Knot Invariants and Configuration Space Integrals

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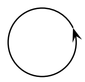
Summary. After a short presentation of the theory of Vassiliev knot invariants, we shall introduce a universal finite type invariant for knots in the ambient space. This invariant is often called the perturbative series expansion of the Chern-Simons theory of links in the euclidean space. It will be constructed as a series of integrals over configuration spaces.

1 First Steps in the Folklore of Knots, Links and Knot Invariants

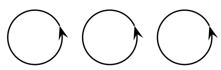
1.1 Knots and Links

Intuitively, a *knot* is a circle embedded in the ambient space up to elastic deformation; a *link* is a finite family of disjoint knots.

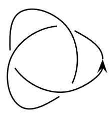
Examples 1.1. Here are some pictures of simple knots and links. More examples can be found in [32].



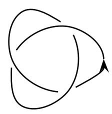
The *trivial knot*



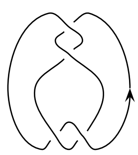
The *trivial 3-component link*



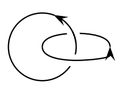
The *right-handed trefoil knot*



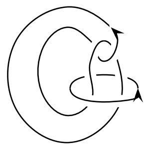
The *left-handed trefoil knot*



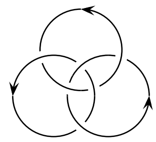
The *figure-eight knot*



The *Hopf link*



The *Whitehead link*



The *Borromean rings*

Let $\coprod^k S^1$ denote the disjoint union of k **oriented** circles. We will represent a *knot* (resp. a k-component link) by a C^∞ embedding[1] of the circle S^1 (resp. of $\coprod^k S^1$) into the ambient space $\mathbf{R}^3$.

Definition 1.2. An *isotopy* of $\mathbf{R}^3$ is a C^∞ map

$$h : \mathbf{R}^3 \times I \longrightarrow \mathbf{R}^3$$

such that $h_t = h(.,t)$ is a diffeomorphism for all $t \in [0,1]$. Two embeddings f and g as above are said to be *isotopic* if there is an isotopy h of $\mathbf{R}^3$ such that $h_0 =$ Identity and $g = h_1 \circ f$. Link isotopy is an equivalence relation. (Checking transitivity requires smoothing... See [17].)

Definition 1.3. A *knot* is an isotopy class of embeddings of S^1 into $\mathbf{R}^3$. A *k-component link* is an isotopy class of embeddings of $\coprod^k S^1$ into $\mathbf{R}^3$.

Definition 1.4. Let $\pi : \mathbf{R}^3 \longrightarrow \mathbf{R}^2$ be the projection defined by $\pi(x,y,z) = (x,y)$. Let $f : \coprod^k S^1 \hookrightarrow \mathbf{R}^3$ be a representative of a link L. The *multiple* (resp. *double*) points of $\pi \circ f$ are the points of $\mathbf{R}^2$ that have several (resp. two) inverse images under $\pi \circ f$. A double point is said to be *transverse* if the two tangent vectors to $\pi \circ f$ at this point generate $\mathbf{R}^2$. $\pi \circ f : \coprod^k S^1 \hookrightarrow \mathbf{R}^2$ is a *regular projection* of L if and only if $\pi \circ f$ is an immersion whose only multiple points are transverse double points.

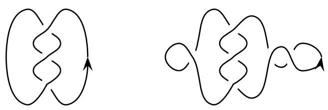
Proposition 1.5. *Any link L has a representative f whose projection $\pi \circ f$ is regular.*

A sketch of proof of this proposition is given in Subsection 7.1.

Definition 1.6. A *diagram* of a link is a regular projection equipped with the additional under/over information: at a double point, the strand that

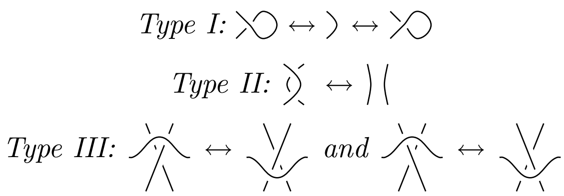
[1] The reader is referred to [17] for the basic concepts of differential topology as well as for more sophisticated ones.

crosses under is broken near the crossing. Note that a link is well-determined by one of its diagrams. The converse is not true as the following diagrams of the right-handed trefoil knot show.



Nevertheless, we have the following theorem whose proof is outlined in Sect. 7.1.

Theorem 1.7 (Reidemeister theorem (1926) [31]). *Up to orientation-preserving diffeomorphism of the plane, two diagrams of a link can be related by a finite sequence of* Reidemeister moves *that are local changes of the following type:*



As it will often be the case during this course, these pictures represent local changes. They should be understood as parts of bigger diagrams that are unchanged outside their pictured parts.

Exercise 1.8. (**) Prove that there are at most 4 knots that can be represented with at most 4 crossings (namely, the trivial knot, the two trefoil knots and the figure-eight knot).

Exercise 1.9. (**) Prove that the set of knots is countable.

A *crossing change* is a local modification of the type

Proposition 1.10. *Any link can be unknotted by a finite number of crossing changes.*

PROOF: At a philosophical level, it comes from the fact that $\mathbf{R}^3$ is simply connected, and that a homotopy $h : S^1 \times [0,1] \longrightarrow \mathbf{R}^3$ that transforms a link into a trivial one can be replaced by a homotopy that is an isotopy except at a finite number of times where it is a crossing change. (Consider $h \times 1_{[0,1]} : (x,t) \mapsto (h(x,t),t) \in \mathbf{R}^3 \times [0,1]$. The homotopy h can be perturbed so that $h \times 1_{[0,1]}$ is an immersion with a finite number of multiple points that are transverse double points –[17, Exercise 1, p.82]–...)

However, an elementary constructive proof of this proposition from a link diagram is given in Subsection 7.1. $\diamond$

The general open problem in knot theory is to find a satisfactory classification of knots, that is an intelligent way of producing a complete and repetition-free list of knots. A less ambitious task is to be able to decide from two knot presentations whether these presentations represent the same knot. This will sometimes be possible with the help of knot invariants.

1.2 Link Invariants

Definition 1.11. A link *invariant* is a map from the set of links to another set.

Such a map can be defined as a function of diagrams that is invariant under the Reidemeister moves.

Definition 1.12. A *positive crossing* in a diagram is a crossing that looks like  (up to rotation of the plane). (The "shortest" arc that goes from the arrow of the top strand to the arrow of the bottom strand turns counterclockwise.) A *negative crossing* in a diagram is a crossing that looks like  .

Definition 1.13. The *linking number* for two-component links is half the number of the positive crossings that involve the two components minus half the number of negative crossings that involve the two components.

Exercise 1.14. Use Reidemeister's theorem to prove that the linking number is a link invariant. Use the linking number to distinguish the Hopf link from the Whitehead link.

1.3 An Easy-to-Compute Non-Trivial Link Invariant: The Jones Polynomial

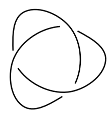
In this section, we will show an example of an easy-to-compute non-trivial link invariant: The Jones polynomial. We will follow the Kauffman approach that is different from the original approach of Jones who discovered this polynomial in 1984.

Let D be an unoriented link diagram. A crossing $\times$ of D can be removed in two different ways:
the left-handed one where $\times$ becomes $)\,($,
(someone walking on the upper strand towards the crossing turns left just before reaching the crossing)
and the right-handed one where $\times$ becomes $\asymp$.
Let $C(D)$ denote the set of crossings of D and let f be a map from $C(D)$ to $\{L, R\}$, then D_f will denote the diagram obtained by removing every crossing x, in the left-handed way if $f(x) = L$ and in the right-handed way if $f(x) = R$. D_f is nothing but a collection of $n(D_f)$ circles embedded in the plane.

We define the *Kauffman bracket* $< D > \in \mathbf{Z}[A, A^{-1}]$ of D as

$$< D >= \sum_{f:C(D)\longrightarrow\{L,R\}} A^{(\sharp f^{-1}(L)-\sharp f^{-1}(R))}\delta^{(n(D_f)-1)}$$

with $\delta = -A^2 - A^{-2}$.

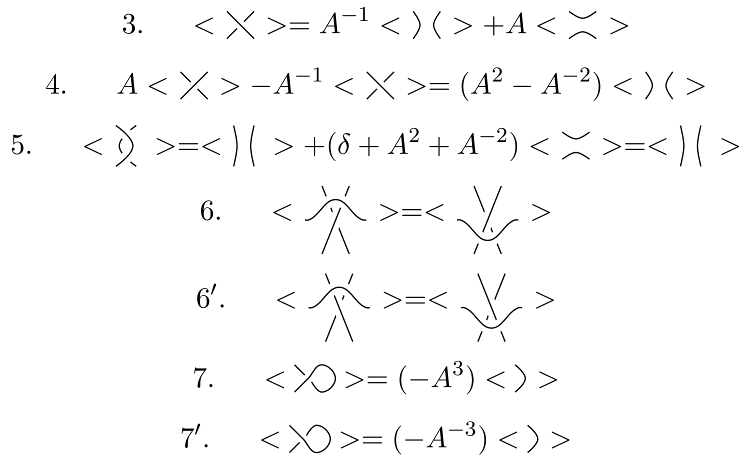
Exercise 1.15. Compute $<$  $>= -A^5 - A^{-3} + A^{-7}$.

The Kauffman bracket satisfies the following properties:

$$1. \qquad < \text{n disjoint circles} >= \delta^{n-1}$$

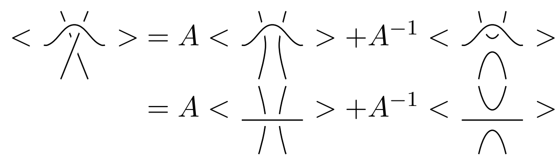
and we have the following equalities that relate brackets of diagrams that are identical anywhere except where they are drawn.

$$2. \qquad < \times >= A <)\,(> + A^{-1} < \asymp >$$

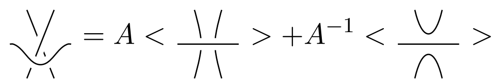


8. The Kauffman bracket of the mirror image of a diagram D is obtained from $< D >$ by exchanging A and A^{-1}.

PROOF: The first five properties of the Kauffman bracket and the eighth one are straightforward. Property 5 shows that δ has been chosen to get invariance under the second Reidemeister move. Let us prove Equality 6.



The second equality comes from the above invariance under the second Reidemeister move. Rotating the picture by π yields:



and concludes the proof of Equality 6. Equality $6'$ is deduced from Equality 6 by mirror image (up to rotation). Equality 7 is easy and Equality $7'$ is its mirror image, too. $\diamond$

Definition 1.16. The *writhe* $w(D)$ of an oriented link diagram D is the number of positive crossings minus the number of negative ones.

Theorem 1.17. *The Jones polynomial $V(L)$ of an oriented link L is the Laurent polynomial of $\mathbf{Z}[t^{1/2}, t^{-1/2}]$ defined from an oriented diagram D of L by*

$$V(L) = (-A)^{-3w(D)} < D >_{A^{-2}=t^{1/2}}$$

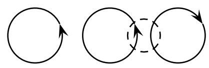
V is an invariant of oriented links. It is the unique invariant of oriented links that satisfies:

1. *V(trivial knot)=1,*
2. *and the* skein relation:

$$t^{-1}V(\quad) - tV(\quad) = (t^{1/2} - t^{-1/2})V(\quad)$$

PROOF: It is easy to check that $(-A)^{-3w(D)} < D > \in \mathbf{Z}[A^2, A^{-2}]$. It is also easy to check that this expression is invariant under the Reidemeister moves since the only Reidemeister moves that change the writhe are the moves I, and the variation of $(-A)^{-3w(D)}$ under these moves makes up for the variation of $< D >$. The skein relation can be easily deduced from the fourth property of the Kauffman bracket, and it is immediate that $V(\text{trivial knot}) = 1$.

Let us prove that these two properties *uniquely* determine V. Applying the skein relation in the case when is the surrounded part of a standard diagram of a trivial n-component link like:



and and are therefore (parts of) diagrams of trivial $(n-1)$-component links shows that

$$(t^{-1} - t)V(\text{trivial (n-1)-comp. link}) = (t^{1/2} - t^{-1/2})V(\text{trivial n-comp. link})$$

and thus determines

$$V(\text{trivial n-component link}) = (-t^{1/2} - t^{-1/2})^{n-1}.$$

By induction on the number of crossings, and, for a fixed number of crossings, by induction on the number of crossings to be changed in the projection to make the projected link trivial (see Proposition 1.10), the link invariant V is determined by its value at the trivial knot and the skein relation.

◇

Exercise 1.18. Compute $V(\text{right-handed trefoil}) = -t^4 + t^3 + t$ and show that the right-handed trefoil is not isotopic to the left-handed trefoil.

Knot invariants are especially interesting when they can be used to derive general properties of knots. The main known application of the Jones polynomial is given and proved in Subsection 7.2. The Chern-Simons series defined in Sect. 4 will be used to prove the fundamental theorem of Vassiliev invariants 2.26.

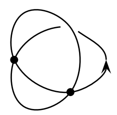
2 Finite Type Knot Invariants

Definition 2.1. A *singular knot* with n double points is represented by an immersion from S^1 to $\mathbf{R}^3$ with n transverse double points that is an embedding when restricted to the complement of the preimages of these double points.

Two such immersions f and g are said to be *isotopic* if there is an isotopy h of $\mathbf{R}^3$ such that $h_0 = \text{Identity}$ and $g = h_1 \circ f$.

A *singular knot* with n double points is an isotopy class of such immersions with n double points.

Example 2.2.



A *singular knot with two double points*

Definition 2.3. Let  be a double point of a singular knot. This double point can disappear in a *positive way* by changing  into  , or in a *negative way* by changing  into  . Note that the sign of such a *desingularisation,* that can be seen in a diagram as above, is defined from the orientation of the ambient space. Choose one strand involved in the double point. Call this strand the first one. Consider the tangent plane to the double point, that is the vector plane equipped with the basis (tangent vector $\boldsymbol{v_1}$ to the first strand, tangent vector $\boldsymbol{v_2}$ to the second one). This basis orients the plane, and allows us to define a positive normal vector $\boldsymbol{n}$ to the plane (that is a vector $\boldsymbol{n}$ orthogonal to $\boldsymbol{v_1}$ and $\boldsymbol{v_2}$ such that the triple $(\boldsymbol{v_1},\boldsymbol{v_2},\boldsymbol{n})$ is an oriented basis of $\mathbf{R}^3$). The n the *positive desingularisation* is obtained by pushing the first strand in the direction of $\boldsymbol{n}$. Note that this definition is independent of the choice of the *first* strand.

Notation 2.4. Let K be a singular knot with n double points numbered by 1,2 ..., n. Let f be a map from $\{1, 2 \dots, n\}$ to $\{+, -\}$. Then K_f denotes the genuine knot obtained by removing every crossing i by the transformation: $_i$ becomes  if $f(i) = +$, and $_i$ becomes  if $f(i) = -$.
Let $\mathcal{K}$ denote the set of knots. Let $\mathbf{Z}[\mathcal{K}]$ denote the free $\mathbf{Z}$-module with basis $\mathcal{K}$. Then $[K]$ denotes the following element of $\mathbf{Z}[\mathcal{K}]$:

$$[K] = \sum_{f:\{1,2\dots,n\}\longrightarrow\{+,-\}} (-1)^{\sharp f^{-1}(-)} K_f$$

where the symbol $\sharp$ is used to denote the cardinality of a set.

Proposition 2.5. *We have the following equality in V:*

[] = [] − []

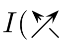
that relates the brackets of three singular knots that are identical outside a ball where they look like in the above pictures.

PROOF: Exercise. ◇

Definition 2.6. Let G be an abelian group. Then any G-valued knot invariant I is extended to $\mathbf{Z}[\mathcal{K}]$, linearly. It is then extended to singular knots by the formula

$$I(K) \stackrel{\text{def}}{=} I([K])$$

Let n be an integer. A G-valued knot invariant I is said to be of *degree less or equal than* n, if it vanishes at singular knots with $(n + 1)$ double points. Of course, such an invariant is of degree n if it is of degree less or equal than n without being of degree less or equal than $(n - 1)$. A G-valued knot invariant I is said to be of *finite type* or of *finite degree* if it is of degree n for some n. Note that we could have defined the extension of a G-valued knot invariant to singular knots by induction on the number of double points using the induction formula:

$I($$) = I($$) - I($$).$

Performing the following change of variables:

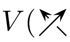
$$t^{1/2} = -\exp(-\frac{\lambda}{2})$$

transforms the Jones polynomial into a series

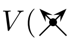
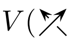
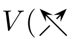
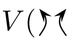
$$V(L) = \sum_{n=0}^{\infty} v_n(L)\lambda^n$$

Proposition 2.7. *The coefficient $v_n(L)$ of the renormalized Jones polynomial is a rational invariant of degree less or equal than n.*

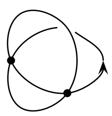
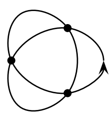
PROOF: Under the above change of variables, the skein relation satisfied by the Jones polynomial becomes

$\exp(\lambda)V($$) - \exp(-\lambda)V($$) = (\exp(\frac{\lambda}{2}) - \exp(-\frac{\lambda}{2}))V($$)$

that is equivalent to

$V($$) = (1-\mathrm{e}^{\lambda})V($$) + (\mathrm{e}^{-\lambda}-1)V($$) + (\mathrm{e}^{\lambda/2}-\mathrm{e}^{-\lambda/2})V($$)$

This equality allows us to see that the series V of a singular knot with n double points has valuation (i.e. degree of the term of minimal degree) at least n by induction on n. $\diamond$

Exercise 2.8. 1. Compute $v_2($$)$ and $v_3($$)$.

2. Show that v_2 and v_3 are exactly of degree 2 and 3, respectively.

Definition 2.9. Let V be a vector space. A *filtration* of V is a decreasing sequence of vector spaces

$$V = V_0 \supseteq V_1 \supseteq \cdots \supseteq V_n \supseteq V_{n+1} \supseteq \dots$$

that begins with $V_0 = V$. The *graded space* associated to such a filtration is the vector space

$$\mathcal{G} = \oplus_{k=1}^{\infty}\mathcal{G}_k((V_n)_{n\in\mathbf{N}}) \text{ where } \mathcal{G}_k((V_n)_{n\in\mathbf{N}}) = \frac{V_k}{V_{k+1}}$$

Let $\mathcal{V}$ denote the $\mathbf{R}$-vector space freely generated by the knots. Let $\mathcal{V}_n$ denote the subspace of $\mathcal{V}$ generated by the brackets of the singular knots with n double points. The *Vassiliev filtration of* $\mathcal{V}$ is the sequence of the $\mathcal{V}_n$. Let $\mathcal{I}_n$ denote the set of real-valued invariants of degree less or equal than n. The set $\mathcal{I}_n$ is nothing but the dual vector space of $\frac{\mathcal{V}}{\mathcal{V}_{n+1}}$ that is the space of linear forms on $\frac{\mathcal{V}}{\mathcal{V}_{n+1}}$.

$$\mathcal{I}_n = \mathrm{Hom}(\frac{\mathcal{V}}{\mathcal{V}_{n+1}}; \mathbf{R}) = \left(\frac{\mathcal{V}}{\mathcal{V}_{n+1}}\right)^*$$

Proposition 2.10.

$$\frac{\mathcal{I}_n}{\mathcal{I}_{n-1}} = \left(\frac{\mathcal{V}_n}{\mathcal{V}_{n+1}}\right)^*$$

PROOF: Exercise.

Exercise 2.11. (*) Let $\lambda \in \mathcal{I}_n$, let $\mu \in \mathcal{I}_m$. Define the invariant $\lambda\mu$ at genuine knots K by $\lambda\mu(K) = \lambda(K)\mu(K)$. Prove that $\lambda\mu \in \mathcal{I}_{n+m}$.

Exercise 2.12. Prove that

$$\dim\left(\frac{\mathcal{V}_2}{\mathcal{V}_3}\right) \geq 1 \quad \text{and} \quad \dim\left(\frac{\mathcal{V}_3}{\mathcal{V}_4}\right) \geq 1.$$

We already know that there exist finite type invariants. Now, we ask the natural question.

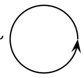
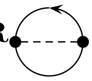
How many finite type invariants does there exist? In other words, what is the dimension of $\frac{\mathcal{V}_n}{\mathcal{V}_{n+1}}$?

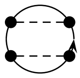
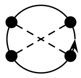
In the next subsection, we are going to bound this dimension from above. This dimension will then be theoretically given by the fundamental theorem of Vassiliev invariants 2.26 that will be roughly proved at the end of Sect 5, as the dimension of a vector space presented by a finite number of generators and relators.

2.1 Chord Diagrams

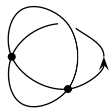
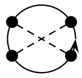
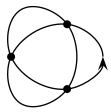
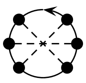
Definition 2.13. An n-chord diagram on a one-manifold M is an isotopy class of embeddings of a finite set U with $2n$ elements, equipped with a partition into n pairs, into the interior of M, up to a permutation of U that preserves the partition.

For us, so far, $M = S^1$. In particular, the datum of the above isotopy class is equivalent to the datum of a cyclic order on U where a *cyclic order* on a finite set U is a cyclic permutation σ of this set, that provides every element u of U with a unique successor (namely $\sigma(u)$). An n-chord diagram on M will be represented by an embedding of M equipped with $2n$ points (the image of the embedding of U) where the points of a pair are related by a dotted chord. The $\mathbf{R}$-vector space freely generated by the n-chord diagrams on M will be denoted by $\mathcal{D}_n(M)$.

Examples 2.14. $\mathcal{D}_0(S^1) = \mathbf{R}$ , $\mathcal{D}_1(S^1) = \mathbf{R}$ ,

$\mathcal{D}_2(S^1) = \mathbf{R}$  $\oplus\, \mathbf{R}$ .

Notation 2.15. Let K be a singular knot with n double points. The *n-chord diagram* $D(K)$ associated to K is the set of the $2n$ inverse images of the double points of K (cyclically ordered by the orientation of S^1), equipped with the partition into pairs where every pair contains the inverse images of one double point.

Examples 2.16. $D($  $) =$ , $D($  $) =$ 

Lemma 2.17. *Every chord diagram on S^1 is the diagram of a singular knot. Furthermore, two singular knots which have the same diagram are related by a finite number of crossing changes (where a* crossing change *is again a modification which transforms*  *into*  *).*

Sketch of proof: Consider a chord diagram D. First embed the neighborhoods in S^1 of the double points (that correspond to the chords) into disjoint balls of $\mathbf{R}^3$. Let C be the complement in $\mathbf{R}^3$ of these disjoint balls. We have enough room in C to join the neighborhoods of the double points by arcs embedded in C, in order to get a singular knot K_0 with associated diagram D.

Now, consider another singular knot K with the same diagram. There exists an isotopy that carries the neighborhoods of its double points on the corresponding neighborhoods for K_0. The arcs that join the double points in the simply connected C are homotopic to the former ones. Therefore, they can be carried to the former ones by a sequence of isotopies and crossing changes. A diagrammatic proof of this lemma is given in Subsection 7.3. $\diamond$

Notation 2.18. Let ϕ_n denote the linear map from $\mathcal{D}_n(S^1)$ to $\frac{\mathcal{V}_n}{\mathcal{V}_{n+1}}$ that maps an n-chord diagram D to the projection in $\frac{\mathcal{V}_n}{\mathcal{V}_{n+1}}$ of a singular knot K such that $D(K) = D$. The above lemma ensures that ϕ_n is well-defined. Indeed, if $D(K) = D(K')$, then, by the above lemma, K and K' are related by a finite number of crossing changes. Therefore, $[K] - [K']$ is an element of $\mathcal{V}_{n+1}$.

Furthermore, it is obvious that ϕ_n is onto. As a consequence, the dimension of $\frac{\mathcal{V}_n}{\mathcal{V}_{n+1}}$ is bounded from above by the number of n-chord diagrams.

Exercise 2.19. Prove that this number is bounded from above by $\frac{(2n)!}{2^n n!}$, and by $1 + (n-1)\frac{(2(n-1))!}{2^{(n-1)}(n-1)!}$ if $n \geq 1$. Improve these upper bounds.

The following lemmas will allow us to improve the upper bound on the dimension of $\frac{\mathcal{V}_n}{\mathcal{V}_{n+1}}$. In a chord diagram, an *isolated chord* is a chord that relates two consecutive points.

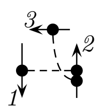
Lemma 2.20. *Let D be a diagram on S^1 that contains an isolated chord. Then $\phi_n(D) = 0$.*

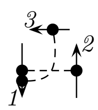
Proof: In $\frac{\mathcal{V}_n}{\mathcal{V}_{n+1}}$, we can write:

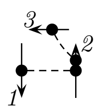
$\phi_n($  $) = [$  $] = [$  $] - [$  $]$

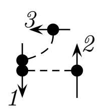
$\diamond$

Lemma 2.21. *Let D^1, D^2, D^3, D^4 be four n-chord diagrams that are identical outside three portions of circles where they look like:*

$D^1 =$ 

$, D^2 =$ 

$, D^3 =$ 

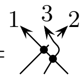
and $D^4 =$ 

.

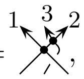
then

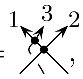
$$\phi_n(-D^1 + D^2 + D^3 - D^4) = 0.$$

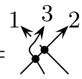
PROOF: We may represent D^1 by a singular knot K^1 with n double points that intersects a ball like

$K^1 =$ 

Let K^2, K^3, K^4 be the singular knots with n double points that coincide with K^1 outside this ball, and that intersect this ball like in the picture:

$K^2 =$ 

, $K^3 =$ 

, $K^4 =$ 

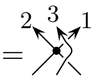
Then $D(K^2) = D^2$, $D(K^3) = D^3$ and $D(K^4) = D^4$. Therefore, $\phi_n(-D^1 + D^2 + D^3 - D^4) = -[K^1] + [K^2] + [K^3] - [K^4]$.

Thus, it is enough to prove that in $\mathcal{V}$ we have

$$-[K^1] + [K^2] + [K^3] - [K^4] = 0$$

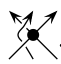
Let us prove this.

Let K_0 be the singular knot with $(n-1)$ double points that intersects our ball like

$K_0 =$ 

and that coincides with K^1 outside this ball.

The strands 1 and 2 involved in the pictured double point are in the horizontal plane and they orient it, the strand 3 is vertical and intersects the horizontal plane in a positive way between the tails of 1 and 2. Now, make 3 turn around the double point counterclockwise, so that it becomes successively the knots with $n-1$ double points:

$K_1 =$  , $K_2 =$  and $K_3 =$  .

On its way, it goes successively through our four knots K^1, K^2, K^3 and K^4 with n double points that appear inside matching parentheses in the following obvious identity in $\mathcal{V}_{n-1}$

$$([K_1] - [K_0]) + ([K_2] - [K_1]) + ([K_3] - [K_2]) + ([K_0] - [K_3]) = 0.$$

Now, $[K^i] = \pm([K_i] - [K_{i-1}])$ where the sign $\pm$ is plus when the vertical strand goes through an arrow from K_{i-1} to K_i and minus when it goes through a tail. Therefore the above equality can be written as

$$-[K^1] + [K^2] + [K^3] - [K^4] = 0$$

and finishes the proof of the lemma. ◇

Notation 2.22. The relation

$$D^2 + D^3 - D^1 - D^4 = 0$$

that involves diagrams D^1, D^2, D^3, D^4 which satisfy the hypotheses of Lemma 2.21 is called the *four-term relation* and is denoted by $(4T)$. Let $\mathcal{A}_n(M)$ denote the quotient of $\mathcal{D}_n(M)$ by the relation $(4T)$, that is the quotient of $\mathcal{D}_n(M)$ by the subspace of $\mathcal{D}_n(M)$ generated by left-hand sides of $(4T)$, that are expressions of the form $(-D^1 + D^2 + D^3 - D^4)$, for a (D^1, D^2, D^3, D^4) satisfying the conditions of Lemma 2.21.

$$\mathcal{A}_n = \frac{\mathcal{D}_n}{4T}$$

Let $\overline{\mathcal{A}_n}(M)$ denote the quotient of $\mathcal{A}_n(M)$ by the subspace of $\mathcal{A}_n(M)$ generated by the projections of the diagrams with an isolated chord. The relation

$$D = 0 \text{ for a diagram } D \text{ with an isolated chord.}$$

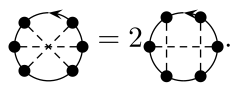
is sometimes called the one-term relation, and it is denoted by $(1T)$.

Lemmas 2.20 and 2.21 show the following proposition.

Proposition 2.23. *The map ϕ_n factors through $\overline{\mathcal{A}_n}(S^1)$ to define the surjective map:*

$$\overline{\phi_n} : \overline{\mathcal{A}_n}(S^1) \longrightarrow \frac{\mathcal{V}_n}{\mathcal{V}_{n+1}}.$$

Exercise 2.24. 1. Show that, in $\overline{\mathcal{A}_3}(S^1)$, we have



2. Show that

$$\dim\left(\frac{\mathcal{V}_0}{\mathcal{V}_1}\right) = 1, \quad \dim\left(\frac{\mathcal{V}_1}{\mathcal{V}_2}\right) = 0, \quad \dim\left(\frac{\mathcal{V}_2}{\mathcal{V}_3}\right) \leq 1 \quad \text{and} \quad \dim\left(\frac{\mathcal{V}_3}{\mathcal{V}_4}\right) \leq 1$$

3. Show that

$$\dim(\overline{\mathcal{A}_2})(S^1) = 1$$

Definition 2.25. The topological vector space $\prod_{k=0}^{\infty} \overline{\mathcal{A}_k}(M)$ will be denoted by $\overline{\mathcal{A}}(M)$. Similarly, the topological vector space $\prod_{k=0}^{\infty} \mathcal{A}_k(M)$ will be denoted by $\mathcal{A}(M)$. The *degree n part* of an element a of $\mathcal{A}(M)$ or $\overline{\mathcal{A}}(M)$ is the natural projection of a in $\mathcal{A}_n(M)$ or $\overline{\mathcal{A}}_n(M)$. It is denoted by a_n.

In the next sections, we shall prove that $\overline{\phi_n}$ is an isomorphism by constructing its inverse given by the Chern-Simons series. More precisely, we shall prove the following theorem.

Theorem 2.26 (Kontsevich, 92). *There exists a linear map*

$$\overline{Z} : \mathcal{V} \longrightarrow \overline{\mathcal{A}}(S^1)$$

such that, for any integer n and for any singular knot K with n double points, if $k < n$, then

$$\overline{Z}_k([K]) = 0,$$

and

$$\overline{Z}_n([K]) = D(K).$$

In other words, the restriction to $\frac{\mathcal{V}_n}{\mathcal{V}_{n+1}}$ of $\overline{Z}_n$ will be the inverse of $\overline{\phi_n}$.

Corollary 2.27. *Any degree n $\mathbf{R}$-valued invariant I is of the form*

$$I = \psi \circ \overline{Z}$$

where ψ is a linear form on $\overline{\mathcal{A}}(S^1)$ that vanishes on $\prod_{k=n+1}^{\infty} \overline{\mathcal{A}_k}(S^1)$.

Proof: The Kontsevich theorem allows us to see, by induction on n, that $\overline{Z}$ induces the isomorphism

$$p_{\leq n} \circ \overline{Z} : \frac{\mathcal{V}}{\mathcal{V}_{n+1}} \longrightarrow \prod_{k=0}^{n} \overline{\mathcal{A}_k}(S^1)$$

where $p_{\leq n} : \overline{\mathcal{A}}(S^1) \longrightarrow \prod_{k=0}^{n} \overline{\mathcal{A}_k}(S^1)$ is the natural projection. Thus, any degree n $\mathbf{R}$-valued invariant I is of the form

$$I = I \circ \left(p_{\leq n} \circ \overline{Z}\right)^{-1} \circ p_{\leq n} \circ \overline{Z} = \psi \circ \overline{Z}$$

where $\psi = I \circ \left(p_{\leq n} \circ \overline{Z}\right)^{-1} \circ p_{\leq n}$ is a linear form on $\overline{\mathcal{A}}(S^1)$ that vanishes on $\prod_{k=n+1}^{\infty} \overline{\mathcal{A}_k}(S^1)$. $\diamond$

In Section 4, we shall construct the Chern-Simons series

$$\overline{Z}^0_{CS} : \mathcal{V} \longrightarrow \overline{\mathcal{A}}(S^1)$$

as a series of integrals of configuration spaces. We shall show how to prove that the Chern-Simons series satisfies the properties of $\overline{Z}$ in the Kontsevich theorem above (2.26) in Sect. 5. Thus we will be able to construct all the real-valued finite type knot invariants as above, theoretically. Unfortunately, there are two problems. First, the Chern-Simons series is hard to compute explicitly. Second, we do not know a canonical basis for $\overline{\mathcal{A}_n}(S^1)$. The dimension of this vector space is unknown for a general n, although it can be computed by an algorithm that lists the finitely many diagrams and the finitely many relations and that computes the dimension of the quotient space. We first give another presentation of the vector spaces $\mathcal{A}_n(M)$ and study some of their properties in the next section.

3 Some Properties of Jacobi-Feynman Diagrams

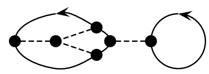
3.1 Jacobi Diagrams

Definition 3.1. Let M be an oriented one-manifold. A *diagram*, or a *Jacobi diagram,* Γ with support M is a finite uni-trivalent graph Γ such that every connected component of Γ has at least one univalent vertex, equipped with:

1. an isotopy class of injections i of the set U of univalent vertices of Γ into the interior of M,
2. an *orientation* of every trivalent vertex, that is a cyclic order on the set of the three half-edges which meet at this vertex,

Such a diagram Γ is again represented by a planar immersion of $\Gamma \cup M$ where the univalent vertices of U are located at their images under i, the one-manifold M is represented by solid lines, whereas the diagram Γ is dashed. The vertices are represented by big points. The local orientation of a vertex is represented by the counterclockwise order of the three half-edges that meet at it.

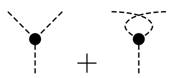
Here is an example of a diagram Γ on the disjoint union $M = S^1 \coprod S^1$ of two circles:

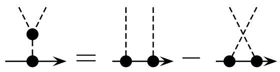


The *degree* of such a diagram is half the number of all the vertices of Γ.

Of course, a chord diagram is a diagram on a one-manifold M without trivalent vertices.

Let $\mathcal{D}_n^t(M)$ denote the real vector space generated by the degree n diagrams on M, and let $\mathcal{A}_n^t(M)$ denote the quotient of $\mathcal{D}_n^t(M)$ by the following relations AS and STU:

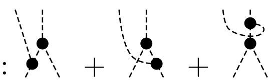
AS :  = 0.

STU :  .

As usual, each of these relations relate diagrams which are identical outside the pictures where they are like in the pictures.

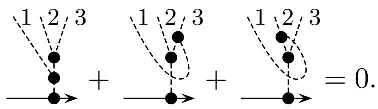
3.2 The Relation IHX in $\mathcal{A}_n^t$

Proposition 3.2. *Let M be a compact one-manifold, then the following relation IHX*

IHX :  = 0

(that relates three diagrams that are represented by three immersions which coincide outside a disk D where they are like in the pictures) is true in $\mathcal{A}_n^t(M)$.

Proof: We want to prove that the relation IHX is true in the quotient of $\mathcal{D}_n^t(M)$ by AS and STU. Consider three diagrams that are represented by three immersions which coincide outside a disk D where they are like in the pictures involved in the relation IHX. Use STU as long as it is possible to remove all trivalent vertices that can be removed without changing the two vertices in D, simultaneously on the three diagrams. This transforms the relation IHX to be shown into a sum of similar relations, where one of the four entries of the disk is directly connected to M. Thus, since the four entries play the same role, we may assume that the relation IHX to be shown is:



Using STU twice and AS transforms the summands of the left-hand side into diagrams that can be represented by three straight lines from the entries 1,2,3 to three fixed points of the horizontal line numbered from left to right. When the entry $i \in \{1, 2, 3\}$ is connected to the point $\sigma(i)$ of the horizontal plain line, where σ is a permutation of of $\{1, 2, 3\}$, the corresponding diagram will be denoted by $(\sigma(1)\sigma(2)\sigma(3))$. Thus, the expansion of the left-hand side of the above equation is

$$\begin{aligned} &((123) - (132) - (231) + (321)) \\ -\ &((213) - (231) - (132) + (312)) \\ -\ &((123) - (213) - (312) + (321)) \end{aligned}$$

that vanishes and the lemma is proved. $\diamond$

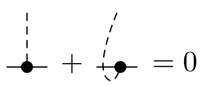
We shall prove the following proposition in Subsection 3.4 below.

Proposition 3.3. *Let M be a compact one-manifold, then the natural map from $\mathcal{D}_n(M)$ to $\mathcal{A}^t_n(M)$ that maps a chord diagram to its class in $\mathcal{A}^t_n(M)$ induces an isomorphism from $\mathcal{A}_n(M)$ to $\mathcal{A}^t_n(M)$.*

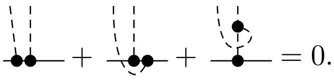
Before, we need a technical lemma that will lead to other fundamental properties of the spaces of diagrams.

3.3 A Useful Trick in Diagram Spaces

We shall first adopt a convention. So far, in a diagram picture, or in a chord diagram picture, the dashed edge of a univalent vertex, has always been attached on the left-hand side of the oriented one-manifold. Now, if k dashed edges are attached on the other side on a diagram picture, then we agree that the corresponding represented element of $\mathcal{A}^t_n(M)$ or $\mathcal{A}_n(M)$ is $(-1)^k$ times the underlying diagram. With this convention, we have the new antisymmetry relation in $\mathcal{A}^t_n(M)$ or in $\mathcal{A}_n(M)$



and the STU relation can be drawn like the IHX relation:



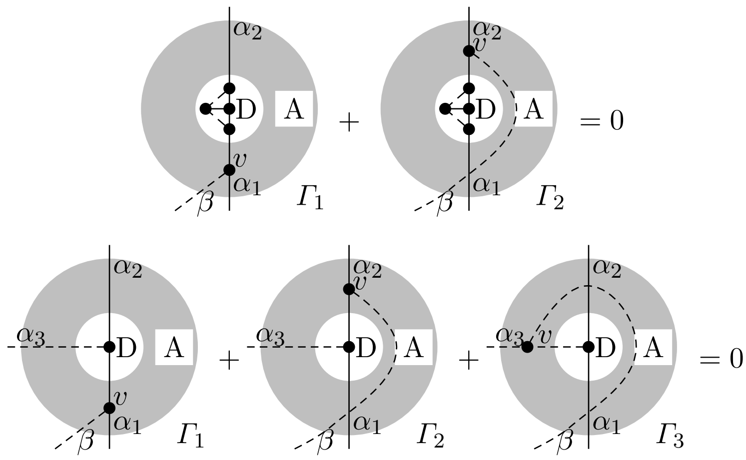
Lemma 3.4. *Let Γ_1 be a diagram (resp. a chord diagram) with support M. Assume that $\Gamma_1 \cup M$ is immersed in the plane so that $\Gamma_1 \cup M$ meets an open annulus A embedded in the plane exactly along $n+1$ embedded arcs α_1, α_2, $\dots$, α_n and β, and one vertex v so that:*

1. *The α_i may be dashed or solid, they run from a boundary component of A to the other one,*
2. *β is a dashed arc which runs from the boundary of A to $v \in \alpha_1$,*
3. *The bounded component D of the complement of A does not contain a boundary point of M.*

Let Γ_i be the diagram obtained from Γ_1 by attaching the endpoint v of β to α_i instead of α_1 on the same side, where the side of an arc is its side when going from the outside boundary component of A to the inside one ∂D. Then, we have in $\mathcal{A}^t(M)$, (resp. in $\mathcal{A}(M)$),

$$\sum_{i=1}^{n} \Gamma_i = 0.$$

Examples 3.5.



Proof: The second example shows that STU is equivalent to this relation when the bounded component D of $\mathbf{R}^2 \setminus A$ intersects Γ_1 in the neighborhood of a univalent vertex on M. Similarly, IHX is easily seen as given by this relation when D intersects Γ_1 in the neighborhood of a trivalent vertex. Also note that AS corresponds to the case when D intersects Γ_1 along a dashed or solid arc. Now for the Bar-Natan [4, Lemma 3.1] proof. See also [35, Lemma 3.3]. Assume without loss that v is always attached on the right-hand-side of the α's. Add to the sum the trivial (by IHX and STU) contribution of the sum of the diagrams obtained from Γ_1 by attaching v to each of the three (dashed or solid) half-edges of each vertex w of $\Gamma_1 \cup M$ in D on the left-hand side when the half-edges are oriented towards w. Now, group the terms of the obtained sum by edges of $\Gamma_1 \cup M$ where v is attached, and observe that the sum is zero edge by edge by AS. $\diamond$

Assume that a one-manifold M is decomposed as a union of two one-manifolds $M = M_1 \cup M_2$ whose interiors in M do not intersect. Define the *product associated to this decomposition*:

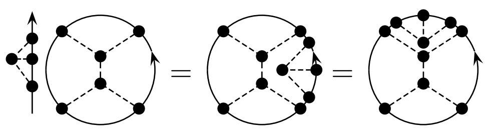
$$\mathcal{A}(M_1) \times \mathcal{A}(M_2) \longrightarrow \mathcal{A}(M)$$

as the continuous bilinear map which maps $([\Gamma_1], [\Gamma_2])$ to $[\Gamma_1 \coprod \Gamma_2]$, if Γ_1 is a diagram with support M_1 and if Γ_2 is a diagram with support M_2, where $\Gamma_1 \coprod \Gamma_2$ denotes their disjoint union. Let $I = [0,1]$ be the compact oriented interval. If $I = M$, and if we identify I to $M_1 = [0, 1/2]$ and to $M_2 = [1/2, 1]$ with respect to the orientation, the above process turns $\mathcal{A}(I)$ into an algebra where the elements with non-zero degree zero part admit an inverse.

With each choice of a connected component C of M, associate an $\mathcal{A}(I)$-*module structure on* $\mathcal{A}(M)$, that is given by the continuous bilinear map:

$$\mathcal{A}(I) \times \mathcal{A}(M) \longrightarrow \mathcal{A}(M)$$

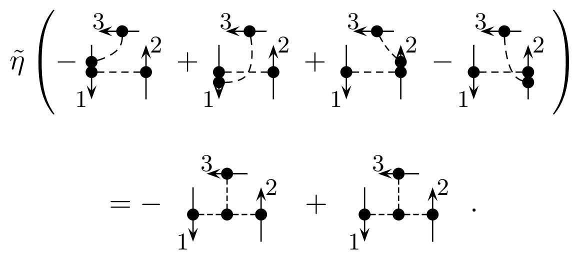
such that: If Γ' is a diagram with support M and if Γ is a diagram with support I, then $([\Gamma], [\Gamma'])$ is mapped to the class of the diagram obtained by inserting Γ along C outside the vertices of Γ, according to the given orientation. For example,



As shown in the first example that illustrates Lemma 3.4, the independence of the choice of the insertion locus is a consequence of Lemma 3.4 where Γ_1 is the disjoint union $\Gamma \coprod \Gamma'$ and intersects D along $\Gamma \cup I$. This also proves that $\mathcal{A}(I)$ is a commutative algebra. Since the morphism from $\mathcal{A}(I)$ to $\mathcal{A}(S^1)$ induced by the identification of the two endpoints of I amounts to quotient out $\mathcal{A}(I)$ by the relation that identifies two diagrams that are obtained from one another by moving the nearest univalent vertex to an endpoint of I near the other endpoint, a similar application of Lemma 3.4 also proves that this morphism is an isomorphism from $\mathcal{A}(I)$ to $\mathcal{A}(S^1)$. (In this application, β comes from the inside boundary of the annulus.) This identification between $\mathcal{A}(I)$ and $\mathcal{A}(S^1)$ will be used several times.

3.4 Proof of Proposition 3.3

Let $\tilde{\eta} : \mathcal{D}_n(M) \rightarrow \mathcal{A}^t_n(M)$ denote the above natural map. Let us show that it factors through $4T$. By STU, we have



Thus, $\tilde{\eta}(4T)$ vanishes and we are done. Note that the induced map η is surjective because STU allows one to express any diagram (whose components contain at least one univalent vertex!) as a combination of chord diagrams.

Let us try to construct an inverse $\overline{\iota}$ to the induced map $\eta : \mathcal{A}_n(M) \rightarrow \mathcal{A}^t_n(M)$. Let $\mathcal{D}_{n,k}(M)$ denote the subspace of $\mathcal{D}^t_n(M)$ generated by the diagrams on M that have at most k trivalent vertices.

We shall define linear maps ι_k from $\mathcal{D}_{n,k}(M)$ to $\mathcal{A}_n(M)$ by induction on k so that

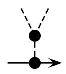
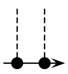
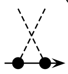
1. ι_0 is induced by the equality $\mathcal{D}_{n,0}(M) = \mathcal{D}_n(M)$,
2. the restriction of ι_k to $\mathcal{D}_{n,k-1}(M)$ is ι_{k-1}, and,
3. ι_k maps all the relations AS and STU that involve only elements of $\mathcal{D}_{n,k}(M)$ to zero.

It is clear that when we have succeeded in such a task, the linear map from $\mathcal{D}_n(M)$ that maps a diagram d with k trivalent vertices to $\iota_k(d)$ will factor through STU and AS, and that the induced map $\overline{\iota}$ will satisfy $\overline{\iota} \circ \eta = \mathrm{Id}$

and therefore provide the wanted inverse since η is surjective. Now, let us succeed!

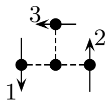
Let $k \geq 1$, assume that ι_{k-1} is defined on $\mathcal{D}_{n,k-1}(M)$ and that ι_{k-1} maps all the relations AS and STU that involve only elements of $\mathcal{D}_{n,k-1}(M)$ to zero. We want to extend ι_{k-1} on $\mathcal{D}_{n,k}(M)$ to a linear map ι_k that maps all the relations AS and STU that involve only elements of $\mathcal{D}_{n,k}(M)$ to zero.

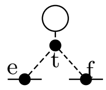
Let d be a diagram with k trivalent vertices, and let e be an edge of d that contains one univalent vertex and one trivalent vertex. Set

$\iota\Big((d,e) =$  $\Big) = \iota_{k-1}\Big($  $-$  $\Big)$

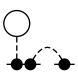
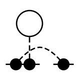
It suffices to prove that $\iota(d,e)$ is independent of our chosen edge e to conclude the proof by defining the linear map ι_k that will obviously satisfy the wanted properties by

$$\iota_k(d) = \iota(d,e)$$

Assume that there are two different edges e and f of d that connect a trivalent vertex to a univalent vertex. We prove that $\iota(d,e) = \iota(d,f)$. If e and f are disjoint, then the fact that ι_{k-1} satifies STU allows us to express both $\iota(d,e)$ and $\iota(d,f)$ as the same combination of 4 diagrams with $(k-2)$ vertices, and we are done. Thus, we assume that e and f are two different edges that share a trivalent vertex t. If there exists another trivalent vertex that is connected to M by an edge g, then $\iota(d,e) = \iota(d,g) = \iota(d,f)$ and we are done. Thus, we furthermore assume that t is the unique trivalent vertex that is connected to M by an edge. So, either t is the unique trivalent vertex, and its component is necessarily like 

and the fact that $\iota(d,e) = \iota(d,f)$ is a consequence of (4T), or the component of t is of the form 

where the circle represents a dashed diagram with only one pictured entry. Thus,

$\iota(d,e) = \iota_{k-1}\Big($  $-$  $\Big)$

Now, this is zero because the expansion of  as a sum of chord diagrams commutes with any vertex in $\mathcal{A}_n(M)$, according to Lemma 3.4. Similarly, $\iota(d,f) = 0$. Thus, $\iota(d,e) = \iota(d,f)$ in this last case and we are done. $\diamond$

Notation 3.6. Because of the canonical isomorphism of Proposition 3.3, $\mathcal{A}_n(M)$ will denote both $\mathcal{A}_n(M)$ and $\mathcal{A}_n^t(M)$ from now on.

4 The "Kontsevich, Bott, Taubes, Bar-Natan, Altschuler, Freidel, D. Thurston" Universal Link Invariant Z

4.1 Introduction to Configuration Space Integrals: The Gauss Integrals.

In 1833, Carl Friedrich Gauss defined the first example of a *configuration space integral* for an oriented two-component link. Let us formulate his definition in a modern language. Consider a smooth (C^∞) embedding

$$L : S_1^1 \sqcup S_2^1 \hookrightarrow \mathbf{R}^3$$

of the disjoint union of two circles $S^1 = \{z \in \mathbf{C} \ \text{ s.t. } \ |z| = 1\}$ into $\mathbf{R}^3$. With an element (z_1, z_2) of $S_1^1 \times S_2^1$ that will be called a *configuration*, we associate the oriented direction

$$\Psi((z_1, z_2)) = \frac{1}{\parallel \overrightarrow{L(z_1)L(z_2)} \parallel} \overrightarrow{L(z_1)L(z_2)} \in S^2$$

of the vector $\overrightarrow{L(z_1)L(z_2)}$. Thus, we have associated a map

$$\Psi : S_1^1 \times S_2^1 \longrightarrow S^2$$

from a compact oriented 2-manifold to another one with our embedding. This map has an integral degree $\deg(\Psi)$ that can be defined in several equivalent ways. For example, it is the *differential degree* $\deg(\Psi, y)$ of any regular value y of Ψ, that is the sum of the ± 1 signs of the Jacobians of Ψ at the points of the preimage of y [28, §5]. Thus, $\deg(\Psi)$ can easily be computed from a regular diagram of our two-component link as the differential degree of a unit vector $\overrightarrow{v}$ pointing to the reader or as the differential degree of $(-\overrightarrow{v})$.

$$\deg(\Psi) = \deg(\Psi, \overrightarrow{v}) = \sharp \times_{2\,1} - \sharp \times_{1\,2} = \deg(\Psi, -\overrightarrow{v}) = \sharp \times_{1\,2} - \sharp \times_{2\,1}.$$

It can also be defined as the following *configuration space integral*

$$\deg(\Psi) = \int_{S^1 \times S^1} \Psi^*(\omega)$$

where ω is the homogeneous volume form on S^2 such that $\int_{S^2}\omega = 1$. Of course, this integral degree is an isotopy invariant of L, and the reader has recognized is nothing but the *linking number* of the two components of L.

We can again follow Gauss and associate the following similar *Gauss integral* $I(K;\theta)$ to a C^∞ embedding $K : S^1 \hookrightarrow \mathbf{R}^3$. (The meaning of θ will be specified later.) Here, we consider the *configuration space* $C(K;\theta) = S^1 \times]0,2\pi[$, and the map

$$\Psi : C(K;\theta) \longrightarrow S^2$$

that maps (z_1,η) to the oriented direction of $\overrightarrow{K(z_1)K(z_1e^{i\eta})}$, and we set

$$I(K;\theta) = \int_{C(K;\theta)} \Psi^*(\omega).$$

Let us compute $I(K;\theta)$ in some cases. First notice that Ψ may be extended to the closed annulus

$$\overline{C}(K;\theta) = S^1 \times [0,2\pi]$$

by the tangent map K' of K along $S^1 \times \{0\}$ and by $(-K')$ along $S^1 \times \{2\pi\}$. Then by definition, $I(K;\theta)$ is the algebraic area (the integral of the differential degree with respect to the measure associated with ω) of the image of the annulus in S^2. Now, assume that K is contained in a horizontal plane except in a neighborhood of crossings where it entirely lies in vertical planes. Such a knot embedding will be called *almost horizontal.* In that case, the image of the annulus boundary has the shape of the following bold line in S^2.

In particular, for each hemisphere, the differential degree of a regular value of Ψ does not depend on the choice of the regular value in the hemisphere. Assume that the orthogonal projection onto the horizontal plane is regular. Then $I(K;\theta)$ is the average of the differential degrees of the North Pole and the South Pole, and it can be computed from the horizontal projection as the writhe of the projection

$$I(K;\theta) = \sharp\nearrow\!\!\!\!\nwarrow - \sharp\nwarrow\!\!\!\!\nearrow.$$

This number can be changed without changing the isotopy class of the knot by local modifications where —— becomes  or . In particular, $I(K;\theta)$ can reach any integral value on a given isotopy class of knots, and since it varies continuously on such a class, it can reach any real value on any given isotopy class of knots. Thus, this Gauss integral is NOT an isotopy invariant.

However, we can follow Guadagnini, Martellini, Mintchev [14] and Bar-Natan [5] and associate configuration space integrals to any embedding L of an oriented one-manifold M and to any uni-trivalent diagram Γ without simple loop like  on M.

4.2 The Chern–Simons Series

Let M be an oriented one-manifold and let

$$L : M \longrightarrow \mathbf{R}^3$$

denote a C^∞ embedding from M to $\mathbf{R}^3$. Let Γ be a Jacobi diagram on M. Let $U = U(\Gamma)$ denote the set of univalent vertices of Γ, and let $T = T(\Gamma)$ denote the set of trivalent vertices of Γ. A *configuration* of Γ is an embedding

$$c : U \cup T \hookrightarrow \mathbf{R}^3$$

whose restriction $c_{|U}$ to U may be written as $L \circ j$ for some injection

$$j : U \hookrightarrow M$$

in the given isotopy class $[i]$ of embeddings of U into the interior of M. Denote the set of these configurations by $C(L;\Gamma)$,

$$C(L;\Gamma) = \left\{c : U \cup T \hookrightarrow \mathbf{R}^3 \ ; \exists j \in [i], c_{|U} = L \circ j\right\}.$$

In $C(L;\Gamma)$, the univalent vertices move along $L(M)$ while the trivalent vertices move in the ambient space, and $C(L;\Gamma)$ is naturally an open submanifold of $M^U \times (\mathbf{R}^3)^T$.

Denote the set of (dashed) edges of Γ by $E = E(\Gamma)$, and fix an orientation for these edges. Define the map $\Psi : C(L;\Gamma) \longrightarrow \left(S^2\right)^E$ whose projection to the S^2 factor indexed by an edge from a vertex v_1 to a vertex v_2 is the direction of $\overrightarrow{c(v_1)c(v_2)}$. This map Ψ is again a map between two orientable manifolds that have the same dimension, namely the number of dashed half-edges of Γ, and we can write the *configuration space integral:*

$$I(L;\Gamma) = \int_{C(L;\Gamma)} \Psi^*(\bigwedge^{E} \omega).$$

Bott and Taubes have proved that this integral is convergent [8]. See also Subsections 5.1, 5.2 below. Thus, this integral is well-defined up to sign. In fact, the orientation of the trivalent vertices of Γ provides $I(L;\Gamma)$ with a well-defined sign. Indeed, since S^2 is equipped with its standard orientation, it is enough to orient $C(L;\Gamma) \subset M^U \times (\mathbf{R}^3)^T$ in order to define this sign. This will be done by providing the set of the natural coordinates of $M^U \times (\mathbf{R}^3)^T$ with some order up to an even permutation. This set is in one-to-one correspondence with the set of (dashed) half-edges of Γ, and the vertex-orientation of the trivalent vertices provides a natural preferred such one-to-one correspondence up to some (even!) cyclic permutations of three half-edges meeting at a trivalent vertex. Fix an order on E, then the set of half-edges becomes ordered by (origin of the first edge, endpoint of the first edge, origin of the second edge, $\dots$, endpoint of the last edge), and this order orients $C(L;\Gamma)$. The property of this sign is that the product $I(L;\Gamma)[\Gamma] \in \mathcal{A}(M)$ depends neither on our various choices nor on the vertex orientation of Γ. Check it as an exercise!

Now, the *perturbative series expansion of the Chern–Simons theory for one-manifold embeddings in* $\mathbf{R}^3$ is the following sum running over all the Jacobi diagrams Γ without vertex orientation[2]:

$$Z_{\mathrm{CS}}(L) = \sum_{\Gamma} \frac{I(L;\Gamma)}{\sharp \mathrm{Aut}\Gamma}[\Gamma] \quad \in \mathcal{A}(M)$$

where $\sharp\mathrm{Aut}\Gamma$ is the number of automorphisms of Γ as a uni-trivalent graph with a given isotopy class of injections of U into M, but without vertex-orientation for the trivalent vertices.

More precisely, let Γ be a diagram on M, let $V(\Gamma)$ be the set of its vertices, let $U(\Gamma)$ be the set of its univalent vertices, and let $E(\Gamma)$ be the set of its edges. Define the set $H(\Gamma)$ of its half-edges as

[2] This sum runs over equivalence classes of Jacobi diagrams, where two diagrams are equivalent if and only if they coincide except possibly for their vertex orientation.

$$H(\Gamma) = \{h = (v(h), e(h)) \in V(\Gamma) \times E(\Gamma); v(h) \in e(h)\}.$$

Let i denote an injection of $U(\Gamma)$ into M in the isotopy class that defines Γ. An *automorphism* of Γ is a permutation σ of $H(\Gamma)$ such that $v(h) = v(h') \Longrightarrow v(\sigma(h)) = v(\sigma(h'))$, $e(h) = e(h') \Longrightarrow e(\sigma(h)) = e(\sigma(h'))$, and i is isotopic to $i \circ \overline{\sigma}$ where $\overline{\sigma}$ denotes the permutation of $U(\Gamma)$ induced by σ.

Remark 4.1. Any configuration $c : U \cup T \hookrightarrow \mathbf{R}^3$ uniquely extends to a map $i(c) : \Gamma \longrightarrow \mathbf{R}^3$ that is linear along the edges of Γ. The configurations c such that two edges of Γ have colinear images under $i(c)$ do not contribute to the integral, because their images under Ψ lie in a codimension 2 subspace of $\left(S^2\right)^E$. (In particular, if two vertices of Γ are related by several edges, then $I(L;\Gamma) = 0$.) If $i(c)$ maps all the edges of Γ to pairwise non colinear segments of $\mathbf{R}^3$, there are exactly $\sharp Aut(\Gamma)$ configurations d of Γ with respect to $L(M)$ such that $i(c)(\Gamma) = i(d)(\Gamma)$. In other words, with the factor $\frac{1}{\sharp \mathrm{Aut}\Gamma}$, the image of an immersed unitrivalent graph contributes exactly once to the expression of $Z_{\mathrm{CS}}(L)$.

Let θ denote the Jacobi diagram

$\theta =$ 

on S^1. When L is a knot K, the degree one part of $Z_{CS}(K)$ is $\frac{I(K;\theta)}{2}[\theta]$ and therefore Z_{CS} is not invariant under isotopy. However, the evaluation Z^0_{CS} at representatives of knots with null Gauss integral is an isotopy invariant that is a universal Vassiliev invariant of knots. (All the real-valued finite type knot invariants factor through it.) This is the content of the following theorem, due independently to Altschuler and Freidel [1], and to D. Thurston [34], after the work of many people including Guadagnini, Martellini and Mintchev [14], Bar-Natan [5], Axelrod and Singer [2, 3], Kontsevich [18, 19], Bott and Taubes [8]...

Theorem 4.2 (Altschuler–Freidel [1], D. Thurston [34], 1995). *If $L = K_1 \cup \cdots \cup K_k$ is a link, then $Z_{CS}(L)$ only depends on the isotopy class of L and on the Gauss integrals $I(K_i;\theta)$ of its components. In particular, the evaluation*

$$Z^0_{CS}(L) \in \prod_{n \in \mathbf{N}} \mathcal{A}_n(\sqcup_{i=1}^k S^1_i)$$

at representatives of L whose components have zero Gauss integrals is an isotopy invariant of L. Furthermore, Z^0_{CS} is a universal Vassiliev invariant of links in the following sense. When L is a singular link with n double points, the degree k part $Z^0_{CS,k}(L)$ vanishes for $k < n$, while $Z^0_{CS,n}(L)$ is nothing but the chord diagram of L.

This theorem implies the fundamental theorem 2.26 of Vasiliev invariants. The main ideas involved in its proof are sketched in the next section.

5 More on Configuration Spaces

In this section, we describe the main ideas involved in the proof of Theorem 4.2, and we sketch the proof of this theorem. First, we need to understand the compactifications of configuration spaces of [11]. We shall present them with the Poirier point of view [30].

5.1 Compactifications of Configuration Spaces

Definition 5.1. A homothety with ratio in $]0, +\infty[$ will be called a *dilation.*

Let $X = \{\xi_1, \xi_2, \ldots, \xi_p\}$ be a finite set of cardinality $p \geq 2$, let k denote a positive integer. Let $C_0(X; \mathbf{R}^k)$ (resp. $C(X; \mathbf{R}^k)$) denote the set of injections (resp. of non-constant maps) f from X to $\mathbf{R}^k$, up to translations and dilations. $C_0(X; \mathbf{R}^k)$ is the quotient of

$$\{(x_1 = f(\xi_1), x_2 = f(\xi_2), \ldots, x_p = f(\xi_p)) \in (\mathbf{R}^k)^p; x_i \neq x_j \text{ if } i \neq j\}$$

by the translations which identify $(x_1, x_2, \ldots, x_p)$ to $(x_1 + T, x_2 + T, \ldots, x_p + T)$ for all $T \in \mathbf{R}^k$ and by the dilations which identify $(x_1, x_2, \ldots, x_p)$ to $(\lambda x_1, \lambda x_2, \ldots, \lambda x_p)$ for all $\lambda > 0$.

Examples 5.2. 1. For example, $C_0(X; \mathbf{R})$ has $p!$ connected components corresponding to the possible orders of the set X. Each of its components can be identified with the interior $\{(x_2, x_3, \ldots, x_{p-1}) \in \mathbf{R}^{p-2}; 0 < x_2 < x_3 < \cdots < x_{p-1} < 1\}$ of a $(p-2)$ simplex.
2. As another example, $C(\{1,2\}; \mathbf{R}^k) = C_0(\{1,2\}; \mathbf{R}^k)$ is homeomorphic to the sphere S^{k-1}.

In general, the choice of a point $\xi \in X$ provides a homeomorphism

$$\begin{array}{rcl} \phi_\xi : C(X, \mathbf{R}^k) & \longrightarrow & S^{kp-k-1} \\ f & \mapsto & \left(x \mapsto \frac{f(x)-f(\xi)}{\|\sum_{i=1}^{p}(f(\xi_i)-f(\xi))\|}\right) \end{array}$$

where S^{kp-k-1} is the unit sphere of $\left(\mathbf{R}^k\right)^{p-1}$. These homeomorphisms equip $C(X, \mathbf{R}^k)$ with an analytic (C^ω) structure and make clear that $C(X, \mathbf{R}^k)$ is compact. There is a natural embedding

$$\begin{array}{rcl} i : C_0(X; \mathbf{R}^k) & \hookrightarrow & \prod_{A \subseteq X; \sharp A \geq 2} C(A; \mathbf{R}^k) \\ c_X & \mapsto & (c_{X|A})_{A \subseteq X; \sharp A \geq 2} \end{array}$$

where $c_{X|A}$ denotes the restriction of c_X to A. Define the compactification $C(X; k)$ of $C_0(X; \mathbf{R}^k)$ as

$$C(X; k) = \overline{i(C_0(X; \mathbf{R}^k))} \subseteq \prod_{A \subseteq X; \sharp A \geq 2} C(A; \mathbf{R}^k)$$

In words, in $C(X; k)$, some points of X are allowed to collide with each other, or to become infinitely closer to each other than they are to other points, but the compactification provides us with the magnifying glasses $C(A; \mathbf{R}^k)$ that allow us to see the infinitely small configurations at the scales of the collisions.

Observe that the elements $(c_A)_{A \subseteq X; \sharp A \geq 2}$ of $C(X; k)$ satisfy the following condition $(\star)$.

$(\star)$: If $B \subset A$, then the restriction $c_{A|B}$ of c_A to B is either constant or equal to c_B.

Indeed, the above condition holds for elements of $i(C_0(X; \mathbf{R}^k))$, and it can be rewritten as the following condition that is obviously closed. *For any two sets A and B such that $B \subset A$, if $x \in B$, the two vectors of $(\mathbf{R}^k)^{B \setminus \{x\}}$, $(c_{A|B}(y) - c_{A|B}(x))_{y \in B \setminus \{x\}}$ and $(c_B(y) - c_B(x))_{y \in B \setminus \{x\}}$, are colinear, and their scalar product is non negative.*

Proposition 5.3. *The set $C(X; k)$ has a natural structure of an analytic manifold with corners*[3] *and*

$$C(X; k) = \{(c_A) \in \prod_{A \subseteq X; \sharp A \geq 2} C(A; \mathbf{R}^k);\ (c_A) \textit{ satisfies } (\star)\}.$$

[3] Every point c of $C(X; k)$ has a neighborhood diffeomorphic to $[0, \infty[^r \times \mathbf{R}^{n-r}$, and the transition maps are analytic.

PROOF: Set

$$\tilde{C}(X;k) = \{(c_A) \in \prod_{A \subseteq X; \sharp A \geq 2} C(A;\mathbf{R}^k);\ (c_A) \text{ satisfies } (\star)\}.$$

We have already proved that $C(X;k) \subseteq \tilde{C}(X;k)$. In order to prove the reversed inclusion, we first study the structure of $\tilde{C}(X;k)$. Let

$$c^0 = (c^0_A)_{A \subseteq X; \sharp A \geq 2} \in \tilde{C}(X;k).$$

Given this point c^0, we construct the rooted tree $\tau(c^0)$, with oriented edges, whose vertices are some subsets of X with cardinality greater than 1, in the following way. The root-vertex is X. The edges starting at a vertex $A \subseteq X$ are in one-to-one correspondence with the maximal subsets B of A with $\sharp B \geq 2$, such that $c^0_{A|B}$ is constant. The edge corresponding to a subset B goes from A to B. Note that the tree structure can be recovered from the set of vertices. Therefore, we identify the tree $\tau^0 = \tau(c^0)$ with its set of vertices and τ^0 is a set of subsets of X. For any strict subset A of X, $\hat{A}$ denotes the smallest element of τ^0 that strictly contains A.

Now, we construct a chart of $\tilde{C}(X;k)$ near c^0.

Let $B \in \tau^0$. Let $C_{\tau^0}(B;\mathbf{R}^k)$ be the subspace of $C(B;\mathbf{R}^k)$ made of maps from B to $\mathbf{R}^k$ such that two elements of B have the same image in $\mathbf{R}^k$ if and only if they belong to a common endpoint (subset of X) of an edge starting at B. (Note that $C_{\tau^0}(B;\mathbf{R}^k)$ is naturally homeomorphic to $C_0(B_{\tau^0};\mathbf{R}^k)$ where B_{τ^0} is the set obtained from B by identifying two elements of B that belong to a common strict subset of B in τ^0.) Let $V \subseteq \prod_{B \in \tau^0} C_{\tau^0}(B;\mathbf{R}^k)$ be an open neighborhood of $(c^0_B)_{B \in \tau^0}$ in $\prod_{B \in \tau^0} C_{\tau^0}(B;\mathbf{R}^k)$. Let $\varepsilon > 0$. When V and ε are small enough, define the map

$$\begin{array}{llll} F: & [0,\varepsilon[^{\tau^0 \setminus \{X\}} \times V & \longrightarrow & \prod_{D \subseteq X; \sharp D \geq 2} C(D;\mathbf{R}^k) \\ & P = \left((\lambda_B)_{B \in \tau^0 \setminus \{X\}}, (c_B)_{B \in \tau^0}\right) & \mapsto & (F(P)_D)_{D \subseteq X; \sharp D \geq 2} \end{array}$$

where $F(P)_D$ will be equal to $F(P)_{\hat{D}|D}$ if $D \notin \tau^0$, and if $A \in \tau_0$, $F(P)_A$ is represented by the map

$$\tilde{F}(P)_A : A \longrightarrow \mathbf{R}^k$$

that maps an element $(x \in A)$ to the vector that admits the following recursive definition. Let

$$\hat{\{x\}} = B_1 \subset B_2 \subset \cdots \subset B_m \subset B_{m+1} = A$$

be the sequence of vertices of τ^0 such that $B_1 = \hat{\{x\}}$ and $B_{r+1} = \hat{B}_r$. Fix a point $\xi(B)$ in any subset $B \in \tau^0$ so that if $B' \in \tau^0$ and if $\xi(B) \in B' \subset B$, then $\xi(B') = \xi(B)$ (the $\xi(B)$ depend on τ^0 that is fixed). Then $\tilde{F}(P)_{B_1}(x) = \phi_{\xi(B_1)}(c_{B_1})(x)$ and

$$\tilde{F}(P)_{B_{k+1}}(x) = \phi_{\xi(B_{k+1})}(c_{B_{k+1}})(x) + \lambda_{B_k}\tilde{F}(P)_{B_k}(x).$$

In particular, the small parameter $\lambda_B \in [0, \varepsilon[$ is the ratio of the scale of the representative $\tilde{F}(P)_{\hat{B}|B}$ in $\hat{B}$ by the scale of $\tilde{F}(P)_B$ in B.

Let $O(\tau^0)$ be the following open subset of $\prod_{A\subseteq X;\sharp A\geq 2} C(A;\mathbf{R}^k)$,

$$O(\tau^0) = \{(f_A)_{A\subseteq X;\sharp A\geq 2}; \forall A, \forall x, y \in A, c^0_A(x) \neq c^0_A(y) \Rightarrow f_A(x) \neq f_A(y)\}.$$

Since $F\left(P_0 = ((0)_{B\in\tau^0\setminus\{X\}}, (c^0_B)_{B\in\tau^0})\right) = c^0 \in O(\tau^0)$, when V and ε are small enough, the image of F is in $O(\tau^0)$. In particular, $F(P)_D$ is never constant and F is well-defined. Furthermore, the image of F is in $\tilde{C}(X;k)$. Assume the following lemma that will be proved later.

Lemma 5.4. *There is an open neighborhood O of c^0 in $O(\tau^0)$, and an analytic map G from O to $\mathbf{R}^{\tau^0\setminus\{X\}} \times V$, such that the restriction of G to $O \cap \tilde{C}(X;k)$ is an inverse for $F : [0, \varepsilon[^{\tau^0\setminus\{X\}} \times V \longrightarrow O \cap \tilde{C}(X;k)$.*

Thus, F is a homeomorphism onto its image that is an open subset of $\tilde{C}(X;k)$. Also notice that the tree (or its vertices set) corresponding to a point in the image of $F((\lambda_B)_{B\in\tau^0\setminus\{X\}}, (c_B)_{B\in\tau^0})$ is obtained from τ^0 by removing the subsets B such that $\lambda_B > 0$. The points of $i\left(C_0(X;\mathbf{R}^k)\right)$ are the points whose tree is reduced to its root X. In particular, the point c^0 we started with is in the closure of $F(]0, \varepsilon[^{B\in\tau^0\setminus\{X\}} \times V) \subseteq i(C_0(X;\mathbf{R}^k))$. This finishes proving that $C(X;k) \subseteq \tilde{C}(X;k)$. Furthermore, since F and G are analytic, the above local homeomorphisms equip $C(X;k)$ with the structure of a C^ω manifold with corners. This ends the proof of Proposition 5.3 assuming Lemma 5.4.$\diamond$

Proof of Lemma 5.4: Let $f = (f_A)_{A\subseteq X;\sharp A\geq 2} \in O(\tau^0)$. Define $G(f) = \left((\mu_B(f))_{B\in\tau^0\setminus\{X\}}, (d_B(f))_{B\in\tau^0}\right) \in \mathbf{R}^{\tau^0\setminus\{X\}} \times \prod_{B\in\tau^0} C_{\tau^0}(B;\mathbf{R}^k)$ as follows. Assume that any f_B is represented by $f_B = \phi_{\xi(B)}(f_B)$. Define $\tilde{d}_B(f)$ by

$$\tilde{d}_B(f)(x) = \begin{cases} f_B(x) & \text{if } \widehat{\{x\}} = B \\ f_B(\xi(B')) & \text{if } x \in B' \text{ and } \hat{B}' = B \end{cases}$$

and $d_B(f) = \phi_{\xi(B)}(\tilde{d}_B(f))$. For $B \in \tau^0$, choose $\xi'(B) \neq \xi(B) \in B$ such that $c^0(\xi'(B)) \neq c^0(\xi(B))$, and either $\{\widehat{\xi'(B)}\} = B$ or there exists a B' such that $\hat{B}' = B$ and $\xi'(B) = \xi(B')$. Note that when $f_B = \phi_{\xi(B)}(\tilde{F}(P)_B)$, then

$$d_B = d_B(f) = \phi_{\xi(B)}(c_B),$$

$$f_B = \frac{\| f_B(\xi'(B)) \|}{\| \tilde{F}(P)_B(\xi'(B)) \|}\tilde{F}(P)_B = \frac{\| f_B(\xi'(B)) \|}{\| d_B(\xi'(B)) \|}\tilde{F}(P)_B,$$

$$\tilde{F}(P)_{\hat{B}}(\xi'(B)) - \tilde{F}(P)_{\hat{B}}(\xi(B)) = \lambda_B \tilde{F}(P)_B(\xi'(B)) = \lambda_B d_B(\xi'(B)),$$

$$\begin{aligned}\lambda_B &= \frac{<\tilde{F}(P)_{\hat{B}}(\xi'(B))-\tilde{F}(P)_{\hat{B}}(\xi(B)),d_B(\xi'(B))>}{\|d_B(\xi'(B))\|^2}\\ &= \frac{\|d_{\hat{B}}(\xi'(\hat{B}))\|}{\|f_{\hat{B}}(\xi'(\hat{B}))\|}\frac{<f_{\hat{B}}(\xi'(B))-f_{\hat{B}}(\xi(B)),d_B(\xi'(B))>}{\|d_B(\xi'(B))\|^2}.\end{aligned}$$

Therefore, we define

$$\mu_B(f) = \frac{\| d_{\hat{B}}(\xi'(\hat{B})) \|}{\| f_{\hat{B}}(\xi'(\hat{B})) \|}\frac{< f_{\hat{B}}(\xi'(B)) - f_{\hat{B}}(\xi(B)), d_B(\xi'(B)) >}{\| d_B(\xi'(B)) \|^2}.$$

Now, it is clear that G is analytic, and that $G \circ F =$ Identity on

$$O = G^{-1}(] - \varepsilon, \varepsilon[^{\tau^0 \setminus \{X\}} \times V).$$

Then it is enough to see that the restriction of $F \circ G$ to $O \cap \tilde{C}(X;k)$ is well-defined ($\forall B, \mu_B \geq 0$), and is the identity. This is left as an exercise for the reader. $\diamond$

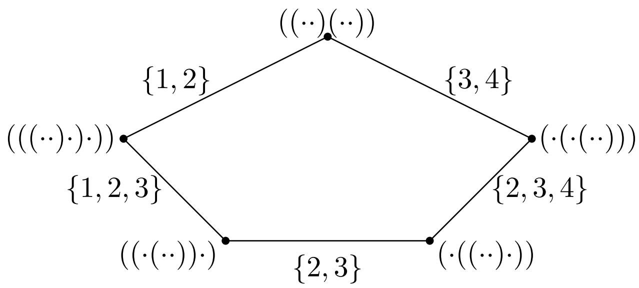
The space $C(X;k)$ is also equipped with a partition by the associated trees of the above proposition proof. Note that the part $F(\tau)$ corresponding to a given tree τ is a submanifold of dimension $\big(\dim(C_0(X;\mathbf{R}^k)) - (\sharp\tau - 1)\big)$ that is homeomorphic to $\prod_{B\in\tau} C_\tau(B;\mathbf{R}^k)$. In particular, the boundary of $C(X;k)$ has a partition into open *faces,* corresponding to trees τ with $\sharp\tau > 1$, of codimension $(\sharp\tau - 1)$.

Example 5.5. Let us again consider the case where $k = 1$. Let us fix one order on X and let us study the corresponding component $C_<(X;1)$ of $C(X;1)$. Here the trees that provide non-empty faces are those which are made of subsets containing consecutive elements, and all the faces are homeomorphic to open balls. When $X = \{1,2,3\}$, $C_<(X;1)$ is an interval whose endpoints correspond to the two trees $\{X,\{1,2\}\}$ and $\{X,\{2,3\}\}$. For $X = \{1,2,\dots,p\}$, we find a *Stasheff polyhedron* that is a polyhedron whose maximal codimension faces are points that can be described as *non-associative words* as in the following definition.

Definition 5.6. A *non-associative word* or *n.a. word* w in the letter $\cdot$ is an element of the free non-associative monoid generated by $\cdot$. The *length* of such a w is the number of letters of w. Equivalently, we can define a *non-associative word* by saying that each such word has an integral *length* $\ell(w) \in \mathbf{N}$, the only word of length 0 is the *empty word*, the only word of length 1 is $\cdot$, the product $w'w''$ of two n.a. words w' and w'' is a n.a. word of length $(\ell(w') + \ell(w''))$, and every word w of length $\ell(w) \geq 2$ can be decomposed in a unique way as the product $w'w''$ of two n.a. words w' and w'' of nonzero length.

Example 5.7. The unique n.a. word of length 2 is $(\cdot\cdot)$. The two n.a. words of length 3 are $((\cdot\cdot)\cdot)$ and $(\cdot(\cdot\cdot))$. There are five n.a. words of length 4 drawn in the following picture of $C_<(\{1,2,3,4\};1)$.
A n.a. word corresponds to the binary tree of subsets of points between matching parentheses.

In particular, $C_<(\{1,2,3,4\};1)$ is the following well-known pentagon, whose edges are labeled by the element of the corresponding $\tau \setminus \{\{1,2,3,4\}\}$.



As another example, the reader can recognize that $C(\{1,2,3\};2)$ is diffeomorphic to the product by S^1 of the complement of two disjoint open discs in the unit two-dimensional disk (or to the exterior of a 3-component Hopf link in S^3, made of three Hopf fibers).

5.2 Back to Configuration Space Integrals for Links

We can use these compactifications in order to study the configuration space integrals defined in Sect. 4.2 as in [30]. Indeed, there is a natural embedding

$$i : C(L;\Gamma) \hookrightarrow M^U \times (S^3 = \mathbf{R}^3 \cup \infty)^T \times C(U \cup T;3).$$

Define the compactification $\overline{C}(L;\Gamma)$ of $C(L;\Gamma)$ as the closure of $i(C(L;\Gamma))$ in this compact space. As before, the compactification can be provided with a structure of a C^∞ manifold with corners, with a stratification that will again be given by trees recording the different relative collapses of points.[4] Furthermore, since Ψ is defined on $C(U \cup T; 3)$ as the projection on $\prod_E C(E;3)$ where an edge E is seen as the pair of its endpoints ordered by the orientation, Ψ extends to $\overline{C}(L;\Gamma)$. This extension is smooth, and we have

$$I(L;\Gamma) = \int_{\overline{C}(L;\Gamma)} \Psi^*(\bigwedge^E \omega).$$

In particular, this shows the convergence of the integrals of Sect. 4.2. The variation of $I(L;\Gamma)$ under a C^∞ isotopy

$$\begin{array}{rcl} L : M \times I & \longrightarrow & \mathbf{R}^3 \\ (m,t) & \mapsto & L^t(M) \end{array}$$

is computed with the help of the Stokes theorem. Since $\Psi^*(\bigwedge^E \omega)$ is a closed form defined on $\cup_{t\in I} C(L^t;\Gamma)$, the variation $(I(L_1;\Gamma) - I(L_0;\Gamma))$ is given by the sum over the codimension one faces $F(\tau)(L;\Gamma)$ of the

$$V(F(\tau)(L;\Gamma)) = \int_{\cup_{t\in I} F(\tau)(L^t;\Gamma)} \Psi^*(\bigwedge^E \omega).$$

The Altschuler-Freidel proof and the Thurston proof that Z^0_{CS} provides a link invariant now rely on a careful analysis of the codimension one faces, and of the variations that they induce. See [1, 34, 30]. This analysis was successfully started by Bott and Taubes [8]. It shows that the faces that indeed contribute in the link case, where $M = \coprod_{i=1}^k S^1_i$, are of four possible forms.

1. Two trivalent vertices joined by an edge collide with each other.
2. Two univalent vertices consecutive on M collide with each other.
3. A univalent vertex and a trivalent vertex that are joined by an edge collide with each other.

[4] There are two main differences with the already studied case, due to the one-manifold embedding L. First, the univalent vertices vary along L, and when they approach each other, their direction that makes sense in the compactification approaches the direction of the tangent vector to L at the point where they meet. Second, there is a preferred observation scale namely the scale of the ambient space where the embedding lies.

4. The *anomalous faces* where some connected component of the dashed graph Γ collapses at one point.

The STU and IHX relation make the first three kinds of variations cancel. Let us see roughly how it works for the first kind of faces. Such a face is homeomorphic to the product of the sphere S^2 by the configuration space of the graph obtained from Γ by identifying the two colliding points (which become a four-valent vertex), where S^2 is the configuration space of the two endpoints of the infinitely small edge. Let Γ_1, Γ_2 and Γ_3 be three graphs related by an IHX relation so that $[\Gamma_1] + [\Gamma_2] + [\Gamma_3] = 0$. Let τ_i be the tree made of $U \cup T$ and the visible edge of Γ_i. Then $V(F(\tau_i)(L;\Gamma_i))$ is independent of i. Therefore,

$$V(F(\tau_1)(L;\Gamma_1))[\Gamma_1] + V(F(\tau_2)(L;\Gamma_2))[\Gamma_2] + V(F(\tau_3)(L;\Gamma_3))[\Gamma_3] = 0.$$

Thus, the sum[5] of these variations plugged into the Chern-Simons series is zero. The STU relation makes the variations of the second kind and the third kind of faces cancel each other in a similar way.

For the anomalous faces, we do not have such a cancellation. But, we are about to see that we have a formula like

$$\frac{\partial}{\partial t}(Z_{CS}(L^t)) = \left(\sum_{i=1}^{k} \frac{\partial}{\partial t} \frac{I(K_i^t;\theta)}{2} \alpha \sharp_i \right) Z_{CS}(L^t). \tag{1}$$

where α is the *anomaly* that is the constant of $\mathcal{A}(\mathbf{R})$ that is defined below, and $\sharp_i$ denotes the $\mathcal{A}(\mathbf{R})$-module structure on $\mathcal{A}(\coprod_{i=1}^k S_i^1)$ by insertion on the i^{th} component. See Subsection 3.3.

5.3 The Anomaly

Let us define the anomaly. Let $v \in S^2$. Let D_v denote the linear map

$$\begin{array}{rcl} D_v : \mathbf{R} & \longrightarrow & \mathbf{R}^3 \\ 1 & \mapsto & v. \end{array}$$

Let Γ be a Jacobi-diagram on $\mathbf{R}$. Define $C(D_v;\Gamma)$ and Ψ as in Sect. 4.2. Let $\hat{C}(D_v;\Gamma)$ be the quotient of $C(D_v;\Gamma)$ by the translations parallel to D_v and

[5] To make this sketchy proof work and to avoid thinking of the $(1/\sharp \mathrm{Aut}\Gamma)$ factor, use Remark 4.1.

by the dilations. Then Ψ factors through $\hat{C}(D_v;\Gamma)$ that has two dimensions less. Now, allow v to run through S^2 and define $\hat{C}(\Gamma)$ as the total space of the fibration over S^2 where the fiber over v is $\hat{C}(D_v;\Gamma)$. The map Ψ becomes a map between two smooth oriented manifolds of the same dimension. Indeed, $\hat{C}(\Gamma)$ carries a natural smooth structure and can be oriented as follows. Orient $C(D_v;\Gamma)$ as before, orient $\hat{C}(D_v;\Gamma)$ so that $C(D_v;\Gamma)$ is locally homeomorphic to the oriented product (translation vector $(0,0,z)$ of the oriented line, ratio of homothety $\lambda \in]0,\infty[$) $\times \hat{C}(D_v;\Gamma)$ and orient $\hat{C}(\Gamma)$ with the (base($=S^2$) $\oplus$ fiber) convention[6]. Then we can again define

$$I(\Gamma) = \int_{\hat{C}(\Gamma)} \Psi^*(\bigwedge^E \omega).$$

Now, the *anomaly* is the following sum running over all connected Jacobi diagrams Γ on the oriented line (again without vertex-orientation and without small loop):

$$\alpha = \sum \frac{I(\Gamma)}{\sharp \mathrm{Aut}\Gamma}[\Gamma] \quad \in \mathcal{A}(\mathbf{R})$$

Its degree one part is

$\alpha_1 = \Big[$  $\Big]$.

Then Formula 1 expresses the following facts. Let Γ be a connected dashed graph on the circle S^1_j. The set $U = \{u_1, u_2, \dots, u_k\}$ of its univalent vertices is cyclically ordered, and the anomalous faces for Γ correspond to the different total orders (which are visible at the scale of the collision) inducing the given cyclic order. Assume that $u_1 < u_2 < \dots < u_k = u_0$ is one of them. Denote the Jacobi diagram on $\mathbf{R}$ obtained by cutting the circle between u_{i-1} and u_i by Γ_i.

The anomalous face where Γ collapses with the total order induced by Γ_i fibers over $[0,1] \times S^1$. The fibration maps a limit configuration $c \in F(\tau)(L^t;\Gamma)$ to (t,z), where the collapse occurs at $K^t_j(z)$. The fiber over (t,z) is $\hat{C}(D_{(K^t_j)'(z)};\Gamma)$. In particular, the contribution of the collapse that orders U like Γ_i to the variation $(I(K^1_j;\Gamma) - I(K^0_j;\Gamma))$ during a knot isotopy $((z,t) \mapsto K^t_j(z))$ is proportional to the area covered by the unit derivative of K_j on S^2 during the isotopy, that is $\frac{I(K^1_j;\theta) - I(K^0_j;\theta)}{2}$. More precisely, it is

[6] This can be summarized by saying that the S^2-coordinates replace (z,λ).

$$\frac{I(K_j^1;\theta)-I(K_j^0;\theta)}{2}I(\Gamma_i).$$

Therefore, the contribution to the variation $\frac{(I(K_j^1;\Gamma)-I(K_j^0;\Gamma))}{\sharp\mathrm{Aut}(\Gamma)}$ of the anomalous faces is

$$\sum_{i=1}^{k}\frac{I(K_j^1;\theta)-I(K_j^0;\theta)}{2\sharp\mathrm{Aut}(\Gamma)}I(\Gamma_i).$$

The group of automorphisms of Γ_i is isomorphic to the subgroup $\mathrm{Aut}_0(\Gamma)$ of $\mathrm{Aut}(\Gamma)$ made of the automorphisms of Γ that fix U pointwise. The quotient $\frac{\mathrm{Aut}(\Gamma)}{\mathrm{Aut}_0(\Gamma)}$ is a subgroup of the cyclic group of the permutations of U that preserve the cyclic order of U, of order $\frac{k}{p}$, for some integer p that divides into k; and Γ_i is isomorphic to Γ_{i+p}, for any integer $i \leq (k-p)$. Thus, the contribution to the variation $\frac{(I(K_j^1;\Gamma)-I(K_j^0;\Gamma))}{\sharp\mathrm{Aut}(\Gamma)}$ of the anomalous faces is

$$\sum_{i=1}^{p}\frac{I(K_j^1;\theta)-I(K_j^0;\theta)}{2\sharp\mathrm{Aut}(\Gamma_i)}I(\Gamma_i).$$

In general, one must multiply the infinitesimal variation due to the collapse of one connected component of the dashed graph by the contributions of the other connected components of the dashed graphs, and it is better to use Remark 4.1 to avoid thinking of the number of automorphisms.

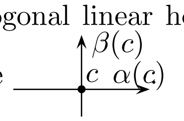
The integration of Formula 1 shows the Altschuler and Freidel formula:

$$Z_{CS}(L)=\exp(\frac{I(K_1;\theta)}{2}\alpha)\sharp_1\exp(\frac{I(K_2;\theta)}{2}\alpha)\sharp_2\dots\exp(\frac{I(K_k;\theta)}{2}\alpha)\sharp_k Z^0_{CS}(L)$$

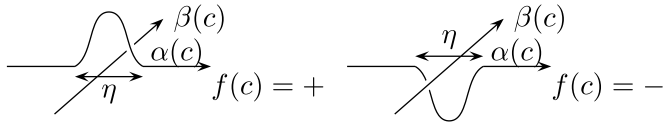
with respect to the structures defined in Subsection 3.3.

5.4 Universality of Z^0_{CS}.

In order to prove that Z^0_{CS} is a universal Vassiliev invariant, it is enough to compute its projection $\overline{Z}^0_{CS,k}(L)$ onto $\overline{\mathcal{A}}_k(S^1)$ when L is a singular link, that is a singular immersion of M with n double points, and when $k \leq n$.

To do this, fix n disjoint balls $B(c)$ of radius 2 in $\mathbf{R}^3$ associated to the double points c of L, fix an almost planar representative of L that intersects each ball $B(c)$ as a pair of orthogonal linear horizontal arcs $\alpha(c)$ and $\beta(c)$ crossing at the center of $B(c)$ like 

The desingularisation of L associated to a map f will be obtained from this embedding by moving $\alpha(c)$ to one of the two following positions depending on the value $f(c)$ of the desingularisation f at c.



The obtained embeddings $L_0(f, \eta)$ depend on the small parameter η that is the diameter of the ball where $\alpha(c)$ has changed. But each of them will have integral Gauss integrals. Assume that the Gauss integrals of the positive desingularisation (where $f(c) = +$ for all c) of L vanish.

For every double point c that is a self-crossing of a component of L, cancel the Gauss integral modification of $L_0(f, \eta)$ if $f(c) = -$ by a modification of $\alpha(c)$ in $B(c)$ where

Let $L(f, \eta)$ denote the obtained almost planar representative of the desingularisation f of L. For any uni-trivalent diagram Γ on M, set

$$I(L(\eta); \Gamma) = \sum_{f:\{1,2\ldots,n\}\longrightarrow\{+,-\}} (-1)^{\sharp f^{-1}(-)} I(L(f, \eta); \Gamma).$$

It is enough to prove that for any given $\varepsilon > 0$, and for any uni-trivalent diagram Γ without isolated chord on M of degree $\leq n$, there exists η such that:
If $\Gamma \neq D(L)$, then $|I(L(\eta); \Gamma)| < \varepsilon$, and,
if $\Gamma = D(L)$, then $|I(L(\eta); \Gamma) - \sharp\mathrm{Aut}(\Gamma)| < \varepsilon$.

Fix a double point c of L and a uni-trivalent diagram Γ on M. The configurations of Γ that map no univalent vertices of Γ to $\alpha(c)$ will contribute in the same way to the integral $I(L(f, \eta); \Gamma)$ corresponding to a desingularisation f and to the integral $I(L(f^c, \eta); \Gamma)$ corresponding to a desingularisation f^c obtained from f by changing $f(c)$ into $(-f(c))$. Therefore, they will not contribute at all to $I(L(\eta); \Gamma)$. Thus, we shall only consider configurations with at least one univalent vertex $u(c)$ on each $\alpha(c)$.

Similarly, the contributions to $I(L(\eta);\Gamma)$ of the configurations where the given neighborhood of a given double point c does not contain any other vertex $u_2(c)$ in $B(c)$ related by an edge to $u(c)$ approach 0 when η approaches 0. Thus, we are left with the diagrams with at least 2 vertices $u(c)$ and $u_2(c)$ in every $B(c)$, and with at most $2n$ vertices. They are diagrams with exactly two vertices in each ball $B(c)$. If some vertex $u_2(c)$ is trivalent, then considering the value of $(u_2(c)-u(c))$ rather than the $\mathbf{R}^3$-position parameter of $u_2(c)$, we can conclude that $I(L(\eta);\Gamma)$ approaches 0 when η approaches 0. Thus, we are left with the case where all the $u_2(c)$ are univalent and therefore Γ must be a chord diagram with one chord between $u(c)$ and $u_2(c)$ for every double point c. Since we only consider diagrams without isolated chords, $u_2(c)$ must be on $\beta(c)$. Thus, we are left with the chord diagrams with one chord between $\alpha(c)$ and $\beta(c)$ for every double point c, that is by definition the chord diagram of the singular link L.

The number of embeddings (up to isotopy within the neighborhoods of the double points) of the vertices of this chord diagram Γ that respect the pairing of the chords is the number of automorphisms of Γ, and the contribution to $I(L(\eta);\Gamma)$ of each isotopy class of such embeddings is the product over the double points of the following algebraic areas. For each double point c, the algebraic area is the difference between the algebraic area covered by the directions of the segments from a point of $\alpha(c)$ to a point of $\beta(c)$ in the positive η-desingularisation of c minus the algebraic area covered by the directions of the segments from a point of $\alpha(c)$ to a point of $\beta(c)$ in the negative η-desingularisation of c. It is not hard to see that this algebraic area approaches one when η goes to zero... and to conclude this sketch of proof. $\diamond$

5.5 Rationality of Z^0_{CS}

As it has been first noticed by Dylan Thurston in [34], Z^0_{CS} is rational. This means that for any integer n, and for any link L, the degree n part $Z^0_{CS,n}(L)$ of $Z^0_{CS}(L)$ is in $\mathcal{A}^{\mathbf{Q}}_n(\coprod_{i=1}^k S^1_i)$ that is the quotient of the *rational* vector space generated by the diagrams on $\coprod_{i=1}^k S^1_i$ by the STU relation. Indeed, if L is almost horizontal, $Z^0_{CS,n}(L)$ may be interpreted as the following differential degree.

Let $e_n = 3n-2$ be a number of edges greater or equal than the number of edges of degree n diagrams that might contribute with a non zero integral to the Chern-Simons series. We wish to interpret $Z^0_{CS,n}(L)$ as the differential degree of a map to $\left(S^2\right)^{e_n}$. We first modify the configuration space $\overline{C}(L;\Gamma)$ of a degree n diagram Γ whose set of edges is $E(\Gamma)$ by

$$\hat{C}(L;\Gamma) = \overline{C}(L;\Gamma) \times \left(S^2\right)^{e_n - \sharp E(\Gamma)}.$$

Next, in order to be able to map it to $\left(S^2\right)^{e_n}$, we *label* Γ, that is we orient the edges of Γ and we define a bijection from $E(\Gamma) \cup \{1,2,\dots,e_n - \sharp E(\Gamma)\}$ to $\{1,2,\dots,e_n\}$. This bijection transforms the map

$$\Psi \times \mathrm{Identity}(\left(S^2\right)^{e_n-\sharp E(\Gamma)}) : \hat{C}(L;\Gamma) \longrightarrow \left(S^2\right)^{E(\Gamma)} \times \left(S^2\right)^{e_n-\sharp E(\Gamma)}$$

into a map

$$\hat{\Psi} : \hat{C}(L;\Gamma) \longrightarrow \left(S^2\right)^{e_n}$$

whose oriented image is independent of a possible labelling of the vertices. For a given degree n diagram Γ, there are $\frac{2^{\sharp E(\Gamma)} e_n!}{\sharp \mathrm{Aut}(\Gamma)}$ labeled diagrams. Now,

$$Z^0_{CS,n}(L) = \sum_{\Gamma \text{ labeled diagram of degree } n} \frac{1}{2^{\sharp E(\Gamma)} e_n!} \int_{\hat{C}(L;\Gamma)} \hat{\Psi}^*(\bigwedge^{e_n} \omega)[\Gamma]$$

Define the differential degree $\deg(\Psi,x)$ of Ψ over the formal union

$$\cup_{\Gamma \text{ labeled diagram of degree } n} \frac{1}{2^{\sharp E(\Gamma)} e_n!}[\Gamma]\hat{C}(L;\Gamma)$$

as follows for a regular[7] point $x \in (S^2)^{e_n}$:

$$\deg(\Psi,x) = \sum_{\Gamma \text{ labeled diagram of degree } n} \frac{1}{2^{\sharp E(\Gamma)} e_n!} \deg(\Psi_{|\hat{C}(L;\Gamma)},x)[\Gamma]$$

where $\deg(\Psi_{|\hat{C}(L;\Gamma)},x)$ is a usual differential degree. Then D. Thurston proved that $\deg(\Psi,x)$ does not vary across the images of the codimension one faces of the $\hat{C}(L;\Gamma)$. See also [30]. In other words, the above weighted union of configuration spaces behaves as a closed $2e_n$-dimensional manifold from the point of view of the differential degree theory. In particular, ω can be replaced by any volume form of S^2 with total volume 1. Computing $Z^0_{CS,n}$ as the

[7] Here, regular means regular with respect to all the $\Psi_{|\hat{C}(L;\Gamma)}$.

degree of a generic point of $\left(S^2\right)^{e_n}$ shows that $Z^0_{CS,n}$ belongs to the lattice of $\mathcal{A}_n^{\mathbf{Q}}(\coprod_{i=1}^k S_i^1)$ generated by the $\frac{(e_n-\sharp E(\Gamma))!}{2^{\sharp E(\Gamma)}e_n!}[\Gamma]$, where the Γ's are the degree n graphs that may produce a nonzero integral. This interpretation is more convenient for computational purposes.

6 Diagrams and Lie Algebras. Questions and Problems

In this section, we are going to show how a Lie algebra equipped with a non-degenerate symmetric invariant bilinear form and some representation induces a linear form on $\mathcal{A}_n(S^1)$. In particular, such a datum allows one to deduce numerical knot invariants from the Chern-Simons series by composition.

First of all, we recall the needed background about Lie algebras.

6.1 Lie Algebras

Definition 6.1. A (finite-dimensional) *Lie algebra* over $\mathbf{R}$ is a vector space g over $\mathbf{R}$ of finite dimension equipped with a *Lie bracket* that is a bilinear map denoted by $[.,.] : g \times g \to g$ that satisfies:
the *antisymmetry relation:*

$$\forall x \in g, \quad [x,x] = 0 \quad (\Longrightarrow \quad \forall (x,y) \in g^2, \quad [x,y] = -[y,x])$$

and the *Jacobi relation:*

$$\forall (x,y,z) \in g^3, \quad [x,[y,z]] + [y,[z,x]] + [z,[x,y]] = 0.$$

Examples 6.2. 1. The vector space of the endomorphisms of $\mathbf{R}^N$, $gl_N = gl(\mathbf{R}^N)$, equipped with the bracket

$$\begin{aligned} [.,.] : gl_N \times gl_N &\to gl_N \\ (x,y) &\mapsto xy - yx \end{aligned}$$

is a Lie algebra.

2. The vector space of the trace zero endomorphisms of $\mathbf{R}^N$, $sl_N = sl(\mathbf{R}^N)$, equipped with the restriction of the above Lie bracket is a Lie subalgebra of gl_N.

Definition 6.3. A (finite-dimensional) *representation* of such a Lie algebra g into a (finite-dimensional) $\mathbf{R}$-vector space E is a *Lie algebra morphism* ρ from g to $gl(E)$, where a Lie algebra morphism is a $\mathbf{R}$-linear map that preserves the Lie bracket $(\rho([x,y]) = [\rho(x),\rho(y)] \stackrel{\text{def}}{=} \rho(x)\rho(y) - \rho(y)\rho(x))$.

Examples 6.4. 1. The inclusion i_N of sl_N into gl_N is called the *standard representation* of sl_N into the vector space $\mathbf{R}^N$.
2. For any Lie algebra g, the morphism

$$\begin{aligned} \text{ad} : g &\longrightarrow gl(g) \\ x &\mapsto \quad (\text{ad}(x) : y \mapsto ad(x)(y) = [x,y]) \end{aligned}$$

is a representation of g (thanks to the Jacobi identity) that is called the *adjoint representation* of g.

Definition 6.5. A bilinear form $\beta : g \times g \longrightarrow \mathbf{R}$ is said to be *ad-invariant* or *invariant* if it satisfies

$$\forall (x,y,z) \in g^3, \quad \beta([x,z],y) + \beta(x,[y,z]) = 0.$$

Example 6.6. When a Lie algebra is equipped with a representation (E,ρ), the *associated bilinear form*

$$\beta(\rho) : g \times g \longrightarrow \mathbf{R}$$

$$(x,y) \mapsto \text{trace}(\rho(x)\rho(y))$$

is a symmetric invariant bilinear form on g.

Definition 6.7. Let $\beta : g \times g \longrightarrow \mathbf{R}$ be a non-degenerate symmetric bilinear invariant form on a Lie algebra g. Then β induces the natural isomorphism $\hat{\beta} : g \longrightarrow g^*$ that maps x to $(\hat{\beta}(x) : y \mapsto \beta(x,y))$. The *Casimir element* Ω_β of β is the inverse of $\hat{\beta}$ viewed as an element of $g \otimes g$ with the help of the following canonical identifications.

$$g \otimes g \cong (g^*)^* \otimes g \cong \text{Hom}(g^*, g)$$

Exercise 6.8. Under the hypotheses of the above definition, let $(e_i)_{i=1,\dots,n}$ and $(e'_i)_{i=1,\dots,n}$ be two dual bases of g with respect to β that are two bases such that $\beta(e_i, e'_j)$ is the Kronecker symbol δ_{ij}. Show that

$$\Omega_\beta = \sum_{i=1}^{n} e_i \otimes e'_i$$

Note that this shows that the right-hand side of the above equality is independent of the choice of the two dual bases.

Proposition 6.9. *Let $(e_i)_{i=1,\dots,N}$ be a basis of $\mathbf{R}^N$, let $(e_i^*)_{i=1,\dots,N}$ denote its dual basis, and let e_{ij} be the element $e_j^* \otimes e_i$ of $gl_N \cong (\mathbf{R}^N)^* \otimes \mathbf{R}^N$. The form β_N of sl_N is non-degenerate,*

$$\Omega_{\beta_N} = \sum_{\substack{(i,j)\in\{1,\dots,N\}^2 \\ i\neq j}} e_{ij} \otimes e_{ji} + \sum_{i=1}^{N} \left(e_{ii} - \frac{1}{N}\sum_{j=1}^{N} e_{jj} \right) \otimes \left(e_{ii} - \frac{1}{N}\sum_{j=1}^{N} e_{jj} \right)$$

and

$$\Omega_{\beta_N} = \sum_{(i,j)\in\{1,\dots,N\}^2} e_{ij} \otimes e_{ji} - \frac{1}{N}\left(\sum_{i=1}^{N} e_{ii}\right) \otimes \left(\sum_{i=1}^{N} e_{ii}\right)$$

Proof: Let β be the form induced by the standard representation on gl_N that is $(x,y) \mapsto \mathrm{trace}(xy)$. The form β is non-degenerate on gl_N because the basis $(e_{ij})_{(i,j)\in\{1,\dots,n\}^2}$ of gl_N is dual to the basis (e_{ji}). Furthermore, $\beta\left(\sum_{i=1}^N e_{ii}, \sum_{i=1}^N e_{ii}\right) = N \neq 0$. Thus, since sl_N is the orthogonal of $\left(\sum_{i=1}^N e_{ii}\right)$ in gl_N, β_N that is the restriction of the symmetric form β to sl_N is non-degenerate. It is easy to check that the two proposed expressions of Ω_{β_N} are equal. The first one makes clear that our candidate belongs to $sl_N \otimes sl_N$. Now, it is enough to evaluate the second expression of our candidate viewed as an element of $\mathrm{Hom}(gl_N^*, gl_N)$ at the $\hat\beta(e_{ji})$, $j \neq i$ and at the $\hat\beta(e_{ii} - e_{jj})$ that are mapped to e_{ji} and $(e_{ii} - e_{jj})$, respectively, as they must be. $\diamond$

6.2 More Spaces of Diagrams

We need to introduce more kinds of diagrams. Namely, we need to consider diagrams with free univalent vertices labeled by a finite set A.

Definition 6.10. Let M be an oriented one-manifold and let A be a finite set. A *diagram* Γ with support $M \cup A$ is a finite uni-trivalent graph Γ such that every connected component of Γ has at least one univalent vertex, equipped with:

1. a partition of the set U of univalent vertices of Γ also called *legs* of Γ into two (possibly empty) subsets U_M and U_A,
2. a bijection f from U_A to A,
3. an isotopy class of injections i of U_M into the interior of M,
4. an *orientation* of every trivalent vertex, that is a cyclic order on the set of the three half-edges which meet at this vertex.

Such a diagram Γ is again represented by a planar immersion of $\Gamma \cup M$ where the univalent vertices of U_M are located at their images under i, the one-manifold M is represented by solid lines, whereas the diagram Γ is dashed. The vertices are represented by big points. The local orientation of a trivalent vertex is again represented by the counterclockwise order of the three half-edges that meet at it.

Let $\mathcal{D}_n(M,A)$ denote the real vector space generated by the degree n diagrams on $M \cup A$, and let $\mathcal{A}_n(M,A)$ denote the quotient of $\mathcal{D}_n(M,A)$ by the relations AS, STU and IHX.

6.3 Linear Forms on Spaces of Diagrams

Notation 6.11. Let g be a finite-dimensional Lie algebra equipped with a finite-dimensional representation (E,ρ) and a non-degenerate bilinear symmetric invariant form β. For any oriented compact one-manifold M, the set of the boundary points of M where the corresponding components start as in $\bullet\!\!\rightarrow$ is denoted by ∂M^- whereas $\partial M \setminus \partial M^-$ is denoted by ∂M^+. When A is a finite set decomposed as the disjoint union of two subsets denoted by A^+ and A^-, and when M is an oriented compact one-manifold define

$$T(g,\rho,\beta)(M, A^- \coprod A^+) = \bigotimes^{\partial M^-} E^* \otimes \bigotimes^{\partial M^+} E \otimes \bigotimes^{A^-} g^* \otimes \bigotimes^{A^+} g$$

$T(g,\rho,\beta)(M, A^- \coprod A^+)$ is the tensor product of $\sharp\partial M^-$ copies of E^* indexed by the elements of ∂M^-, $\sharp\partial M^+$ copies of E indexed by the elements of ∂M^+, and copies of g^* and g indexed by the elements of A. In particular, when the boundary of M is empty $T(g,\rho,\beta)(M,\emptyset) = \mathbf{R}$.

Set

$$T(g,\rho,\beta)\left(\longrightarrow\ \in \mathcal{A}([0,1])\right) = \text{Identity} \in gl(E)$$

where $gl(E) \cong E^* \otimes E = T(g,\rho,\beta)([0,1])$,

$$T(g,\rho,\beta)\left(\underset{\longrightarrow}{\overset{\overset{x}{\bullet}}{\vdots}}\ \in \mathcal{A}([0,1],\{x\}^-)\right) = \rho \in \mathrm{Hom}(g, gl(E))$$

where $\mathrm{Hom}(g, gl(E)) \cong g^* \otimes E^* \otimes E = T(g,\rho,\beta)([0,1],\{x\}^-)$,

$$T(g,\rho,\beta)\left(x\bullet\text{---}\bullet y\ \in \mathcal{A}(\{x,y\}^+)\right) = \Omega_\beta \in g \otimes g$$

where $g \otimes g = T(g,\rho,\beta)(\{x,y\}^+)$,

$$T(g,\rho,\beta)\left(y\bullet\text{-}\overset{\overset{x}{\bullet}}{\bullet}\text{-}\bullet z \in \mathcal{A}(\{x,y,z\}^-)\right) = \hat{\beta} \circ [.,.] \in \mathrm{Bil}(g_x \times g_y, g_z^*)$$

where $\mathrm{Bil}(g_x \times g_y, g_z^*) \cong (g_x \otimes g_y)^* \otimes g_z^* \cong g_x^* \otimes g_y^* \otimes g_z^*$.

Note that $T(g,\rho,\beta)\left(x\bullet\text{---}\bullet y\right)$ is symmetric with respect to the permutation of the two factors. Also note that $T(g,\rho,\beta)\left(y\bullet\text{-}\overset{\overset{x}{\bullet}}{\bullet}\text{-}\bullet z\right)$ can be canonically identified with the trilinear form

$$g_x \times g_y \times g_z \longrightarrow \mathbf{R}$$

$$(a,b,c) \mapsto \beta([a,b],c)$$

In particular, it is antisymmetric with respect to any transposition of two factors g^*. Therefore, the above definitions make sense.

For any $\mathbf{R}$-vector space E, recall the natural linear *contraction map* from $E^* \otimes E$ to $\mathbf{R}$ that maps $f \otimes e$ to $f(e)$.

Theorem 6.12 (Bar-Natan [4]). *Let g be a finite-dimensional Lie algebra equipped with a finite-dimensional representation (E,ρ) and a non-degenerate bilinear symmetric invariant form β. Then there exists a unique family of linear maps*

$$T(g,\rho,\beta) : \mathcal{A}_n(M, A^- \coprod A^+) \longrightarrow T(g,\rho,\beta)(M, A^- \coprod A^+)$$

for any $n \in \mathbf{N}$, for any oriented compact one-manifold M, and for any finite set $A^- \coprod A^+$ such that:

1. *$T(g,\rho,\beta)$ takes the above values at the four diagrams above.*

2. *For any $(M, A^- \coprod A^+)$ as above, and for any $x^- \in \partial M^-$, and $x^+ \in \partial M^+$, the following diagram commutes:*

$$\begin{tikzcd}
\mathcal{A}_n(M, A^- \coprod A^+) \arrow[r, "T(g,\rho,\beta)"] \arrow[d, "\pi_*"] & T(g,\rho,\beta)(M, A^- \coprod A^+) \arrow[d, "\substack{\text{Contraction of}\\ \text{the factors of}\\ x^+ \text{ and } x^-}"] \\
\mathcal{A}_n(\tilde{M}, A^- \coprod A^+) \arrow[r, "T(g,\rho,\beta)"] & T(g,\rho,\beta)(\tilde{M}, A^- \coprod A^+)
\end{tikzcd}$$

where $\tilde{M} = M/(x^+ \sim x^-)$ denotes the compact oriented one-manifold obtained from M by identifying x^+ and x^-, $\pi : M \longrightarrow \tilde{M}$ is the associated quotient map, and π_* is the induced map on diagram spaces.

3. *For any $(M, A^- \coprod A^+)$ as above, and for any $a^- \in A^-$, and $a^+ \in A^+$, the following diagram commutes:*

$$\begin{tikzcd}
\mathcal{A}_n(M, A^- \coprod A^+) \arrow[r, "T(g,\rho,\beta)"] \arrow[d, "\iota"] & T(g,\rho,\beta)(M, A^- \coprod A^+) \arrow[d, "\substack{\text{Contraction of}\\ \text{the factors of}\\ a^+ \text{ and } a^-}"] \\
\mathcal{A}_{n-1}(M, \tilde{A}) \arrow[r, "T(g,\rho,\beta)"] & T(g,\rho,\beta)(M, \tilde{A})
\end{tikzcd}$$

where $\tilde{A} = (A^- \setminus \{a^-\}) \coprod (A^+ \setminus \{a^+\})$ and the map ι consists in identifying the two univalent vertices labeled by a^+ and a^-.

Proof: We first define $T = T(g, \rho, \beta)$ in a consistent way for diagrams, and next, we show that it factors through the relations AS, IHX, and STU. Except for the diagrams that have components like $x\bullet\text{---}\bullet y$ where x (by symmetry) belongs to the set labeled by $-$, any diagram can be decomposed into finitely many pieces like the four pieces where T has already been defined. The decomposition is unique when the number of $\longrightarrow$ is required to be minimal. Then the behaviour of T under gluings compels us to define T at any diagram as the tensor obtained by contracting the elementary tensors associated to the elementary pieces of such a decomposition, by the contractions corresponding to the gluings. To complete the definition, set

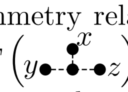
$$T(g, \rho, \beta)\left(x\bullet\text{---}\bullet y \in \mathcal{A}(\{x, y\}^-)\right) = \beta \in g^* \otimes g^*$$

and

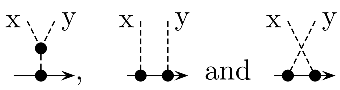
$$T(g, \rho, \beta)\left(x\bullet\text{---}\bullet y \in \mathcal{A}(\{x\}^- \coprod \{y\}^+)\right) = (\text{Identity} : g \to g) \in g^* \otimes g$$

Now, T is defined at any diagram.

When U, V and W are three $\mathbf{R}$-vector spaces, if $f \in \mathrm{Hom}(U, V) \cong U^* \otimes V$ and $g \in \mathrm{Hom}(V, W) \cong V^* \otimes W$, then the contraction of $V \otimes V^*$ maps $(f \otimes g) \in U^* \otimes V \otimes V^* \otimes W$ to $g \circ f$. In particular, inserting trivial pieces like $\longrightarrow$ in the decomposition of a diagram does not change the resulting tensor, and the behaviour under the identification of boundary points is the desired one for diagrams that do not contain exceptional components like $x\bullet\text{---}\bullet y$. For the other ones, it is enough to notice that a contraction of $g^* \otimes g$ maps the tensor product of the two symmetric tensors $\left(\Omega_\beta = \hat{\beta}^{-1}\right) \otimes \left(\beta \cong \hat{\beta}\right)$ to (Identity : $g \to g$) $\in g^* \otimes g$ that is mapped to the Identity map of g^* by the permutation of the factors. In fact, this just amounts to say that g and g^* are always identified via $\hat{\beta}$.

Now, T is consistently defined for diagrams. The antisymmetry relation for trivalent vertices comes from the total antisymmetry of T .

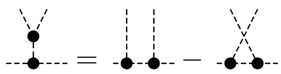
Let us show that T factors through STU. To do that, we view the values of T at the three local parts of STU



that are tensors in $g_x^* \otimes g_y^* \otimes E^* \otimes E$ as three bilinear maps from $g_x \times g_y$ to $gl(E)$, that map $(a,b) \in g_x \times g_y$ to $\rho([b,a])$, $\rho(b) \circ \rho(a)$ and $\rho(a) \circ \rho(b)$, respectively.

Thus, T factors through STU because the representation ρ represents.

By AS, the relation IHX can be redrawn as

IHX : 

and treated as a particular case of STU using the adjoint representation instead of ρ. Thus, T factors through IHX because the adjoint representation represents, and that is because of the Jacobi identity. ◇

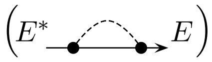
Remark 6.13. We could slightly generalize the maps T by *colouring* every component of the plain one-manifold with a different representation of g. Everything works similarly. Here gluings must respect colours.

Examples 6.14. Let us fix $N \in \mathbf{N} \setminus \{0,1\}$, and let us compute some examples when $g = sl_N$, $E = \mathbf{R}^N$, ρ is the standard representation that is the inclusion i_N of sl_N into gl_N, and $\beta = \beta_N$ is the associated invariant form. Set $T_N = T(sl_N, i_N, \beta_N)$.

1. The contraction from $gl(E) \cong E^* \otimes E$ to $\mathbf{R}$ is nothing but the trace of endomorphisms. In particular,

$$T_N(\bigcirc) = \text{trace(Identity of } \mathbf{R}^N) = N$$

2. Consider the following diagram $\Gamma_{11} \in \mathcal{A}(\uparrow)$ with only one chord.

$$\Gamma_{11} = $$ 

Let us compute $T_N(\Gamma_{11})$ viewed as an endomorphism of E. It is computed from the Casimir

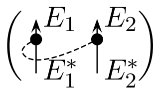
$$\Omega_{\beta_N} = \sum_{(i,j)\in\{1,\dots,N\}^2} e_{ij} \otimes e_{ji} - \frac{1}{N}\left(\sum_{i=1}^{N} e_{ii}\right) \otimes \left(\sum_{i=1}^{N} e_{ii}\right)$$

as

$$\begin{aligned} T_N(\Gamma_{11}) &= \textstyle\sum_{(i,j)\in\{1,\dots,N\}^2} e_{ji} \circ e_{ij} - \frac{1}{N}\left(\sum_{i=1}^{N} e_{ii}\right) \circ \left(\sum_{i=1}^{N} e_{ii}\right) \\ &= \left(N - \frac{1}{N}\right) (\text{Identity} = \textstyle\sum_{i=1}^{N} e_{ii}). \end{aligned}$$

Thus, $T_N(\Gamma_{11})$ is nothing but the multiplication by the number $(N^2-1)/N$.

3. Now, consider the following diagram $\Gamma_{12} \in \mathcal{A}(\uparrow\uparrow)$ with only one horizontal chord between the two strands.

$$\Gamma_{12} = $$ 

Let us compute $T_N(\Gamma_{12})$ viewed as an endomorphism of $(E_1 = E)\otimes(E_2 = E)$. It is computed from the Casimir Ω_{β_N} as

$$T_N(\Gamma_{12})(e_k \otimes e_l) = \sum_{(i,j)\in\{1,\dots,N\}^2} e_{ij}(e_k) \otimes e_{ji}(e_l) - \frac{1}{N}(e_k \otimes e_l)$$

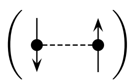
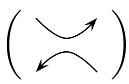
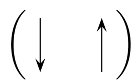
Thus,

$$T_N(\Gamma_{12})(e_k \otimes e_l) = e_l \otimes e_k - \frac{1}{N}(e_k \otimes e_l)$$

and, in general,

$$T_N(\Gamma_{12}) = \tau - \frac{1}{N}\text{Identity}$$

where $\tau : E \otimes E \longrightarrow E \otimes E$ is the transposition of the two factors that maps $(x \otimes y)$ to $(y \otimes x)$. This can be written as

$$T_N$$  $$= T_N$$  $$- \frac{1}{N} T_N$$ 

and allows for a simple recursive computation of the evaluation of T_N at chord diagrams on disjoint union of circles by induction on the number of chords starting with

$$T_N\left(1 \in \mathcal{A}(\coprod^k S^1)\right) = N^k.$$

4. Let $\wedge^2(E)$ be the subspace of $E \otimes E$ generated by the antisymmetric elements of the form $(x \otimes y - y \otimes x)$, and let $S^2(E)$ be the subspace of $E \otimes E$ generated by the symmetric elements of the form $(x \otimes y + y \otimes x)$. Since $x \otimes y = \frac{1}{2}((x \otimes y - y \otimes x) + (x \otimes y + y \otimes x))$, $E \otimes E = \wedge^2(E) + S^2(E)$. Furthermore, the restriction of $T_N(\Gamma_{12})$ to $S^2(E)$ is the multiplication by $(1 - \frac{1}{N})$ while the restriction of $T_N(\Gamma_{12})$ to $\wedge^2(E)$ is the multiplication by $(-1 - \frac{1}{N})$. In particular, $E \otimes E = \wedge^2(E) \oplus S^2(E)$, and this decomposes $E \otimes E$ into the two eigenspaces associated to the eigenvalues $(1 - \frac{1}{N})$ and $(-1 - \frac{1}{N})$. Let Γ_{12}^n denote the chord diagram with n horizontal chords between the two strands in $\mathcal{A}(\uparrow\uparrow)$. Then $T_N(\Gamma_{12}^n) = (1 - \frac{1}{N})^n p_{S^2(E)} + (-1 - \frac{1}{N})^n p_{\wedge^2(E)}$, and

$$\begin{aligned} T_N(\Gamma_{12}^n)(x \otimes y) = & \tfrac{(N-1)^n + (-N-1)^n}{2N^n}(x \otimes y) \\ & + \tfrac{(N-1)^n - (-N-1)^n}{2N^n}(y \otimes x) \end{aligned}$$

Remark 6.15. We can construct an injection Δ from $\overline{\mathcal{A}}_n(S^1)$ into $\mathcal{A}_n(S^1)$ as in [20] in order to be able to deduce linear forms on $\overline{\mathcal{A}}_n(S^1)$ from the above linear forms by composition.

We first contruct the linear map $\tilde{\delta} : \mathcal{D}_n(S^1) \longrightarrow \mathcal{A}_{n-1}(S^1)$ that maps a chord diagram D with n chords to the sum (over these n chords) of the n diagrams obtained by deleting one chord from D. This map factors through $\mathcal{A}_n(S^1)$. Let $\delta : \mathcal{A}_n(S^1) \longrightarrow \mathcal{A}_{n-1}(S^1)$ be the induced map. Then for any $(d, d') \in \mathcal{A}(S^1)^2$, with the algebra structure of Subsection 3.3,

$$\delta(dd') = \delta(d)d' + d\delta(d').$$

Now, define the linear map $\tilde{\Delta} : \mathcal{A}(S^1) \longrightarrow \mathcal{A}(S^1)$ that maps an element d of $\mathcal{A}_n(S^1)$ to

$$\tilde{\Delta}(d) = \sum_{k=0}^{n} \frac{(-1)^k}{k!}\theta^k \delta^k(d) \in \mathcal{A}_n(S^1).$$

Note that $\tilde{\Delta}$ is a morphism of algebras that maps θ to 0. Therefore $\tilde{\Delta}$ factors through $\overline{\mathcal{A}}_n(S^1)$. Let Δ be the induced map and let $P : \mathcal{A}(S^1) \longrightarrow \overline{\mathcal{A}}(S^1)$ be the canonical projection. Since $P \circ \Delta$ is the identity, Δ is an injection.

6.4 Questions

Question 1: *Do real-valued finite type invariants distinguish knots?*

Note that this question is equivalent to the following one. Does the Chern-Simons series distinguish knots? The answer to this question is unknown. Nevertheless, very interesting results obtained by Goussarov [12], Habiro [15] and Stanford [33] present modifications on knots that do not change their invariants of degree less than any given integer n and such that any two knots with the same Vassiliev invariants of degree less than n can be obtained from one another by a sequence of these explicit modifications.

Question 2: *Relate the Chern-Simons series to other invariants.*

There is another famous universal Vassiliev link invariant –that is a possibly different map $\overline{Z}$ that satisfies the conclusions of Theorem 2.26– that is called the *Kontsevich integral* and will be denoted by Z_K (see [4, 10, 23]). The HOMFLY polynomial that is a generalization of the Jones polynomial for links discovered shortly after independently by Hoste, Ocneanu, Millet, Freyd, Lickorish, Yetter, Przytycki and Traczyk, can be expressed as a function of Z_K as $P = T^N \circ Z_K$ where

$$T^N \left(\sum_{n \in \mathbf{N}} d_n \in \mathcal{A}(M) \right) = \sum_{n \in \mathbf{N}} \lambda^n T_N(d_n)$$

with the notation of Subsection 6.3. See [23]. The Homfly polynomial belongs to $\mathbf{R}[N][[\lambda]]$. The specialization of P at $N = 2$ is equal to the Jones polynomial. Setting $N = 0$ and performing the change of variables $t^{1/2} = \exp(\lambda/2)$ yields the famous *Alexander-Conway polynomial* discovered by Alexander in the beginning of the twentieth century. Le and Murakami [25] proved that all quantum invariants for knots can be obtained as the composition by Z_K of a linear map constructed as in Subsection 6.3.

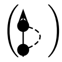
Is is conjectured but still unproved that the Kontsevich integral coincides with the Chern-Simons series. Sylvain Poirier proved that if the anomaly vanishes in degree greater than 6, then these two universal invariants coincide. In [24], using results of Poirier, I give the form of an isomorphism of $\overline{\mathcal{A}}$ that transforms one invariant into the other.

Let $\beta = (\beta_n)_{n \in \mathbf{N}}$ be an element of $\mathcal{A}(\emptyset, \{1, 2\})$ that is symmetric with respect to the exchange of 1 and 2. (In fact, according to [35, Corollary 4.2], all two-leg elements are symmetric with respect to this symmetry modulo the standard AS and IHX relations. This is reproved as Lemma 7.18 below.) If Γ is a chord diagram, then $\Psi(\beta)(\Gamma)$ is defined by replacing each chord by β. By Lemma 7.16, $\Psi(\beta)$ is a well-defined morphism of topological vector spaces from $\mathcal{A}(M)$ to $\mathcal{A}(M)$ for any one-manifold M, and $\Psi(\beta)$ is an isomorphism as soon as $\beta_1 \neq 0$.

Theorem 6.16 ([24]). *There exists*

$$\beta = (\beta_n)_{n \in \mathbf{N}} \in \mathcal{A}(\emptyset, \{1, 2\})$$

such that

- *the anomaly α reads $\alpha = \Psi(\beta)$* *,*
- *for any (zero-framed) link L, the Chern-Simons series $Z^0_{CS}(L)$ is equal to $\Psi(\beta)(Z_K(L))$.*

Of course, the following question is still open.

Question 3: *Compute the anomaly.*

After the articles of Axelrod, Singer [2, 3], Bott and Cattaneo [6, 7, 9], Greg Kuperberg and Dylan Thurston have constructed a universal finite type invariant for 3-dimensional homology spheres in the sense of [22, 29] as a series of configuration space integrals similar to Z^0_{CS}, in [21]. Their construction yields two natural questions:

Question 4: *Find a surgery formula for the Kuperberg-Thurston invariant in terms of the above Chern–Simons series.*

Question 5: *Compare the Kuperberg–Thurston invariant to the LMO invariant constructed in [26].*

7 Complements

7.1 Complements to Section 1

Remark 7.1. Definition 1.2 of isotopic embeddings is equivalent to the following one:

A *link isotopy* is a C^∞ map

$$h : \coprod^{k} S^1 \times I \longrightarrow \mathbf{R}^3$$

such that $h_t = h(.,t)$ is an embedding for all $t \in [0,1]$. Two embeddings f and g as above are said to be *isotopic* if there is an isotopy h such that $h_0 = f$ and $h_1 = g$.
The non-obvious implication of the equivalence comes from the isotopy extension theorem [17, Theorem 1.3, p.180].

Exercise 7.2. (**) Prove that for any C^∞ embedding $f : S^1 \longrightarrow \mathbf{R}^3$, there exists a continuous map $h : S^1 \times I \longrightarrow \mathbf{R}^3$ such that $h_t = h(.,t)$ is a C^∞ embedding for all $t \in [0,1]$, h_0 is a representative of the trivial knot, and $h_1 = f$. (Hint: Put the complicated part of f in a box, and shrink it.)

Sketch of proof of Proposition 1.5: In fact, it could be justified with the help of [17] that when the space of representatives of a given link is equipped with a suitable topology, the representatives whose projection is regular form a dense open subspace of this space. The reader can also complete the following sketch of proof. A *PL* or *piecewise linear* link representative is an embedding of a finite family of polygons whose restrictions to the polygon edges are linear. Such a PL representative can be *smoothed* by replacing a neighbborhood of a vertex like  by  in the same plane. It is a representative of our given link if the smooth representatives obtained by smoothing close enough to the vertices are representatives of our link. A planar linear projection of such a PL representative is *regular* if there are only finitely many multiple points which are only double points without vertices in their inverse image. Observe that an orthogonal projection of a generic PL representative is regular if the direction of the kernel of the projection avoids:
I. the vector planes parallel to the planes containing one edge and one vertex outside that edge.
II. the directions of the lines that meet the interiors of 3 distinct edges.
Fix a triple of pairwise non coplanar edges. Then for every point in the third edge there is at most one line intersecting this point and the two other edges. One can even see that the set of directions of lines intersecting these three edges is a dimension one compact submanifold of the projective plane $\mathbf{R}P^2$ parametrized by subintervals of this third edge. Thus, the set of allowed oriented directions for the kernel of the projection is the complement of a finite number of one-dimensional submanifolds of the sphere S^2. Therefore it is an open dense subset in S^2 according to a weak version of the Morse-Sard

theorem [17, Proposition 1.2, p.69] or [28, p.16]. Note that changing the direction of the projection amounts to composing the embedding by a rotation of $SO(3)$. Now, it is easy to smooth the projection, and to get a smooth representative whose projection is regular. $\diamond$

Lack of proof of the Reidemeister theorem 1.7: It could be proved by studying the topology of the space of representatives of a given link, that a generic path between two representatives whose projections are regular in this space (i.e. a generic isotopy) only meets a finite number of times three walls made of singular representatives. The first wall that leads to the first Reidemeister move is made of the representatives that have one vertical tangent vector (and nothing else prevents their projections from being regular). The second wall that leads to the second move is made of embeddings whose projections have one non-transverse double point while the third wall (that leads to RIII) is made of embeddings whose projections have one triple point. $\diamond$

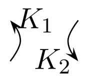
Remark 7.3. The unproved Reidemeister theorem is not needed in Sects. 2 to 6 of the course. Nevertheless, the following exercise can help understanding a proof idea of the Reidemeister theorem.

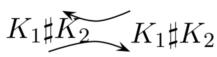
Exercise 7.4. (**) Say that a PL representative of a link is *generic* if its only pairs of coplanar edges are the pairs of edges that share one vertex. Consider a generic piecewise linear link representative. Prove that for any pair $(\rho, \sigma) \in SO(3)^2$ such that $\pi \circ \rho \circ f$ and $\pi \circ \sigma \circ f$ are regular, (smoothings of) $\pi \circ \rho \circ f$ and $\pi \circ \sigma \circ f$ are related by a finite sequence of Reidemeister moves.

Elementary proof of Proposition 1.10: Number the components of the link from 1 to m. Choose an open arc of every component K_i of the link whose projection does not meet any crossing, and choose two distinct points b_i and a_i on this oriented arc so that a_i follows b_i on this arc. Then change the crossings in your link diagram if necessary so that:
1. If $i < j$, K_i crosses K_j under K_j.
2. When we follow the component K_i from a_i to b_i, we meet the lowest preimage of the crossing before meeting the corresponding highest preimage.
After these (possible) modifications, we get a diagram of a (usually different)

link that is represented by an embedding whose first two coordinates can be read on the projection and whose third coordinate is given by a real-valued height function h that can be chosen so that $h(a_i) = 2i$, $h(b_i) = 2i + 1$, and h is strictly increasing from a_i to b_i and strictly decreasing from b_i to a_i. (This is consistent with the above assumptions on the crossings.) Then the obtained link is a disjoint union of components (separated by horizontal planes) that have at most two points at each height and that can therefore not be knotted. $\diamond$

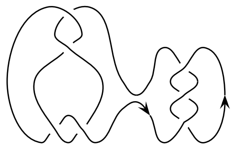
Definition 7.5. The *disjoint union* $K_1 \coprod K_2$ of two knots K_1 and K_2 is represented by two representatives of the knots sitting in two disjoint balls of the ambient space. The disjoint union of two regular projections of K_1 and K_2 -where the two projections lie in disjoint disks of the plane $\mathbf{R}^2$- is a regular projection of $K_1 \coprod K_2$. The local change in such a projection of $K_1 \coprod K_2$:



becomes


transforms $K_1 \coprod K_2$ into the *connected sum* $K_1 \sharp K_2$ of K_1 and K_2. The connected sum of knots is a commutative well-defined operation. (Prove it as an exercise!) A knot is said to be *prime* if it cannot be written as a connected sum of two non-trivial knots. Modulo commutativity, every knot can be expressed in a unique way as the connected sum of a finite number of prime knots. (See [27, Theorem 2.12].) Let K be a knot represented by an embedding f from S^1 into $\mathbf{R}^3$. The *reverse* $-K$ of K is the knot represented by the embedding $f \circ \mathrm{conj}$ where conj is the complex conjugation acting on the unit circle S^1 of the complex plane. The *mirror image* $\overline{K}$ of K is the knot represented by the embedding $\sigma \circ f$ where σ is the reflection of $\mathbf{R}^3$ such that $\sigma(x, y, z) = (x, y, -z)$. If K is presented by a diagram D, $\overline{K}$ is presented by the *mirror image* $\overline{D}$ of D that is obtained from D by changing all its crossings.

Examples 7.6. The two trefoil knots are mirror images of each other. The figure-eight knot is its own mirror image. (You have surely proved it when solving Exercise 1.8!) There are knots which are not equivalent to their reverses, like the eight-crossing knot 8_{17} in [27, Table 1.1, p.5]. Here is a picture of the connected sum of the figure-eight knot and the right-handed trefoil knot.



The *connected sum of the figure-eight knot and the right-handed trefoil knot*

A table of prime knots with at most 9 crossings is given in [32]. In this table, knots are not distinguished from their reverses and their mirror images. According to Thistlethwaithe, with the same conventions, the table of prime knots with 15 crossings contains 253293 items [27, Table 1.2, p.6].

Remark 7.7. Every knot bounds an oriented[8] embedded surface in $\mathbf{R}^3$. See [32, p.120] or [27, Theorem 2.2]. Such a surface is called a Seifert surface of the knot. The linking number of two knots could be defined as the algebraic intersection number of a knot with a Seifert surface of the other one... and there are lots of other definitions like the original Gauss definition that is given in Sect. 4.1.

Examples 7.8. There are numerical knot invariants that are easy to define but difficult to compute like:

• the *minimal number* of crossings $m(L)$ in a projection,

• the *unknotting number* of a knot that is the minimal number of crossing changes to be performed in $\mathbf{R}^3$ to unknot the knot (i.e. to make it equivalent to the trivial knot),

• the *genus* of a knot that is the minimal genus of an oriented embedded surface bounded by the knot.

An invariant is said to be *complete* if it is injective. The knot itself is a complete invariant. There are invariants coming from algebraic topology like the fundamental group of the complement of the link. A *tubular neighborhood* of a knot is a solid torus $S^1 \times D^2$ embedded in $\mathbf{R}^3$ such that its core $S^1 \times \{0\}$ is (a representative of) the knot. (See [17, Theorem 5.2, p.110] for the existence of tubular neighborhoods.) A *meridian* of a knot is the boundary of a small disk that intersects the knot once transversally and positively. A *longitude* of the knot is a curve on the boundary of a tubular neighborhood of the knot that is parallel to the knot. Up to isotopy of the pair (knot, longitude), the

[8] Boundaries are always oriented with the "outward normal first" convention.

longitudes of a knot are classified by their linking number with the knot. The *preferred longitude* of a knot is the one such that its linking number with the knot is zero. According to a theorem of Waldhausen [36], the fundamental group equipped with two elements that represent the oriented *meridian* of the knot and the preferred *longitude* is a complete invariant of the knot. (See also [16, Chap. 13]). According to a more recent difficult theorem of Gordon and Luecke [13], the *knot complement*, that is the compact 3-manifold that is the closure of the complement of a knot tubular neighborhood (viewed up to orientation-preserving homeomorphism), determines the knot up to orientation. Nevertheless, these meaningful invariants are hard to manipulate.

Exercise 7.9. The *segments* of a link diagram are the connected components of the link diagram that are segments between two undercrossings where the diagram is broken. An *admissible 3-colouring* of a link diagram is a function from the set of segments of a diagram to the three-element set {Blue, Red, Yellow} such that, for any crossing, the image of the set of (usually three) segments that meet at the crossing contains either one or three elements (exactly). Prove that the number of admissible 3-colourings is a link invariant. Use this invariant number to distinguish the trefoil knots from the figure-eight knot, and the Borromean link from the trivial 3-component link. (In fact the admissible 3-colourings of a link are in one-to-one correspondence with the representations of the fundamental group of the link complement to the group of permutations of 3 elements that map the link meridians to transpositions.)

The following additional properties of the Jones polynomial are not hard to check.

Proposition 7.10. *The Jones polynomial V satisfies the additional properties:*

3. For any link L,

$$V(\overline{L})(t) = V(L)(t^{-1}).$$

4. For any two links L_1 and L_2,

$$V(L_1 \coprod L_2) = -(t^{1/2} + t^{-1/2})V(L_1)V(L_2).$$

5. For any two knots K_1 and K_2,

$$V(K_1 \sharp K_2) = V(K_1)V(K_2).$$

7.2 An Application of the Jones Polynomial to Alternating Knots

Definition 7.11. A link diagram is said to be *alternating* if the over-crossings and the under-crossings alternate as one travels along the link components. In this subsection 7.2, a *connected* link diagram is a diagram whose underlying knot projection is connected. In a connected link diagram, a crossing is said to be *separating* if one of the two transformations of the link diagram “ becomes ” or “ becomes ” makes the diagram disconnected, or, equivalently, if the transformation “ becomes ” makes the diagram disconnected.

The Jones polynomial allowed Kauffman and Murasugi to prove the following theorem in 1988, independently. This answered a Tait conjecture of 1898.

Theorem 7.12 (Kauffman-Murasugi, 1988). *When a knot K has a connected alternating diagram without separating crossing with c crossings, then c is the minimal number of crossings $m(K)$ of K.*

This theorem is a direct consequence of Proposition 7.14 about the properties of the breadth of the Jones polynomial.

Definition 7.13. The *breadth* $B(P)$ of a Laurent polynomial P is the difference between the maximal degree and the minimal degree occuring in the polynomial.

Proposition 7.14. *When a link L has a connected diagram with c crossings, then $B(V(L)) \leq c$. When a link L has a connected alternating diagram without separating crossing with c crossings, then $B(V(L)) = c$.*

The proof of this proposition will also yield the following obstruction for a link to have a connected alternating diagram without separating crossing.

Proposition 7.15. *When a link L has a connected alternating diagram without separating crossing, the coefficients of the terms of extremal degrees in its Jones polynomial $V(L)$ are ± 1. Furthermore, the product of these two coefficients is $(-1)^{B(V(L))}$.*

In particular, this obstruction shows that the eight-crossing knot 8_{21} has no connected alternating diagram without separating crossing. See [27, Table 1.1, p.5; Table 3.1, p.27].

PROOF OF PROPOSITIONS 7.14 AND 7.15: Let L be a link, and let D be one of its diagrams with c crossings. Observe that

$$B(V(L)) = \frac{B(< D >)}{4}$$

We will refer to the construction of the Kauffman bracket at the beginning of Subsection 1.3. Let $C(D)$ denote the set of crossings of D. Let $f^L : C(D) \longrightarrow \{L, R\}$ be the constant map that maps every crossing to L, and let $f^R : C(D) \longrightarrow \{L, R\}$ be the constant map that maps every crossing to R. Set $n_L = n(D_{f^L})$ and $n_R = n(D_{f^R})$. We shall prove:

(i) $B(< D >) \leq 2c + 2n_L + 2n_R - 4$

(ii) The above inequality is an equality when D is a connected alternating diagram without separating crossing.

(iii) If D is a connected alternating diagram, then $n_L + n_R - 2 = c$.

(iv) If D is a connected diagram, then $n_L + n_R - 2 \leq c$.

It is clear that these four properties imply Proposition 7.14. Let us prove these properties.

A map f from $C(D)$ to $\{L, R\}$ gives rise to the term

$$A^{(\sharp f^{-1}(L) - \sharp f^{-1}(R))} \delta^{(n(D_f)-1)}$$

in $< D >$. This summand is a polynomial in A and A^{-1}
whose highest degree term is $(-1)^{(n(D_f)-1)} A^{h(f)}$ with

$$h(f) = \sharp f^{-1}(L) - \sharp f^{-1}(R) + 2(n(D_f) - 1)$$

and whose lowest degree term is $(-1)^{(n(D_f)-1)} A^{\ell(f)}$ with

$$\ell(f) = \sharp f^{-1}(L) - \sharp f^{-1}(R) - 2(n(D_f) - 1).$$

Observe that $h(f^L) = c + 2n_L - 2$ and that $\ell(f^R) = -c - 2n_R + 2$.

Therefore, in order to prove (i), it is enough to show that for any map f from $C(D)$ to $\{L, R\}$, we have

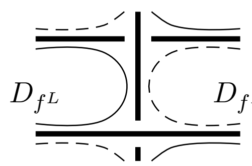
$$h(f) \leq h(f^L) \text{ and } \ell(f) \geq \ell(f^R)$$

Notice that if f and g are two maps from $C(D)$ to $\{L,R\}$ that coincide at every crossing but one, then $n(D_f)=n(D_g)\pm 1$. This property allows us to prove $h(f)\leq h(f^L)$ for all f by induction on $\sharp f^{-1}(R)$. Indeed, changing a value L of a map f into the value R removes 2 from $(\sharp f^{-1}(L)-\sharp f^{-1}(R))$ whereas $2(n(D_f)-1)$ cannot increase by more than 2 by the above property. Thus, for all f, $h(f)\leq h(f^L)$. Similarly, $\ell(f)\geq \ell(f^R)$ and (i) is proved.

We prove (ii) and Proposition 7.15 for a connected alternating diagram D without separating crossing with c crossings. In view of the above arguments (and by definition of the Jones polynomial for Proposition 7.15), it is enough to prove that for any non constant map f from $C(D)$ to $\{L,R\}$, we have

$$h(f)<h(f^L) \text{ and } \ell(f)>\ell(f^R).$$

We consider the underlying projection of D in the one-point compactification of the plane $\mathbf{R}^2$ that is the sphere S^2. The *faces* of the diagram D will be the connected components of $S^2\setminus D$. Since D is connected, these components have only one boundary component. Therefore they are topological disks. The alternating nature of D allows us to push each connected component of D_{f^L} or D_{f^R} inside one face as the following picture shows:



This pushing defines a one-to-one correspondence between the faces of D and the connected components of $D_{f^L}\coprod D_{f^R}$. Choose a crossing x of D. Since x is not separating, the two parts of D_{f^L} near x bound distinct faces of D, and thus they belong to different components of D_{f^L}. Therefore, changing the value of f^L at x into R, changes D_{f^L} into a diagram D_f such that $n(D_f)=n_L-1$, where f maps $(C(D)\setminus x)$ to L and x to R. Thus, $h(f)<h(f^L)$ for all the maps f such that $\sharp f^{-1}(R)=1$, and for all the maps such that $\sharp f^{-1}(R)>0$ by induction on $\sharp f^{-1}(R)$. Similarly, $\ell(f)>\ell(f^R)$ for all the maps f different from f^R. Thus (ii) and the first part of Proposition 7.15 are proved. The second part of 7.15 is a consequence of the above arguments and (iii) below.

(iii) is obtained by computing the Euler characteristic of the sphere as the number (n_L+n_R) of faces of D plus the number (c) of crossings of D minus the number $(2c)$ of edges of the projection –that contain exactly two crossings which are at their extremities–. (It could also be proved by induction on c.)

We prove (iv). Let D be a connected diagram with c crossings. We want to prove:

$$n_L + n_R - 2 \leq c$$

by induction on c. This is true for the only connected diagram without crossing that is the diagram of the unknot. Let D be a connected diagram with $c \geq 1$ crossings. Let x be one of its crossings, and let D' be the diagram obtained from D by removing x in the left-handed way.
If D' is connected, then $n(D_{fL}) = n(D'_{f'L})$ while $D'_{f'R}$ is equal to D_f where f maps $C(D) \setminus x$ to R and x to L, thus $n(D_{fR}) = n(D'_{f'R}) \pm 1$. Therefore, the inequality that is true for the diagram D' that has $(c-1)$ crossings, by induction hypothesis, implies that the inequality holds for D.
If D' is not connected, then the diagram $\overline{D}'$ obtained from the mirror image $\overline{D}$ of D by removing x in the left-handed way is connected, the inequality holds for $\overline{D}$ by the above argument, therefore it is true for D.

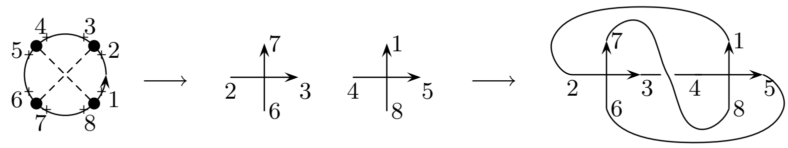
$\diamond$

7.3 Complements to Section 2

Diagrammatic proof of Lemma 2.17: Though the first assertion is very easy, we prove it because its proof is the beginning of the proof of the second assertion. Let d be an n-chord diagram on S^1. Put 4n *cutting points,* numbered from 1 to 4n along S^1, on the support S^1 of d, one near each extremity of the 2n intervals separated by the vertices, so that $2i-1$ and $2i$ are on the same interval. Next embed the neighborhoods of the n double points, bounded by the 4n cutting points, into n fixed disjoint disks in $\mathbf{R}^2 = \mathbf{R}^2 \times \{0\}$ so that the cutting points lie on the boundaries of these disks. Choose n disjoint 3-dimensional balls that intersect $\mathbf{R}^2$ along these n disks. Let C be the closure of the complement of these n fixed balls in $\mathbf{R}^3$.

Then, in order to construct our first representative K^0 of d, it is enough to notice that we have enough room to embed the *remaining 2n intervals* of S^1 (the $[2i-1, 2i]$) into C. Next, the proof could be "concluded" as follows: Let K be another representative of d. After an isotopy, we may assume that K intersects our n balls like K^0 does. Then, since $\pi_1(C)$ is trivial, there is a boundary-fixing homotopy in C that maps the remaining 2n intervals for K to the remaining 2n intervals for K^0. Such a homotopy may be approximated by a finite sequence of (isotopies and) crossing changes, and we are done. However, we will again give a planar elementary proof.

We may demand that the orthogonal projection π of K^0 is regular and that the projections of the intervals $]2i-1, 2i[$ are embeddings that avoid the fixed neighborhoods of the double points, and that the interval $]2i-1, 2i[$ is under $]2j-1, 2j[$ if $i < j$. The projections of these three steps are represented in the following example:



Let K be another representative of d. After an isotopy, we may assume that K intersects our n balls like K^0 does and that the projection of K is regular and avoids the fixed neighborhoods of the double points. After some crossing changes, we may assume that $]1, 2[$ is above the other $]2i-1, 2i[$, we may unknot it as in the proof of Proposition 1.10 and we may assume that its projection coincides with the restriction of the projection of K^0. Do the same for $]3, 4[$: put it above everything else, unknot it, and make its projection coincide with the corresponding one for K^0, then for $]5, 6[$, ..., and finish with $]4n-1, 4n[$.

$\diamond$

7.4 Complements to Subsection 6.4

Let β be an element of $\mathcal{A}(\emptyset, \{1, 2\})$. Let Γ be a diagram with support M as in Definition 3.1. We define $\Psi(\beta)(\Gamma)$ to be the element of $\mathcal{A}(M)$ obtained by inserting β d times on each degree d component of Γ (where a *component* of Γ is a connected component of the dashed graph).

Lemma 7.16. *$\Psi(\beta)(\Gamma)$ does not depend on the choice of the insertion loci.*

PROOF: It is enough to prove that moving β from an edge of Γ to another one does not change the resulting element of $\mathcal{A}(M)$, when the two edges share some vertex v. Since this move amounts to slide v through β, it suffices to prove that sliding a vertex from some leg of a two leg-diagram to the other one does not change the diagram modulo AS, IHX and STU, this is a direct consequence of Lemma 3.4 when the piece of diagram inside D is β. $\diamond$

It is now easy to check that $\Psi(\beta)$ is compatible with the relations IHX, STU and AS. This allows us to define continuous vector space endomorphisms $\Psi(\beta)$ of the $\mathcal{A}(M)$ such that, for any diagram Γ:

$$\Psi(\beta)([\Gamma]) = \Psi(\beta)(\Gamma).$$

Say that $(d_n)_{n\in\mathbf{N}} \in \mathcal{A}(M)$ is *of filtration at least* d if $d_k = 0$ for any $k < d$. $\Psi(\beta)$ satisfies the following properties:

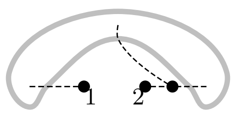
Lemma 7.17. *1. $\Psi(\beta)$ is compatible with the products of Subsection 3.3. ($\Psi(\beta)(xy) = \Psi(\beta)(x)\Psi(\beta)(y)$.)*

2. If $\beta_1 \neq 0$, $\Psi(\beta)$ is an isomorphism of topological vector spaces such that $\Psi(\beta)$ and $\Psi(\beta)^{-1}$ map elements of filtration at least d to elements of filtration at least d.

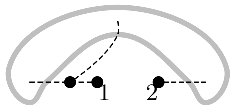
PROOF: The first property is obvious. For the second one, first note that $\beta_1 = b_1$ for some non zero number b_1. Thus, for $x = \sum_{i=d}^{\infty} x_i$, $\Psi(\beta)(x) - (b_1^d x_d)$ is of filtration at least $d+1$. This shows that $\Psi(\beta)$ is injective and allows us to construct a preimage for any element by induction on the degree, proving that Ψ is onto. $\diamond$

Lemma 7.18 (Vogel). *Elements of $\mathcal{A}(\emptyset, \{1,2\})$ are symmetric with respect to the exchange of 1 and 2.*

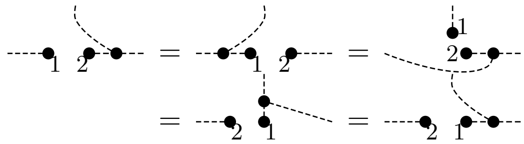
PROOF: Since a chord is obviously symmetric, we can restrict ourselves to a two-leg diagram with at least one trivalent vertex and whose two univalent vertices are respectively numbered by 1 and 2. We draw it as



where the dashed trivalent part inside the thick topological circle is not represented. Applying Lemma 3.4 where the annulus is a neighborhood of the thick topological circle that contains the pictured trivalent vertex shows that this diagram is equivalent to



This yields the relations



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More references on finite type invariants may be found in Dror Bar-Natan's bibliography on Vassiliev invariants at http://www.ma.huji.ac.il/~drorbn.