Syntactical Content of Finite Approximations of Partial Algebras

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[**Abstract.**] **In this paper we give a syntactical answer to the following question: What do we actually know about a partial algebra when we know its set of weak or relative subalgebras with cardinal smaller than a fixed bound, if we do not have any information on how they are linked to each other within the algebra?**

1 Introduction

The "roughness" of a theory consists essentially in the fact that, under some data system, objects can only be described approximately. Thus, different objects may be indistinguishable from each other by the means available in the system. This situation is made precise by the notion of indiscernibility.

This paper is intended to pursue the idea of structural approximation for algebras. Given a weak or relative subalgebra of an algebra (total or partial), it can be seen as an approximation of the structure of the latter. This leads to the following notion of indiscernibility. Let $\mathcal{K}$ be a class of algebras of a given signature. We define $S_w(\mathcal{K})$ as the class of all weak subalgebras of algebras in $\mathcal{K}$. Then for any class $\mathcal{M}$ of algebras we say that two algebras $\mathbf{A}$ and $\mathbf{B}$ in $\mathcal{K}$ are $\mathcal{M}$-*indiscernible* iff for every $\mathbf{D} \in \mathcal{M}$, $\mathbf{D} \in S_w(\mathbf{A})$ if and only if $\mathbf{D} \in S_w(\mathbf{B})$. Clearly, a similar notion can be defined for relative subalgebras, too. Thus two algebras are indiscernible when they cannot be distinguished from each other by the available approximations.

In this paper we are concerned with approximations of an algebra $\mathbf{A}$ given by finite weak or relative subalgebras (or even by subalgebras of at most some fixed finite cardinality). Notice that such finite approximations are ubiquitous in $\mathbf{A}$ (each finite subset of the carrier of $\mathbf{A}$ supports at least one weak subalgebra and exactly one relative subalgebra of $\mathbf{A}$), and they determine completely $\mathbf{A}$, provided we know the way they glue (a partial algebra is the direct limit of

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its directed system of finite weak, or finite relative, subalgebras [2, Cor. 4.4.7]). But what do we actually know about a partial algebra when we know its finite approximations (up to isomorphisms), if no information on how they are linked to each other within the algebra is available?

We give here a syntactical answer to this question. We define the *syntactical content* of a type of finite approximations as, roughly, a set of formulas of the form "conjunction implies disjunction" that are 'captured' by these approximations (see Def. 2 below), and we determine it for weak and relative subalgebras (with a fixed bound on the cardinality of their carrier) of partial and total algebras.

Notice that partial algebras are often used as models of objects appearing in soft computing, such as graphs, relational systems and data bases. For instance, a binary relation on a set can be understood as a partial binary operation given by the first projection defined only on the pairs in the relation. Our results allow us to characterize syntactically, for instance, the knowledge of all its weak subsystems (weak subalgebras of that binary algebra) with less than a fixed number of elements. Another way of looking at the problem considered in this paper is to ask what knowledge on an algebraic structure can be derived from finite experiments, which thus brings us close to machine learning.

This note is born in part from the desire to better understand some of the results on similar problems obtained in [1], to be published elsewhere.

To simplify things, we only deal here with partial algebras over *finite* (i.e., with finitely many operation symbols) and *homogeneous* (i.e., one-sorted) signatures, but the results we obtain in this case are easily generalized to more general cases; cf. again [1].

2 Preliminaries and notations

For the convenience of the reader, in this section we recall the basic definitions on partial algebras, used in this paper (except for weak and relative subalgebras defined at the beginning of the next section); for any notion not defined here, as well as for more details about those defined, see [2]. In this section we also fix some notation and conventions to be used throughout the paper.

We fix for the rest of this paper a *signature* $\Sigma = (\Omega, \eta)$, where Ω is a *finite* set of *operation symbols* and $\eta : \Omega \to \mathbb{N}$ is the *arity mapping*. We set $\Omega^{(n)} = \{\varphi \in \Omega \mid \eta(\varphi) = n\}$ for every $n \in \mathbb{N}$.

A *partial Σ-algebra* (an *algebra*, for short) is a structure $\mathbf{A} = (A, (\varphi^{\mathbf{A}})_{\varphi\in\Omega})$, where A is a set, called the *carrier* of the algebra, and for every $\varphi \in \Omega$, $\varphi^{\mathbf{A}} : A^{\eta(\varphi)} \to A$ is a partial mapping with domain $\mathrm{dom}\,\varphi^{\mathbf{A}} \subseteq A^{\eta(\varphi)}$. We denote the class of all such algebras by Alg_Σ.

Given an algebra denoted by a capital letter in boldface type ($\mathbf{A}$, $\mathbf{B}$, etc.), we always denote, unless otherwise stated, its carrier set by the same capital letter in slanted type (A, B, etc.).

An algebra is *finite* when its carrier is finite. The *cardinal* $|\mathbf{A}|$ of a finite algebra $\mathbf{A}$ is the cardinal of its carrier.

An algebra $\mathbf{A}$ is *total* when $\varphi^{\mathbf{A}}$ is a total mapping, for every $\varphi \in \Omega$; we denote by TAlg_{Σ} the class of all total algebras.

Two algebras $\mathbf{A} = (A, (\varphi^{\mathbf{A}})_{\varphi\in\Omega})$ and $\mathbf{B} = (B, (\varphi^{\mathbf{B}})_{\varphi\in\Omega})$ are *isomorphic* when there exists a bijection $h : A \to B$ (an *isomorphism*) such that for every $\varphi \in \Omega$ and for every $a, a_1, \ldots, a_{\eta(\varphi)} \in A$, $\varphi^{\mathbf{A}}(a_1, \ldots, a_{\eta(\varphi)}) = a$ iff $\varphi^{\mathbf{B}}(h(a_1), \ldots, h(a_{\eta(\varphi)})) = h(a)$.

We fix henceforth a countably infinite *set of variables* $\mathcal{X} = \{x_i \mid i \in \mathbb{N}\}$, disjoint from Ω. The set $\mathrm{T}_{\Sigma}(\mathcal{X})$ of *(Σ-)terms with variables in* $\mathcal{X}$ is defined as the least set T such that $\mathcal{X} \cup \Omega^{(0)} \subseteq \mathrm{T}$ and, if $\varphi \in \Omega$ and $\mathbf{t}_1, \ldots, \mathbf{t}_{\eta(\varphi)} \in \mathrm{T}$, then $\varphi(\mathbf{t}_1, \ldots \mathbf{t}_{\eta(\varphi)}) \in \mathrm{T}$.

Given a term $\mathbf{t} \in \mathrm{T}_{\Sigma}(\mathcal{X})$ and an algebra $\mathbf{A}$, we define the (partial) *term function* $\mathbf{t}^{\mathbf{A}} : A^{\mathcal{X}} \to A$ (where $A^{\mathcal{X}}$ denotes the set of all *valuations* $v : \mathcal{X} \to A$) as follows:

– If $\mathbf{t} = x_i \in \mathcal{X}$, then $\mathbf{t}^{\mathbf{A}}(v) = v(x_i)$ for every $v : \mathcal{X} \to A$.
– If $\mathbf{t} = \varphi \in \Omega^{(0)}$, then $\mathbf{t}^{\mathbf{A}}(v) = \varphi^{\mathbf{A}}$ for every $v : \mathcal{X} \to A$ if $\varphi^{\mathbf{A}}$ is defined[1], and $\mathrm{dom}\,\mathbf{t}^{\mathbf{A}} = \emptyset$ otherwise.
– If $\mathbf{t} = \varphi(\mathbf{t}_1, \ldots \mathbf{t}_n)$ for some $\varphi \in \Omega^{(n)}$ and terms $\mathbf{t}_1, \ldots, \mathbf{t}_n$, then $v \in \mathrm{dom}\,\mathbf{t}^{\mathbf{A}}$ iff $v \in \bigcap_{i=1}^{n} \mathrm{dom}\,\mathbf{t}_i^{\mathbf{A}}$ and $(\mathbf{t}_1^{\mathbf{A}}(v), \ldots, \mathbf{t}_n^{\mathbf{A}}(v)) \in \mathrm{dom}\,\varphi^{\mathbf{A}}$, and if $v \in \mathrm{dom}\,\mathbf{t}^{\mathbf{A}}$ then $\mathbf{t}^{\mathbf{A}}(v) = \varphi^{\mathbf{A}}(\mathbf{t}_1^{\mathbf{A}}(v), \ldots, \mathbf{t}_n^{\mathbf{A}}(v))$.

Notice that the definedness and value of $\mathbf{t}^{\mathbf{A}}(v)$ only depend on the images under v of the variables appearing in $\mathbf{t}$. Moreover, if $\varphi \in \Omega^{(n)}$, then the term function associated to $\varphi(x_1, \ldots, x_n)$ is (essentially) the operation $\varphi^{\mathbf{A}} : A^n \to A$.

To simplify the notation, and unless otherwise stated, when we write in the sequel $\mathbf{t}^{\mathbf{A}}(v)$, we always assume that it is defined, i.e., that $v \in \mathrm{dom}\,\mathbf{t}^{\mathbf{A}}$.

An *existence equation*, an *equation* for short, is a pair $(\mathbf{p}, \mathbf{q}) \in \mathrm{T}_{\Sigma}(\mathcal{X})^2$ of terms, and will be written $\mathbf{p} \approx \mathbf{q}$ in the sequel.

Given an algebra $\mathbf{A}$ and a valuation $v : \mathcal{X} \to A$, the equation $\mathbf{p} \approx \mathbf{q}$ is *satisfied* in $\mathbf{A}$ w.r.t. v, in symbols $(\mathbf{A}, v) \models \mathbf{p} \approx \mathbf{q}$, when $v \in \mathrm{dom}\,\mathbf{p}^{\mathbf{A}} \cap \mathrm{dom}\,\mathbf{q}^{\mathbf{A}}$ and $\mathbf{p}^{\mathbf{A}}(v) = \mathbf{q}^{\mathbf{A}}(v)$. So, for instance, $(\mathbf{A}, v) \models \mathbf{p} \approx \mathbf{p}$ means simply that $\mathbf{p}^{\mathbf{A}}(v)$ is defined; therefore, we denote the equation $\mathbf{p} \approx \mathbf{p}$ by $\exists\mathbf{p}$.

Using equations as atoms, and the connectives $\neg, \vee, \wedge, \Rightarrow, \ldots$ (with their usual logical meaning), we can build up formulas and define their satisfaction in a partial algebra w.r.t. a given valuation; see [2, §7.1] for details. In this paper we are only interested in a special type of such formulas.

A *quasi-existence equation of type* Σ is a formula of the form $\left(\bigwedge_{i\in I} \mathbf{p}_i \approx \mathbf{q}_i\right) \Rightarrow \mathbf{p} \approx \mathbf{q}$ with I a finite set. A *disjunctive quasi-existence equation*, a $\vee$-*equation* for short, *of type* Σ is a formula of the form

$$\Big(\bigwedge_{i\in I} \mathbf{p}_i \approx \mathbf{q}_i\Big) \Rightarrow \Big(\bigvee_{j\in J} \mathbf{p}'_j \approx \mathbf{q}'_j\Big)$$

with I and J finite sets; so, $\vee$-equations include, as special cases, quasi-existence equations and disjunctions of equations (taking $|J| = 1$ and $I = \emptyset$, respectively).

[1] If $\varphi \in \Omega^{(0)}$, we say that $\varphi^{\mathbf{A}}$ is *defined* when $\varphi^{\mathbf{A}} : A^0 \to A$ is total, and then we use the same symbol $\varphi^{\mathbf{A}}$ to denote the image of this mapping.

To simplify the notation, we usually omit the brackets embracing the premise and the conclusion in $\vee$-equations.

We denote by $\mathcal{L}$ the set of all $\vee$-equations of some previously fixed type.

Let now $\Phi = \bigwedge_{i\in I} \mathbf{p}_i \approx \mathbf{q}_i \Rightarrow \bigvee_{j\in J} \mathbf{p}'_j \approx \mathbf{q}'_j$ be a $\vee$-equation. Then Φ is *satisfied* in an algebra $\mathbf{A}$ w.r.t. a valuation $v : \mathcal{X} \to A$, in symbols $(\mathbf{A}, v) \models \Phi$, iff the following condition holds:

> If $(\mathbf{A}, v) \models \mathbf{p}_i \approx \mathbf{q}_i$ for every $i \in I$, then $(\mathbf{A}, v) \models \mathbf{p}'_j \approx \mathbf{q}'_j$ for some $j \in J$.

So, $(\mathbf{A}, v) \not\models \Phi$ iff $\mathbf{p}_i^{\mathbf{A}}(v) = \mathbf{q}_i^{\mathbf{A}}(v)$ for every $i \in I$ but, for every $j \in J$, either $v \notin \operatorname{dom} \mathbf{p}'^{\mathbf{A}}_j$, or $v \notin \operatorname{dom} \mathbf{q}'^{\mathbf{A}}_j$, or $\mathbf{p}'^{\mathbf{A}}_j(v) \neq \mathbf{q}'^{\mathbf{A}}_j(v)$.

Now, an algebra $\mathbf{A}$ *(globally) satisfies* a $\vee$-equation Φ, in symbols $\mathbf{A} \models \Phi$, when $(\mathbf{A}, v) \models \Phi$ for every $v : \mathcal{X} \to A$.

It is clear that two isomorphic algebras satisfy exactly the same $\vee$-equations (as we will see later, the converse implication is false, even for total algebras).

We say that an equation $\mathbf{p} \approx \mathbf{q}$ is a *consequence* of a finite set of equations $\{\mathbf{p}_i \approx \mathbf{q}_i\}_{i\in I}$ when the quasi-equation $\bigwedge_{i\in I} \mathbf{p}_i \approx \mathbf{q}_i \Rightarrow \mathbf{p} \approx \mathbf{q}$ is a *tautology* (i.e., it is satisfied by *all* algebras); it is equivalent to say that $\mathbf{p} \approx \mathbf{q}$ is deduced from $\{\mathbf{p}_i \approx \mathbf{q}_i\}_{i\in I}$ through Burmeister's deduction rules for existence equations [2, §6.4.8].

3 Main results

We begin by recalling the definitions of weak and relative subalgebras.

[**Definition 1.**] *Let* $\mathbf{A} = (A, (\varphi^{\mathbf{A}})_{\varphi\in\Omega})$ *and* $\mathbf{B} = (B, (\varphi^{\mathbf{B}})_{\varphi\in\Omega})$ *be two algebras, with* $B \subseteq A$.

i) $\mathbf{B}$ *is a* weak subalgebra *of* $\mathbf{A}$ *when, for every* $\varphi \in \Omega$*, if* $\underline{b} \in \operatorname{dom} \varphi^{\mathbf{B}}$ *then* $\underline{b} \in \operatorname{dom} \varphi^{\mathbf{A}}$ *and* $\varphi^{\mathbf{B}}(\underline{b}) = \varphi^{\mathbf{A}}(\underline{b})$.

ii) $\mathbf{B}$ *is a* relative subalgebra *of* $\mathbf{A}$ *when it is a weak subalgebra and, for every* $\varphi \in \Omega$*, if* $\underline{b} \in \operatorname{dom} \varphi^{\mathbf{A}} \cap B^{\eta(\varphi)}$ *and* $\varphi^{\mathbf{A}}(\underline{b}) \in B$ *then* $\underline{b} \in \operatorname{dom} \varphi^{\mathbf{B}}$.

Notice that every subset B of the carrier of an algebra $\mathbf{A}$ supports (in principle) many weak subalgebras of $\mathbf{A}$, but only one such relative subalgebra, namely the greatest possible weak subalgebra of $\mathbf{A}$ supported on B.

Given an algebra $\mathbf{A}$, let $S_w^{(n)}(\mathbf{A})$ and $S_r^{(n)}(\mathbf{A})$ be the classes of all algebras of cardinal at most n that are isomorphic to weak and relative subalgebras of $\mathbf{A}$, respectively. Let also $S_w^f(\mathbf{A})$ and $S_r^f(\mathbf{A})$ be the classes of all finite algebras that are isomorphic to weak and relative subalgebras of $\mathbf{A}$, respectively. These are the *finite approximations* of $\mathbf{A}$ we consider in this paper.

[**Definition 2.**] *Let* $\mathcal{C}$ *be a class of algebras and let* $\tilde{S}$ *be an algebraic operator corresponding to some type of finite subalgebras (for instance, one of those defined above).*

A set $\tilde{\mathcal{L}}$ *of* $\vee$*-equations is the* syntactical content *of* $\tilde{S}$ *for* $\mathcal{C}$ *when it is the greatest subset of* $\mathcal{L}$ *satisfying the following three conditions:*

[i)] *If* $\bigwedge_{i\in I}\mathbf{p}_i \approx \mathbf{q}_i \Rightarrow \bigvee_{j\in J}\mathbf{p}'_j \approx \mathbf{q}'_j$ *belongs to* $\tilde{\mathcal{L}}$ *then* $\tilde{\mathcal{L}}$ *also contains every* $\vee$*-equation of the form* $\bigwedge_{i\in I}\mathbf{p}_i \approx \mathbf{q}_i \Rightarrow \bigvee_{j\in J'}\mathbf{p}'_j \approx \mathbf{q}'_j$ *with* $J' \subseteq J$ *(we say then that* $\tilde{\mathcal{L}}$ *is* well-formed*).*

[ii)] *For every formula* $\Phi \in \tilde{\mathcal{L}}$ *there exists a non-empty finite set* $C_{\tilde{S},\mathcal{C}}(\Phi)$ *of finite algebras such that, for every algebra* $\mathbf{A} \in \mathcal{C}$*,*

$$\mathbf{A} \not\models \Phi \textit{ iff there exists some } \mathbf{A}_0 \in C_{\tilde{S},\mathcal{C}}(\Phi) \cap \tilde{S}(\mathbf{A}).$$

[iii)] *For every finite algebra* $\mathbf{A}_0$*, there exists a formula* $\Phi_{\tilde{S},\mathcal{C}}(\mathbf{A}_0) \in \tilde{\mathcal{L}}$ *such that, for every algebra* $\mathbf{A} \in \mathcal{C}$*,*

$$\mathbf{A}_0 \in \tilde{S}(\mathbf{A}) \textit{ iff } \mathbf{A} \not\models \Phi_{\tilde{S},\mathcal{C}}(\mathbf{A}_0).$$

Thus, a well-formed set $\tilde{\mathcal{L}}$ of $\vee$-equations is the syntactical content of $\tilde{S}$ for a class $\mathcal{C}$ when it is the greatest such set such that, for every $\mathbf{A} \in \mathcal{C}$, the knowledge of $\tilde{S}(\mathbf{A})$ is equivalent to the knowledge of

$$\tilde{\mathcal{L}}(\mathbf{A}) = \{\Phi \in \tilde{\mathcal{L}} \mid \mathbf{A} \models \Phi\}.$$

Indeed, notice that, for every $\mathbf{A} \in \mathcal{C}$:

- To know whether $\mathbf{A}$ satisfies a formula $\Phi \in \tilde{\mathcal{L}}$, one has only to check whether some algebra in the finite set $C_{\tilde{S},\mathcal{C}}(\Phi)$ belongs to $\tilde{S}(\mathbf{A})$;
- To know whether a given finite algebra $\mathbf{A}_0$ belongs to $\tilde{S}(\mathbf{A})$, one only has to check the non-satisfaction of $\Phi_{\tilde{S},\mathcal{C}}(\mathbf{A}_0)$ by $\mathbf{A}$.

In particular, the following result holds.

[**Proposition 1.**] *Let* $\tilde{\mathcal{L}}$ *be the syntactical content of the operator* $\tilde{S}$ *for a class* $\mathcal{C}$ *of algebras. Then, given any two algebras* $\mathbf{A}, \mathbf{B} \in \mathcal{C}$*,* $\tilde{S}(\mathbf{A}) = \tilde{S}(\mathbf{B})$ *iff* $\tilde{\mathcal{L}}(\mathbf{B}) = \tilde{\mathcal{L}}(\mathbf{A})$*.*

Consider now the following definition.

[**Definition 3.**] *Given a non-tautological* $\vee$*-equation* Φ*, we shall call its* complexity *the greatest cardinal* $\kappa(\Phi)$ *of an algebra* $\mathbf{A}$ *such that* $\mathbf{A} \not\models \Phi$ *but* $\mathbf{A}' \models \Phi$ *for every strict weak subalgebra of* $\mathbf{A}$*. We adopt the convention that tautological* $\vee$*-equations have complexity 0.*

Let $\mathcal{L}^{(n)}$ be the set of all $\vee$-equations of complexity at most n, and, for every $\tilde{\mathcal{L}} \subseteq \mathcal{L}$, set $\tilde{\mathcal{L}}^{(n)} = \tilde{\mathcal{L}} \cap \mathcal{L}^{(n)}$.

Notice that the complexity of a $\vee$-equation Φ is always smaller or equal than the cardinal of the least initial segment[2] of $\mathrm{T}_\Sigma(\mathcal{X})$ containing all terms appearing in it.

[2] A subset $Y \subseteq \mathrm{T}_\Sigma(\mathcal{X})$ is an *initial segment* when, for every $\varphi \in \Omega$ and $\mathbf{t}_1, \dots \mathbf{t}_{\eta(\varphi)} \in \mathrm{T}_\Sigma(\mathcal{X})$, $\varphi(\mathbf{t}_1, \dots \mathbf{t}_{\eta(\varphi)}) \in Y$ implies $\mathbf{t}_1, \dots \mathbf{t}_{\eta(\varphi)} \in Y$.

[**Proposition 2.**] *Let* $\mathcal{L}_w$ *denote the set of all* $\vee$*-equations of the form*

$$\bigwedge_{i\in I} \mathbf{p}_i \approx \mathbf{q}_i \Rightarrow (\bigvee_{(j_1,j_2)\in J} x_{j_1} \approx x_{j_2}) \vee (\bigvee_{k\in K} \mathbf{p}'_k \approx \mathbf{q}'_k)$$

such that, for every $k \in K$, $\exists \mathbf{p}'_k, \exists \mathbf{q}'_k$ *are consequences of* $\bigwedge_{i\in I} \mathbf{p}_i \approx \mathbf{q}_i$.

i) For every $n \in \mathbb{N}$, $\mathcal{L}_w^{(n)}$ *is the syntactical content of* $S_w^{(n)}$ *for* Alg_Σ.

ii) $\mathcal{L}_w$ *is the syntactical content of* S_w^f *for* Alg_Σ.

[*Proof.*]We will prove only (i), since (ii) follows immediately. To do that, $\mathcal{L}_w^{(n)}$ being clearly well-formed, we check points (ii) and (iii) in the definition of syntactical content, and then we show that $\mathcal{L}_w^{(n)}$ is the greatest well-formed set of $\vee$-equations satisfying point (ii) therein.

a) For every $\Phi = \bigwedge_{i\in I} \mathbf{p}_i \approx \mathbf{q}_i \Rightarrow (\bigvee_{(j_1,j_2)\in J} x_{j_1} \approx x_{j_2}) \vee (\bigvee_{k\in K} \mathbf{p}'_k \approx \mathbf{q}'_k)$ in $\mathcal{L}_w^{(n)}$, let $C_{S_w^{(n)},\mathrm{Alg}_\Sigma}(\Phi)$ be a (minimal) set containing one, and only one, representative of every isomorphism class of algebras $\mathbf{A}'$ such that $\mathbf{A}' \not\models \Phi$ and $|\mathbf{A}'| \leq \kappa(\Phi) \leq n$. This set is clearly finite (and it is empty iff Φ is a tautology). We will show that it satisfies the property required in Definition 2.

Let $\mathbf{A}$ be an algebra such that $\mathbf{A} \not\models \Phi$, and let $v : \mathcal{X} \to A$ be a valuation such that $(\mathbf{A}, v) \not\models \Phi$, i.e., such that $\mathbf{p}_i^{\mathbf{A}}(v) = \mathbf{q}_i^{\mathbf{A}}(v)$ for every $i \in I$ but $\mathbf{p}'^{\mathbf{A}}_k(v) \neq \mathbf{q}'^{\mathbf{A}}_k(v)$ for every $k \in K$ (notice that all $\mathbf{p}'^{\mathbf{A}}_k(v)$ and $\mathbf{q}'^{\mathbf{A}}_k(v)$ are defined because $\exists \mathbf{p}'_k$ and $\exists \mathbf{q}'_k$ are consequences of $\bigwedge_{i\in I} \mathbf{p}_i \approx \mathbf{q}_i$) and $v(x_{j_1}) \neq v(x_{j_2})$ for every $(j_1, j_2) \in J$ with $j_1 \neq j_2$.

Let V be the set of all variables appearing (explicitly) in the terms of Φ, and let $\mathbf{A}'$ be the least finite weak subalgebra of $\mathbf{A}$ containing $v(V)$ and such that $\mathbf{p}_i^{\mathbf{A}'}(v)$ and $\mathbf{q}_i^{\mathbf{A}'}(v)$ are defined for every $i \in I$. Then, for any valuation $v' : \mathcal{X} \to A'$ that coincides with v on V we have $(\mathbf{A}', v') \not\models \Phi$, hence $\mathbf{A}' \not\models \Phi$.

Since any strict weak subalgebra of $\mathbf{A}'$ satisfies Φ, we have that $|\mathbf{A}'| \leq \kappa(\Phi)$ and thus it has an isomorphic copy $\mathbf{A}'_0$ in $C_{S_w^{(n)},\mathrm{Alg}_\Sigma}(\Phi)$. This shows that if $\mathbf{A} \not\models \Phi$ then there exists some $\mathbf{A}'_0$ in $C_{S_w^{(n)},\mathrm{Alg}_\Sigma}(\Phi) \cap S_w^{(n)}(\mathbf{A})$.

Conversely, let $\mathbf{A}'$ be a finite algebra of cardinality less then $\kappa(\Phi)$ such that $\mathbf{A}' \not\models \Phi$ and let $\mathbf{A}$ be an algebra containing $\mathbf{A}'$ as a weak subalgebra. Let $v : \mathcal{X} \to A'$ be a valuation such that $(\mathbf{A}', v) \not\models \Phi$: then $\mathbf{p}_i^{\mathbf{A}'}(v) = \mathbf{q}_i^{\mathbf{A}'}(v)$ for every $i \in I$ but $\mathbf{p}'^{\mathbf{A}'}_k(v) \neq \mathbf{q}'^{\mathbf{A}'}_k(v)$ for every $k \in K$ (remember that $\exists \mathbf{p}'_k$ and $\exists \mathbf{q}'_k$ are consequences of $\bigwedge_{i\in I} \mathbf{p}_i \approx \mathbf{q}_i$) and $v(x_{j_1}) \neq v(x_{j_2})$ for every $(j_1, j_2) \in J$ with $j_1 \neq j_2$.

Taking as $v : \mathcal{X} \to A$ the same valuation with target set A, we also have $\mathbf{p}_i^{\mathbf{A}}(v) = \mathbf{q}_i^{\mathbf{A}}(v)$ for every $i \in I$ (because they are already defined, and equal, in $\mathbf{A}'$), $\mathbf{p}'^{\mathbf{A}'}_k(v) \neq \mathbf{q}'^{\mathbf{A}'}_k(v)$ for every $k \in K$ (because they are already defined, and different, in $\mathbf{A}'$), and $v(x_{j_1}) \neq v(x_{j_2})$ for every $(j_1, j_2) \in J$ with $j_1 \neq j_2$, so $(\mathbf{A}, v) \not\models \Phi$ and consequently $\mathbf{A} \not\models \Phi$.

b) Every partial algebra has an empty weak subalgebra; thus, we can take as $\Phi_{S_w^{(n)},\mathrm{Alg}_\Sigma}(\emptyset)$ the equation $x_1 \approx x_1$.

Assume now that $\mathbf{A}_0$ is a non-empty partial algebra of cardinality $m \geq 1$, with carrier $A_0 = \{a_1, \ldots, a_m\}$. Let $I(\mathbf{A}_0)$ be the set of equations

$$I(\mathbf{A}_0) = \{\varphi(x_{i_1}, \ldots, x_{i_n}) \approx x_{i_0} \mid \varphi \in \Omega^{(n)}, n \geq 0, i_0, \ldots, i_n \in \{1, \ldots, m\}, \\ \varphi^{\mathbf{A}_0}(a_{i_1}, \ldots, a_{i_n}) = a_{i_0}\}$$

and take as $\Phi_{S_w^{(n)}, \mathrm{Alg}_\Sigma}(\mathbf{A}_0)$ the $\vee$-equation

$$\bigwedge I(\mathbf{A}_0) \Rightarrow \bigvee_{1 \leq j_1 < j_2 \leq m} x_{j_1} \approx x_{j_2}$$

Notice that $(\mathbf{A}_0, v) \not\models \Phi_{S_w^{(n)}, \mathrm{Alg}_\Sigma}(\mathbf{A}_0)$ for any valuation $v : \mathcal{X} \to A_0$ such that $v(x_i) = a_i$, $i = 1, \ldots, m$. Therefore, if $\mathbf{A}_0 \in S_w^f(\mathbf{A})$, then $\mathbf{A} \not\models \Phi_{S_w^{(n)}, \mathrm{Alg}_\Sigma}(\mathbf{A}_0)$.

Conversely, assume that $\mathbf{A} \not\models \Phi_{S_w^{(n)}, \mathrm{Alg}_\Sigma}(\mathbf{A}_0)$, and let $v : \mathcal{X} \to A$ be a valuation such that $(\mathbf{A}, v) \not\models \Phi_{S_w^{(n)}, \mathrm{Alg}_\Sigma}(\mathbf{A}_0)$. Taking $a'_i = v(x_i)$ for every $i = 1, \ldots, m$, we have that

- if $\varphi(x_{i_1}, \ldots, x_{i_n}) \approx x_{i_0} \in I(\mathbf{A}_0)$, i.e., if $\varphi^{\mathbf{A}_0}(a_{i_1}, \ldots, a_{i_n}) = a_{i_0}$, then $\varphi^{\mathbf{A}}(a'_{i_1}, \ldots, a'_{i_n}) = a'_{i_0}$;
- if $1 \leq j_1 < j_2 \leq m$ then $a'_{j_1} \neq a'_{j_2}$.

Let $A'_0 = \{a'_1, \ldots, a'_m\}$, and take the weak subalgebra $\mathbf{A}'_0 = (A'_0, (\varphi^{\mathbf{A}'_0})_{\varphi \in \Omega})$ of $\mathbf{A}$ with

$$\operatorname{dom} \varphi^{\mathbf{A}'_0} = \{(a'_{i_1}, \ldots, a'_{i_n}) \mid (a_{i_1}, \ldots, a_{i_n}) \in \operatorname{dom} \varphi^{\mathbf{A}_0}\}, \quad \varphi \in \Omega^{(n)}, n \geq 0.$$

Then the mapping $h : A_0 \to A'_0$ defined by $h(a_i) = a'_i$, $i = 1, \ldots, m$, is an isomorphism of $\mathbf{A}_0$ onto $\mathbf{A}'_0$, and therefore $\mathbf{A}_0 \in S_w^{(n)}(\mathbf{A})$.

Notice that the complexity of $\Phi_{S_w^{(n)}, \mathrm{Alg}_\Sigma}(\mathbf{A}_0)$ is exactly $|\mathbf{A}_0| = m$. Therefore, for every algebra $\mathbf{A}_0$ of cardinality at most n we have constructed a $\vee$-equation $\tilde{\Phi}_{S_w^{(n)}, \mathrm{Alg}_\Sigma}(\mathbf{A}_0)$ of complexity at most n such that, for every $\mathbf{A} \in \mathcal{C}$, $\mathbf{A}_0 \in S_w^{(n)}(\mathbf{A})$ iff $\mathbf{A} \not\models \tilde{\Phi}_{S_w^{(n)}, \mathrm{Alg}_\Sigma}(\mathbf{A}_0)$, as we wanted.

c) Assume that there is some well-formed set of equations $\tilde{\mathcal{L}}$, not contained in $\mathcal{L}_w^{(n)}$, and satisfying (ii) in Definition 2 w.r.t. the operator $S_w^{(n)}$ and the class Alg_Σ. Then, $\tilde{\mathcal{L}}$ will contain a formula Φ of the form $\bigwedge_{i \in I} \mathbf{p}_i \approx \mathbf{q}_i \Rightarrow \mathbf{p} \approx \mathbf{q}$, where, say, $\exists \mathbf{p}$ is not a consequence of the premise.

Let $\mathbf{A}$ be a minimal algebra not satisfying $\bigwedge_{i \in I} \mathbf{p}_i \approx \mathbf{q}_i \Rightarrow \mathbf{p} \approx \mathbf{q}$, and therefore Φ either. Then by (ii) $\mathbf{A}$ has a finite weak subalgebra $\mathbf{A}_0$ that belongs to $C_{S_w^{(n)}, \mathrm{Alg}_\Sigma}(\Phi)$; however, $\mathbf{A}$ and thus also $\mathbf{A}_0$ has an extension satisfying Φ, in contradiction with (ii). □

In the proofs of the next propositions, we shall only give the corresponding sets $C_{\tilde{S}, \mathcal{C}}(\Phi)$ and $\vee$-equations $\Phi_{\tilde{S}, \mathcal{C}}(\mathbf{A}_0)$; the proofs of the desired properties are similar to those in the previous proposition, already presented in detail.

[**Proposition 3.**] *a)* $\mathcal{L}^{(n)}$ *is the syntactical content of* $S_w^{(n)}$ *for* TAlg_Σ.
b) $\mathcal{L}$ *is the syntactical content of* S_w^f *for* TAlg_Σ.

[*Proof.*]Given an arbitrary $\vee$-equation $\Phi = \bigwedge_{i\in I} \mathbf{p}_i \approx \mathbf{q}_i \Rightarrow \bigvee_{j\in J} \mathbf{p}'_j \approx \mathbf{q}_j$ in $\mathcal{L}^{(n)}$, let $C_{S_w^{(n)},\mathrm{TAlg}_\Sigma}(\Phi)$ be a (minimal) set containing one, and only one, representative of every isomorphism class of algebras $\mathbf{A}'$ of cardinality at most $\kappa(\Phi)$ that do not satisfy Φ for some valuation $v : \mathcal{X} \to A'$.

Moreover, given a finite algebra $\mathbf{A}_0$, let $\Phi_{S_w^{(n)},\mathrm{TAlg}_\Sigma}(\mathbf{A}_0) = \Phi_{S_w^{(n)},\mathrm{Alg}_\Sigma}(\mathbf{A}_0)$. □

[**Proposition 4.**] *Let* $\mathcal{L}_r$ *denote the set of all* $\vee$*-equations of the form*

$$\bigwedge_{i\in I} \mathbf{p}_i \approx \mathbf{q}_i \Rightarrow (\bigvee_{j\in J} \mathbf{t}_j \approx x_{i_j}) \vee (\bigvee_{k\in K} \mathbf{p}'_k \approx \mathbf{q}'_k)$$

where every $\mathbf{t}_j$ *is either a variable or a term of the form* $\varphi(x_{i_1}, \ldots, x_{i_{\eta(\varphi)}})$ *for some* $\varphi \in \Omega$*, and, for every* $k \in K$*,* $\exists \mathbf{p}'_k, \exists \mathbf{q}'_k$ *are consequences of* $\bigwedge_{i\in I} \mathbf{p}_i \approx \mathbf{q}_i$.

a) For every $n \in \mathbb{N}$*,* $\mathcal{L}_r^{(n)}$ *(resp.* $\mathcal{L}^{(n)}$*) is the syntactical content of* $S_r^{(n)}$ *for* Alg_Σ *(resp.* TAlg_Σ*).*

b) $\mathcal{L}_r$ *(resp.* $\mathcal{L}$*) is the syntactical content of* S_w^f *for* Alg_Σ *(resp.* TAlg_Σ*).*

[*Proof.*]For every $\Phi \in \mathcal{L}_r^{(n)}$ take $C_{S_r^{(n)},\mathrm{Alg}_\Sigma}(\Phi) = C_{S_w^{(n)},\mathrm{Alg}_\Sigma}(\Phi)$ and $C_{S_r^{(n)},\mathrm{TAlg}_\Sigma}(\Phi)$ $= C_{S_w^{(n)},\mathrm{TAlg}_\Sigma}(\Phi)$. Moreover, for every finite algebra $\mathbf{A}_0$:

- $\Phi_{S_r^{(n)},\mathrm{Alg}_\Sigma}(\emptyset) = \Phi_{S_r^{(n)},\mathrm{TAlg}_\Sigma}(\emptyset) = x_1 \approx x_1$;
- If $\mathbf{A}_0$ has carrier $A_0 = \{a_1, \ldots, a_m\}$, $m \geq 1$, then let $I(\mathbf{A}_0)$ be the set of equations associated to $\mathbf{A}_0$ as in the proof of Proposition 2, and let

$$I^c(\mathbf{A}_0) = \{\varphi(x_{i_1}, \ldots, x_{i_n}) \approx x_{i_0} \mid \varphi \in \Omega^{(n)}, n \geq 0, i_0, \ldots, i_n \in \{1, \ldots, m\}, (a_{i_1}, \ldots, a_{i_n}) \notin \mathrm{dom}\, \varphi^{\mathbf{A}} \text{ or } \varphi^{\mathbf{A}}(a_{i_1}, \ldots, a_{i_n}) \neq a_{i_0}\}$$

Then take $\Phi_{S_r^{(n)},\mathrm{Alg}_\Sigma}(\mathbf{A}_0) = \Phi_{S_r^{(n)},\mathrm{TAlg}_\Sigma}(\mathbf{A}_0)$ to be the $\vee$-equation

$$\bigwedge I(\mathbf{A}_0) \Rightarrow (\bigvee I^c(\mathbf{A}_0) \vee \bigvee_{1\leq j_1<j_2\leq m} x_{j_1} \approx x_{j_2})$$

□

References

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