A crash course in (higher) topos theory I

André Joyal

UQÀM

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I would like to thank the organisers

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for their invitation.

In this first talk, I will introduce the notion of (Grothendieck) topos with examples.

Topoi (or toposes) were introduced in algebraic geometry by Grothendieck.

I will use the term "logos" instead of "topos" for reasons that will be become clear.

What is logos ?

Definition

We say that a category $\mathcal{E}$ is a *logos* (or a *topos*) if the following conditions hold:

1. $\mathcal{E}$ is presentable;
2. Colimits are stable by base changes (distributive law);
3. The descent principle holds for monomorphisms.

A *homomorphism* of logoi $\phi : \mathcal{E} \to \mathcal{E}'$ is a cocontinuous functor which is left exact (=preserves finite limits).

The category of logoi **Logos** is a 2-category.

The category of topoi **Topos** is the opposite 2-category $\textbf{Logos}^{op}$.

Logos vs commutative ring

Logos	*commutative ring*
colimits	sums
finite limits	products
$u^\star : \mathcal{E}/B \to \mathcal{E}/A$ cocontinuous	distributive law
homomorphism of logoi $\phi : \mathcal{E} \to \mathcal{E}'$	homomorphism of rings $\phi : A \to B$

Logos	*commutative ring*
Set	integers $\mathbb{Z}$
Set$[U]$	polynomial ring $\mathbb{Z}[X]$
Slice extension $\mathcal{E} \to \mathcal{E}/B$	monogenic extension $A \to A[x]$
lex localisation $\rho : \mathcal{E} \to Sh(\mathcal{E}, \Sigma)$	quotient rings $\rho : A \to A/J$

Examples

1. The category of sets *Set* is a logos;
2. If $\mathcal{E}$ is a logos, then so is the category $\mathcal{E}/B$ for any object B;
3. The category of presheaves $Psh(\mathcal{K}) = [\mathcal{K}^{op}, Set]$ on a small category $\mathcal{K}$ is a logos;
4. If $\mathcal{E}$ is a logos and $\mathcal{K}$ is a small category, then the category $\mathcal{E}^{\mathcal{K}} = [\mathcal{K}, \mathcal{E}]$ of diagrams $\mathcal{K} \to \mathcal{E}$ is a logos
5. $\mathcal{E}$ is a logos, then so the category of sheaves $Sh(\mathcal{E}, \Sigma)$ for any set of monomorphisms $\Sigma \subseteq \mathcal{E}$;
6. Every logos is equivalent to a category of sheaves on a small category.

Plan

1. Presentable categories
2. The distributive law
3. The descent principle
4. The logos $\mathit{Set}[U]$
5. Sheaves and left exact localisations

1.1 Complete and complete categories

Roughly speaking, $\mathcal{E}$ is said to *presentable* if it is cocomplete and admits a presentation by generators and relations (see the appendix).

Recall that a category $\mathcal{E}$ is said to be *complete* if every diagram $F : \mathcal{J} \to \mathcal{E}$ has a limit $\varprojlim_{\mathcal{J}} F \in \mathcal{E}$.

A functor between complete categories $\phi : \mathcal{E} \to \mathcal{F}$ is said to be *continuous* if it preserves limits.

A category $\mathcal{E}$ is said to be *cocomplete* if every diagram $F : \mathcal{J} \to \mathcal{E}$ has a colimit $\varinjlim_{\mathcal{J}} F \in \mathcal{E}$.

A functor between cocomplete categories $\phi : \mathcal{E} \to \mathcal{F}$ is said to be *cocontinuous* if it preserves colimits.

The category of presheaves $Psh(\mathcal{K}) = [\mathcal{K}^{op}, Set]$ on a small category $\mathcal{K}$ is complete and cocomplete.

1.2. The Yoneda functor

The presheaves on $\mathcal{K}$ form a category $Psh(\mathcal{K}) := [\mathcal{K}^{op}, Set]$.

The *Yoneda functor* $Y : \mathcal{K} \to Psh(\mathcal{K})$ is defined by putting $Y(k) = Map(-, k) : \mathcal{K}^{op} \to Set$ for every object $k \in \mathcal{K}$.

Lemma

[Yoneda] *The presheaf $Y(k)$ is freely generated by the element $1_k \in Y(k)(k)$.*

In other words, for every $F \in Psh(\mathcal{K})$ and every element $a \in F(k)$ there exists a unique natural transformation $\phi : Y(k) \to F$ such that $\phi(1_k) = a$. It follows that

$$Hom(Y(k), F) \simeq F(k)$$

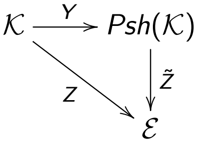
.

1.3. The category $Psh(\mathcal{K})$

Theorem

The category $Psh(\mathcal{K})$ is cocomplete and freely generated by the Yoneda functor $Y : \mathcal{K} \to Psh(\mathcal{K})$.

In other words, for any cocomplete category $\mathcal{E}$ and any functor $Z : \mathcal{K} \to \mathcal{E}$ there exists a cocontinuous functor $\tilde{Z} : Psh(\mathcal{K}) \to \mathcal{E}$ such that $\tilde{Z} \circ Y = Z$.



The functor $\tilde{Z}$ is unique up to unique isomorphism. We have

$$\tilde{Z}(F) = \int^{k \in \mathcal{K}} F(k) \times Z(k)$$

1.4 Localisation

An object X in a category $\mathcal{E}$ is said to be *local* with respect to a morphism $u : A \to B$ if the map

$$Map(u, X) : Map(B, X) \to Map(A, X)$$

is invertible.

An object $X \in \mathcal{E}$ is said to be *local* with respect to a set of morphisms $\Sigma \subseteq \mathcal{E}$ if it is local with respect to every map in Σ.

We shall denote by $\mathcal{E}^{\Sigma}$ the full subcategory of $\mathcal{E}$ spanned by the Σ-local objects of $\mathcal{E}$.

We shall say that the subcategory $\mathcal{E}^{\Sigma}$ is a *localisation* if the inclusion functor $\mathcal{E}^{\Sigma} \subseteq \mathcal{E}$ has a left adjoint

$$\rho : \mathcal{E} \to \mathcal{E}^{\Sigma}$$

called the *reflector*.

1.5 Presentable categories

Theorem

[Gabriel & Ulmer] *If $\mathcal{K}$ is a small category and $\Sigma \subseteq Psh(\mathcal{K})$ is a set of maps, then the subcategory $Psh(\mathcal{K})^{\Sigma} \subseteq Psh(\mathcal{K})$ is reflective.*

The category $Psh(\mathcal{K})^{\Sigma}$ is cocomplete and the reflector $\rho : Psh(\mathcal{K}) \to Psh(\mathcal{K})^{\Sigma}$ is cocontinuous.

Definition

A cocomplete category $\mathcal{E}$ is said to be *presentable* if it is equivalent to a category $Psh(\mathcal{K})^{\Sigma}$ for a small category $\mathcal{K}$ and a set of maps $\Sigma \subseteq Psh(\mathcal{K})$.

For example, the category $Psh(\mathcal{K})$ is presentable.

1.6. Presentable categories

A presentable category $\mathcal{E}$ is complete and cocomplete.

If a category $\mathcal{E}$ is presentable then every continuous functor $F : \mathcal{E}^{op} \to Set$ is representable.

Theorem

If $\mathcal{E}$ is presentable and $\mathcal{C}$ is cocomplete, then every cocontinuous functor $\phi : \mathcal{E} \to \mathcal{C}$ has a right adjoint $\phi_\star : \mathcal{C} \to \mathcal{E}$.

In particular, a cocontinuous functor between presentable categories $\phi : \mathcal{E} \to \mathcal{F}$ has a right adjoint $\phi_\star : \mathcal{F} \to \mathcal{E}$.

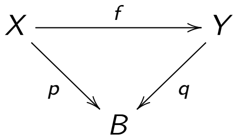
2.1. The slice category $\mathcal{E}/B$

Recall that if B is an object in a category $\mathcal{E}$, then $\mathcal{E}/B$ is a category whose objects are the maps $p : X \to B$ in $\mathcal{E}$.

The pair (X, p) is an object of $\mathcal{E}$ *over* B.

The map $p : X \to B$ is called the *structure map* of (X, p)

A *morphism* $f : (X, p) \to (Y, q)$ in $\mathcal{E}/B$ is a map $f : X \to Y$ in $\mathcal{E}$ such that $qf = p$.



2.2. Internal families

In anticipation to Eric's talk, I introduce some notation of type theory.

If the category $\mathcal{E}$ has pullbacks, then the *fiber* of a map $p : X \to B$ at an element $b : 1 \to B$ is the object $X(b) := p^{-1}(b)$ defined by the pullback square

$$\begin{array}{ccc} X(b) & \longrightarrow & X \\ \downarrow & & \downarrow{\scriptstyle p} \\ 1 & \xrightarrow{\;b\;} & B \end{array}$$

In type theory, the object (X, p) is viewed as a *family of objects* $(X(b)|b : B)$ of $\mathcal{E}$ parametrised by "internal elements" $b : B$.

The object X is the "total space" of its fibers. Symbolically:

$$X = \sum_{b:B} X(b)$$

2.3. The pushforward functor

If $u : A \to B$ is a map in a category $\mathcal{E}$, then the *pushforward functor*

$$u_! : \mathcal{E}/A \to \mathcal{E}/B$$

is defined by putting $u_!(X, p) = (X, up)$ for every object $X = (X, p)$ in $\mathcal{E}/A$.

The fiber of $u_!(X)$ at $b : 1 \to B$ is a "sum" of fibers of X:

$$u_!(X)(b) = (up)^{-1}(b) = p^{-1}(u^{-1}(b)) = \sum_{a:A(b)} X(a)$$

where $A(b) := u^{-1}(b)$.

2.4. The base change functor

If the category $\mathcal{E}$ has finite limits, then the functor $u_!$ has a right adjoint

$$u^\star : \mathcal{E}/B \to \mathcal{E}/A$$

defined by putting $u^\star(Y) = (A \times_B Y, p_1)$

$$\begin{array}{ccc} A \times_B Y & \xrightarrow{\;p_2\;} & Y \\ {\scriptstyle p_1}\downarrow & & \downarrow{\scriptstyle q} \\ A & \xrightarrow{\;u\;} & B \end{array}$$

for every $Y = (Y, q) \in \mathcal{E}/B$. The functor $u^\star$ is said to be the *pullback functor*, or the *base change functor* along u.

The fiber of $u^\star(Y)$ at $a : A$ is the fiber of Y at $u(a)$,

$$u^\star(Y)(a) = Y(u(a))$$

2.5. The distributive law

If $\mathcal{E}$ is a logos, then the base change functor $u^\star : \mathcal{E}/B \to \mathcal{E}/A$ is cocontinuous for any map $u : A \to B$. by the second axiom.

In particular, if $(g_i : Y_i \to B)$ is a family of objects of $\mathcal{E}/B$ indexed by a set I, then the canonical map

$$A \times_B \coprod_{i \in I} Y_i \to \coprod_{i \in I} A \times_B Y_i$$

is invertible. This is like the distributive law of product over sum in a commutative ring:

$$a \cdot (\sum_{i \in I} y_i) = \sum_{i \in I} a \cdot y_i$$

2.6. Internal products

Every cocontinous functor between presentable categories has a right adjoint. Hence the base change functor $u^\star : \mathcal{E}/B \to \mathcal{E}/A$ has a right adjoint

$$u_\star : \mathcal{E}/A \to \mathcal{E}/B$$

for every map $u : A \to B$ in a logos $\mathcal{E}$.

The functor $u_\star$ takes an object $X = (X, p)$ of $\mathcal{E}/A$ to an object $u_\star(X) = u_\star(X, p)$ of $\mathcal{E}/B$.

In type theory notation, we have

$$u_\star(X)(b) = \prod_{a:A(b)} X(a)$$

for every $b : B$, where $A(b) = u^{-1}(b)$.

The *exponential* B^A is the product of A copies of B:

$$B^A = \prod_{a:A} B$$

3.1. Subobjects

Recall that a map $u : A \to B$ in a category $\mathcal{E}$ is said to be a *monomorphism* if the map $Map(K, u) : Map(K, A) \to Map(K, B)$ is injective for every object $K \in \mathcal{E}$.

Two monomorphisms $u : A \to B$ and $u' : A' \to B$ are *equivalent* if they are isomorphic in the category $\mathcal{E}/B$.

Definition

A *subobject* $S \subseteq B$ is an equivalence class of monomorphisms $A \to B$.

Let us denote the set of subobjects of A by $Sub(A)$.

The pullback of a monomorphism along any map is a monomorphism.

This defines a contravariant functor $Sub : \mathcal{E} \to Set$.

3.2. Descent for monomorphisms

If $\mathcal{E}$ is a logos, then the third axiom asserts that the contravariant functor

$$Sub : \mathcal{E} \to Set$$

takes colimits to limits.

In particular, have

$$Sub(\bigsqcup_{i \in I} X_i) = \prod_{i \in I} Sub(X_i)$$

for every family of objects $(X_i : i \in I)$ in $\mathcal{E}$.

It follows that the canonical map $\mathrm{in}_i : X_i \to \bigsqcup_{i \in I} X_i$ is a monomorphism for every $i \in I$.

3.3. The Lawvere object Ω

The third axiom of a logos $\mathcal{E}$ implies that the functor $Sub : \mathcal{E}^{op} \to Set$ is representable.

Hence there exists an object $\Omega \in \mathcal{E}$ together with a subobject $\Omega_\bullet \subseteq \Omega$ such that for every subobject $S \subseteq B$ of an object $B \in \mathcal{E}$ there exists a unique map $\chi : B \to \Omega$ such that $\chi^{-1}(\Omega_\bullet) = S$.

$$\begin{array}{ccc} S & \longrightarrow & \Omega_\bullet \\ \downarrow & & \downarrow \\ B & \xrightarrow{\ \chi\ } & \Omega \end{array}$$

It turns out that the subobject $\Omega_\bullet \to \Omega_\bullet$ is represented by a map $t : 1 \to \Omega$.

The exponential Ω^B is fundamental in an *elementary topos*.

3.4. The logos *Set*

The category *Set* is a logos. In this case we have $\Omega = \{0, 1\}$.

For every subset $S \subseteq B$ of a set B there exists a unique map $\chi : B \to \{0, 1\}$ such that $\chi^{-1}(1) = S$.

$$\begin{array}{ccc} S & \longrightarrow & 1 \\ \downarrow & & \downarrow {\scriptstyle \{1\}} \\ B & \xrightarrow{\chi} & \{0,1\} \end{array}$$

3.5. The logos $Set^{[1]}$

The poset $[1] = \{0 < 1\}$ is a category with a unique arrow $i : 0 \to 1$. A functor $X : [1] \to Set$ is a map $X(i) : X_0 \to X_1$ between two sets. A morphism $f : X \to Y$ in $Set^{[1]}$ is a commutative square

$$\begin{array}{ccc} X_0 & \xrightarrow{f_0} & Y_0 \\ {\scriptstyle X(i)}\downarrow & & \downarrow{\scriptstyle Y(i)} \\ X_1 & \xrightarrow{f_1} & Y_1 \end{array}$$

The category $Set^{[1]}$ is a logos, since it is a presheaf category. The Lawvere object $\Omega \in Set^{[1]}$ is a map $\Omega(i) : \Omega_0 \to \Omega_1$.

Show that $\Omega_0 = \{0, u, 1\}$, $\Omega_1 = \{0, 1\}$ and that

$$\Omega(i)(0) = 0, \quad \Omega(i)(u) = 1 \quad \text{and} \quad \Omega(i)(1) = 1$$

3.6. The logos of simplicial sets $Psh(\Delta) = [\Delta^{op}, Set]$

Recall that the *simplex category* Δ has for objects the non-empty ordinals $[n] = \{0, \ldots, n\}$ $(n \geq 0)$ and for morphisms the poset maps.

A *simplicial set* is a presehaf $X : \Delta^{op} \to Set$. The set $X([n])$ is denoted X_n.

The representable functor $Hom(-, [n]) : \Delta^{op} \to Set$ is denoted $\Delta[n]$.

The Lawvere object is a simplicial set $\Omega : \Delta^{op} \to Set$ such that

$$\Omega_n = Sub(\Delta[n])$$

for every $n \geq 0$. An element of Ω_n is a sub-complex of the simplicial complex $P_0([n])$.

4.1. The logos $Set[U]$

If Fins is the category of finite sets $\underline{n} = \{1, \ldots, n\}$ $(n \geq 0)$, then the category $[\text{Fins}, Set] = Psh(\text{Fins}^{op})$ is a logos.

By Yoneda, every functor $F : \text{Fins} \to Set$ is a colimit of representables

$$F = \int^{\underline{n} \in Fins} F(n) \times Hom(\underline{n}, -)$$

If $U := Hom(1, -)$, then $U^n = Hom(\underline{n}, -)$. Thus,

$$F = \int^{\underline{n} \in Fins} F(n) \times U^n$$

This a "polynomial expansion" of the functor F in terms of the functor $U^n : \text{Fins} \to Set$. Let us put

$$Set[U] = [Fins, Set]$$

4.2 The logos $Set[U]$

Theorem

If $\mathcal{E}$ is a logos then for every object $A \in \mathcal{E}$ there exists a unique homomorphism of logoi $ev_A : Set[U] \to \mathcal{E}$ such that $ev_A(U) = A$.

By construction,

$$ev_A(F) = \int^{\underline{n} \in Fins} F(n) \times A^n$$

for every $F \in Set[U]$. Let us put $F(A) := ev_A(F)$.

This defines a *polynomial functor* $F : \mathcal{E} \to \mathcal{E}$.

A functor $F : Set \to Set$ is polynomial if and only if it is finitary.

$Set[U]$ is equivalent to the category of finitary functors $Set \to Set$.

4.3 The logos $\mathcal{E}/B = \mathcal{E}[x_B]$

The base change functor $p_B^\star : \mathcal{E} \to \mathcal{E}/B$ along the map $p_B : B \to 1$ is a homomorphism of logoi.

By construction, $p_B^\star(X) = (B \times X, p_1)$ for every object $X \in \mathcal{E}$.

In particular $p_B^\star(B) = (B \times B, p_1)$ and $p_B^\star(1) = (B, 1_B) = 1_B$.

Notice that the diagonal $\Delta(B) := (1_B, 1_B) : B \to B \times B$ is a morphism $x_B : 1_B \to p_B^\star(B)$ in $\mathcal{E}/B$.

Theorem

If $\phi : \mathcal{E} \to \mathcal{F}$ is a homomorphism of logoi then for every object $B \in \mathcal{E}$ and every every element $b : 1 \to \phi(B)$ there exists a unique morphism of logoi $\phi_b : \mathcal{E}/B \to \mathcal{F}$ such that $\phi_b(x_B) = b$.

Hence the extension $\iota := p_B^\star : \mathcal{E} \to \mathcal{E}/B$ is freely generated by the element $x_B : 1_B \to \iota(B)(= B)$.

5.1. Sheaves

Ley $u : A \to B$ be a monomorphism in a logos $\mathcal{E}$. We shall say that an object $X \in \mathcal{E}$ is a *u-sheaf* if the map

$$Map(u', X) : Map(B', X) \to Map(A', X)$$

is invertible for every base change $u' : A' \to B'$ of u.

If $\Sigma \subseteq \mathcal{E}$ is a set of monomorphisms, we shall say that X is a Σ-*sheaf* if it is a u-sheaf for every $u \in \Sigma$.

We shall denote by $Sh(\mathcal{E}, \Sigma)$ the full subcategory of $\mathcal{E}$ spanned by the Σ-sheaves.

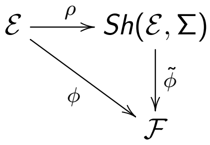
The subcategory $Sh(\mathcal{E}, \Sigma) \subseteq \mathcal{E}$ is reflective and the reflector $\rho : \mathcal{E} \to Sh(\mathcal{E}, \Sigma)$ inverts every map in Σ.

5.2. Left exact localisation $\mathcal{E} \to Sh(\mathcal{E}, \Sigma)$

Theorem

Let $\Sigma \subseteq \mathcal{E}$ be a set of monomorphisms in a logos $\mathcal{E}$. Then the category $Sh(\mathcal{E}, \Sigma)$ is a logos. The functor $\rho : \mathcal{E} \to Sh(\mathcal{E}, \Sigma)$ is a homomorphism of logoi and it inverts the maps in Σ universally .

The last sentence means if $\phi : \mathcal{E} \to \mathcal{F}$ is a homomorphism of logoi which inverts every map in Σ, then there exists a unique homomorphism of logoi $\tilde{\phi} : Sh(\mathcal{E}, \Sigma) \to \mathcal{F}$ such that $\tilde{\phi} \circ \rho = \phi$.



Thank you for your attention!
See you tomorrow for course II on higher toposes.