is not a trace on the operator $(Q^{-1}A)^n$ for $n < p$. It follows that, *in the sense of formal power series in* A, the differential of $W_\tau(A)$ is given by the finite sum

$$
\begin{aligned}
s\,W_\tau(A) = \sum_{n=1}^{m}\frac{(-1)^{n+1}}{n}\,s\,\tau\,(Q^{-1}A)^n& \qquad (327)\\
+(-1)^m\mathrm{Tr}((\omega_+ - Q^{-1}\omega_- Q)(Q^{-1}A)^m)&\ .
\end{aligned}
$$

This is by definition the chiral anomaly $\Delta(g)$ associated to the loop g. It is always given a local formula. Let us state without proof the explicit computation of the anomaly. It is the sum of residues

$$
\begin{aligned}
\Delta(g) = \operatorname*{Res}_{z=0}\left(\frac{1}{z}\mathrm{Tr}((\omega_+ - \omega_-)Q^{-2z})\right) + \sum_{n\geq 1, k\geq 0}(-)^{n+k}c(k)\times \qquad (328)\\
\times\ \operatorname*{Res}_{z=0}\left(\frac{\Gamma(z+n+k)}{z\Gamma(z)}\mathrm{Tr}((\omega_+ Q - Q\omega_-)A^{(k_1)}QA^{(k_2)}Q\ldots A^{(k_n)}Q^{-2(z+n+k)})\right)\ ,
\end{aligned}
$$

where $k=(k_1,\ldots,k_n)$ is a multi-index, and

$$
c(k)^{-1} = (k_1!\ldots k_n!)(k_1+1)(k_1+k_2+2)\ldots(k_1+\ldots+k_n+n)\ . \qquad (329)
$$

Observe the analogy with the Connes-Moscovici local index formula. Remark also that if τ is really a trace on the algebra $\Psi(\mathcal{A})$ (this happens for example when $\mathcal{H}$ is finite-dimensional), then the cancellation term by term holds down to the first term of the series, and the anomaly reduces to the difference of representations of the Lie algebra of chiral transformations:

$$
\Delta(g) = s\,W_\tau(A) = \mathrm{Tr}(\omega_+ - \omega_-)\ . \qquad (330)
$$

This is what we expect from a finite-dimensional theory. In contrast, when $\mathcal{H}$ is infinite-dimensional the lack of tracial property for τ implies the existence of additional counterterms, which, according to the discussion above, are necessarily local.

Since the anomaly is a finite sum of one-forms, one can integrate it over the circle to get a number

$$\frac{1}{2\pi i}\oint \Delta(g) \ \in \mathbb{C}\ . \tag{331}$$

Although $\Delta(g)$ looks like a coboundary, one should not conclude that its integral is zero. Indeed, the free energy $W_\tau(A)$ is not a convergent series in general, but its differential is a well-defined one-form with non-vanishing integral. When g is an idempotent loop, we will actually show that this number coincides with the index of a Dirac operator, hence is an integer. This corresponds to the winding number of the determinant function $\det(Q^{-1}Q_A)$ over the circle, as we have already seen in the first chapter.

Theorem 4.12 *Let $(\mathcal{A},\mathcal{H},D)$ be a p-summable spectral triple of even degree. Let W_τ be the free energy of the corresponding massive fermionic quantum field theory, renormalized by means of any linear extension $\tau : \Psi(\mathcal{A}) \to \mathbb{C}$ of the operator trace. Let $e \in \mathcal{A}$ be an idempotent and $g = 1+(\beta-1)e$ the corresponding idempotent loop. Then the integral of the anomaly $\Delta(g) = sW_\tau$ along this loop is the integer*

$$\frac{1}{2\pi i}\oint \Delta(g) = \mathrm{Ind}(D,e) \ \in \mathbb{Z} \tag{332}$$

corresponding to the index of the Dirac operator D against the idempotent.

Before giving the proof of the theorem, it is worth mentioning that the integral of the anomaly is necessarily independent of the choice of extension τ. Indeed, if $\tau' : \Psi(\mathcal{A}) \to \mathbb{C}$ is any other extension, the difference of renormalized free energies $W_{\tau'}(A) - W_\tau(A)$ is a finite sum because both τ and τ' coincide with the operator trace in all the terms of the series except in low degree. Hence the difference of the corresponding anomalies

$$\Delta'(g) - \Delta(g) = s(W_{\tau'}(A) - W_\tau(A)) = s(\text{finite counterterms}) \tag{333}$$

is a de Rham coboundary in $\Omega^1(S^1)$. This shows the cohomology class of the anomaly is independent of the given extension.

We will give a proof of this result using cyclic cohomology. It follows from purely algebraic considerations, once the extension τ is given. Consider the algebra $\mathcal{B} = C^\infty(S^1) \otimes \widetilde{\mathcal{A}}$, and the de Rham complex of smooth differential forms on the circle $\Omega(S^1)$. We shall work in a tricomplex mixing the cyclic cohomology of $\mathcal{B}$ with the de Rham cohomology of S^1. This is done as follows. For any $n \in \mathbb{N}$ and $k = 0, 1$, consider the space $C^k_n(\mathcal{B}, S^1)$ of linear maps

$$C^k_n(\mathcal{B}, S^1) = \mathrm{Hom}(\Omega^n\mathcal{B}, \Omega^k(S^1)) \tag{334}$$

from the noncommutative n-forms over $\mathcal{B}$ to the (ordinary, commutative) de Rham k-forms over S^1. By collecting the spaces $C^k_n(\mathcal{B}, S^1)$ for all possible values of n, k we get a tricomplex $C^*_*(\mathcal{B}, S^1)$. The first differential is the Hochschild operator $b : C^k_n(\mathcal{B}, S^1) \to C^k_{n+1}(\mathcal{B}, S^1)$, obtained by dualizing the Hochschild operator on noncommutative forms:

$$b\varphi^k_n(\omega_{n+1}) = \varphi^k_n(b\omega_{n+1})\ , \quad \forall\ \varphi^k_n \in C^k_n\ ,\ \omega_{n+1} \in \Omega^{n+1}\mathcal{B}\ . \tag{335}$$

The second differential is given by the Connes boundary $B : C^k_n(\mathcal{B}, S^1) \to C^k_{n-1}(\mathcal{B}, S^1)$

$$B\varphi^k_n(\omega_{n-1}) = \varphi^k_n(B\omega_{n-1}) \ . \tag{336}$$

The last differential is the de Rham coboundary $s : C^k_n(\mathcal{B}, S^1) \to C^{k+1}_n(\mathcal{B}, S^1)$

$$s\varphi^k_n(\omega_n) = (-)^{n+1} s(\varphi^k_n(\omega_n)) \ . \tag{337}$$

We have chosen signs such that all the differentials anticommute:

$$sb + bs = sB + Bs = bB + Bb = 0 \ , \quad s^2 = b^2 = B^2 = 0 \ . \tag{338}$$

We will construct cochains of the tricomplex from the massive operator $Q = D + \gamma m$ of the fermionic quantum field theory associated to the spectral triple $(\mathcal{A}, \mathcal{H}, D)$. This is done in much the same way as the Chern-Connes character of section 4.1. On the $\mathbb{Z}_2$-graded Hilbert space $\mathcal{H}' = \mathcal{H} \oplus \mathcal{H}$, consider the odd operator

$$F = \begin{pmatrix} 0 & Q^{-1} \\ Q & 0 \end{pmatrix} \tag{339}$$

and the following representation of any element $a \in \mathcal{B}$:

$$a = \begin{pmatrix} a'_+ & 0 \\ 0 & a'_- \end{pmatrix} \ , \ \text{with} \quad a'_+ = \begin{pmatrix} a_+ & 0 \\ 0 & 0 \end{pmatrix} \ , \quad a'_- = \begin{pmatrix} 0 & 0 \\ 0 & a_- \end{pmatrix} . \tag{340}$$

Here $a'_\pm$ (resp. $a_\pm$) are elements of $C^\infty(S^1) \otimes \mathcal{L}(\mathcal{H})$ (resp. $C^\infty(S^1) \otimes \mathcal{L}(\mathcal{H}_\pm)$). Hence F and a may be considered as elements of the tensor product $C^\infty(S^1) \otimes M_2(\Psi(\mathcal{A}))$. More generally, we will work in the algebra $\Omega(S^1) \otimes M_2(\Psi(\mathcal{A}))$, endowed with the differential

$$s : \Omega(S^1) \otimes M_2(\Psi(\mathcal{A})) \to \Omega(S^1) \otimes M_2(\Psi(\mathcal{A})) \tag{341}$$

acting on the first factor. Here we take into account the $\mathbb{Z}_2$-graduations of $\Omega(S^1)$ and $M_2(\Psi(\mathcal{A}))$, so that their tensor product is actually a DG algebra. There is a second differential given by the graded commutator $[F, \ ]$ (recall F is odd). Since F is a constant operator above the circle, its derivative w.r.t. s vanishes

$$sF = 0 \ , \tag{342}$$

which implies the differentials s and $[F, \ ]$ anticommute. Finally, remark that any linear extension $\tau : \Psi(\mathcal{A}) \to \mathbb{C}$ of the trace induces a linar extension τ_s of the supertrace,

$$\tau_s : M_2(\Psi(\mathcal{A})) \to \mathbb{C} \tag{343}$$

by setting

$$\tau_s \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \tau(a) - \tau(d) \ . \tag{344}$$

This in turn induces a linear map commuting with the de Rham differential s

$$\tau_s : \Omega(S^1) \otimes M_2(\Psi(\mathcal{A})) \to \Omega(S^1) \ . \tag{345}$$

Now for any positive odd integer $n = 2k+1$ we introduce the following cochains $\mathrm{cs}^0_n \in C^0_n(\mathcal{B}, S^1)$:

$$\mathrm{cs}^0_n(a_0 da_1 \dots da_n) = \frac{1}{2} \frac{k!}{n!} \tau_s(F a_0 [F, a_1] \dots [F, a_n]) \tag{346}$$

for any $a_i \in \mathcal{B}$. Hence cs^0_n is a linear map $\Omega^n\mathcal{B} \to C^\infty(S^1)$. On the other hand, for any positive even integer $n = 2k+2 \geq 2$ we introduce the following cochains $\mathrm{cs}^1_n \in C^1_n(\mathcal{B}, S^1)$:

$$\begin{aligned} \mathrm{cs}^1_n(a_0da_1\dots da_n) &= \frac{1}{2}\frac{k!}{(n+1)!}\sum_{i=1}^n \tau_s(Fa_0[F,a_1]\dots sa_i\dots[F,a_n] \\ &\quad -F[F,a_n]a_0[F,a_1]\dots sa_i\dots[F,a_{n-1}]+\dots \\ &\quad \dots + F[F,a_1]\dots sa_i\dots[F,a_n]a_0)\ . \end{aligned} \tag{347}$$

Hence the latter formula differs from cs^0_n just by replacing some commutators $[F,a_i]$ by the differential $sa_i \in \Omega(S^1)\otimes\mathcal{L}(\mathcal{H}')$, and performing some cyclic permutations of terms. cs^1_n is therefore a linear map $\Omega^n\mathcal{B}\to\Omega^1(S^1)$. Following the discussion of section 4.1, the operator under the extension τ_s is trace-class for n sufficiently large, typically when the number of commutators $[F,a]$ is larger than the summability degree p of the spectral triple. In this case τ_s coincides with the supertrace Tr_s, and the following lemma computes the coboundaries of the cochains cs^0_n and cs^1_n in the tricomplex:

Lemma 4.13 *For any integer $n \geq p+1$, with p the summability degree of the spectral triple, one has*

$$\begin{aligned} b\mathrm{cs}^1_n(a_0da_1\dots da_{n+1}) &= -\frac{(k+1)!}{(n+1)!}\sum_{i=1}^{n+1}\mathrm{Tr}_s(a_0[F,a_1]\dots sa_i\dots[F,a_{n+1}]) \\ B\mathrm{cs}^1_n(a_0da_1\dots da_{n-1}) &= \frac{1}{2}\frac{k!}{(n-1)!}\sum_{i=0}^{n-1}\mathrm{Tr}_s(F[F,a_0]\dots sa_i\dots[F,a_{n-1}]) \end{aligned}$$

if $n=2k+2$ is even, and

$$\begin{aligned} s\mathrm{cs}^0_n(a_0da_1\dots da_n) &= \frac{k!}{n!}\sum_{i=1}^n\mathrm{Tr}_s(a_0[F,a_1]\dots sa_i\dots[F,a_n]) \\ &\quad -\frac{1}{2}\frac{k!}{n!}\sum_{i=0}^n\mathrm{Tr}_s(F[F,a_0]\dots sa_i\dots[F,a_n]) \end{aligned} \tag{348}$$

if $n=2k+1$ is odd.

Proof: Direct computation. ∎

This lemma is not true when $n < p+1$, because then τ_s is no longer a supertrace. For any odd integer n, let us define the cochain φ^1_n by

$$\varphi^1_n = s\mathrm{cs}^0_n + B\mathrm{cs}^1_{n+1} + b\mathrm{cs}^1_{n-1}\ . \tag{349}$$

The previous lemma implies $\varphi^1_n = 0$ whenever $n \geq p+2$. This can be conveniently written as a descending chain of equations:

$$\begin{aligned}
\varphi_1^1 &= s\mathrm{cs}_1^0 + B\mathrm{cs}_2^1 \\
\varphi_3^1 &= s\mathrm{cs}_3^0 + B\mathrm{cs}_4^1 + b\mathrm{cs}_2^1 \\
&\vdots \\
\varphi_m^1 &= s\mathrm{cs}_m^0 + B\mathrm{cs}_{m+1}^1 + b\mathrm{cs}_{m-1}^1 \\
0 &= s\mathrm{cs}_{m+2}^0 + B\mathrm{cs}_{m+3}^1 + b\mathrm{cs}_{m+1}^1 \\
&\vdots \\
0 &= s\mathrm{cs}_n^0 + B\mathrm{cs}_{n+1}^1 + b\mathrm{cs}_{n-1}^1 \\
&\vdots
\end{aligned} \tag{350}$$

where m is the largest odd integer $< p+2$. The sum

$$\varphi^1 := \sum_{n=1}^{m} \varphi_n^1 = s\sum_{n=1}^{m} \mathrm{cs}_n^0 + (b+B)\sum_{n=2}^{m-1} \mathrm{cs}_n^1 + B\mathrm{cs}_{m+1}^1 \tag{351}$$

is an inhomogeneous cochain in $C_*^1(\mathcal{B}, S^1)$, that is, a linear map $\Omega\mathcal{B} \to \Omega^1(S^1)$. If we compose it with the integration map over the circle

$$\oint : \Omega^1(S^1) \to \mathbb{C} \ , \tag{352}$$

we just get a periodic cochain of odd degree over $\mathcal{B}$

$$\oint \varphi^1 : \Omega\mathcal{B} \to \mathbb{C} \tag{353}$$

whose components of degree $n \geq p+2$ vanish. Moreover, since the integral of an s-coboundary vanishes, one has

$$\oint \varphi^1 = \oint (b+B)\sum_{n=2}^{m-1} \mathrm{cs}_n^1 + \oint B\mathrm{cs}_{m+1}^1 \ , \tag{354}$$

where the last term is a cochain of degree m over the algebra $\mathcal{B} = C^\infty(S^1) \otimes \widetilde{\mathcal{A}}$. It turns out that it is the cup-product, in cyclic cohomology, of the following cocycles. The first one if the cyclic 1-cocycle $[S^1]$ over the algebra $C^\infty(S^1)$ corresponding to the fundamental class of the circle (i.e. the integration map)

$$[S^1](f_0 df_1) = \oint f_0\, sf_1 \tag{355}$$

for any smooth functions $f_i \in C^\infty(S^1)$. The second one is the cyclic cocycle of degree $m-1$ over $\widetilde{\mathcal{A}}$ corresponding to the Chern character of the p-summable spectral triple $(\mathcal{A}, \mathcal{H}, D)$ (remark one has necessarily $m-1 > p-1$):

$$\mathrm{ch}_{m-1}(Q)(a_0 da_1 \dots da_{m-1}) = \frac{1}{2}\frac{((m-1)/2)!}{(m-1)!}\mathrm{Tr}_s(F[F,a_0][F,a_1]\dots[F,a_{m-1}]) \tag{356}$$

for any $a_i \in \widetilde{\mathcal{A}}$. Indeed, by lemma 4.13 one has

$$\oint B\mathrm{cs}^1_{m+1}(a_0 da_1 \dots da_m) = \frac{1}{2}\frac{k!}{m!}\sum_{i=0}^{m} \oint \mathrm{Tr}_s(F[F,a_0]\dots sa_i \dots [F,a_m]) \tag{357}$$

with $m = 2k+1$ and $a_i \in \mathcal{B}$. This is precisely the expression for the cup-product

$$\oint B\mathrm{cs}^1_{m+1} = [S^1]\#\mathrm{ch}_{m-1}(Q) \text{ in } HC^m(C^\infty(S^1)\otimes\widetilde{\mathcal{A}}) \ . \tag{358}$$

Therefore, equation (354) implies $\oint \varphi^1$ is a periodic cocycle of odd degree over $\mathcal{B}$, cohomologous to this cup-product. Given any invertible element $g \in \mathcal{B}$, its K-theory class $[g] \in K_1(\mathcal{B})$ can be paired with the cocycle $\oint \varphi^1$ as explained in section 4.1. The proposition below computes this pairing in the case of idempotent loops.

Proposition 4.14 *Let $e \in \mathcal{A}$ be a projector and $g \in \mathcal{B} = C^\infty(S^1)\otimes\widetilde{\mathcal{A}}$ be the corresponding idempotent loop. Then the pairing of the K-theory class $[g] \in K_1(\mathcal{B})$ with the periodic cocycle $\oint \varphi^1$ computes the index pairing of the Dirac operator with e:*

$$\frac{1}{2\pi i}\langle \oint \varphi^1, \mathrm{ch}(g)\rangle = \mathrm{Ind}(D,e) \ \in \mathbb{Z} \ . \tag{359}$$

Proof: This comes from the compatibility cup products regarding pairings, see proposition 4.7. ∎

It remains to link the cocycle φ^1 with the chiral anomaly. For any invertible element $g \in \mathcal{B}$, define the potential $V \in C^\infty(S^1)\otimes\mathcal{L}(\mathcal{H}')$ by

$$F + V = g^{-1}Fg \ , \quad V = g^{-1}[F,g]. \tag{360}$$

Then V is odd for the $\mathbb{Z}_2$-grading of $\mathcal{H}'$ and may be matricially expressed in terms of the potential A of equation (319) as

$$F + V = \begin{pmatrix} 0 & (Q+A)^{-1} \\ Q+A & 0 \end{pmatrix} \ . \tag{361}$$

Next for any odd integer n we can evaluate the chain cs^0_n on the nth component of the Chern character of g: we write

$$\mathrm{cs}^0_n(g) := \mathrm{cs}^0_n(\mathrm{ch}(g)) = (-)^k k!\,\mathrm{cs}^0_n(g^{-1}dg(dg^{-1}dg)^k) \tag{362}$$

with $n = 2k+1$.

Lemma 4.15 *For any odd integer $n = 2k+1$ one has*

$$\mathrm{cs}^0_n(g) = \frac{1}{2}\frac{(k!)^2}{n!}\tau_s(FV^n) \ . \tag{363}$$

Proof: Immediate using $[F,g^{-1}] = -g^{-1}[F,g]g^{-1}$. ∎

Consider now the infinite sum

$$\sum_{k=0}^{\infty} \mathrm{cs}_{2k+1}^0(g) = \sum_{k=0}^{\infty} \frac{1}{2}\, \frac{(k!)^2}{(2k+1)!}\, \tau_s(FV^{2k+1}) \ . \tag{364}$$

This sum is divergent in general, so we need to view it as a formal series with respect to the powers of some parameter. Remark that V itself is a formal power series with respect to the connection A. Indeed as a formal series one has

$$(Q+A)^{-1} = Q^{-1} - Q^{-1}AQ^{-1} + Q^{-1}AQ^{-1}AQ^{-1} - \dots \tag{365}$$

so that

$$\begin{aligned} V &= \begin{pmatrix} 0 & (Q+A)^{-1} - Q^{-1} \\ A & 0 \end{pmatrix} \\ &= \begin{pmatrix} 0 & -Q^{-1}AQ^{-1} \\ A & 0 \end{pmatrix} + \sum_{n=2}^{\infty} (-)^n \begin{pmatrix} 0 & Q^{-1}(AQ^{-1})^n \\ 0 & 0 \end{pmatrix} \ . \end{aligned} \tag{366}$$

The important remark is that at the lowest order V depends linearly on A (there is no "constant" term). Hence, it is not difficult to see that any formal power series in V gives also unambiguously a formal power series in A. From now on we will regard $\sum_{k=0}^{\infty} \mathrm{cs}_{2k+1}^0(g)$ as a formal power series in A.

Proposition 4.16 *For any invertible element $g \in \mathcal{B}$ and the corresponding connection $A = g_-^{-1}Qg_+ - Q$, one has the equality of formal power series*

$$\sum_{k=0}^{\infty} \mathrm{cs}_{2k+1}^0(g) = W_\tau(A) + R(A) \tag{367}$$

where $R(A)$ is a finite sum of counterterms depending polynomially on A, and $W_\tau(A)$ is the renormalized free energy

$$W_\tau(A) = \tau \ln(1 + Q^{-1}A) \ . \tag{368}$$

Hence the series $\sum_k \mathrm{cs}_{2k+1}^0(g)$ provides another renormalization of the free energy.

Proof: This is left as an exercise to the reader. ■

Consequently, the anomaly $\Delta(g) = sW_\tau(A)$ may be computed, modulo the de Rham coboundary of counterterms, by the s-variation of the formal series $\sum_k \mathrm{cs}_{2k+1}^0(g)$. This variation is a finite sum of one-forms depending polynomially on A, and by integration over the circle one deduces

$$\oint \Delta(g) = \oint s \sum_k \mathrm{cs}_{2k+1}^0(g) \ . \tag{369}$$

But if we evaluate the descent equations (350) on the Chern character $\mathrm{ch}(g)$, one gets

$$\begin{array}{rcl}
\varphi_1^1(\mathrm{ch}(g)) & = & s\mathrm{cs}_1^0(g) + \mathrm{cs}_2^1(B\mathrm{ch}(g)) \\
\varphi_3^1(\mathrm{ch}(g)) & = & s\mathrm{cs}_3^0(g) + \mathrm{cs}_4^1(B\mathrm{ch}(g)) + \mathrm{cs}_2^1(b\mathrm{ch}(g)) \\
 & \vdots & \\
\varphi_m^1(\mathrm{ch}(g)) & = & s\mathrm{cs}_m^0(g) + \mathrm{cs}_{m+1}^1(B\mathrm{ch}(g)) + \mathrm{cs}_{m-1}^1(b\mathrm{ch}(g)) \\
0 & = & s\mathrm{cs}_{m+2}^0(g) + \mathrm{cs}_{m+3}^1(B\mathrm{ch}(g)) + \mathrm{cs}_{m+1}^1(b\mathrm{ch}(g)) \\
 & \vdots & \\
0 & = & s\mathrm{cs}_n^0(g) + \mathrm{cs}_{n+1}^1(B\mathrm{ch}(g)) + \mathrm{cs}_{n-1}^1(b\mathrm{ch}(g)) \\
 & \vdots &
\end{array} \tag{370}$$

Hence adding all these equations yields the following equality of formal series

$$\varphi^1(\mathrm{ch}(g)) = s\sum_{k=0}^{\infty} \mathrm{cs}_{2k+1}^0(g) + \sum_{k=1} \mathrm{cs}_{2k}^1((b+B)\mathrm{ch}(g)) \ , \tag{371}$$

so that the cycle condition $(b+B)\mathrm{ch}(g) = 0$ implies

Corollary 4.17 *For any invertible element $g \in \mathcal{B}$ the integral of the anomaly computes the pairing*

$$\oint \Delta(g) = \langle \oint \varphi^1, \mathrm{ch}(g) \rangle \tag{372}$$

in the cyclic cohomology of $\mathcal{B}$. ∎

Theorem 4.12 follows from proposition 4.14 and corollary 4.17.

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