EXERCISES

Exercise 1. Let X be a complex variety. For a vector bundle $\pi\colon \mathrm{E} \to \mathrm{X}$ consider the sheaf $\mathscr{E}$ on X defined for an open $\mathrm{U} \subset \mathrm{X}$ as

$$\mathscr{E}(\mathrm{U}) \;=\; \{\text{sections of } \pi^{-1}(\mathrm{U}) \to \mathrm{U}\}.$$

Show that $\mathscr{E}$ is a locally free $\mathscr{O}_\mathrm{X}$-module of finite rank. Moreover, any locally free $\mathscr{O}_\mathrm{X}$-module of finite rank arises in this way. (More precisely, the association $\mathrm{E} \rightsquigarrow \mathscr{E}$ sets up an equivalence between the category of vector bundles over X and that of locally free $\mathscr{O}_\mathrm{X}$-modules of finite rank.)

Exercise 2. Let X be a complex projective variety. Show that any complex closed submanifold (or more generally complex subspace) of $\mathrm{X}(\mathbb{C})$ is algebraic. (Hint: use Serre's GAGA theorem.)

Exercise 3. Let X and Y be (smooth) complex projective varieties. Show that any holomorphic map $\mathrm{X}(\mathbb{C}) \to \mathrm{Y}(\mathbb{C})$ is algebraic. (Hint: use Serre's GAGA theorem.)

Exercise 4. Show that any algebraic line bundle on $\mathbb{C}^n$ and on $(\mathbb{C}^*)^n$ is trivial.

Exercise 5. Let E and F be vector bundles over an algebraic variety X. Show that the cohomology group $\mathrm{H}^1(\mathrm{X}, \mathscr{H}om(\mathrm{E}, \mathrm{F}))$ parametrizes the isomorphism classes of extensions of E by F. Adapt the statement and the proof for holomorphic objects.

Exercise 6. ($\star$) Describe explicitly all holomorphic line bundles on $(\mathbb{C}^*)^n$. (Hint: write $\mathbb{C}^* \cong \mathbb{C}/\mathbb{Z}$ and get inspired by the Appell-Humbert theorem.)

Exercise 7. ($\star\star$) Let X be a smooth complex affine curve. Show that any algebraic or holomorphic line bundle on X is trivial.

Exercise 8. Write down a basis of the cohomology group $\mathrm{H}^1(\mathrm{X}, \mathscr{O}_\mathrm{X})$ of the algebraic variety $\mathrm{X} = \mathbb{C}^2 \smallsetminus \{0\}$.

Exercise 9. Let ∇ and ∇' be algebraic connections on a vector bundle E on a complex smooth variety X. Show that there is a global section ω of the vector bundle $\Omega^1_\mathrm{X} \otimes \mathscr{E}nd(\mathrm{E})$ such that $\nabla' = \nabla + \omega$.

Exercise 10. Write down all algebraic connections on the line bundle $\mathrm{E} = \mathrm{X} \times \mathbb{C}^n$ on an open subset $\mathrm{X} = \mathbb{P}^1(\mathbb{C})$.

Exercise 11. ($\star$) Let X be a topological space. Given a complex vector space V, the constant sheaf $\mathscr{V}$ on X of value V is defined, for an open $\mathrm{U} \subset \mathrm{X}$, as

$$\mathscr{V}(\mathrm{U}) \;:=\; \{\text{locally constant maps } \mathrm{U} \to \mathrm{V}\}.$$

A sheaf $\mathscr{F}$ on X is said to be *locally constant* or a *local system* if any $x \in \mathrm{X}$ admits an open neighbourhood on which $\mathscr{F}$ is constant. Suppose that X is connected, locally arc connected and locally simply connected. For $x \in \mathrm{X}$ set up an equivalence between the category of local systems on X and that of representation of the fundamental group $\pi_1(\mathrm{X}, x)$.

Exercise 12. Let E be a holomorphic vector bundle endowed with a holomorphic connection ∇ on a Riemann surface X. Using Cauchy's theorem on the solution of differential equations with holomorphic coefficients show that $\mathrm{E}^{\nabla} := \mathrm{Ker}\,\nabla$ is a local system on X. Show that any local system of finite rank arises in this way. (Better, this sets an equivalence between the category of holomorphic vector bundle with a connection on X and that of local systems of finite rank on X.)