The One-Variable Fragment of Corsi Logic

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Abstract. The one-variable fragment of the first-order logic of linear intuitionistic Kripke models, referred to here as Corsi logic, is shown to have as its modal counterpart the many-valued modal logic $\mathsf{S5}(\mathbf{G})$. It is also shown that $\mathsf{S5}(\mathbf{G})$ can be interpreted in the crisp many-valued modal logic $\mathsf{S5}(\mathbf{G})^{\mathsf{C}}$, the modal counterpart of the one-variable fragment of first-order Gödel logic. Finally, an algebraic finite model property is proved for $\mathsf{S5}(\mathbf{G})^{\mathsf{C}}$ and used to establish co-NP-completeness for validity in the aforementioned modal logics and one-variable fragments.

1 Introduction

One-variable fragments of first-order logics are often studied as propositional modal logics, where each unary predicate $P(x)$ is replaced with a propositional variable p and quantifiers $(\forall x)$ and $(\exists x)$ are replaced with modalities $\Box$ and $\Diamond$, respectively. This shift in perspective can be useful in obtaining axiomatization, finite model property, and complexity results both for the fragments and for corresponding classes of algebraic models. In particular, the modal logic $\mathsf{S5}$ and intuitionistic modal logic MIPC (corresponding to monadic Boolean algebras and monadic Heyting algebras) are modal counterparts of the one-variable fragments of first-order classical logic and intuitionistic logic, respectively. Both these modal logics have the finite model property and are decidable. The correspondence between one-variable fragments of first-order intermediate logics and varieties of monadic Heyting algebras has been considered in some depth in [14, 15, 2]. Decidability and complexity results have also been obtained for intermediate modal logics viewed as fragments of classical bimodal logics (see [8] for details).

In this paper, we investigate the one-variable fragment of the first-order logic of linear intuitionistic Kripke models, axiomatized by Corsi in [7] as the extension of first-order intuitionistic logic with the prelinearity axiom schema $(\alpha \to \beta) \vee (\beta \to \alpha)$, and referred to here as *Corsi logic*. In particular, we prove that the modal counterpart of this one-variable fragment is the many-valued modal logic $\mathsf{S5}(\mathbf{G})$, with propositional connectives interpreted using the standard semantics of Gödel logic and $\Box$ and $\Diamond$ interpreted as infima and suprema relative to $[0,1]$-valued accessibility relations. It has been shown in [6] that an axiomatization of $\mathsf{S5}(\mathbf{G})$ is obtained by extending MIPC with the prelinearity axiom schema, and that adding also the axiom schema $\Box(\Box\varphi \vee \psi) \to (\Box\varphi \vee \Box p)$ yields an axiomatization of the crisp version $\mathsf{S5}(\mathbf{G})^{\mathsf{C}}$ of $\mathsf{S5}(\mathbf{G})$, obtained by restricting to $\{0,1\}$-valued accessibility relations. The logic $\mathsf{S5}(\mathbf{G})^{\mathsf{C}}$ is the modal counterpart of the one-variable fragment of first-order Gödel logic or, equivalently (see [16, 1]), the first-order logic of linear intuitionistic Kripke models with constant domains.

The logic $\mathsf{S5}(\mathbf{G})$ lacks the finite model property with respect to its standard Kripke semantics, but is complete with respect to a variety of monadic Heyting algebras that has this property (see [2]) and is hence decidable. We provide here an alternative decidability proof that also establishes co-NP-completeness. First, we give an interpretation of the one-variable fragment of Corsi logic in the one-variable fragment of first-order Gödel logic, yielding an interpretation of $\mathsf{S5}(\mathbf{G})$ in $\mathsf{S5}(\mathbf{G})^{\mathsf{C}}$. Although $\mathsf{S5}(\mathbf{G})^{\mathsf{C}}$ also lacks the finite model property, decidability (indeed co-NP-completeness) has been established in [5] using an alternative Kripke semantics that does have the property. We show here that this rather ad hoc alternative semantics emerges naturally from a well-known representation of monadic Heyting algebras (see [2]). Finally, an algebraic finite model property is established for $\mathsf{S5}(\mathbf{G})^{\mathsf{C}}$, and used to prove co-NP-completeness for the two many-valued modal logics and their associated one-variable fragments.

2 The One-Variable Fragments

In this section, we present the one-variable fragments of first-order intermediate logics defined over all linear Kripke models and the linear Kripke models that have constant domains. For convenience, we restrict our definitions here to the set Fm_1 of one-variable first-order formulas $\alpha, \beta, \ldots$, built inductively as usual from a countably infinite set of unary predicates $\{P_i\}_{i\in\mathbb{N}}$, propositional connectives $\wedge, \vee, \to, \bot, \top$, a fixed variable x, and quantifiers $\forall, \exists$.

A *monadic intuitionistic Kripke model* (for short, IK_1*-model*) is a 4-tuple $\mathfrak{M} = \langle W, \preceq, \{D_w\}_{w\in W}, \{I_w\}_{w\in W}\rangle$ consisting of a non-empty poset $\langle W, \preceq\rangle$, a non-empty set D_w for each $w \in W$ called the *domain* of w, and functions $\{I_w\}_{w\in W}$ mapping each P_i to some $I_w(P_i) \subseteq D_w$, satisfying for all $w, v \in W$ and $i \in \mathbb{N}$,

$$w \preceq v \implies D_w \subseteq D_v \text{ and } I_w(P_i) \subseteq I_v(P_i).$$

Satisfaction in $\mathfrak{M}$ is then defined inductively as follows for $w \in W$ and $a \in D_w$:

$$\begin{array}{lcl}
\mathfrak{M}, w \models^a \bot & \iff & \text{never} \\
\mathfrak{M}, w \models^a \top & \iff & \text{always} \\
\mathfrak{M}, w \models^a P_i(x) & \iff & a \in I_w(P_i) \\
\mathfrak{M}, w \models^a \alpha \wedge \beta & \iff & \mathfrak{M}, w \models^a \alpha \text{ and } \mathfrak{M}, w \models^a \beta \\
\mathfrak{M}, w \models^a \alpha \vee \beta & \iff & \mathfrak{M}, w \models^a \alpha \text{ or } \mathfrak{M}, w \models^a \beta \\
\mathfrak{M}, w \models^a \alpha \to \beta & \iff & \mathfrak{M}, v \models^a \alpha \text{ implies } \mathfrak{M}, v \models^a \beta \text{ for all } v \succeq w \\
\mathfrak{M}, w \models^a (\forall x)\alpha & \iff & \mathfrak{M}, v \models^b \alpha \text{ for all } v \succeq w \text{ and } b \in D_v \\
\mathfrak{M}, w \models^a (\exists x)\alpha & \iff & \mathfrak{M}, w \models^b \alpha \text{ for some } b \in D_w.
\end{array}$$

Let us call $\mathfrak{M}$ an IKL_1-*model* if $\preceq$ is linear, a CDIK_1-*model* if it has constant domains (i.e., $D_w = D_v$ for all $v, w \in W$), and a CDIKL_1-*model* if both these conditions are satisfied. We say that $\alpha \in \mathrm{Fm}_1$ is *valid* in $\mathfrak{M}$ if $\mathfrak{M}, w \models^a \alpha$ for all $w \in W$ and $a \in D_w$. Given $\mathsf{L} \in \{\mathsf{IK}_1, \mathsf{IKL}_1, \mathsf{CDIK}_1, \mathsf{CDIKL}_1\}$, we say that $\alpha \in \mathrm{Fm}_1$ is L-*valid*, denoted by $\models_{\mathsf{L}} \alpha$, if it is valid in all L-models.

Let $\mathcal{IQC}$ be an axiomatization for first-order intuitionistic logic and consider the following axiom schema for all (i.e., not just one-variable) first-order formulas α and β, where x is not free in β for (cd):

$$\text{(prl) } (\alpha \to \beta) \vee (\beta \to \alpha) \quad \text{and} \quad \text{(cd) } (\forall x)(\alpha \vee \beta) \to ((\forall x)\alpha \vee \beta).$$

By known completeness results for first-order logics, we obtain for any $\alpha \in \mathrm{Fm}_1$:

$$\models_{\mathsf{IK}_1} \alpha \iff \vdash_{\mathcal{IQC}} \alpha \quad [12]; \qquad \models_{\mathsf{CDIK}_1} \alpha \iff \vdash_{\mathcal{IQC}+\text{(cd)}} \alpha \quad [9];$$

$$\models_{\mathsf{IKL}_1} \alpha \iff \vdash_{\mathcal{IQC}+\text{(prl)}} \alpha \quad [7]; \qquad \models_{\mathsf{CDIKL}_1} \alpha \iff \vdash_{\mathcal{IQC}+\text{(cd)}+\text{(prl)}} \alpha \quad [16].$$

Now let $\mathrm{Fm}_{\Box\Diamond}$ be the set of modal formulas $\varphi, \psi, \ldots$, built inductively over a set of propositional variables $\{p_i\}_{i\in\mathbb{N}}$, propositional connectives $\wedge, \vee, \to, \bot, \top$, and modal connectives $\Box, \Diamond$. Recall also the standard translations $(-)^*$ and $(-)^\circ$ between $\mathrm{Fm}_{\Box\Diamond}$ and Fm_1, where $\star \in \{\wedge, \vee, \to\}$, $c \in \{\bot, \top\}$:

$$\begin{aligned} (P_i(x))^* &= p_i & p_i^\circ &= P_i(x) \\ c^* &= c & c^\circ &= c \\ (\alpha \star \beta)^* &= \alpha^* \star \beta^* & (\varphi \star \psi)^\circ &= \varphi^\circ \star \psi^\circ \\ ((\forall x)\alpha)^* &= \Box\alpha^* & (\Box\varphi)^\circ &= (\forall x)\varphi^\circ \\ ((\exists x)\alpha)^* &= \Diamond\alpha^* & (\Diamond\varphi)^\circ &= (\exists x)\varphi^\circ. \end{aligned}$$

Clearly $(\alpha^*)^\circ = \alpha$ for any $\alpha \in \mathrm{Fm}_1$ and $(\varphi^\circ)^* = \varphi$ for any $\varphi \in \mathrm{Fm}_{\Box\Diamond}$.

Let $\mathcal{MIPC}$ be an axiomatization of intuitionistic propositional logic extended with the necessitation rule $\varphi/\Box\varphi$ and the axiom schema

$$\begin{array}{ll} \Box(\varphi \to \psi) \to (\Box\varphi \to \Box\psi) & \Diamond(\varphi \vee \psi) \to (\Diamond\varphi \vee \Diamond\psi) \\ \Box\varphi \to \varphi & \varphi \to \Diamond\varphi \\ \Diamond\varphi \to \Box\Diamond\varphi & \Diamond\Box\varphi \to \Box\varphi \\ \Box(\varphi \to \psi) \to (\Diamond\varphi \to \Diamond\psi), & \end{array}$$

and consider the additional axiom schema

$$\text{(prl) } (\varphi \to \psi) \vee (\psi \to \varphi) \quad \text{and} \quad \text{(cd)}_\Box \ \Box(\Box\varphi \vee \psi) \to (\Box\varphi \vee \Box\psi).$$

The following completeness results are known:

$$\begin{aligned} &\models_{\mathsf{IK}_1} \alpha \iff \vdash_{\mathcal{MIPC}} \alpha^* && [4]; \\ &\models_{\mathsf{CDIK}_1} \alpha \iff \vdash_{\mathcal{MIPC}+\text{(cd)}_\Box} \alpha^* && [13]; \\ &\models_{\mathsf{CDIKL}_1} \alpha \iff \vdash_{\mathcal{MIPC}+\text{(cd)}_\Box+\text{(prl)}} \alpha^* && [6]. \end{aligned}$$

In Section 3 of this paper, we establish the missing result for Corsi logic.

Theorem 1. *For any* $\alpha \in \mathrm{Fm}_1$, $\models_{\mathsf{IKL}_1} \alpha \iff \vdash_{\mathcal{MIPC}+(\mathrm{prl})} \alpha^*$.

The one-variable fragment of first-order intuitionistic logic IK_1 has the finite model property and is decidable [13], but the precise complexity is not known, whereas CDIKL_1, the one-variable fragment of first-order Gödel logic [16, 1], lacks the finite model property but is co-NP-complete [5]. The one-variable fragment IKL_1 of Corsi logic also lacks the finite model property. For example, the formula $(\forall x)\neg\neg P_0(x) \to \neg\neg(\forall x)P_0(x)$ (where $\neg\alpha$ is defined as $\alpha \to \bot$) is valid in all finite IKL_1-models, but not in the infinite IKL_1-model $\mathfrak{M} = \langle \mathbb{N}, \leq, \{D_n\}_{n\in\mathbb{N}}, \{I_n\}_{n\in\mathbb{N}}\rangle$ with $D_n = \{a_0, \ldots, a_n\}$ and $I_n(P_0) = \{a_0, \ldots, a_{n-1}\}$ for each $n \in \mathbb{N}$. On the other hand, it is known (see [2]) that the variety of monadic Heyting algebras corresponding to the axiom system $\mathcal{MIPC} + (\mathrm{prl})$ has the finite model property, implying, by Theorem 1, that validity in IKL_1 is decidable. We prove a stronger result here, giving first an interpretation of IKL_1 in CDIKL_1 (Section 4) and then establishing an algebraic finite model property for CDIKL_1 (Section 5), to obtain the following complexity bound.

Theorem 2. *Validity in* IKL_1 *is co-NP-complete.*

3 The Many-Valued Modal Logics

In this section, we prove that the one-variable fragment of Corsi logic has as its modal counterpart the many-valued modal logic $\mathsf{S5}(\mathbf{G})$. Since the latter was axiomatized in [6] as an extension of $\mathcal{MIPC}$ with the prelinearity axiom schema (prl), this result yields a proof of Theorem 1.

Consider first the *standard Gödel algebra* $\mathbf{G} = \langle [0,1], \wedge, \vee, \to, 0, 1\rangle$ where $x \to y$ is y if $x > y$, and 1 otherwise. An $\mathsf{S5}(\mathbf{G})$-*model* is a triple $\mathfrak{M} = \langle W, R, V\rangle$ consisting of a non-empty set W, a map $R\colon W \times W \to [0,1]$ satisfying $Rww = 1$ (reflexivity); $Rwv = Rvw$ (symmetry); and $Rwv \wedge Rvu \leq Rwu$ (transitivity), and a map $V\colon \{p_i\}_{i\in\mathbb{N}} \times W \to [0,1]$. The map V is extended to $V\colon \mathrm{Fm}_{\Box\Diamond} \times W \to [0,1]$ inductively by the clauses $V(\bot, w) = 0$, $V(\top, w) = 1$, $V(\varphi \star \psi, w) = V(\varphi, w) \star V(\psi, w)$ for $\star \in \{\wedge, \vee, \to\}$, and

$$V(\Box\varphi, w) = \bigwedge\{Rwv \to V(\varphi, v) \mid v \in W\}$$
$$V(\Diamond\varphi, w) = \bigvee\{Rwv \wedge V(\varphi, v) \mid v \in W\}.$$

If $Rvw \in \{0,1\}$ for all $v, w \in W$, then $\mathfrak{M}$ is called an $\mathsf{S5}(\mathbf{G})^{\mathsf{C}}$-*model*. A formula $\varphi \in \mathrm{Fm}_{\Box\Diamond}$ is said to be *valid* in $\mathfrak{M}$ if $V(\varphi, w) = 1$ for all $w \in W$, and L-*valid* for $\mathsf{L} \in \{\mathsf{S5}(\mathbf{G}), \mathsf{S5}(\mathbf{G})^{\mathsf{C}}\}$, written $\models_{\mathsf{L}} \varphi$, if φ is valid in all L-models.

An $\mathsf{S5}(\mathbf{G})^{\mathsf{C}}$-model $\mathfrak{M} = \langle W, R, V\rangle$ is called *universal* if $Rwv = 1$ for all $w, v \in W$; we then write $\mathfrak{M} = \langle W, V\rangle$, since the conditions for $\Box, \Diamond$ simplify to

$$V(\Box\varphi, w) = \bigwedge\{V(\varphi, v) \mid v \in W\} \quad \text{and} \quad V(\Diamond\varphi, w) = \bigvee\{V(\varphi, v) \mid v \in W\}.$$

It is easily proved that $\models_{\mathsf{S5}(\mathbf{G})^{\mathsf{C}}} \varphi$ if and only if φ is valid in all universal $\mathsf{S5}(\mathbf{G})^{\mathsf{C}}$-models, and that this holds if and only if φ° is valid in first-order Gödel logic. The equivalence between first-order Gödel logic and the logic of linear Kripke models with constant domains (see [16, 1]) therefore yields the following correspondence.

Theorem 3. *For any* $\alpha \in \mathrm{Fm}_1$, $\models_{\mathsf{CDIKL_1}} \alpha \iff \models_{\mathsf{S5}(\mathbf{G})^{\mathsf{C}}} \alpha^*$.

The rest of this section is devoted to proving the analogous result for $\mathsf{IKL_1}$.

Theorem 4. *For any* $\alpha \in \mathrm{Fm}_1$, $\models_{\mathsf{IKL_1}} \alpha \iff \models_{\mathsf{S5}(\mathbf{G})} \alpha^*$.

We consider first the right-to-left direction. Proceeding contrapositively, let (without loss of generality) $\varphi \in \mathrm{Fm}_{\Box\Diamond}$ and suppose that $\not\models_{\mathsf{IKL_1}} \varphi^\circ$. Then there exists a countable $\mathsf{IKL_1}$-model $\mathfrak{M} = \langle W, \preceq, \{D_w\}_{w \in W}, \{I_w\}_{w \in W}\rangle$, $w_0 \in W$, and $a_0 \in D_{w_0}$ such that $\mathfrak{M}, w_0 \not\models^{a_0} \varphi^\circ$. Let $\mathrm{Up}(\langle W, \preceq\rangle)$ be the complete linearly ordered set of upsets of $\langle W, \preceq\rangle$ ordered by inclusion with W and $\emptyset$ as greatest and least elements, respectively. Since W is countable, there exists a complete (i.e., preserving all suprema and infima) order-embedding of $\mathrm{Up}(\langle W, \preceq\rangle)$ into $\langle [0,1], \leq\rangle$ (see [1]) and we may therefore implicitly identify $\mathrm{Up}(\langle W, \preceq\rangle)$ with a subset of $[0, 1]$.

Let $W^* = \bigcup_{v \in W} D_v$ and for each $a \in W^*$, let $U(a) = \{v \in W \mid a \in D_v\}$, i.e., $U(a)$ is the largest (with respect to $\subseteq$) $U \in \mathrm{Up}(\langle W, \preceq\rangle)$ such that $a \in \bigcap_{v \in U} D_v$. We define an $\mathsf{S5}(\mathbf{G})$-model $\mathfrak{M}^* = \langle W^*, R, V\rangle$ where for all $a, b \in W^*$,

$$Rab = \begin{cases} W & a = b \\ U(a) \cap U(b) & a \neq b \end{cases} \quad \text{and} \quad V(p_i, a) = \{v \in W \mid a \in I_v(P_i)\}.$$

Note that each $V(p_i, a)$ is an upset of $\langle W, \preceq\rangle$ since $u \preceq v$ implies $I_u(P_i) \subseteq I_v(P_i)$, and that $Raa = W$, $Rab = Rba$, and $Rab \wedge Rbc \leq Rac$ for all $a, b, c \in W^*$.

The following lemma yields $V(\varphi, a_0) \neq W$ and hence $\not\models_{\mathsf{S5}(\mathbf{G})} \varphi$ as required.

Lemma 1. *For any* $\varphi \in \mathrm{Fm}_{\Box\Diamond}$, $w \in W$, *and* $a \in D_w$,

$$\mathfrak{M}, w \models^a \varphi^\circ \iff w \in V(\varphi, a).$$

Proof. The following observation will be useful. If $a \in D_w$ then for any $b \in W^*$, $b \in D_w$ if and only if $w \in Rab$. Indeed, if $b = a$, this is trivial. If $b \neq a$, then as $a \in D_w$, $w \in U(a) \cap U(b)$ if and only if $w \in U(b)$, i.e., $b \in D_w$.

We prove the claim by induction on the length of φ. The base cases for $\bot$, $\top$, and p_i are immediate from the definitions. The cases for $\wedge$ and $\vee$ are also straightforward, so let us just consider the cases when φ is of the form $\psi_1 \to \psi_2$, $\Box\psi$, or $\Diamond\psi$. Let $w \in W$ and $a \in D_w$, and set $[w) = \{v \in W \mid v \succeq w\}$.

- Suppose that $\varphi = \psi_1 \to \psi_2$.

$$\begin{aligned}
\mathfrak{M}, w \models^a (\psi_1 \to \psi_2)^\circ &\iff \mathfrak{M}, v \models^a \psi_1^\circ \text{ implies } \mathfrak{M}, v \models^a \psi_2^\circ \text{ for all } v \succeq w \\
&\iff v \in V(\psi_1, a) \text{ implies } v \in V(\psi_2, a) \text{ for all } v \succeq w \\
&\iff [w) \cap V(\psi_1, a) \subseteq V(\psi_2, a) \\
&\iff [w) \subseteq (V(\psi_1, a) \to V(\psi_2, a)) \\
&\iff w \in V(\psi_1 \to \psi_2, a).
\end{aligned}$$

- Suppose that $\varphi = \Box\psi$.

$$\begin{aligned}
\mathfrak{M}, w \models^a (\Box\psi)^\circ &\iff \mathfrak{M}, v \models^b \psi^\circ \text{ for all } v \succeq w \text{ and } b \in D_v \\
&\iff v \in V(\psi, b) \text{ for all } v \succeq w \text{ such that } v \in Rab \\
&\iff [w) \cap Rab \subseteq V(\psi, b) \text{ for all } b \in W^* \\
&\iff w \in (Rab \to V(\psi, b)) \text{ for all } b \in W^* \\
&\iff w \in V(\Box\psi).
\end{aligned}$$

- Suppose that $\varphi = \Diamond\psi$.

$$\begin{aligned}
\mathfrak{M}, w \models^a (\Diamond\psi)^\circ &\iff \mathfrak{M}, w \models^b \psi^\circ \text{ for some } b \in D_w \\
&\iff w \in Rab \text{ and } w \in V(\psi, b) \text{ for some } b \in D_w \\
&\iff w \in \bigvee\{Rab \cap V(\psi, b) \mid b \in W^*\} \\
&\iff w \in V(\Diamond\psi, b).
\end{aligned}$$

  The second-to-last equivalence follows from the fact that in $\mathrm{Up}(\langle W, \preceq\rangle)$ suprema are interpreted as unions. □

For the left-to-right direction, we also proceed contrapositively. For technical reasons, however, we show first that we can restrict our attention to a restricted class of $\mathsf{S5}(\mathbf{G})$-models. We say that an $\mathsf{S5}(\mathbf{G})$-model $\mathfrak{M} = \langle W, R, V\rangle$ is *irrational* if $V(\varphi, w)$ is irrational, 0, or 1 for all $\varphi \in \mathrm{Fm}_{\Box\Diamond}$ and $w \in W$.

Lemma 2. *For any countable* $\mathsf{S5}(\mathbf{G})$*-model* $\mathfrak{M} = \langle W, R, V\rangle$*, there exists an irrational* $\mathsf{S5}(\mathbf{G})$*-model* $\mathfrak{M}' = \langle W, R', V'\rangle$ *such that for all* $\varphi, \psi \in \mathrm{Fm}_{\Box\Diamond}$*,* $w \in W$*:*

$$V(\varphi, w) < V(\psi, w) \iff V'(\varphi, w) < V'(\psi, w).$$

Proof. By [10, Lemma 3.7], there exists a complete order-embedding f from the countable set $S = \{V(\varphi, w) \mid w \in W,\ \varphi \in \mathrm{Fm}_{\Box\Diamond}\} \cup R[W \times W]$ into $\mathbb{Q} \cap [0, 1]$. Now for each $q \in \mathbb{Q} \cap [0, 1]$, let

$$g(q) = \begin{cases} \frac{\pi}{3}q & q \leq \frac{1}{2} \\ \frac{\pi}{6} + (2 - \frac{\pi}{3})(q - \frac{1}{2}) & q > \frac{1}{2}. \end{cases}$$

Then g is a complete order-embedding from $\mathbb{Q} \cap [0, 1]$ into $([0, 1]\backslash\mathbb{Q}) \cup \{0, 1\}$ with $g(0) = 0$, $g(1) = 1$. So $h = g \circ f$ is a complete order-embedding from S into $([0, 1]\backslash\mathbb{Q}) \cup \{0, 1\}$ with $h(0) = 0$, $h(1) = 1$. Finally, let $\mathfrak{M}' = \langle W, R', V'\rangle$ where $R'wv = h(Rwv)$ and $V'(p_i, w) = h(V(p_i, w))$ for $w, v \in W$ and $i \in \mathbb{N}$. A straightforward induction on formula length yields $V'(\varphi, w) = h(V(\varphi, w))$ for any $\varphi \in \mathrm{Fm}_{\Box\Diamond}$ and $w \in W$ and the claim follows immediately. □

Now let $\mathfrak{M} = \langle W, R, V\rangle$ be any irrational $\mathsf{S5}(\mathbf{G})$-model and fix $w_0 \in W$. Let $(0, 1)_{\mathbb{Q}}$ denote $(0, 1) \cap \mathbb{Q}$. We define the $\mathsf{IKL_1}$-model

$$\mathfrak{M}^\circ = \langle (0, 1)_{\mathbb{Q}}, \geq, \{D_q\}_{q \in (0,1)_{\mathbb{Q}}}, \{I_q\}_{q \in (0,1)_{\mathbb{Q}}}\rangle$$

such that for each $q \in (0, 1)_{\mathbb{Q}}$ and unary predicate P_i,

$$D_q = \{v \in W \mid Rw_0v \geq q\} \quad \text{and} \quad I_q(P_i) = \{v \in W \mid V(p_i, v) \geq q\} \cap D_q.$$

Lemma 3. *For any $\varphi \in \mathrm{Fm}_{\Box\Diamond}$, $q \in (0,1)_{\mathbb{Q}}$, and $w \in D_q$,*

$$\mathfrak{M}^\circ, q \models^w \varphi^\circ \iff V(\varphi, w) \geq q.$$

Proof. We prove the claim by induction on the length of φ. The base cases follow by definition and the cases of the propositional connectives are straightforward. We consider the modal cases.

- For $\varphi = \Box\psi$, observe first that

$$\begin{aligned}\mathfrak{M}^\circ, q \models^w (\forall x)\psi^\circ &\iff \mathfrak{M}^\circ, r \models^v \psi^\circ \text{ for all } r \leq q \text{ and } v \in D_r\\ &\iff V(\psi, v) \geq r \text{ for all } r \leq q \text{ and } v \in D_r;\end{aligned}$$

$$\begin{aligned}V(\Box\psi, w) \geq q &\iff \bigwedge\{Rwv \to V(\psi, v) \mid v \in W\} \geq q\\ &\iff Rwv \to V(\psi, v) \geq q \text{ for all } v \in W\\ &\iff V(\psi, v) \geq q \wedge Rwv \text{ for all } v \in W.\end{aligned}$$

  For the left-to-right direction suppose that $V(\psi, v) \geq r$ for all $r \leq q$ and $v \in D_r$. By assumption, $w \in D_q$, so $Rw_0w \geq q$. Let $v \in W$. If $q \leq Rwv$, then, by symmetry and transitivity, $Rw_0v \geq q$, i.e., $v \in D_q$, and hence $V(\psi, v) \geq q = q \wedge Rwv$. If $q > Rwv$, then $Rw_0w \geq q > Rwv$. By symmetry and transitivity, $Rw_0v = Rwv$. For all $r \in (0,1)_{\mathbb{Q}}$ such that $r \leq Rw_0v$, it holds that $v \in D_r$, so $V(\psi, v) \geq r$. Since $(0,1)_{\mathbb{Q}}$ is dense in $(0,1)\backslash\mathbb{Q}$, we have $\sup\{r \in (0,1)_{\mathbb{Q}} \mid Rw_0v \geq r\} = Rw_0v$ and hence $V(\psi, v) \geq Rw_0v = Rwv$.

  For the right-to-left direction, suppose that $V(\psi, v) \geq q \wedge Rwv$ for all $v \in W$. Let $r \leq q$ and $v \in D_r$. Then $Rw_0v \geq r$. Since $w \in D_q$, also $Rw_0w \geq q \geq r$, and by symmetry and transitivity, $Rwv \geq r$. Hence $V(\psi, v) \geq q \wedge Rwv \geq r$.

- For $\varphi = \Diamond\psi$, observe first that since $\mathfrak{M}$ is irrational and $q \in (0,1)_{\mathbb{Q}}$, $V(\varphi, w) \geq q$ if and only if $V(\varphi, w) > q$. Now observe that

$$\begin{aligned}\mathfrak{M}^\circ, q \models^w (\exists x)\psi^\circ &\iff \mathfrak{M}^\circ, q \models^v \psi^\circ \text{ for some } v \in D_q\\ &\iff V(\psi, v) \geq q \text{ for some } v \in D_q;\end{aligned}$$

$$\begin{aligned}V(\Diamond\psi, w) \geq q &\iff \bigvee\{Rwv \wedge V(\psi, v) \mid v \in W\} \geq q\\ &\iff \bigvee\{Rwv \wedge V(\psi, v) \mid v \in W\} > q\\ &\iff Rwv \wedge V(\psi, v) \geq q \text{ for some } v \in W.\end{aligned}$$

  For the left-to-right direction, suppose that $V(\psi, v) \geq q$ for some $v \in D_q$. Since $w, v \in D_q$, by transitivity, $Rwv \geq q$ and hence $Rwv \wedge V(\psi, v) \geq q$. For the right-to-left direction, suppose that there exists $v \in W$ such that $Rwv \wedge V(\psi, v) \geq q$, i.e., $Rwv \geq q$ and $V(\psi, v) \geq q$. Since $w \in D_q$, also $Rwv \geq q$, so $v \in D_q$ and $V(\psi, v) \geq q$. □

To conclude the proof of Theorem 4 suppose that $\not\models_{\mathsf{S5}(\mathbf{G})} \varphi$. It follows that there exist an $\mathsf{S5}(\mathbf{G})$-model $\langle W, R, V\rangle$ and $w \in W$ such that $V(\varphi, w) < 1$. By Lemma 2, there exist an irrational $\mathsf{S5}(\mathbf{G})$-model $\mathfrak{M} = \langle W, R', V'\rangle$ and $r \in (0,1)_{\mathbb{Q}}$ such that $V'(\varphi, w) < r < 1$. But then, by Lemma 3, for the IKL_1-model $\mathfrak{M}^\circ$ defined above, $\mathfrak{M}^\circ, r \not\models^w \varphi^\circ$. That is, $\not\models_{\mathsf{IKL}_1} \varphi^\circ$.

4 An Interpretation of $\mathsf{S5}$(G) in $\mathsf{S5}(\mathrm{G})^{\mathsf{C}}$

In this section, we provide an interpretation of the one-variable fragment of Corsi logic in the one-variable fragment of first-order Gödel logic, thereby obtaining also an interpretation of $\mathsf{S5}(\mathbf{G})$ in $\mathsf{S5}(\mathbf{G})^{\mathsf{C}}$. The key idea of this interpretation is to use a distinguished unary predicate P_0 for a $\mathsf{CDIKL_1}$-model to describe the domains of a corresponding $\mathsf{IKL_1}$-model. To this end, we let $\mathrm{Fm}_1^r \subseteq \mathrm{Fm}_1$ denote the set of one-variable first-order formulas not containing P_0, and define an $\mathsf{IKL_1^r}$-model to be an $\mathsf{IKL_1}$-model $\mathfrak{M} = \langle W, \preceq, \{D_w\}_{w\in W}, \{I_w\}_{w\in W}\rangle$ such that the functions $\{I_w\}_{w\in W}$ are restricted to $\{P_i\}_{i\in\mathbb{N}^+}$.

With every $\mathsf{CDIKL_1}$-model $\mathfrak{M} = \langle W, \preceq, \{D\}, \{I_w\}_{w\in W}\rangle$, we associate an $\mathsf{IKL_1^r}$-model $\mathfrak{M}^r = \langle W, \preceq, \{D_w\}_{w\in W}, \{I_w^r\}_{w\in W}\rangle$, where for each $w \in W$ and $i \in \mathbb{N}^+$,

$$D_w = I_w(P_0) \quad \text{and} \quad I_w^r(P_i) = I_w(P_i) \cap D_w.$$

Notice that $\mathfrak{M} \mapsto \mathfrak{M}^r$ is a surjective map from $\mathsf{CDIKL_1}$-models to $\mathsf{IKL_1^r}$-models.

Now for each $\alpha \in \mathrm{Fm}_1^r$, we define $\alpha^c \in \mathrm{Fm}_1$ by relativizing quantifiers to the unary predicate P_0. Inductively, we let $(P_i(x))^c = P_i(x)$ for each $i \in \mathbb{N}^+$, $\bot^c = \bot$, $\top^c = \top$, $(\alpha \star \beta)^c = \alpha^c \star \beta^c$ for $\star \in \{\wedge, \vee, \to\}$,

$$((\forall x)\alpha)^c = (\forall x)(P_0(x) \to \alpha^c), \quad \text{and} \quad ((\exists x)\alpha)^c = (\exists x)(P_0(x) \wedge \alpha^c).$$

Lemma 4. *Given any $\alpha \in \mathrm{Fm}_1^r$, $\mathsf{CDIKL_1}$-model $\mathfrak{M} = \langle W, \preceq, \{D\}, \{I_w\}_{w\in W}\rangle$, $w \in W$, and $a \in I_w(P_0)$,*

$$\mathfrak{M}^r, w \models^a \alpha \iff \mathfrak{M}, w \models^a \alpha^c.$$

Proof. We prove the claim by induction on the length of α. For the base case, for each $i \in \mathbb{N}^+$, using the assumption that $a \in D_w$,

$$\mathfrak{M}^r, w \models^a P_i(x) \iff a \in I_w^r(P_i) \iff a \in I_w(P_i) \iff \mathfrak{M}, w \models^a P_i(x).$$

The cases for the propositional connectives follow easily using the induction hypothesis and the definition of α^c, so we just check the cases for the quantifiers:

$$\begin{aligned}
\mathfrak{M}^r, w \models^a (\forall x)\beta &\iff \mathfrak{M}^r, v \models^b \beta \text{ for all } v \succeq w \text{ and } b \in D_v \\
&\iff \mathfrak{M}, v \models^b \beta^c \text{ for all } v \succeq w \text{ and } b \in I_v(P_0) \\
&\iff (\mathfrak{M}, v \models^b P_0(x) \Rightarrow \mathfrak{M}, v \models^b \beta^c) \text{ for all } v \succeq w \text{ and } b \in D \\
&\iff \mathfrak{M}, v \models^b P_0(x) \to \beta^c \text{ for all } v \succeq w \text{ and } b \in D \\
&\iff \mathfrak{M}, w \models^a (\forall x)(P_0(x) \to \beta^c) \\
&\iff \mathfrak{M}, w \models^a ((\forall x)\beta)^c.
\end{aligned}$$

$$\begin{aligned}
\mathfrak{M}^r, w \models^a (\exists x)\beta &\iff \mathfrak{M}^r, w \models^b \beta \text{ for some } b \in D_w \\
&\iff \mathfrak{M}, w \models^b \beta^c \text{ for some } b \in I_w(P_0) \\
&\iff (\mathfrak{M}, w \models^b P_0(x) \text{ and } \mathfrak{M}, w \models^b \beta^c) \text{ for some } b \in D \\
&\iff \mathfrak{M}, w \models^b P_0(x) \wedge \beta^c \text{ for some } b \in D \\
&\iff \mathfrak{M}, w \models^a (\exists x)(P_0(x) \wedge \beta^c) \\
&\iff \mathfrak{M}, w \models^a ((\exists x)\beta)^c. \quad \square
\end{aligned}$$

Corollary 1. *For any sentence* $\alpha \in \mathrm{Fm}_1^r$, $\models_{\mathsf{IKL}_1} \alpha \iff \models_{\mathsf{CDIKL}_1} \alpha^c$.

Proof. Consider a CDIKL_1-model $\mathfrak{M} = \langle W, \preceq, \{D\}, \{I_w\}_{w \in W} \rangle$ and any $a \in D$. Since $\alpha \in \mathrm{Fm}_1^r$ is a sentence, $\mathfrak{M} \models \alpha^c$ if and only if $\mathfrak{M}, w \models^a \alpha^c$ for all $w \in W$. So, by the previous lemma, $\mathfrak{M} \models \alpha^c$ if and only if $\mathfrak{M}^r, w \models^a \alpha$ for all $w \in W$, which holds, since $\alpha \in \mathrm{Fm}_1^r$ is a sentence, if and only if $\mathfrak{M}^r \models \alpha$. The result now follows immediately using the fact that the map $\mathfrak{M} \mapsto \mathfrak{M}^r$ is surjective. $\square$

Now let $\mathrm{Fm}_{\Box\Diamond}^r \subseteq \mathrm{Fm}_{\Box\Diamond}$ denote the set of modal formulas not containing p_0. For each $\varphi \in \mathrm{Fm}_{\Box\Diamond}^r$, we define $\varphi^c \in \mathrm{Fm}_1$ by relativizing modalities to p_0. Inductively, we let $(p_i)^c = p_i$ for each $i \in \mathbb{N}^+$, $\bot^c = \bot$, $\top^c = \top$, $(\varphi \star \psi)^c = \varphi^c \star \psi^c$ for $\star \in \{\wedge, \vee, \to\}$,

$$(\Box\varphi)^c = \Box(p_0 \to \varphi^c), \quad \text{and} \quad (\Diamond\varphi)^c = \Diamond(p_0 \wedge \varphi^c).$$

The main result of this section then follows directly using Theorems 3 and 4 and Corollary 1.

Theorem 5. *For all* $\varphi \in \mathrm{Fm}_{\Box\Diamond}$, $\models_{\mathsf{S5}(\mathbf{G})} \varphi \iff \models_{\mathsf{S5}(\mathbf{G})^{\mathsf{C}}} (\Box\varphi)^c$.

Let us remark that the above proof generalizes in a straightforward way to provide an interpretation of the full first-order Corsi logic in the first-order logic of linear Kripke models with constant domains, or, equivalently, first-order Gödel logic. Moreover, the predicate used in this interpretation corresponds exactly to the existence predicate considered in the context of Scott logics by Iemhoff in [11] and is closely related also to the normalized probability distribution used for the possibilistic logic studied in [3]. We intend to investigate these connections in more detail in future work.

5 A Complexity Result

As has been mentioned already, neither $\mathsf{S5}(\mathbf{G})$ nor $\mathsf{S5}(\mathbf{G})^{\mathsf{C}}$ admits the finite model property with respect to their standard Kripke semantics. It is known, however, that $\mathsf{S5}(\mathbf{G})$ does admit the finite model property with respect to its algebraic semantics (see [2]), and we prove here that the same result holds also for $\mathsf{S5}(\mathbf{G})^{\mathsf{C}}$. We then use this finite model property to give a new proof that validity in $\mathsf{S5}(\mathbf{G})^{\mathsf{C}}$ is co-NP-complete (first proved in [5]), and hence also, by Theorem 5, the same result for $\mathsf{S5}(\mathbf{G})$.

An algebra $\langle H, \wedge, \vee, \to, \bot, \top, \Box, \Diamond \rangle$ (also shortened to $\langle \mathbf{H}, \Box, \Diamond \rangle$) is called a *monadic Heyting algebra* if $\mathbf{H} = \langle H, \wedge, \vee, \to, \bot, \top \rangle$ is a Heyting algebra and $\Box, \Diamond$ are unary operators on H satisfying for all $a, b \in H$,

(1a) $\Box a \leq a$	(1b) $a \leq \Diamond a$
(2a) $\Box(a \wedge b) = \Box a \wedge \Box b$	(2b) $\Diamond(a \vee b) = \Diamond a \vee \Diamond b$
(3a) $\Box\top = \top$	(3b) $\bot = \Diamond\bot$
(4a) $\Box\Diamond a = \Diamond a$	(4b) $\Diamond\Box a = \Box a$
(5a) $\Diamond(\Diamond a \wedge b) = \Diamond a \wedge \Diamond b$.	

If a monadic Heyting algebra satisfies the prelinearity law $(x \to y) \vee (y \to x) \approx \top$, then we call it a *monadic linear Heyting algebra*, and if it satisfies also the constant domain law $\Box(\Box x \vee y) \approx \Box x \vee \Box y$, we call it a *monadic Gödel algebra*. It is straightforward to prove that the varieties (equivalently, equational classes) of monadic linear Heyting algebras and monadic Gödel algebras provide equivalent algebraic semantics for $\mathsf{S5}(\mathbf{G})$ and $\mathsf{S5}(\mathbf{G})^{\mathsf{C}}$, respectively. Indeed, the lattices of axiomatic extensions of $\mathcal{MIPC}$ and varieties of monadic Heyting algebras are dual (see [2]).

These algebras also admit a useful alternative representation. For any monadic Heyting algebra $\langle \mathbf{H}, \Box, \Diamond \rangle$, the set $H_0 = \{\Box a \mid a \in H\} = \{\Diamond a \mid a \in H\}$ forms a subuniverse of $\mathbf{H}$ satisfying for all $a \in H$,

$$\Box a = \bigvee \{b \in H_0 \mid b \leq a\} \quad \text{and} \quad \Diamond a = \bigwedge \{b \in H_0 \mid b \geq a\}.$$

Conversely, call any subuniverse H_0 of a Heyting algebra $\mathbf{H}$ where all such suprema and infima exist in H_0 *relatively complete*. Defining $\Box$ and $\Diamond$ as above for any relatively complete subuniverse H_0 of a Heyting algebra $\mathbf{H}$ yields a monadic Heyting algebra $\langle \mathbf{H}, \Box, \Diamond \rangle$.

Theorem 6 (cf. [2]). *There exists a one-to-one correspondence between monadic Heyting algebras $\langle \mathbf{H}, \Box, \Diamond \rangle$ and pairs $\langle \mathbf{H}, H_0 \rangle$ of Heyting algebras where H_0 is a relatively complete subuniverse of $\mathbf{H}$.*

We use this alternative representation to establish the finite model property for the variety of monadic Gödel algebras. Let us call a monadic Gödel algebra *standard* if it is of the form $\langle \mathbf{G}^W, \Box, \Diamond \rangle$, where W is any non-empty set, $\mathbf{G}^W$ is the Heyting algebra with universe $[0,1]^W$ and operations defined pointwise, and for each $f \in [0,1]^W$ and $w \in W$,

$$\Box(f)(w) = \bigwedge \{f(v) \mid v \in W\} \quad \text{and} \quad \Diamond(f)(w) = \bigvee \{f(v) \mid v \in W\}.$$

Using the completeness results of [6], a formula $\varphi \in \mathrm{Fm}_{\Box\Diamond}$ is $\mathsf{S5}(\mathbf{G})^{\mathsf{C}}$-valid if and only if $\varphi \approx \top$ is valid in all standard monadic Gödel algebras. However, this equivalence fails when restricted to standard monadic Gödel algebras $\langle \mathbf{G}^W, \Box, \Diamond \rangle$ where W is finite.

Observe now that for any standard monadic Gödel algebra $\langle \mathbf{G}^W, \Box, \Diamond \rangle$, the subuniverse $\{\Box f \mid f \in [0,1]^W\}$ consists of all constant functions for $r \in [0,1]$,

$$f_r \colon W \to [0,1]; \quad w \mapsto r.$$

We broaden the class of standard monadic Gödel algebras by considering also subuniverses consisting of only some of these constant functions.

Lemma 5. *For any complete sublattice T of $[0,1]$ containing $\{0,1\}$, the set $\{f_r \mid r \in T\}$ is a relatively complete subuniverse of $\mathbf{G}^W$ and yields a monadic Gödel algebra with modal operators*

$$\Box f(w) = \bigvee \{r \in T \mid r \leq \bigwedge \{f(v) \mid v \in W\}\}$$
$$\Diamond f(w) = \bigwedge \{r \in T \mid r \geq \bigvee \{f(v) \mid v \in W\}\}.$$

Proof. It is easy to check that $\{f_r \mid r \in T\}$ is a subuniverse of $\mathbf{G}^W$. To show that it is relatively complete, consider $\bigvee\{f_r \mid f_r \leq g, r \in T\}$ for some $g \in \mathbf{G}^W$. Then $f_r \leq g$ for $r \in T$ amounts to $r \leq g(v)$ for all $v \in W$, i.e., $r \leq \bigwedge\{g(v) \mid v \in W\}$. So $\bigvee\{f_r \mid f_r \leq g, r \in T\} = \bigvee\{f_r \mid r \leq \bigwedge\{g(v) \mid v \in W\}, r \in T\}$, which exists in T since T is complete. Similarly, $\bigwedge\{f_r \mid f_r \geq g\}$ exists in T, so $\{f_r \mid r \in T\}$ is relatively complete. Hence $\langle \mathbf{G}^W, \{f_r \mid r \in T\}\rangle$ corresponds to a monadic Heyting algebra. Clearly, this algebra also satisfies the prelinearity law and it is easy to check that the constant domain law is satisfied using properties of T and relative completeness. $\square$

Note that the algebras described in the previous lemma correspond exactly to the alternative semantics used in [5] to prove decidability and complexity results for $\mathsf{S5}(\mathbf{G})^{\mathsf{C}}$. Here we obtain simpler proofs of these results (avoiding a rather complicated "squeezing" of truth values argument) by establishing a finite model property with respect to this class of monadic Gödel algebras.

Lemma 6. *Suppose that the equation $\varphi \approx \top$ is not valid in a standard monadic Gödel algebra $\langle \mathbf{G}^W, \Box, \Diamond\rangle$ for some $\varphi \in \mathrm{Fm}_{\Box\Diamond}$ of length $n-2$. Then there exist a non-empty set $W' \subseteq W$, a set $T \subseteq [0,1]$ with $\{0,1\} \subseteq T$, and a subalgebra $\mathbf{A}$ of $\mathbf{G}$ with $T \subseteq A$ satisfying $|W'| \leq n$, $|T| \leq n$, and $|A| \leq n^2$, such that $\varphi \approx \top$ is not valid in the finite monadic Gödel algebra corresponding to $\langle \mathbf{A}^{W'}, \{f_r \mid r \in T\}\rangle$.*

Proof. Suppose that $\varphi \approx \top$ is not valid in some standard monadic Gödel algebra $\langle \mathbf{G}^W, \Box, \Diamond\rangle$ for some $\varphi \in \mathrm{Fm}_{\Box\Diamond}$ of length $n-2$. Then there exists an evaluation e from $\mathrm{Fm}_{\Box\Diamond}$ to G^W satisfying $e(\varphi)(w) < 1$ for some $w \in W$. Let $\Sigma \subseteq \mathrm{Fm}_{\Box\Diamond}$ be the set of subformulas of φ, noting that $|\Sigma| \leq n-2$. We define

$$T = \{0,1\} \cup \{e(\Box\psi)(w) \mid \Box\psi \in \Sigma\} \cup \{e(\Diamond\psi)(w) \mid \Diamond\psi \in \Sigma\}.$$

Clearly, $|T| \leq n$. For each $\Box\psi \in \Sigma$ and $\Diamond\psi \in \Sigma$, we pick a witness $v_{\Box\psi} \in W$ or $v_{\Diamond\psi} \in W$, respectively, such that

$$e(\Box\psi)(w) = \bigvee\{r \in T \mid r \leq e(\psi)(v_{\Box\psi})\}$$
$$e(\Diamond\psi)(w) = \bigwedge\{r \in T \mid r \geq e(\psi)(v_{\Diamond\psi})\}.$$

We define

$$W' = \{w\} \cup \{v_{\Box\psi} \mid \Box\psi \in \Sigma\} \cup \{v_{\Diamond\psi} \mid \Diamond\psi \in \Sigma\} \quad \text{and} \quad e' = e\restriction_{G^{W'}}.$$

Clearly, $|W'| \leq n$. Moreover, $e'(\varphi)(w) = e(\varphi)(w) < 1$ and hence $\varphi \approx \top$ is not valid in the monadic Gödel algebra corresponding to $\langle \mathbf{G}^{W'}, \{f_r \mid r \in T\}\rangle$. Finally, we define

$$A = \{0,1\} \cup \bigcup_{\psi \in \Sigma} e(\psi)[W'].$$

Clearly, $T \subseteq A$ and $|A| \leq n^2$. Moreover, $A^{W'}$ is a finite subuniverse of $\mathbf{G}^{W'}$, and $\varphi \approx \top$ is not valid in the finite monadic Gödel algebra corresponding to $\langle A^{W'}, \{f_r \mid r \in T\}\rangle$. $\square$

An analysis of the number of steps needed to find a finite countermodel yields an upper complexity bound for checking validity in $\mathsf{S5}(\mathbf{G})^{\mathsf{C}}$.

Theorem 7.

(a) *The variety of monadic Gödel algebras has the finite model property.*

(b) *Checking validity in* $\mathsf{S5}(\mathbf{G})^{\mathsf{C}}$ *is co-NP-complete.*

Proof. (a) The variety of monadic Gödel algebras is generated by its standard members, and hence the finite algebras described in the previous lemma also generate this variety.

(b) To check the non-validity of an equation $\varphi \approx \top$, we fix sets A and W' that we may identify with $K = \{1, 2, \ldots, n^2\}$, where $n - 2$ is the length of φ, letting $\mathbf{A}$ denote the unique Gödel algebra induced by the standard order. It suffices to find a relative subuniverse $T \subseteq A$ and an evaluation $e\colon \mathrm{Pr}(\Sigma) \to A^{W'}$ (where $\mathrm{Pr}(\Sigma)$ is the set of propositional variables occurring in Σ that we may also identify with K) and check $e(\varphi) < \top$ when evaluated in the algebra $\langle A^{W'}, \{f_r \mid r \in T\}\rangle$. Finding such T and e is equivalent to finding a characteristic function $\widetilde{T}\colon A \to \{0,1\}$ and a function $\tilde{e}\colon \mathrm{Pr}(\Sigma) \times W' \to A$; that is, finding a pair of sequences of length n^2 and n^4 respectively with entries in K. The tasks of guessing non-deterministically these sequences and checking $e(\varphi) < \top$ in the resulting algebra can be performed in polynomial time. Hence checking non-validity is in NP. But also non-modal formulas are valid in $\mathsf{S5}(\mathbf{G})^{\mathsf{C}}$ if and only if they are valid in Gödel logic, which is known to be co-NP-complete. Hence checking validity in $\mathsf{S5}(\mathbf{G})^{\mathsf{C}}$ is co-NP-complete. ☐

Using the interpretation of $\mathsf{S5}(\mathbf{G})$ in $\mathsf{S5}(\mathbf{G})^{\mathsf{C}}$, provided by Theorem 5, that is linear in the length of the input formula, it follows that also checking validity in $\mathsf{S5}(\mathbf{G})$ is co-NP-complete. The correspondence between validity in $\mathsf{S5}(\mathbf{G})$ and the one-variable fragment of Corsi logic provided by Theorem 1, again linear in the length of the input formula, then completes the proof of Theorem 2.

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