Tropical geometry

Exercises (second session)

Exercise 1 (a) Prove that the only root of the tropical polynomial $P(x) = “x”$ is $-\infty$.

(b) Prove that x_0 is a root of order k of a tropical polynomial $P(x)$ if and only if there exists a tropical polynomial $Q(x)$ such that $P(x) = “(x + x_0)^k Q(x)”$ and x_0 is not a root of $Q(x)$.

(c) Prove that the tropical semi-field $\mathbb{T}$ is algebraically closed, in other words, every tropical polynomial in one variable of positive degree d has exactly d roots when counted with multiplicities.

Exercise 2 Let d be a positive integer number, and let $T_d \subset \mathbb{R}^n$ be the n-dimensional simplex with vertices $(0, 0, 0, \ldots, 0, 0)$, $(d, 0, 0, \ldots, 0, 0)$, $(0, d, 0, \ldots, 0, 0)$, $\ldots$, $(0, 0, 0, \ldots, 0, d)$. Prove that a tropical hypersurface X in $\mathbb{R}^n$ with Newton polytope T_d has at most d^n vertices, and that X is non-singular if and only if equality holds.

Exercise 3 Let $T_2 \subset \mathbb{R}^3$ be the tetrahedron with vertices $(0, 0, 0)$, $(2, 0, 0)$, $(0, 2, 0)$, and $(0, 0, 2)$, and let S be a non-singular tropical surface in $\mathbb{R}^3$ with Newton polytope T_2. Show that S has a unique compact facet.

Exercise 4 Let d be a positive integer, and let $T_d \subset \mathbb{R}^3$ be the tetrahedron with vertices $(0, 0, 0)$, $(d, 0, 0)$, $(0, d, 0)$, and $(0, 0, d)$. Consider a real non-singular tropical surface in $\mathbb{R}^3$ with Newton polytope T_d, and denote by F_t the corresponding family of homogeneous Viro polynomials in three variables. Assuming that t is sufficiently big, calculate the Euler characteristic of the zero locus of F_t in $\mathbb{R}P^3$.