Tropical geometry

Exercises (first session)

Exercise 1 Let $ABCD$ be a square. Find a convex triangulation τ_1 of the triangle ABD and a convex triangulation τ_2 of the triangle BCD such that τ_1 and τ_2 form together a non-convex triangulation of the square $ABCD$.

Exercise 2 Let k be a positive integer number. Using the combinatorial patchworking construct

(a) a non-singular curve of degree $2k$ in $\mathbb{R}P^2$ such that the real point set of this curve is empty ;

(b) a non-singular curve of degree $2k-1$ in $\mathbb{R}P^2$ such that the real point set of this curve is connected.

Exercise 3 Let d be a positive integer number, and let $T_d \subset \mathbb{R}^2$ be the triangle with vertices $(0,0)$, $(d,0)$, and $(0,d)$.

(a) Assume that each integer point (i,j) of T_d such that i and j are both even is equipped with the sign "-", and assume that all other integer points of T_d are equipped with the sign "+". Choose any convex primitive triangulation of T_d. Prove that the chosen triangulation and the chosen distribution of signs produce, via the combinatorial patchworking, a maximal curve of degree d in $\mathbb{R}P^2$.

(b) Assume now that each integer point of T_d is equipped with the sign "+". Find a convex primitive triangulation of T_d such that this triangulation and the chosen distribution of signs produce, via the combinatorial patchworking, a maximal curve of degree d in $\mathbb{R}P^2$.

Exercise 4 Let $P(x,y) = \text{“}\sum_{i,j} a_{i,j}x^iy^j\text{”}$ be a tropical polynomial. The *Newton polygon* of $P(x,y)$, denoted by $\Delta(P)$, is defined by

$$\Delta(P) = Conv\{(i,j) \in (\mathbb{Z}_{\geq 0})^2 \mid a_{i,j} \neq -\infty\} \subset \mathbb{R}^2.$$

Given a point $(x_0,y_0) \in \mathbb{R}^2$, let

$$\Delta_{(x_0,y_0)} = Conv\{(i,j) \in (\mathbb{Z}_{\geq 0})^2 \mid P(x_0,y_0) = \text{“}a_{i,j}x_0^iy_0^j\text{”}\} \subset \Delta(P).$$

The tropical curve C defined by $P(x,y)$ induces a polyhedral decomposition Θ of $\mathbb{R}^2$, and the polygon $\Delta_{(x_0,y_0)}$ only depends on the cell $F \ni (x_0,y_0)$ of Θ. Thus, we define $\Delta_F = \Delta_{(x_0,y_0)}$ for $(x_0,y_0) \in F$.

Prove that the polyhedra Δ_F form a subdivision of the Newton polygon $\Delta(P)$ and that this subdivision is dual to Θ in the following sense :

— $\Delta(P) = \bigcup_F \Delta_F$, where the union is taken over all cells F of the decomposition Θ ;
— $\dim F = \operatorname{codim} \Delta_F$ for each cell F of Θ ;
— Δ_F and F are orthogonal for each cell F of Θ ;
— $\Delta_F \subset \Delta_{F'}$ if and only if $F' \subset F$; furthermore, in this case Δ_F is a face of $\Delta_{F'}$;
— Δ_F is contained in the boundary of $\Delta(P)$ if and only if F is unbounded.

Exercise 5 (a) Let d be a positive integer. Show that a tropical curve defined in $\mathbb{R}^2$ by a polynomial of degree d has at most d^2 vertices.

(b) Show that the first Betti number of a non-singular tropical curve in $\mathbb{R}^2$ is equal to the number of integer points contained in the interior of the Newton polygon of a tropical polynomial defining this curve.

Exercise 6 A finite rectilinear graph $\Gamma \subset \mathbb{R}^2$ whose edges have rational slopes and are equipped with positive integer weights is called a *balanced graph* if Γ satisfies the balancing condition at each vertex. Prove that any balanced graph in $\mathbb{R}^2$ represents a tropical curve.