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# A SIMPLE SOLUTION TO FRIEDMAN'S FOURTH PROBLEM

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**Abstract.** It is shown that Friedman's problem, whether there exists a proper extension of first order logic satisfying the compactness and interpolation theorems, has extremely simple positive solutions if one considers extensions by generalized (finitary) propositional connectives. This does not solve, however, the problem of whether such extensions exist which are also closed under relativization of formulas.

It is well known that the classical propositional connectives form a complete set in the sense that any Boolean function is definable from them. However, if one considers connectives as generalized quantifiers (cf. [4]), then the possibilities multiply since a given connective may vary its meaning according to the size of the universe. Consider, for example, the connective $\between$ defined by

$$\mathfrak{A} \models \varphi \between \psi \Leftrightarrow (|A| < \omega \text{ and } \mathfrak{A} \models \varphi \wedge \psi) \text{ or } (|A| \geq \omega \text{ and } \mathfrak{A} \models \varphi \vee \psi).$$

The classical connectives are thus the "constant" generalized connectives.

It turns out that many of the logics obtained by adding these generalized connectives to first order logic satisfy both the compactness and interpolation theorem, providing an extremely simple solution to Friedman's problem 4 in [2]. In fact, for these logics countable compactness and interpolation are equivalent properties. We characterize them and give examples.

Since generalized connectives may be considered generalized quantifiers of monadic type, and we have shown elsewhere [1] that proper extensions of $L_{\omega\omega}$ by quantifiers of monadic type do not satisfy many-sorted interpolation, and if they enjoy relativizations they do not satisfy (plain) interpolation, we conclude that

$$\begin{aligned} \text{compactness} + \text{interpolation} &\not\Rightarrow \text{many-sorted interpolation} \\ &\not\Rightarrow \text{closure under relativizations.} \end{aligned}$$

To our knowledge, all attempts to construct a (countably) compact proper extension of $L_{\omega\omega}$ satisfying interpolation and closed under relativizations have failed so far.

Generalized connectives are implicit in Lindström's general definition of quantifier [4]. They correspond to classes of structures of type $\langle 0, \ldots, 0 \rangle$ (0-ary relations identified with truth values) closed under isomorphism. However, nowhere in the literature is there a treatment of these specific quantifiers and their properties.

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Notice that new infinitary propositional connectives may be added to infinitary logic, $L_{\omega_1\omega}$, preserving interpolation and other pleasant properties (cf. [3]).

DEFINITION. Let $n \in \omega$. A *generalized n-ary propositional connective* is a function $c$: Cardinals $\to \{0,1\}^{\{0,1\}^n}$ (this allows for 0-ary connectives $c$: Card $\to \{0,1\}$).

$L_{\omega\omega}(c)$ will be the logic obtained by allowing formulas of the form $c(\varphi_1,\dots,\varphi_n)$ with the semantics given by

$$\mathfrak{A} \models_s c(\varphi_1,\dots,\varphi_n) \Leftrightarrow c(|A|)(\varphi_1^{\mathfrak{A}}[s],\dots,\varphi_n^{\mathfrak{A}}[s]) = 1,$$

where $\varphi^{\mathfrak{A}}[s] = 1$ if $\mathfrak{A} \models_s \varphi$, 0 otherwise. Analogously, one may define $L_{\omega\omega}(c_i \mid i \in I)$ for any family $\{c_i \mid i \in I\}$ of generalized connectives.

DEFINITION. If $c$ is an $n$-ary generalized connective, define for each Boolean function $f: \{0,1\}^n \to \{0,1\}$ the class $E_f(c) = \{\kappa \in \text{Card} \mid c(\kappa) = f\}$. Note that we may always write a sentence $\sigma_f(c)$ in $L_{\omega\omega}(c)$ such that

$$\mathfrak{A} \models \sigma_f(c) \Leftrightarrow |A| \in E_f(c).$$

For example, if $c = \text{✡}$, as described before, and $f = \wedge$, then $E_\wedge(\text{✡}) = \omega$ and we may choose as $\sigma_\wedge(\text{✡})$ the sentence

$$\text{✡}(T,T) \wedge \neg\,\text{✡}(T,\neg T) \wedge \neg\,\text{✡}(\neg T, T) \wedge \neg\,\text{✡}(\neg T,\neg T),$$

where $T$ is any valid formula.

THEOREM 1. *$L_{\omega\omega}(c_1,\dots,c_k)$ satisfies the interpolation theorem if and only if for any $f_1,\dots,f_k$ (of the appropriate arities) the class $\bigcap_{i=1}^k E_{f_i}(c_i)$ is finite or has infinite cardinals*

PROOF. "$\Leftarrow$" For each formula $\varphi \in L_{\omega\omega}(c_1,\dots,c_k)$ and Boolean functions $f_1,\dots,f_k$ of the corresponding arities, let $\varphi_{f_1\cdots f_k} \in L_{\omega\omega}$ be the result of replacing every occurrence of $c_i$ in $\varphi$ by a classical propositional schema representing the function $f_i$. Then we have

$$|A| \in \bigcap_{i=1}^{k} E_{f_i}(c_i) \Rightarrow \mathfrak{A} \models \varphi \leftrightarrow \varphi_{f_1\cdots f_k}$$

and, *if $\varphi$ is valid,*

(1) $$|A| \in \bigcap_{i=1}^{k} E_{f_i}(c_i) \Rightarrow \mathfrak{A} \models \varphi_{f_1\cdots f_k}.$$

If $\bigcap_{i=1}^k E_{f_i}(c_i)$ has infinite cardinals, then by the Löwenheim-Skolem theorem in $L_{\omega\omega}$, (1) implies

(2) $$\mathfrak{A} \models \varphi_{f_1\cdots f_k} \quad \text{for all infinite } \mathfrak{A}.$$

By compactness of $L_{\omega\omega}$, (2) implies that there is $n(f_1\cdots f_n) \in \omega$ such that

(3) $$\exists^{\geq n(f_1\cdots f_k)} x(x = x) \models \varphi_{f_1\cdots f_k}.$$

Let

$$I = \left\{(f_1\cdots f_k) \,\middle|\, \bigcap_{i=1}^{k} E_{f_i}(c_i) \text{ has infinite cardinals}\right\}.$$

Obviously $I$ is nonempty. By (3) and the assumption of the theorem, there exist $n \in \omega$

such that

(4) $$\exists^{\geq n}x(x = x) \models \varphi_{\vec{f}} \quad \text{for all } \vec{f} \in I,$$

whenever $\varphi$ is a valid formula, and

(5) $$\exists^{\geq n}x(x = x) \models \bigvee_{\vec{f} \in I} \left( \bigwedge_{i=1}^{k} \sigma_{f_i}(c_i) \right).$$

Now, assume $\rho \models \psi$ in $L_{\omega\omega}(c_1, \ldots, c_k)$ with $\mathscr{L}(\rho) = L$ and $\mathscr{L}(\psi) = L'$; then, by (4),

$$\exists^{\geq n}x(x = x) \models \rho_{\vec{f}} \to \psi_{\vec{f}} \quad \text{for all } \vec{f} \in I.$$

For each $\vec{f}$, find an interpolant $\tau^{\vec{f}} \in L_{\omega\omega}$ with $\mathscr{L}(\tau^{\vec{f}}) \subseteq L \cap L'$:

(6) $$\exists^{\geq n}x(x = x) \wedge \rho_{\vec{f}} \models \tau^{\vec{f}} \models \psi_{\vec{f}}.$$

Since

$$\bigwedge_{i=1}^{k} \sigma_{f_i}(c_i) \models (\rho \leftrightarrow \rho_{\vec{f}}) \wedge (\psi \leftrightarrow \psi_{\vec{f}}),$$

then by (5) and (6) we have the validity of

$$\exists^{\geq n}x(x = x) \wedge \rho \models \bigvee_{\vec{f} \in I} \left( \bigwedge_{i=1}^{k} \sigma_{f_i}(c_i) \wedge \tau^{\vec{f}} \right) \models \psi.$$

Now, consider the finite structures of type $L \cap L'$ and size $< n$ which may be expanded to a model of $\rho$. Since there are finitely many of them, they may be described as the models of a sentence $\theta(L \cap L')$ in $L_{\omega\omega}$. Obviously $\rho \wedge \exists^{<n}x(x = x) \models \theta \models \psi$. Hence

$$\rho \models \theta \vee \left[ \bigvee_{\vec{f} \in I} \left( \bigwedge_{i=1}^{k} \sigma_{f_i}(c_i) \wedge \tau^{\vec{f}} \right) \right] \models \psi$$

and we have our interpolant in $L_{\omega\omega}(c_1 \cdots c_k)$.

"$\Rightarrow$" Assume that there is $\vec{f} = (f_1 \cdots f_k)$ such that

$$E_{\vec{f}} = \bigcap_{i=1}^{k} E_{f_i}(c_i) \subseteq \omega$$

and $E_{\vec{f}}$ has arbitrarily large finite cardinals. Consider the following classes of structures:

$$K_1 = \{(A, E) \mid |A| \in E_{\vec{f}} \text{ and } E \text{ is an equivalence relation with an } \textit{even} \text{ number of equivalence classes}\},$$

$$K_2 = \{(A, E) \mid |A| \in E_{\vec{f}} \text{ and } E \text{ is an equivalence relation with an } \textit{odd} \text{ number of equivalence classes}\}.$$

These two classes are obviously disjoint and PC in $L_{\omega\omega}(c_1, \ldots, c_k)$. For each $n \in \omega$ we may find $A$ with $|A| \geq n$ and equivalence relations $E_1, E_2$ such that $(A, E_1) \in K_1$, $(A, E_2) \in K_2$ and $E_1$, $E_2$ divide $A$ into $\geq n$ classes of size $\geq n$. Hence, by a straightforward back-and-forth argument, the two structures are elementarily equivalent with respect to sentences of quantifier depth less than $n$ in $L_{\omega\omega}$. In symbols:

$$(A, E_1) \equiv (A, E_2) \quad \text{in } L^n_{\omega\omega}.$$

As each sentence $\varphi$ of $L_{\omega\omega}(c_1, \ldots, c_k)$ is equivalent to $\varphi_{\vec{f}}$ in both structures and these sentences have the same quantifier depth, we have

$$(A, E_1) \equiv (A, E_2) \quad \text{in } L^n_{\omega\omega}(c_1, \ldots, c_k).$$

This shows that $K_1$ and $K_2$ cannot be separated in $L_{\omega\omega}(c_1, \ldots, c_k)$. Q.E.D.

EXAMPLE 1. $L_{\omega\omega}(\,\diamondsuit\,)$, as defined in the introduction, does not satisfy interpolation, since $E_\wedge(\,\diamondsuit\,) = \omega$ is not finite and does not contain infinite cardinals.

EXAMPLE 2. Define the binary connective $\Diamond$: Card $\to \{0, 1\}^{\{0,1\}^2}$:

$$\Diamond(\kappa) = \begin{cases} \wedge & \text{if } \kappa \text{ is regular (or finite)}, \\ \vee & \text{if } \kappa \text{ is singular.} \end{cases}$$

Then $L_{\omega\omega}(\Diamond)$ satisfies interpolation, since $E_\vee(\Diamond)$ and $E_\wedge(\Diamond)$ have infinite cardinals and the other $E_{\vec{f}}(\Diamond)$ are empty.

EXAMPLE 3. Let $q_\alpha$ mean "*the universe has size* $\geq \omega_\alpha$", this is:

$$q_\alpha(\kappa) = \begin{cases} 1 & \text{if } \kappa \geq \omega_\alpha, \\ 0 & \text{otherwise.} \end{cases}$$

If $\alpha_i \geq 1$ then the logic $L_{\omega\omega}(q_{\alpha_1}, \ldots, q_{\alpha_k})$ satisfies interpolation, because for any sequence $f_1, \ldots, f_k$ of zeros and ones, $\bigcap_{i=1}^k E_{f_i}(q_{\alpha_i})$ has one of the forms $\{\kappa \mid \omega_{\alpha_i} \leq \kappa < \omega_{\alpha_j}\}$, $\{\kappa \mid \kappa < \omega_{\alpha_j}\}$ or $\{\kappa \mid \kappa \geq \omega_{\alpha_i}\}$, which are either empty or have infinite cardinals. For example, $L_{\omega\omega}(q_1)$ satisfies interpolation.

Surprisingly, the same condition required for interpolation in these logics is necessary and sufficient for countable compactness (see Corollary 2). First we give a characterization of $\lambda$-compactness for arbitraty $\lambda$.

Recall that a logic $L$ satisfies *$\lambda$-compactness* if any set of sentences in $L$ of power $\lambda$, for which all finite parts have models, has itself a model. $L$ satisfies *(full) compactness* if it is $\lambda$-compact for all cardinals $\lambda$.

If $C$ is a family of generalized connectives, $S$ is a set of sentences in $L_{\omega\omega}(c \mid c \in C)$ and $\vec{f} = (f_c \mid c \in C)$ is a sequence of Boolean functions, $f_c$ of the same arity as $c$, let

$$S_{\vec{f}} = \{\varphi_{\vec{f}} \mid \varphi \in S\} \subseteq L_{\omega\omega}$$

be the result of replacing the occurrences of the $c$'s in the sentences of $S$ by the propositional connectives $f_c$.

REMARK. We could just consider only 0-ary generalized connectives (that is, classes of cardinals) since they are a "complete" class of connectives. In fact,

$$L_{\omega\omega}(c_1, \ldots, c_k) \equiv L_{\omega\omega}(\sigma_{f_1}(c_1), \ldots, \sigma_{f_k}(c_k) \mid (f_1 \cdots f_k) \in J),$$

where $J$ consists of the finitely many $k$-tuples of Boolean functions with $f_i$ of the same arity as $c_i$, and the $\sigma_f(c)$ are obviously 0-ary connectives; any $\varphi$

$\in L_{\omega\omega}(c_1,\dots,c_k)$ may be expressed as

$$\varphi \equiv \bigvee_{\vec{f}\in J}\left(\bigwedge_{i=1}^{k}\sigma_{f_i}(c_i)\wedge\varphi_{\vec{f}}\right).$$

Notice that in the following theorem we do not put any restriction in the size of the language (first-order nonlogical symbols). Also the family $\mathscr{C}$ of connectives may be a proper class. By the remark above, there is no loss of generality if we assume all the connectives to be 0-ary.

THEOREM 2. *$L_{\omega\omega}(c \mid c\in\mathscr{C})$ is $\lambda$-compact [compact] if and only if for any set $C\subseteq\mathscr{C}$ with $|C|\le\lambda$ [of any size] and any sequence $(f_c \mid c\in C)$ of zeros and ones, either*

(i) *for some finite $D\subseteq C$, $\{\kappa \mid \forall c\in D\colon c(\kappa)=f_c\}$ is a finite set of finite cardinals, or*

(ii) *$\{\kappa \mid \forall c\in C\colon c(\kappa)=f_c\}$ has cardinals $\ge\lambda$ [arbitrarily large cardinals].*

PROOF. Assume $\lambda$-compactness of the logic; then if (i) does not hold, each finite subset of the set

$$\{\sigma_{f_c}(c) \mid c\in C\}\cup\{a_\alpha\neq a_\beta \mid \alpha<\beta<\lambda\}$$

has a model and (ii) follows (the condition is just a restatement of compactness for these sets).

To show sufficiency, assume $S$ is finitely satisfiable, with $|S|\le\lambda$, and let $C\subseteq\mathscr{C}$ be the set of connectives occurring in sentences of $S$.

*Claim. There is a fixed $\vec{f}=(f_c \mid c\in C)$ such that any finite $F\subseteq S$ has models in $\bigcap_{c\in C\restriction F}E_{f_c}(c)$, where $C\restriction F$ denotes the connectives occurring in $F$.*

To prove the claim, let $\mathfrak{J}=\prod_{c\in C}\{0,1\}$, with the product topology induced by discrete $\{0,1\}$. The sets

$$\mathfrak{J}_D=\{\vec{g}\in\mathfrak{J} \mid \forall F\subseteq S,\ F\text{ finite, has models in }\textstyle\bigcap_{c\in D}E_{\vec{g}(c)}(c)\},$$

where $D\subseteq C$ is finite, are closed. Moreover, they are nonempty. Otherwise, if $D=\{c_1,\dots,c_k\}$ for each one of the $k$-tuples $\vec{g}^j=(g_i^j\cdots g_k^j)\in\{0,1\}^k$, $j=1,\dots,M=2^k$, we could find finite subsets $F_1,\dots,F_M\subseteq S$, with $F_j$ not having any models in $\bigcap_{i=1}^k E_{g_i^j}(c_i)$, and so the finite subset $F_1\cup\cdots\cup F_M$ of $S$ would not have models at all. Finally, they have the finite intersection property because $\mathfrak{J}_D\cap\mathfrak{J}_{D'}\supseteq\mathfrak{J}_{D\cup D'}$. By compactness, there is

$$\vec{f}\in\bigcap_{\substack{D\subseteq C\\ D\text{ finite}}}\mathfrak{J}_D,$$

which satisfies the claim.

Now we consider two possibilities.

*Case* 1. For some finite $F\subseteq S$, all its models in $\bigcap_{c\in C\restriction F}E_{f_c}(c)$ have size $\le m$ for some fixed $m\in\omega$, then any other finite $F'\subseteq S$ will have models of size $\le m$ in $\bigcap_{c\in C\restriction F'}E_{f_c}(c)$ (just consider $F\cup F'$). Hence, there exists a single $m'\in\bigcap_{c\in C}E_{f_c}(c)$ such that all finite $F\subseteq S$ have a model of size $m'$. Then the theory

$$S_{\vec{f}}\cup\{\exists^{=m'}x(x=x)\}$$

is finitely satisfiable. By $L_{\omega\omega}$-compactness, it has a model; this model being of size $m'$ (so that $c$ means $f_c$), it is a model of $S$.

*Case* 2. If Case 1 does not hold then, given $F \subseteq S$, there is no finite bound on the size of its models in $\bigcap_{c \in C \restriction F} E_{f_c}(c)$.

Hence, by compactness and the Löwenheim-Skolem theorems in $L_{\omega\omega}$, $S_{\hat{f}}$ has models of any size $\geq \lambda$. On the other hand, condition (i) of Theorem 4 cannot hold, and so by condition (ii), there is $\kappa \geq \lambda$, $\kappa \in \bigcap_{c \in C} E_{f_c}(c)$. Choose a model of $S_{\hat{f}}$ of size $\kappa$; it will then be a model of $S$. Q.E.D.

COROLLARY 1. *$L_{\omega\omega}(c_1, \ldots, c_k)$ satisfies $\lambda$-compactness [compactness] if and only if for any Boolean functions $f_1, \ldots, f_k$, the class $\bigcap_{i=1}^{k} E_{f_i}(c_i)$ is a finite set of finite cardinals or it contains cardinals $\geq \lambda$ [arbitrarily large cardinals].*

COROLLARY 2. *$L_{\omega\omega}(c_1, \ldots, c_k)$ satisfies interpolation if and only it is countably compact.*

EXAMPLE 4. Let $c_\alpha$ mean "*the universe has cofinality* $\alpha$" ($\alpha$ an infinite cardinal). Then the logic

$$L_{\omega\omega}(c_\alpha \mid \alpha \in \text{Inf.Card.})$$

is fully compact and obviously satisfies interpolation because for *any* set $L \subseteq$ Inf. Card. and sequence $(f_\alpha \mid \alpha \in L)$:

$$\{\kappa \mid \forall \alpha \in L: c_\alpha(\kappa) = f_\alpha\} = \begin{cases} \varnothing & \text{if } f_\alpha = 1 \text{ twice or more,} \\ \{\kappa \mid \operatorname{cof} \kappa = \beta\} & \text{if } f_\beta = 1 \text{ and } f_\alpha = 0\ \forall \alpha \neq \beta, \\ \{\kappa \mid \operatorname{cof} \kappa \notin L\} & \text{if } f_\alpha = 0\ \forall \alpha \in L. \end{cases}$$

EXAMPLE 5. $L_{\omega\omega}(r)$ is fully compact, and satisfies interpolation, where $r$ is the 0-ary connective: "*the universe has regular* (or *finite*) *cardinality*"; since $E_1(r) = \{\kappa \mid \kappa$ regular$\}$ and $E_0(r) = \{\kappa \mid \kappa$ singular$\}$, both have arbitrarily large cardinals.

EXAMPLE 6. The same is true of $L_{\omega\omega}(r_N)$ where $N \subseteq \omega$ and $r_N$ is the 0-ary connective: "*the size of the universe is regular infinite or belongs to* $N$".

If we take $N$ of a given degree of unsolvability, validity in $L_{\omega\omega}(r_N)$ will have at least this degree of unsolvability (consider the sentences $\exists^{=n}x(x = x) \to r_N$). In this way, we get a continuum of nonequivalent compact logics satisfying interpolation.

Notice that this last example extends properly $L_{\omega\omega}$ *even for finite models* (cf. [2], problem 4).

EXAMPLE 7. Let $q_\alpha$ be as in Example 3. Then, if $\omega \leq \lambda < \min(\omega_{\alpha_1}, \ldots, \omega_{\alpha_k})$, the logic $L_{\omega\omega}(q_{\alpha_1}, \ldots, q_{\alpha_k})$ is $\lambda$-compact and satisfies interpolation. Hence, for example, $L_{\omega\omega}(q_1, q_2)$ is a countably compact sublogic of $L_{\omega\omega}(Q_1, Q_2)$, in spite of the fact that the last logic is not known to be countably compact without additional set-theoretical hypotheses (cf. [5]).

EXAMPLE 8. $L_{\omega_1\omega}(Q_1)$ contains uncountably many mutually incomparable sublogics satisfying countable compactness and the interpolation theorem. For each $N \subseteq \omega$ define the connective $q_N$ as "*the size of the universe is uncountable or belongs to* $N$". A simple cardinality argument shows that there are uncountably many sublogics of $L_{\omega_1\omega}(Q_1)$ of the form $L_{\omega\omega}(q_N)$.

The logic generated by an $n$-ary connective $c$ is equivalent to that generated by the quantifier of monadic type defined by

$$Qx_1 \cdots x_n(\varphi_1, \ldots, \varphi_n) \equiv c(\forall x_1 \varphi_1, \ldots, \forall x_n \varphi_n),$$

because $c$ may be recovered as

$$c(\varphi_1,\ldots,\varphi_n) \equiv Qx'_1 \cdots x'_n(\varphi_1,\ldots,\varphi_n),$$

where $x'_1$ is taken not free in $\varphi_i$, $i = 1,\ldots,n$. Therefore, from the above examples and results in [1], we obtain

COROLLARY 3. *In abstract model theory*:

*Recursive syntax* + *Compactness* + *Interpolation* $\not\Rightarrow$
—*Closure under relativizations* (cf. [1], Theorem 2.2),
—*Many sorted interpolation* (cf. [1], Theorem 2.3),
—*Axiomatizability* (Example 6).

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