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Standard Gödel Modal Logics

Abstract. We prove strong completeness of the $\Box$-version and the $\Diamond$-version of a Gödel modal logic based on Kripke models where propositions at each world and the accessibility relation are both infinitely valued in the standard Gödel algebra [0,1]. Some asymmetries are revealed: validity in the first logic is reducible to the class of frames having two-valued accessibility relation and this logic does not enjoy the finite model property, while validity in the second logic requires truly fuzzy accessibility relations and this logic has the finite model property. Analogues of the classical modal systems D, T, S4 and S5 are considered also, and the completeness results are extended to languages enriched with a discrete well ordered set of truth constants.

Keywords: many-valued modal logics, Gödel-Dummett logic, fuzzy Kripke semantics, strong completeness.

1. Introduction

Sometimes it is needed in approximate reasoning to deal simultaneously with both fuzziness of propositions and modalities, for instance one may try to assign a degree of truth to propositions like "John is possibly tall" or "John is necessarily tall", where "John is tall" is presented as a fuzzy proposition. Fuzzy logic should be a suitable tool to model not only vagueness but also these and other kinds of information features like certainty, belief or similarity, which have a natural interpretation in terms of modalities. In this context, it is natural to interpret fuzzy modal operators by means of Kripke models over fuzzy frames.

We address in this paper the case of the pure modal operators (necessitation and possibility) for standard Gödel logic, one of the main systems of fuzzy logic arising from Hájek´s classification in [14]. For this purpose we consider a many-valued version of Kripke semantics for modal logic where both propositions at each world and the accessibility relation are infinitely valued in the standard Gödel algebra [0,1].

We provide strongly complete axiomatizations for the $\Box$-fragment and the $\Diamond$-fragment of the resulting minimal logic. These fragments are shown to behave quite asymmetrically. Validity in the first one is determined by

Special Issue: Algebra and Probability in Many-Valued Reasoning
Edited by **Ioana Leuştean and Vincenzo Marra**

Studia Logica (2010) 94: 189–214
DOI: 10.1007/s11225-010-9230-1

the class of frames having a crisp (that is, two-valued) accessibility relation, while validity in the second requires truly fuzzy frames. In addition, the □-fragment does not enjoy the finite model property with respect to the number of worlds or the number of truth values while the ◇-fragment does.

We consider also the Gödel analogues of the classical modal systems D, T, S4 and S5 for each modal operator and show that the first three are characterized by the many-valued versions of the frame properties which characterize their classical counterparts. Finally, we extend the strong completeness results to Pavelka-style languages enriched with a set of explicit truth constants denoting a discrete well ordered set of truth values.

Our approach is related to Fitting´s [10] who considers Kripke models taking values in a fixed finite Heyting algebra; however, Fitting´s proof systems and completeness proofs depend essentially on finiteness of the algebra and the fact that his languages contain constants for all the truth values of the algebra. We must relay on completely different methods.

Modal logics with an (intermediate) intuitionistic basis and Kripke style semantics have been investigated in a number of relevant papers (see Ono [18], Fischer Servi [8], Bošzic and Došen [5], Font [11], Wolter [20], from an extensive literature), but in all cases the models carry two or more crisp accessibility relations satisfying some commuting properties: a pre-order to account for the intuitionistic connectives and one or more binary relations to account for the modal operators. Our semantics has, instead, a single fuzzy accessibility relation and does not seem reducible to those multi-relational semantics since the latter enjoy the finite model property for □ (cf. Grefe [13]).

Recently, Metcalfe and Olivetti [17] have given a proof of weak completeness of a calculus of sequents of relations for the □-fragment of our logic, showing that it is decidable and PSPACE complete. The decidability of the ◇-fragment follows from the finite model property we prove later.

Bou, Esteva, and Godo survey in [4] modal logics with analogue [0,1]-valued Kripke semantics under different choices of the t-norm. However, our methods and results do not generalize to the corresponding modal versions of Łukasiewicz or product logics because they relay on the richness of endomorphisms of the Gödel algebra [0,1]. In fact, we do not know any completeness result for these logics without extra conditions on the frames, strong completeness being known to be untenable.

Classical modal logics are inter-translatable with description logics [1]. Our Gödel-Kripke semantics for the modal operators is similarly translatable into fuzzy (Gödel) description logic (cf. [15]), and thus our results throw light on various fragments of this logic and certain Pavelka-style expansions of them.

We assume the reader is acquainted with modal and Gödel logics and the basic laws of Heyting algebras (cf. Chagrov and Zakharyaschev [6]).

2. Gödel-Kripke models

The language $\mathcal{L}_{\Box\Diamond}$ of propositional *Gödel modal logic* is built from a set Var of propositional variables, logical connectives symbols $\wedge, \rightarrow, \bot$, and the modal operator symbols $\Box$ and $\Diamond$. Other connectives are defined:

$$\top := \varphi \rightarrow \varphi$$
$$\neg\varphi := \varphi \rightarrow \bot$$
$$\varphi \vee \psi := ((\varphi \rightarrow \psi) \rightarrow \psi) \wedge ((\psi \rightarrow \varphi) \rightarrow \varphi)$$
$$\varphi \leftrightarrow \psi := (\varphi \rightarrow \psi) \wedge (\psi \rightarrow \varphi).$$

$\mathcal{L}_{\Box}$ and $\mathcal{L}_{\Diamond}$ will denote, respectively, the $\Box$-fragment and the $\Diamond$-fragment of the language.

As stated before, the semantics of Gödel modal logic will be based on fuzzy Kripke models where the valuations at each world and also the accessibility relation between worlds are $[0,1]$-valued. The symbols $\cdot$ and $\Rightarrow$ will denote the Gödel t-norm in $[0,1]$ and its residuum, respectively:

$$a \cdot b = \min\{a, b\}, \qquad a \Rightarrow b = \begin{cases} 1, & \text{if } a \leq b \\ b, & \text{otherwise} \end{cases}$$

The maximum is definable, $\max\{a, b\} := ((a \Rightarrow b) \Rightarrow b) \cdot ((b \Rightarrow a) \Rightarrow a)$, and the pseudo-complement is denoted $-a := a \Rightarrow 0$. This yields the standard Gödel algebra; that is, the unique Heyting algebra structure in the linearly ordered interval.

DEFINITION 2.1. A *Gödel-Kripke model* (*GK-model*) will be a structure $M = \langle W, S, e\rangle$ where:

- W is a non-empty set of objects that we call *worlds* of M.
- $S : W \times W \rightarrow [0,1]$ is an arbitrary function $(x, y) \longmapsto Sxy$.
- $e : W \times Var \rightarrow [0,1]$ is an arbitrary function $(x, p) \longmapsto e(x, p)$.

The evaluations $e(x, -) : Var \rightarrow [0,1]$ are extended simultaneously to all formula in $\mathcal{L}_{\Box\Diamond}$ by defining inductively at each world x:

$$e(x, \bot) := 0$$
$$e(x, \varphi \wedge \psi) := e(x, \varphi) \cdot e(x, \psi)$$
$$e(x, \varphi \rightarrow \psi) := e(x, \varphi) \Rightarrow e(x, \psi)$$
$$e(x, \Box\varphi) := \inf_{y \in W}\{Sxy \Rightarrow e(y, \varphi)\}$$

$e(x, \Diamond\varphi) := \sup_{y \in W}\{Sxy \cdot e(y, \varphi)\}$.

It follows that $e(x, \varphi \vee \psi) = \max\{e(x, \varphi), e(x, \psi)\}$ and $e(x, \neg\varphi) = -e(x, \varphi)$.

The notions of a formula φ being true at a world x, valid in a model $M = \langle W, S, e\rangle$, or universally valid, are the usual ones:

φ is *true in* M *at* x, written $M \models_x \varphi$, iff $e(x, \varphi) = 1$.
φ is *valid in* M, written $M \models \varphi$, iff $M \models_x \varphi$ at any world x of M.
φ is GK-*valid*, written $\models_{GK} \varphi$, if it is valid in all the GK-models.

Clearly, all valid schemas of Gödel logic are GK-valid. In addition,

Proposition 2.1. *The following modal schemas are GK-valid:*

$\mathbf{K}_\Box$	$\Box(\varphi \to \psi) \to (\Box\varphi \to \Box\psi)$
$\mathbf{Z}_\Box$	$\neg\neg\Box\theta \to \Box\neg\neg\theta$
$\mathbf{K}_\Diamond$	$\Diamond(\varphi \vee \psi) \to (\Diamond\varphi \vee \Diamond\psi)$ (in fact, an equivalence)
$\mathbf{Z}_\Diamond$	$\Diamond\neg\neg\varphi \to \neg\neg\Diamond\varphi$
$\mathbf{F}_\Diamond$	$\neg\Diamond\bot$

Proof. Let $M = \langle W, S, e\rangle$ be an arbitrary GK-model and $x \in W$.

($\mathbf{K}_\Box$) By Definition 2.1 and properties of the residuum we have for any $y \in W$: $e(x, \Box(\varphi \to \psi)) \cdot e(x, \Box\varphi) \leq (Sxy \Rightarrow (e(y, \varphi) \Rightarrow e(y, \psi)) \cdot (Sxy \Rightarrow e(y, \varphi)) \leq (Sxy \Rightarrow e(y, \psi))$. Taking the meet over y in the last expression: $e(x, \Box(\varphi \to \psi)) \cdot e(x, \Box\varphi) \leq e(x, \Box\psi)$; thus, $e(x, \Box(\varphi \to \psi)) \leq e(x, \Box\varphi \to \Box\psi)$.

($\mathbf{Z}_\Box$) Utilizing the Heyting algebra identity: $--(x \Rightarrow y) = (x \Rightarrow --y)$, we have: $e(x, \neg\neg\Box\theta) = --e(x, \Box\theta) \leq (Sxy \Rightarrow --e(y, \theta)) = (Sxy \Rightarrow e(y, \neg\neg\theta))$. Taking the meet over y, $e(x, \neg\neg\Box\theta) \leq e(x, \Box\neg\neg\theta)$.
($\mathbf{K}_\Diamond$) By properties of suprema and distributivity of $\cdot$ over max, $e(\Diamond(\varphi \vee \psi)) = \sup_y\{Sxy \cdot \max\{e(y, \varphi), e(y, \psi)\}\} = \max\{\sup_y\{Sxy \cdot e(y, \varphi)\}, \sup_y\{Sxy \cdot e(y, \psi)\}\}$.
($\mathbf{Z}_\Diamond$) $Sxy \cdot e(y, \neg\neg\varphi) \leq --Sxy \cdot --e(y, \varphi) = --(Sxy \cdot e(y, \varphi)) \leq --e(x, \Diamond\varphi) = e(x, \neg\neg\Diamond\varphi)$.

($\mathbf{F}_\Diamond$) $e(x, \Diamond\bot) = \sup_y\{Sxy \cdot 0\} = 0$. ■

The Modus Ponens rule preserves truth at every world of any GK-model. On the other hand, the classical introduction rules for the modal operators

$$\mathbf{RN}_\Box : \frac{\varphi}{\Box\varphi} \qquad \mathbf{RN}_\Diamond : \frac{\varphi \to \psi}{\Diamond\varphi \to \Diamond\psi} .$$

do not preserve truth at a fixed world. However, Proposition 2.2. *$\mathbf{RN}_\Box$ and $\mathbf{RN}_\Diamond$ preserve validity in any given model, thus they preserve GK-validity.*

Proof. Let $\langle W, S, e\rangle$ be a GK-model. ($\mathbf{RN}_\Box$) If $e(x,\varphi) = 1$ for all $x \in W$ then $e(x, \Box\varphi) = \inf_y\{Sxy \Rightarrow e(y,\varphi)\} = \inf\{1\} = 1$ for all x. ($\mathbf{RN}_\Diamond$) If $e(x, \varphi \to \psi) = 1$ for all $x \in W$ then $Sxy \cdot e(y,\varphi) \leq Sxy \cdot e(y,\psi) \leq e(x, \Diamond\psi)$ for any $y \in W$ Taking the join over y in the left hand side of the last inequality, $e(x, \Diamond\varphi) \leq e(x, \Diamond\psi)$. ■

Semantic consequence is defined for any theory $T \subseteq \mathcal{L}_{\Box\Diamond}$, as follows:

Definition 2.2. $T \models_{GK} \varphi$ if and only if for any GK-model M and any world x of M, $M \models_x T$ implies $M \models_x \varphi$.

Note that Modus Ponens preserves consequence but this is not the case of the inference rules $\mathbf{RN}_\Box$ and $\mathbf{RN}_\Diamond$.

An alternative notion of logical consequence arises naturally. Set $e(x,T) = \{e(x,\varphi) : \varphi \in T\}$ then:

Definition 2.3. $T \models_{\leq GK} \varphi$ if and only if for any GK-model M and any world x in M, $\inf e(x,T) \leq e(x,\varphi)$.

Clearly, $\models_{\leq GK}$ implies $\models_{GK}$, and it will follow from our completeness theorems that both notions are equivalent for countable theories. This fact has been already observed for pure Gödel logic by Baaz and Zach in [3].

3. On strong completeness of Gödel logic

To prove strong completeness of the modal fragments $\mathcal{L}_\Box$ and $\mathcal{L}_\Diamond$ we will reduce the problem to pure Gödel propositional logic.

In the rest of this paper $\mathcal{L}(X)$ will denote the language built from a set X of propositional variables and the connectives $\wedge, \to, \bot$. Let $\mathcal{G}$ be a fixed axiomatic calculus for Gödel logic (also called Dummett's LC) on this language, say the following one given by Hájek ([14], Def. 4.2.3.):

$$(\varphi \to \psi) \to ((\psi \to \chi) \to (\varphi \to \chi))$$
$$(\varphi \wedge \psi) \to \varphi$$
$$(\varphi \wedge \psi) \to (\psi \wedge \varphi)$$
$$(\varphi \to (\psi \to \chi)) \longleftrightarrow ((\varphi \wedge \psi) \to \chi)$$
$$((\varphi \wedge \psi) \to \chi) \longleftrightarrow (\varphi \to (\psi \to \chi))$$
$$\varphi \to (\varphi \wedge \varphi)$$
$$((\varphi \to \psi) \to \chi) \to (((\psi \to \varphi) \to \chi) \to \chi)$$
$$\bot \to \varphi$$

MP: From φ and $\varphi \to \psi$, infer ψ

The symbol $\vdash$ will denote deductive inference in this calculus. It is well known that $\mathcal{G}$ is deductively equivalent to the intermediate logic obtained by adding to Heyting calculus the pre-linearity schema: $(\varphi \to \psi) \vee (\psi \to \varphi)$.

Given a valuation $v : X \to [0, 1]$, let $\overline{v}$ denote the extension of v to $\mathcal{L}(X)$ according to the Gödel interpretation of the connectives. We will need the following strong form of standard completeness for Gödel logic:

Proposition 3.1. *Let T be a countable theory and U a countable set of formulas of $\mathcal{L}(X)$ such that for every finite $S \subseteq U$ we have $T \nvdash \bigvee S$ then there is a valuation $v : X \to [0, 1]$ such that $\overline{v}(\alpha) = 1$ for all $\alpha \in T$ and $\overline{v}(\beta) < 1$ for each $\beta \in U$.*

Proof. Extend T to a prime theory T' (that is, $T' \vdash \alpha \vee \beta$ implies $T' \vdash \alpha$ or $T' \vdash \beta$) satisfying the same hypothesis with respect to U (this is standard). The Lindenbaum algebra $\mathcal{L}(X)/_{\equiv T'}$ of T' is linearly ordered since by primality and the pre-linearity schema $T' \vdash \alpha \to \beta$ or $T' \vdash \beta \to \alpha$. Moreover, the valuation $v : X \to \mathcal{L}(X)/_{T'}$, $v(x) = x/_{\equiv T'}$ is such that $v(T) = 1$, $v(\beta) < 1$ for all $\beta \in U$. As T' is countable we may assume X is countable and thus, being also countable, $\mathcal{L}(X)/_{\equiv T'}$ is embeddable in the Gödel algebra $[0, 1]$, therefore, we may assume $v : X \to [0, 1]$. ∎

Taking $S = \{\varphi\}$ we obtain the usual formulation of completeness. We can not expect strong standard completeness of $\mathcal{G}$ for uncountable theories, as the following example illustrates.

Example. Set $T = \{(p_\beta \to p_\alpha) \to q : \alpha < \beta < \omega_1\}$ where ω_1 is the first uncountable cardinal, then $T \nvdash q$. Otherwise we would have $\Sigma \vdash q$, for some finite $\Sigma = \{(p_{\alpha_{i+1}} \to p_{\alpha_i}) \to q : 1 \leq i < n\}$, but this is not possible by soundness of $\mathcal{G}$, because the valuation $v(q) = \frac{1}{2}$, $v(p_{\alpha_i}) = \frac{1}{2}(1 - \frac{1}{i+1})$ for $1 \leq i \leq n$, makes $v(p_{\alpha_i}) < v(p_{\alpha_{i+1}}) < \frac{1}{2}$ and thus $\overline{v}((p_{\alpha_{i+1}} \to p_{\alpha_i}) \to q) = 1$ for $1 \leq i < n$, while $v(q) < 1$. However, there is no valuation v such that $\overline{v}(T) = 1$ and $v(q) < 1$, because that would imply $\overline{v}(p_\beta \to p_\alpha) < 1$ for all $\alpha < \beta < \omega_1$, and thus the set $\{v(p_\alpha) : \alpha < \omega_1\}$ would be ordered in type ω_1, which is impossible because any well ordered subset of $([0, 1], <)$ is at most countable.

4. Completeness of the $\Box$-fragment

Let $\mathcal{G}_\Box$ be the formal system on the language $\mathcal{L}_\Box$ which is obtained by adding to the system $\mathcal{G}$ for Gödel logic (applied to $\mathcal{L}_\Box$) the following axiom schemas and rule:

K$_\Box$: $\Box(\varphi \to \psi) \to (\Box\varphi \to \Box\psi)$
Z$_\Box$: $\neg\neg\Box\theta \to \Box\neg\neg\theta$
NR$_\Box$: From φ infer $\Box\varphi$

$\vdash_{\mathcal{G}_\Box} \varphi$ will express theoremhood in this logic. Proofs with assumptions will be allowed, with the restriction that **NR**$_\Box$ is to be applied to theorems only (or, what amounts to the same, to previous steps of the proof not depending on the assumptions). $T \vdash_{\mathcal{G}_\Box} \varphi$ will express that there is such a proof of φ with assumptions from the set T.

The deduction theorem follows readily by induction in the length of proofs:

DT: *$T \cup \{\alpha\} \vdash_{\mathcal{G}_\Box} \varphi$ implies $T \vdash_{\mathcal{G}_\Box} \alpha \to \varphi$.*

Applying consecutively **DT**, **NR**$_\Box$, **K**$_\Box$, and **MP**, we obtain the derived rule:

Lemma 4.1. *If $\mu_1, ..., \mu_k \vdash_{\mathcal{G}_\Box} \varphi$ then $\Box\mu_1, ..., \Box\mu_k \vdash_{\mathcal{G}_\Box} \Box\varphi$.*

We have also soundness of $\mathcal{G}_\Box$:

Lemma 4.2. *$T \vdash_{\mathcal{G}_\Box} \varphi$ implies $T \models_{\leq GK} \varphi$, hence, $T \models_{GK} \varphi$.*

Proof. By the deduction theorem, $T \vdash_{\mathcal{G}_\Box} \varphi$ implies $\vdash_{\mathcal{G}_\Box} (\wedge\Sigma \to \varphi)$ for some finite $\Sigma \subseteq T$. Since the axioms of $\mathcal{G}_\Box$ are valid in all GK-models (Prop. 2.1) and **MP**, **NR**$_\Box$, **K**$_\Box$ preserve validity (Prop. 2.2) then $\models_{GK} (\wedge\Sigma \to \varphi)$. Therefore, $\inf e(x, T) \leq e(x, \wedge\Sigma) \leq e(x, \varphi)$ for any world x in any GK-model. ■

Let $T\mathcal{G}_\Box = \{\theta : T \vdash_{\mathcal{G}_\Box} \theta\}$ Since all uses of **NR**$_\Box$ in a proof of $T \vdash_{\mathcal{G}_\Box} \varphi$ produce theorems of $\mathcal{G}_\Box$, the proof may be seen as one in which Modus Ponens is the only rule utilized and $T\mathcal{G}_\Box$ is part of the assumptions. Thus,

Lemma 4.3. *$T \vdash_{\mathcal{G}_\Box} \varphi$ if and only if $T \cup T\mathcal{G}_\Box \vdash \varphi$ in pure Gödel logic.*

To prove strong completeness of $\mathcal{G}_\Box$ we will define a canonical GK-model with the property that for any countable theory T and any formula φ such that $T \nvdash_{\mathcal{G}_\Box} \varphi$, there is a world x in the model which assigns the value 1 to T but less than 1 to φ. A surprising fact will be that this may be achieved with a model where the accessibility relation is crisp.

Let $\Box\mathcal{L}_\Box = \{\Box\theta : \theta \in \mathcal{L}_\Box\}$ be the set of formulas in $\mathcal{L}_\Box$ which start with the connective $\Box$. Then any formula in $\mathcal{L}_\Box$ may be seen as a formula of the pure Gödel language built from $X = Var \cup \Box\mathcal{L}_\Box$ by means of $\wedge, \neg, \bot$. That is, we may consider the formulas in $\Box\mathcal{L}_\Box$ as additional propositional variables for Gödel logic.

Canonical model $\mathcal{M}_\Box = (W^*, S^*, e^*)$:

• The set of worlds W^* will consist of those valuations $v : Var \cup \Box\mathcal{L}_\Box \to [0,1]$ which satisfy $\overline{v}(T\mathcal{G}_\Box) = 1$ when extended to $\overline{v} : \mathcal{L}_\Box = \mathcal{L}(Var \cup \Box\mathcal{L}_\Box) \to [0,1]$ according to the Gödel interpretation of $\wedge, \to, \bot$.

• The accessibility relation between worlds in $\mathcal{M}_\Box$ will be given by

$$S^*vw = \begin{cases} 1, & \text{if } v(\Box\theta) \leq \overline{w}(\theta), \text{ for all } \theta \in \mathcal{L}_\Box \\ 0, & \text{otherwise} \end{cases},$$

• The valuation associated to the world v will be $v \restriction Var$. That is, $e^*(v, p) = v(p)$ for any $p \in Var$.

For the sake of simplicity, we will write from now on: $v(\varphi)$ for $\overline{v}(\varphi)$.

Lemma 4.4. *For any world v in the canonical model $M_\Box$ and any formula φ, $e^*(v, \varphi) = v(\varphi)$.*

Proof. This is proved by induction in the complexity of φ seen again as a formula of $\mathcal{L}_\Box = \mathcal{L}_\Box(Var)$ The atomic step and the inductive steps for the Gödel connectives being straightforward, it is enough to verify inductively $e^*(v, \Box\varphi) = v(\Box\varphi)$. By the induction hypothesis we may assume $e^*(w, \varphi) = w(\varphi)$ for any w, and thus we must show

$$v(\Box\varphi) = \inf_w \{w(\varphi) : S^*vw = 1\}$$

By definition, $S^*vw = 1$ implies $v(\Box\varphi) \leq w(\varphi)$, hence

$$v(\Box\varphi) \leq \inf_w \{w(\varphi) : S^*vw = 1\},$$

and equality above is trivial for $v(\Box\varphi) = 1$. Thus it remains only to show in case $v(\Box\varphi) = \alpha < 1$ that

$$\inf_w \{w(\varphi) : S^*vw = 1\} \leq \alpha. \tag{1}$$

That is, for any $\epsilon > 0$ there is w such that $S^*vw = 1$ and $w(\varphi) < \alpha + \epsilon$. To achieve this we prove first:

Claim. *Let v be a world of $\mathcal{M}_\Box$ and φ be such that $v(\Box\varphi) = \alpha < 1$, then there exists a world u of $\mathcal{M}_\Box$ such that $u(\varphi) < 1$ and*
(i) $u(\theta) = 1$ *if* $v(\Box\theta) > \alpha$
(ii) $u(\theta) > 0$ *if* $v(\Box\theta) > 0$.

Proof. Assume $v(\Box\varphi) = \alpha < 1$ and set

$$T_{\varphi,v} = \{\theta : v(\Box\theta) > \alpha\} \cup \{\neg\neg\theta : v(\Box\theta) > 0\}$$

Notice that $v(\Box\mu) > \alpha$ for any $\mu \in T_{\varphi,v}$ because $v(\Box\theta) > 0$ implies $v(\neg\neg\Box\theta) = 1$, and thus $v(\Box\neg\neg\theta) = 1$ since v satisfies axiom $\mathbf{Z}_\Box$. This implies that $T_{\varphi,v} \nvdash_{\mathcal{G}_\Box} \varphi$. Otherwise, $\mu_1, ..., \mu_k \vdash_{\mathcal{G}_\Box} \varphi$ for some $\mu_i \in T_{\varphi,v}$ and thus

$$\Box\mu_1, ..., \Box\mu_k \vdash_{\mathcal{G}_\Box} \Box\varphi$$

by Lemma 4.1. Hence, by Lemma 4.2 and the previous observations,

$$\alpha < \min\{\Box\mu_1, ..., \Box\mu_k\} \leq v(\Box\varphi),$$

a contradiction. By Lemma 4.3 we have $T_{\varphi,v} \cup T\mathcal{G}_\Box \nvDash \varphi$ and by the countability of $T_{\varphi,v} \cup T\mathcal{G}_\Box$ we may use the completeness theorem of Gödel logic (Proposition 3.1) to get a Gödel valuation $u : Var \cup \Box\mathcal{L}_\Box \to [0,1]$ such that $u(T_{\varphi,v}) = 1$ and $u(\varphi) < 1$. Then $u \in \mathcal{M}_\Box$ and (i) holds by construction. Moreover, (ii) is satisfied because $u(\neg\neg\theta) = 1$ and thus $u(\theta) > 0$ if $v(\Box\theta) > 0$. This ends the proof of the claim.

Pick now an strictly increasing function $g : [0,1] \to [0,1]$ such that

$$g(1) = 1, \;\; g(0) = 0, \;\; \text{and } g[(0,1)] = (\alpha, \alpha + \epsilon).$$

As g is a homomorphism of Heyting algebras, the valuation $w = g \circ u$ preserves the value 1 of the formulas in $T\mathcal{G}_\Box$ and thus it belongs to $\mathcal{M}_\Box$. Moreover, $v(\Box\theta) \leq w(\theta)$ for all θ:

- if $v(\Box\theta) > \alpha$ because $w(\theta) = g(u(\theta)) = g(1) = 1$ by (i) above.

- if $0 < v(\Box\theta) \leq \alpha$ because then $0 < u(\theta) \leq 1$ by (ii) above, and thus $w(\theta) = g(u(\theta)) \in (\alpha, \alpha + \epsilon) \cup \{1\}$.

This means $S^*vw = 1$, and since $u(\varphi) < 1$ we have, $w(\varphi) = g(u(\varphi)) < \alpha + \epsilon$, which shows (1). ■

Definition 4.1. Call a GK-model *accessibility crisp* (*a-crisp* in short) if $S : W \times W \to \{0,1\}$, and write $T \models_{Crisp} \varphi$ if the consequence relation holds at each node of any a-crisp GK-model.

Theorem 4.2. *For any countable theory T and formula φ in $\mathcal{L}_\Box$ the following are equivalent:*

(i) $T \vdash_{\mathcal{G}_\Box} \varphi$

(ii) $T \models_{\leq GK} \varphi$

(iii) $T \models_{GK} \varphi$

(iv) $T \models_{Crisp} \varphi$.

Proof. By Lemma 4.2, it is enough to show (iv) $\Rightarrow$ (i). If $T \nvdash_{\mathcal{G}_\Box} \varphi$ then $T \cup T\mathcal{G}_\Box \nvdash \varphi$ by Lemma 4.3, and by strong completeness of Gödel logic there is a valuation $v : Var \cup \Box\mathcal{L}_\Box \to [0,1]$ such that $\overline{v}(T) = \overline{v}(T\mathcal{G}_\Box) = 1$ and $\overline{v}(\varphi) < 1$. Hence, $v \in W^*$ by definition, $e^*(v,T) = \overline{v}(T) = 1$, and $e^*(v,\varphi) = \overline{v}(\varphi) < 1$ by Lemma 4.4, showing that $\mathcal{M}_\Box \models_v T$ but $\mathcal{M}_\Box \not\models_v \varphi$. That is, $T \not\models_{Crisp} \varphi$ because the canonical model is a-crisp. ■

After the example in Section 3 we can not expect completeness with respect to uncountable theories.

5. $\mathcal{G}_\Box$ does not have the finite model property

The following example shows that $\mathcal{G}_\Box$ does not have the finite model property with respect to GK-models. The reciprocal of axiom $\mathbf{Z}_\Box$:

$$\Box\neg\neg\theta \to \neg\neg\Box\theta,$$

fails in the (a-crisp) model $\mathcal{M} = (\mathbb{N}, S, e)$, where

$$Smn = 1 \text{ for all } m, n, \qquad e(n,p) = \tfrac{1}{n+1} \text{ for all } n.$$

Indeed, $e(n, \neg\neg p) = --\frac{1}{n+1} = 1$ for all n and thus, $e(0, \Box\neg\neg p) = \inf\{1\} = 1$. But $e(0, \Box p) = \inf_{n\in\mathbb{N}}\{1 \Rightarrow \frac{1}{n+1}\} = 0$, and thus $e(0, \neg\neg\Box p) = 0$. However,

Proposition 5.1. *$\Box\neg\neg\theta \to \neg\neg\Box\theta$ is valid in any GK-model $\langle W, S, e\rangle$ with finite W.*

Proof. Assume $e(x, \Box\neg\neg\theta) > e(x, \neg\neg\Box\theta)$ then $- - e(x, \Box\theta) < 1$ and thus $e(x, \Box\theta) = 0$. This implies the existence of a sequence $\{y_n\}_n \subseteq W$ such that $Sxy_n > e(y_n, \theta)$ for all $n \in \mathbb{N}$ and $\{e(y_n, \theta)\}_n$ converges to 0. If W is finite so is the range of the latter sequence and there must exist n such that $e(y_n, \theta) = 0$. Then $(Sxy_n \Rightarrow e(y_n, \neg\neg\theta)) = (Sxy_n \Rightarrow - - 0) = 0$ and thus $e(x, \Box\neg\neg\theta) = 0$, a contradiction. ■

Remark. The proof of the previous proposition shows that $\Box\neg\neg\theta \to \neg\neg\Box\theta$ is valid in all 0-*witnessed* GK-models, those where $e(x, \Box\theta) = 0$ implies the existence of y such that $e(y, \theta) = 0 < Sxy$. In fact, $\mathcal{G}_\Box + \{\Box\neg\neg\theta \to \neg\neg\Box\theta\}$ is strongly complete for 0-witnessed (a-crisp) models. To see this, notice that if any world v of the canonical model is asked to satisfy the new schema, $\mathcal{M}_\Box$ becomes 0-witnessed because then $v(\Box\theta) = 0$ implies $v(\Box\neg\neg\theta) = 0$, and thus by the last line in the proof of Lemma 4.4 there is w such that $S^*vw = 1$ and $w(\neg\neg\theta) < \varepsilon$; hence, $w(\theta) = 0$. We do not know if this system has the finite model property.

6. Completeness of the $\Diamond$-fragment

The system $\mathcal{G}_\Diamond$ results by adding to $\mathcal{G}$ the following axiom schemas and rule in the language $\mathcal{L}_\Diamond$:

K$_\Diamond$: $\Diamond(\varphi \vee \psi) \to (\Diamond\varphi \vee \Diamond\psi)$
Z$_\Diamond$: $\Diamond\neg\neg\varphi \to \neg\neg\Diamond\varphi$
F$_\Diamond$: $\neg\Diamond\bot$
RN$_\Diamond$: *From* $\varphi \to \psi$ *infer* $\Diamond\varphi \to \Diamond\psi$

As in the case of the $\Box$-fragment, in proofs with assumptions the rule **RN**$_\Diamond$ is to be used in theorems only, and thus we have the deduction theorem **DT**, hence the rule:

LEMMA 6.1. *If* $\varphi \vdash_{\mathcal{G}_\Diamond} \psi$ *then* $\Diamond\varphi \vdash_{\mathcal{G}_\Diamond} \Diamond\psi$.

Also the soundness theorem:

LEMMA 6.2. $T \vdash_{\mathcal{G}\Diamond} \varphi$ *implies* $T \models_{\leq GK} \varphi$*, hence,* $T \models_{GK} \varphi$.

Moreover, if $T\mathcal{G}_\Diamond$ is the set of theorems of $\mathcal{G}_\Diamond$ then:

LEMMA 6.3. $T \vdash_{\mathcal{G}_\Diamond} \varphi$ *if and only if* $T \cup T\mathcal{G}_\Diamond \vdash \varphi$ *in Gödel logic.*

Canonical model $\mathcal{M}_\Diamond = (W^*, S^*, e^*)$. Let $\Diamond\mathcal{L}_\Diamond = \{\Diamond\theta : \theta \in \mathcal{L}_\Diamond\}$, then:

• W^* is the set of valuations $v : Var \cup \Diamond\mathcal{L}_\Diamond \to [0,1]$ such that $v(T\mathcal{G}_\Diamond) = 1$ and its positive values have a positive lower bound:

$$\inf_{\theta \in \mathcal{L}_\Diamond} \{v(\theta) : v(\theta) > 0\} = \delta > 0 \tag{2}$$

when v is extended to $\mathcal{L}_\Diamond = \mathcal{L}(Var \cup \Diamond\mathcal{L}_\Diamond)$ as a Gödel valuation.
• The fuzzy relation between worlds in $\mathcal{M}_\Diamond$ is given by

$$S^*vw := \inf_{\varphi \in \mathcal{L}_\Diamond} \{w(\theta) \Rightarrow v(\Diamond\theta)\}.$$

• $e^*(v, p) := v(p)$ for any $p \in Var$.

LEMMA 6.4. *For any world v in the canonical model $\mathcal{M}_\Diamond$ and any $\varphi \in \mathcal{L}_\Diamond$ we have* $e^*(v, \varphi) = v(\varphi)$.

PROOF. The only non trivial step in a proof by induction on complexity of formulas of $\mathcal{L}_\Diamond$ is that of $\Diamond$. By induction hypothesis, $e^*(v, \Diamond\varphi)$

$= \sup_w\{S^*vw \cdot e^*(w,\varphi)\} = \sup_w\{S^*vw \cdot w(\varphi)\}$, then we must show $\sup_w\{S^*vw \cdot w(\varphi)\} = v(\Diamond\varphi)$. By definition

$$S^*vw \leq w(\varphi) \Rightarrow v(\Diamond\varphi),$$

for any $\varphi \in \mathcal{L}_\Diamond$ and $w \in W^*$, then $S^*vw \cdot w(\varphi) \leq v(\Diamond\varphi)$, which yields taking the join over w:

$$e^*(v, \Diamond\varphi) \leq v(\Diamond\varphi).$$

The other inequality is trivial if $v(\Diamond\varphi) = 0$. For the case $v(\Diamond\varphi) > 0$, let w be given as in the following claim then $v(\Diamond\varphi) = \alpha = S^*vw \cdot w(\varphi) \leq e^*(v, \Diamond\varphi)$, concluding the proof of the lemma.

Claim. *If v is a world of $M_\Diamond$ such that $v(\Diamond\varphi) = \alpha > 0$, there exists a world w of $M_\Diamond$ such that $w(\varphi) = 1$ and $S^*vw = \alpha$.*

PROOF. Assume $v(\Diamond\varphi) = \alpha > 0$ and define

$$\Gamma_{\varphi,v} = \{\theta \in \mathcal{L}_\Diamond : v(\Diamond\theta) < \alpha\} \cup \{\neg\neg\mu : \mu \in \mathcal{L}_\Diamond,\ v(\Diamond\mu) = 0\}.$$

This set is not empty because $v(\Diamond\bot) = 0$ by axiom $\mathbf{F}_\Diamond$. Moreover, for any finite subset of $\Gamma_{\varphi,v}$, say $\{\theta_1, ..., \theta_n\} \cup \{\neg\neg\mu_1, ..., \neg\neg\mu_m\}$, we have

$$\varphi \nvdash_{\mathcal{G}_\Diamond} \theta_1 \vee ... \vee \theta_n \vee \neg\neg\mu_1 \vee ... \vee \neg\neg\mu_m.$$

Otherwise, we would have (Cf. Lemma 6.1)

$$\begin{array}{lll} \Diamond\varphi \vdash_{\mathcal{G}_\Diamond} & \Diamond(\theta_1 \vee ... \vee \theta_n \vee \neg\neg\mu_1 \vee ... \vee \neg\neg\mu_m) & \mathbf{RN}_\Diamond \\ & \Diamond\theta_1 \vee ... \vee \Diamond\theta_n \vee \Diamond\neg\neg\mu_1 \vee ... \vee \Diamond\neg\neg\mu_m & \mathbf{K}_\Diamond \\ & \Diamond\theta_1 \vee ... \vee \Diamond\theta_n \vee \neg\neg\Diamond\mu_1 \vee ... \vee \neg\neg\Diamond\mu_m & \mathbf{Z}_\Diamond, \end{array}$$

which would imply by Lemma 6.2

$$v(\Diamond\varphi) \leq \max(\{v(\Diamond\theta_i) : 1 \leq i \leq n\} \cup \{v(\neg\neg\Diamond\mu_i) : 1 \leq i \leq m\}) < \alpha,$$

a contradiction. Therefore, we have by Lemma 6.3

$$T\mathcal{G}_\Diamond, \varphi \nvdash \theta_1 \vee ... \vee \theta_n \vee \neg\neg\mu_1 \vee ... \vee \neg\neg\mu_m;$$

By Proposition 3.1 there is a Heyting algebra valuation $u : Var \cup \Diamond\mathcal{L}_\Diamond \rightarrow [0,1]$ such that $u(\varphi) = u(T\mathcal{G}_\Diamond) = 1$ and $u(\theta) < 1$ for all $\theta \in \Gamma_{\varphi,v}$. Thus, u satisfies the further conditions:

(i) $u(\varphi) = 1$
(ii) $u(\theta) < 1$ *if* $v(\Diamond\theta) < \alpha$, because then $\theta \in \Gamma_{\varphi,v}$

(iii) $u(\theta) = 0$ *if* $v(\Diamond\theta) = 0$, because then $\neg\neg\theta \in \Gamma_{\varphi,v}$ and so $u(\neg\neg\theta) < 1$ which implies $u(\theta) = 0$.

Let $g : [0,1] \to [0,1]$ be the strictly increasing function:

$$g(x) = \begin{cases} 1 & \text{if } x = 1 \\ \delta(x+1)/2 & \text{if } 0 < x < 1 \\ 0 & \text{if } x = 0 \end{cases}$$

where δ is given by (2). Clearly the valuation $w = g \circ u$ inherits the properties (i), (ii) (iii) of u, with (ii) in the stronger form:

(ii′) $w(\theta) < \delta$ *if* $v(\Diamond\theta) < \alpha$

Moreover, $w(\theta) > 0$ implies $w(\theta) > \delta/2$, by construction, and $w(T\mathcal{G}_\Diamond) = 1$ because g is a homomorphism of Heyting algebras, hence, w belongs to $\mathcal{M}_\Diamond$. To see that $S^*vw = \alpha$, note that $w(\theta) \leq v(\Diamond\theta)$ whenever $v(\Diamond\theta) < \alpha$. If $0 < v(\Diamond\theta)$ because then $w(\theta) < \delta \leq v(\Diamond\theta)$ by (ii′) and definition of δ. If $v(\Diamond\theta) = 0$ because then $w(\theta) = 0$ by (iii). Since $(w(\theta) \Rightarrow v(\Diamond\theta)) \geq \alpha$ for $v(\Diamond\theta) \geq \alpha$, and $(w(\varphi) \Rightarrow v(\Diamond\varphi)) = (1 \Rightarrow \alpha) = \alpha$, we have $S^*vw = \inf_{\varphi\in\mathcal{L}_\Diamond}\{w(\varphi) \Rightarrow v(\Diamond\varphi)\} = \alpha$. ■

THEOREM 6.1. *For any countable theory T and formula φ in $\mathcal{L}_\Diamond$, the following are equivalent*
(i) $T \vdash_{\mathcal{G}_\Diamond} \varphi$
(ii) $T \models_{\leq GK} \varphi$
(iii) $T \models_{GK} \varphi$.

PROOF. Assume $T \nvdash_{\mathcal{G}_\Diamond} \varphi$, then $T \cup T\mathcal{G}_\Diamond \nvdash \varphi$. By the strong completeness of Gödel logic, there is a Heyting algebra valuation v such that $v(T \cup T\mathcal{G}_\Diamond) = 1$ and $v(\varphi) < 1$. Since v might not be a world in $\mathcal{M}_\Diamond$ compose it with the Heyting algebra homomorphism: $g(x) = (x+1)/2$ for $x > 0$, $g(0) = 0$. Then $v' = g \circ v$ belongs to $\mathcal{M}_\Diamond$ and we still have $v'(T) = 1$, $v'(\varphi) < 1$. Applying Lemma 6.4 to v' we have $e^*(v', T) = 1$, $e^*(v', \varphi) < 1$. That is, $\mathcal{M}_\Diamond \models_{v'} T$ and $\mathcal{M}_\Diamond \not\models_{v'} \varphi$. Hence, $T \not\models_{GK} \varphi$. By Lemma 6.2 this is enough. ■

$\models_{GK}$ no longer coincides with $\models_{Crisp}$ for the language $\mathcal{L}_\Diamond$ as the following example illustrates.

EXAMPLE. The schema $\neg\neg\Diamond\varphi \to \Diamond\neg\neg\varphi$ is not a theorem of $\mathcal{G}_\Diamond$ because it fails in the two worlds model $\langle\{a, b\}, S, e\rangle$ where

$$Sab = \tfrac{1}{2}, \quad e(a, p) = e(b, p) = 1.$$

since $e(x, \neg\neg\Diamond p) = 1$, and $e(y, \Diamond\neg\neg\varphi) = \frac{1}{2}$. However, it holds in all a-crisp models since then $e(x, \neg\neg\Diamond\varphi) > 0$ implies the existence of y such that Sxy, $e(y, \varphi) > 0$.Hence, $Sxy = 1$ and thus $e(y, \Diamond\neg\neg\varphi) \geq Sxy \cdot (-- e(y, \varphi)) = 1$.

Remark. An interesting question raised by one referee is whether the logic of $\Diamond$ in a-crisp GK-models is axiomatizable. This is granted in the abstract sense (recursive enumerability of valid formulas) because the logic may be interpreted faithfully in Gödel predicate logic which is axiomatizable (cf. [14], [16]). We do not know an explicit axiomatization but the system $\mathcal{G}_\Diamond \cup \{\neg\neg\Diamond\varphi \to \Diamond\neg\neg\varphi\}$ is a candidate because the new schema characterizes crisp frames: if (W, S) is not crisp pick x, $y \in S$ such that $0 < Sxy < 1$, then the valuation $e(y, p) = 1$ and $e(z, p) = 0$ for every $z \neq y$, provides a counterexample to the schema since $e(x, \neg\neg\Diamond p) = -- Sxy = 1$ but $e(x, \Diamond\neg\neg p) = Sxy < 1$.

7. $\mathcal{G}_\Diamond$ has the finite model property

For any sentence φ such that $\nvdash_{\mathcal{G}_\Diamond} \varphi$ we may construct a finite counter-model inside $\mathcal{M}_\Diamond$.

Theorem 7.1. *If $\nvdash_{\mathcal{G}_\Diamond} \varphi$ then there is a GK-model M with finitely many worlds such that $M \nvDash \varphi$.*

Proof. It follows from the Claim in the proof of Lemma 6.4 that for all θ and $v \in \mathcal{M}_\Diamond$ there is $w \in \mathcal{M}_\Diamond$ such that $v(\Diamond\theta) = S^*vw \cdot w(\theta)$. (if $v(\Diamond\theta) = 0$ any w works). Given θ, let $f_\theta(v)$ be a function choosing one such w for each v. For any formula θ let $\mathbf{r}(\theta)$ (*rank* of θ) be the nesting degree of $\Diamond$ in θ, that is, the length of a longest chain of occurrences of $\Diamond$ in the tree of θ.

Given φ and a world (valuation) v_0 in $\mathcal{M}_\Diamond$, let S_j be the set of subformulas of φ of rank j, for each $j \leq n = \mathbf{r}(\varphi)$, and define inductively the following sets of valuations:

$$\begin{aligned} M_0 &= \{v_0\} \\ M_{i+1} &= \{f_\theta(v) : v \in M_i, \quad \Diamond\theta \in S_{n-i}\} \end{aligned}$$

Clearly, $M = \cup_{i \leq n} M_i$ has finitely many worlds. Consider the model M_{φ, v_0} induced in M by restricting e^* and S^* of $\mathcal{M}_\Diamond$ to $M \times Var$ and $M \times M$ respectively, that we will call M for simplicity. Then for any formula $\Diamond\theta \in S_j$ and $v \in M_{n-j}$ there is $w \in M_{n-(j-1)}$ such that $v(\Diamond\theta) = S^*vw \cdot w(\theta)$, and thus

$$v(\Diamond\theta) \leq \sup\{S^*vw \cdot w(\theta) : w \text{ is a world in } M\}.$$

This allows us to show by induction on $j \leq n$ that for all $\theta \in S_j$, $v \in M_{n-j}$ we have $v(\theta) = e_M(v, \theta)$. In particular, if $\nvdash_{\mathcal{G}_\Diamond} \varphi$, and v_0 is a world in $\mathcal{M}_\Diamond$ such that $v_0(\varphi) < 1$ then $e_M(v_0, \varphi) < 1$, which shows $M \not\models \varphi$. ■

The proof of the previous theorem still works if we assume the worlds of M are valuations defined in the variables of φ only and the accessibility relation of M is defined by using subformulas of φ:

$$S^*_M vw := \min_{\theta \in \cup S_i} \{w(\theta) \Rightarrow v(\Diamond\theta)\}.$$

This means that M takes values in a fixed finite subalgebra of $[0, 1]$ depending only on φ. Thus there are only finitely many models to consider and the decidability of the fragment $\mathcal{G}_\Diamond$ follows.

8. Modal extensions

The modal systems we have considered so far correspond to minimal modal logic K, the logic of Gödel-Kripke models with an arbitrary accessibility fuzzy relation. We may consider also the fuzzy analogues of the classical modal systems D, T, $S4$ and $S5$ for each modal operator, usually presented syntactically as combinations of the following axioms:

$\mathbf{D}_\Box$:	$\neg\Box\bot$	$\mathbf{D}_\Diamond$:	$\Diamond\top$
$\mathbf{T}_\Box$:	$\Box\varphi \to \varphi$	$\mathbf{T}_\Diamond$:	$\varphi \to \Diamond\varphi$
$\mathbf{4}_\Box$:	$\Box\varphi \to \Box\Box\varphi$	$\mathbf{4}_\Diamond$:	$\Diamond\Diamond\varphi \to \Diamond\varphi$
$\mathbf{B}_\Box$:	$\varphi \to \Box\neg\Box\neg\varphi$	$\mathbf{B}_\Diamond$:	$\varphi \to \neg\Diamond\neg\Diamond\varphi$

and semantically by asking the frames of the Kripke models to satisfy corresponding structural properties. Notice that $\mathbf{D}_\Box$ follows from $\mathbf{T}_\Box$ and $\mathbf{D}_\Diamond$ from $\mathbf{T}_\Diamond$.

Call a fuzzy frame $\langle W, S\rangle$ *serial* if $\forall x \in W\ \exists y \in W : Sxy = 1$, *reflexive* if $Sxx = 1$ for all $x \in W$, *(min)transitive* if $Sxy \cdot Syz \leq Sxz$ for all x, y, z, and *symmetric* if $Sxy = Syx$ for all $x, y \in W$.

Let Ser, $Refl$, $Trans$, and $Symm$ denote the classes of GK-models over frames satisfying, respectively, each one of the above properties, and let $\models_{\mathcal{C}}$ denote validity and consequence with respect to models in the class $\mathcal{C}$.

PROPOSITION 8.1. *$\models_{Ser}$ $\mathbf{D}_\Box$,$\mathbf{D}_\Diamond$; $\models_{Refl}$ $\mathbf{T}_\Box$,$\mathbf{T}_\Diamond$; $\models_{Trans}$ $\mathbf{4}_\Box$,$\mathbf{4}_\Diamond$; and $\models_{Symm}$ $\mathbf{B}_\Box$,$\mathbf{B}_\Diamond$.*

PROOF. The validity of $\mathbf{D}_\Box$,$\mathbf{D}_\Diamond$ in serial models is immediate because $e(x, \Box\bot) = \inf_y\{-Sxy\}$ and $e(x, \Diamond\top) = \sup_y Sxy$.

Assume $Sxx = 1$ for all x. ($\mathbf{T}_\Box$): $e(x, \Box\varphi) \le (Sxx \Rightarrow e(x,\varphi)) = e(x,\varphi)$. ($\mathbf{T}_\Box$): $e(x, \Diamond\varphi) \ge Sxx \cdot e(x,\varphi) = e(x,\varphi)$.

Assume $Sxy \cdot Syz \le Sxz$ for all x, y, z. ($\mathbf{4}_\Box$): $e(x, \Box\varphi) \cdot Sxy \cdot Syz \le (Sxz \Rightarrow e(z,\varphi)) \cdot Sxz \le e(z,\varphi)$. Hence, $e(x,\Box\varphi) \cdot Sxy \le (Syz \Rightarrow e(z,\varphi))$. Taking the meet over z in the right hand side: $e(x,\Box\varphi) \cdot Sxy \le e(y, \Box\varphi)$; hence, $e(x,\Box\varphi) \le (Sxy \Rightarrow e(y,\Box\varphi))$ for all y and thus $e(x,\Box\varphi) \le e(x,\Box\Box\varphi)$. ($\mathbf{4}_\Diamond$): for any x, y, z, $Sxy \cdot Syz \cdot e(z,\varphi) \le Sxz \cdot e(z,\varphi) \le e(x,\Diamond\varphi)$, hence, $Syz \cdot e(z,\varphi) \le (Sxy \Rightarrow e(x,\Diamond\varphi))$. Taking the join over z in the left, $e(x,\Diamond\varphi) \le (Sxy \Rightarrow e(x,\Diamond\varphi))$, thus $Sxy \cdot e(x,\Diamond\varphi) \le e(x,\Diamond\varphi))$. Taking the join again in the left, $e(x,\Diamond\Diamond\varphi) \le e(x,\Diamond\varphi)$.

Assume $Sxy = Syx$ for all x, y. ($\mathbf{B}_\Box$): we prove the validity of the stronger schema $\neg\varphi \to \Box\neg\Box\varphi$. Assume $e(x,\neg\varphi) > 0$ then $e(x,\varphi) = 0$. Take any y such that $Sxy > 0$, then $e(y,\Box\varphi) \le (Syx \Rightarrow e(x,\varphi)) = (Sxy \Rightarrow e(x,\varphi)) = 0$. Therefore, $e(y,\neg\Box\varphi) = 1$, and $(Sxy \Rightarrow e(y,\neg\Box\varphi)) = 1$. This shows that $x(\Box\neg\Box\varphi) = 1$. ($\mathbf{B}_\Diamond$): suppose $e(x,\varphi) > e(x,\neg\Diamond\neg\Diamond\varphi)$ then $e(x,\neg\Diamond\neg\Diamond\varphi) = 0$ and $e(x,\Diamond\neg\Diamond\varphi) = 1$. This means that there is y such that $Sxy \cdot e(x,\neg\Diamond\varphi) > 0$ thus $Sxy > 0$ and $e(x,\neg\Diamond\varphi) = 1$, hence $e(y,\Diamond\varphi) = 0$, therefore, $Syx \cdot e(x,\varphi) = 0$ which is absurd because $Syx = Sxy > 0$ and $e(x,\varphi) > 0$ by construction. ■

Theorem 8.1.

(i) $\mathcal{G}_\Box + \mathbf{D}_\Box$ *and* $\mathcal{G}_\Diamond + \mathbf{D}_\Diamond$ *are strongly complete for* $\models_{Ser}$.

(ii) $\mathcal{G}_\Box + \mathbf{T}_\Box$ *and* $\mathcal{G}_\Diamond + \mathbf{T}_\Diamond$ *are strongly complete for* $\models_{Refl}$.

(iii) $\mathcal{G}_\Box + \mathbf{4}_\Box$ *and* $\mathcal{G}_\Diamond + \mathbf{4}_\Diamond$ *are strongly complete for* $\models_{Trans}$.

(iv) $\mathcal{G}S4_\Box := \mathcal{G}_\Box + \mathbf{T}_\Box + \mathbf{4}_\Box$ *and* $\mathcal{G}S4_\Diamond := \mathcal{G}_\Diamond + \mathbf{T}_\Diamond + \mathbf{4}_\Diamond$ *are strongly complete for* $\models_{Refl \cap Trans}$.

Proof. Soundness follows from Proposition 8.1. Completeness follows, in each case, by asking the worlds of the canonical models $\mathcal{M}_\Box$ and $\mathcal{M}_\Diamond$ introduced in the completeness proofs of $\mathcal{G}_\Box$ and $\mathcal{G}_\Diamond$ to satisfy the corresponding schemas. The key fact is that the schemas force the accessibility relations $S^*_\Box vw$ and $S^*_\Diamond vw$ to satisfy the respective properties.

(i) If $v(\mathbf{D}_\Box) = 1$ in $\mathcal{M}_\Box$ then then $e^*(v, \Box\bot) = v(\Box\bot) = 0$ for any world v of $\mathcal{M}_\Box$ and necessarily there is w such that $S^*_\Box vw > 0$, but the model is a-crisp thus $S^*_\Diamond vw = 1$. If $v(\mathbf{D}_\Diamond) = v(\Diamond\top) = 1$ in $\mathcal{M}_\Diamond$ then by the Claim in the proof of Lemma 6.4 there is w such that $S^* vw = 1$.

(ii) If $v(\mathbf{T}_\Box) = 1$ then $S^*_\Box vv = \inf_{\varphi \in \mathcal{L}_\Box} \{v(\Box\varphi \to \varphi)\} = 1$. If $v(\mathbf{T}_\Diamond) = 1$ then $S^*_\Diamond vv = \inf_{\varphi \in \mathcal{L}_\Box} \{v(\varphi \to \Diamond\varphi)\} = 1$.

(iii) If $v(\mathbf{4}_\Box) = 1$ then $v(\Box\varphi) \le v(\Box\Box\varphi)$ and so

$$\begin{aligned} S^*_\Box vv' \cdot S^*_\Box v'v'' &\leq [(v(\Box\Box\varphi) \Rightarrow v'(\Box\varphi)) \cdot (v'(\Box\varphi) \Rightarrow v''(\varphi))] \\ &\leq (v(\Box\Box\varphi) \Rightarrow v''(\varphi)) \leq (v(\Box\varphi) \Rightarrow v''(\varphi)) \end{aligned}$$

Taking the meet over φ in the last formula we get: $S^*_\Box vv' \cdot S^*_\Box v'v'' \leq S^*_\Box vv''$.

(iv) If $v(\mathbf{4}_\Diamond) = 1$ then $v(\Diamond\Diamond\varphi) \leq v(\Diamond\varphi)$ and thus

$$\begin{aligned} S^*_\Diamond vv' \cdot S^*_\Diamond v'v'' &\leq [(v'(\Diamond\varphi) \Rightarrow v(\Diamond\Diamond\varphi)) \cdot (v''(\varphi) \Rightarrow v'(\Diamond\varphi))] \\ &\leq (v''(\varphi) \Rightarrow v(\Diamond\Diamond\varphi)) \leq (v''(\varphi) \Rightarrow v(\Diamond\varphi)) \end{aligned}$$

Taking the meet over φ in the last formula we get $S^*_\Diamond vv' \cdot S^*_\Diamond v'v'' \leq S^*_\Diamond vv''$. ■

It follows from the proof of Theorem 8.1 that for the given extensions of $\mathcal{G}_\Box$ we get completeness also with respect to the a-crisp models of the respective class.

Call a fuzzy frame $\langle W, S\rangle$ *weakly serial* if it satisfies $\forall x \exists y\ Sxy > 0$, and let *WSerial* be the class of GK-models over weakly serial frames. Then it is easily seen that $\mathcal{G}_\Box + D_\Box$ is sound (and thus strongly complete) for $\models_{WSerial}$ but $\mathcal{G}_\Diamond + D_\Diamond$ is not. However, $\models_{WSerial}$ is axiomatized by $\mathcal{G}_\Diamond + \{\neg\neg\Diamond\top\}$.

Remark. An original motivations of the second author to study fuzzy modal logics was to interpret the possibility operator $\Diamond$ in the class of Gödel frames $Refl \cap Trans \cap Symm$ as a notion of similarity in the sense of Godo and Rodríguez [12], and a reasonable conjecture was that $\mathcal{G}S5_\Diamond = \mathcal{G}S4_\Diamond + \mathbf{B}_\Diamond$ would axiomatize validity in models over these frames. Unfortunately, the axioms $\mathbf{B}_\Box$, $\mathbf{B}_\Diamond$ do not seem to force symmetry in the canonical models and we have not been able to show completeness of $\mathcal{G}_\Box + \mathbf{B}_\Box$ or $\mathcal{G}_\Diamond + \mathbf{B}_\Diamond$ for $\models_{Symm}$, nor completeness of $\mathcal{G}S5_\Diamond$ or the analogue $\mathcal{G}S5_\Box$ with respect to $\models_{Refl \cap Trans \cap Symm}$. Perhaps stronger symmetry axioms such as

$(\varphi \to \Box\theta) \to \Box(\Box\varphi \to \theta)$
$\Diamond(\Diamond\varphi \to \theta) \to (\varphi \to \Diamond\theta)$,

which characterize symmetric frames, would do.

9. Adding truth constants

The previous results on strong completeness may be generalized to Pavelka-style languages [19] with a set $Q \subseteq [0, 1]$ of truth values added as logical constants, provided Q is well-ordered under the usual order of $[0, 1]$ and discrete in the usual topology of $[0, 1]$. These conditions are satisfied by finite sets and force Q to be at most countable.

Without loss of generality, we assume Q contains 0 and 1, to be identified with $\bot$ and $\top$, respectively. The logical constant corresponding to $r \in Q$ will be denoted by r itself.

Let $\mathcal{L}_{\Box Q}$ be $\mathcal{L}_\Box$ enriched with elements of Q as atomic constituents, and let $\mathcal{G}_\Box(Q)$ be the system in this language obtained by adding to the axioms and rule of $\mathcal{G}_\Box$ the axiom schemas R1 - R4 below.

For all $r, s \in Q$:
R1. (book-keeping axioms)
$0 \to \bot,\ \top \to 1$
$r \to s,$ if $r \leq s$
$(r \to s) \to s,$ if $s < r$
R2. $r \to \Box r$
R3. $(r \to \Box\theta) \to \Box(r \to \theta)$
R4. $((\Box\theta \to r) \to r) \to \Box((\theta \to r) \to r)$

The system $\mathcal{G}_\Diamond(Q)$ in the analogue language $\mathcal{L}_{\Diamond Q}$ is defined similarly by adding R1 and R5 - R7 below to $\mathcal{G}_\Diamond$.

R5. $\Diamond r \to r$
R6. $\Diamond(r \to \varphi) \to (r \to \Diamond\varphi)$
R7. $\Diamond((\varphi \to r) \to r) \to ((\Diamond\varphi \to r) \to r)$.

The double negation shift axioms $\mathbf{Z}_\Box$ and $\mathbf{Z}_\Diamond$ become superfluous in the extended systems due to R4 and R7, respectively; also $\mathbf{F}_\Diamond$ is superfluous due to R5.

GK-models are extended by defining $e(x, r) = r$ at each world x, and validity $\models_{GK} \varphi$ is defined as before in terms of 1-satisfaction. Then R1 to R7 are easily seen to be valid.

However, the consequence notion $T \models_{GK} \varphi$ given in Definition 2.2 is too rough if there are two or more truth constants (consider $\frac{1}{2} \models_{GK} 0$). We will utilize the finer relation $T \models_{\leq GK} \varphi$ for which it may be shown that $\mathcal{G}_\Box(Q)$ and $\mathcal{G}_\Diamond(Q)$ are strongly complete for countable theories if Q is well ordered and discrete, and the same holds for the logics mentioned in Theorem 8.1. No conditions are required on Q to obtain weak completeness. Thus these results extend substantially a result of Esteva, Godo and Nogera [7] on weak completeness of Gödel logic with rational truth constants.

Discreteness of Q is necessary for strong completeness: if r is a limit point of Q then there is a strictly increasing or decreasing sequence of Q converging to r, say $\{r_n\}$ increases to $\sup r_n = r$, then

$$\{r_1 \to \theta, r_2 \to \theta, r_3 \to \theta....\} \models_{\leq GK} r \to \theta$$

but no finite subset of premises can grants this, thus no formal proof is possible. But discreteness alone is not enough, since $Q = \{r_1 < r_2 < ...$ $... < q_2 < q_1\}$ with $\sup r_i = \inf q_i$ is discrete and

$$\{r_1 \to \theta,\ r_2 \to \theta, ... \quad , \psi \to q_1,\ \psi \to q_2, ...\} \models_{\leq GK} \psi \to \theta$$

but no finite subset of the premises yields the same consequence. Thus well order or a related conditions is needed.

We give next a proof of strong completeness for $\mathcal{G}_\Box(Q)$, a refinement of that given for $\mathcal{G}_\Box$. The deduction theorem and lemmas 4.1 and 4.2 extend readily to the system $\mathcal{G}_\Box(Q)$. Moreover, any formula of $\mathcal{L}_{\Box Q}$ may be seen as a formula of Gödel logic over the vocabulary $Var \cup Q \cup \Box\mathcal{L}_{\Box Q}$, and after defining

$$T\mathcal{G}_\Box(Q) = \{\theta : \theta \text{ is an axiom of } \mathcal{G}_\Box(Q)\} \cup \{\Box\theta : \ \vdash_{\mathcal{G}_\Box(Q)} \theta\}$$

it may be shown that

$$T \vdash_{\mathcal{G}_\Box(Q)} \varphi \textit{ if and only if } T \cup T\mathcal{G}_\Box(Q) \vdash \varphi \textit{ in Gödel logic.}$$

Definition 9.1. Call a Gödel valuation $v : Var \cup Q \cup \Box\mathcal{L}_{\Box Q} \to [0,1]$ *normal* if $v(r) = r$ for all $r \in Q$.

Having $\overline{v}(T\mathcal{G}_\Box(Q)) = 1$ does not make v normal. However, the next lemmas show how to transform such a valuation to a normal one still satisfying $T\mathcal{G}_\Box(Q)$ and some other useful properties. As before, we will write v for the Gödel extension $\overline{v}$.

Lemma 9.1. *Let $v : Var \cup Q \cup \Box\mathcal{L}_{\Box Q} \to [0,1]$ be extended to all formulas of $\mathcal{L}_{\Box Q}$ according to the Gödel operations in $[0,1]$. If $v(\mathrm{R1}) = 1$, then $v(0) = 0$, $v(1) = 1$, and*

$$\beta = \min\{r \in Q : v(r) = 1\} > 0.$$

Moreover, v is weakly increasing in Q and strictly increasing in $Q \cap [0, \beta]$.

Proof. $v(0) = 0$ and $v(1) = 1$ by R1, β exists and is positive due to the well ordering of Q. If $r \leq s$ in Q then $v(r) \leq v(s)$ by R1. If $r < s \leq \beta$ in Q then $v(r) < 1$ and so $v(s \to r) \leq v(r) < 1$ by R1, thus $v(s) > v(r)$. ■

Call a formula of $\mathcal{L}_{\Box Q}$ *shy* if any occurrence of $r \in Q \setminus \{0, 1\}$ in the formula is under the scope of an occurrence of $\Box$.

For positive $r \in Q$, let r^- be the supremum (in $[0,1]$) of its predecessors in Q. Necessarily $r^- < r$ because r is isolated, but r^- may not belong to Q.

LEMMA 9.2. (Normalization). *Let* $v : Var \cup Q \cup \Box\mathcal{L}_{\Box Q} \to [0,1]$ *be Gödel valuation satisfying* R1 *and let* β *be defined as in the previous lemma, then there is a normal* $w : Var \cup Q \cup \Box\mathcal{L}_{\Box Q} \to [0,1]$ *such that for any* φ :
1. $v(\varphi) = 1 \Rightarrow w(\varphi) \geq \beta$
2. $v(\varphi) < 1 \Rightarrow w(\varphi) < \beta$
3. $v(\varphi) \leq v(\psi) < 1 \Rightarrow w(\varphi) \leq w(\psi)$
4. $v(\varphi) < v(\psi) \Rightarrow w(\varphi) < w(\psi)$
5. *For any shy formula* φ, $v(\varphi) = 1$ *implies* $w(\varphi) = 1$, *and* $v(\theta \to \varphi) = 1$ *implies* $w(\theta \to \varphi) = 1$.
6. $v(T\mathcal{G}_{\Box}(Q)) = 1$ *implies* $w(T\mathcal{G}_{\Box}(Q)) = 1$.
7. *If* δ_r^-, δ_r *are given so that* $r^- < \delta_r^- < \delta_r < r$ *for each positive* $r \in Q$, *then* w *may be chosen so that* $w(\mathcal{L}_{\Box Q}) \subseteq Q \cup \bigcup_{r\in Q}[\delta_r^-, \delta_r)$. *Hence, no* $r^- \notin Q$ *belongs to the image of* w.

PROOF. Given $0 < r \in Q$, let $v(r)^- = \sup_{[0,1]}\{v(s) : s < r,\ s \in Q\}$. Clearly, $v(r)^- \leq v(r)$; moreover, $v(r)^- = v(s)$, $s \in Q$, if and only if $s = r$ or $s = r^-$. It should be clear also that

$$[0,1] = \{v(r)^-, v(r) : r \in Q\} \cup \bigcup_{r\in Q\cap(0,\beta]} (v(r)^-, v(r))$$

is a partition because $v \restriction Q \cap [0,\beta]$ is strictly increasing by the previous lemma and $v(\beta) = 1$. Given δ_r^-, δ_r as in **7** choose an strictly increasing function $g : [0,1] \to [0,\beta) \cup \{1\}$ satisfying.

$g(1) = 1$
$g(v(r)) = r$ for $r \in Q$, $v(r) < 1$ (hence $r < \beta$).
$g((v(r)^-, v(r))) = (\delta_r^-, \delta_r)$ if $v(r)^- < v(r) < 1$.
$g(v(r)^-) = \delta_r^-$ if $v(r)^- \notin v(Q)$ (hence $v(r)^- < v(r)$)

Define $w : Var \cup Q \cup \Box\mathcal{L}_{\Box Q} \to [0,1]$ as follows:

$w(\rho) = g(v(\rho))$ if $\rho \in Var \cup \Box\mathcal{L}_{\Box Q}$
$w(r) = r$ for $r \in Q$.

Property **7** is insured for elements of $Var \cup Q \cup \Box\mathcal{L}_{\Box Q}$ by construction and it extends to all of $\mathcal{L}_{\Box Q}$ because the value of a formula under a Gödel valuation is identical to $0, 1$, or the value of one of its atomic constituents. For the other properties:

1-2. By simultaneous induction in Gödel connectives, we show **1** and the following strengthening **2′** of **2**: $v(\varphi) < 1 \Rightarrow w(\varphi) = g(v(\varphi)) < \beta$.

Atomic. For $\varphi \in Var \cup \Box\mathcal{L}_{\Box Q}$ by definition of w and g. For $\varphi = r \in Q$: $v(r) < 1$ if and only if $w(r) = r < \beta$ by definition of β, and in the later case $w(r) = r = g(v(r))$ by the way g was chosen.

Conjunction. If $v(\varphi \wedge \psi) = 1$ then $v(\varphi) = v(\psi) = 1$ and by inductive hypothesis $w(\varphi \wedge \psi) = w(\varphi) \cdot w(\psi) \geq \beta$. If $v(\varphi \wedge \psi) < 1$, say $v(\varphi) \leq v(\psi)$, then $v(\varphi) < 1$ and thus $w(\varphi) = g(v(\varphi)) < \beta$ by induction hypothesis. Moreover, $w(\psi) \geq w(\varphi)$: if $v(\psi) = 1$ because $w(\psi) \geq \beta$, if $v(\psi) < 1$ because $w(\psi) = g(v(\psi)) \geq g(v(\varphi))$. Thus $w(\varphi \wedge \psi) = w(\varphi) = g(v(\varphi \wedge \psi)) < \beta$.

Implication. Assume $v(\varphi \rightarrow \psi) = 1$. If $v(\psi) < 1$ then $v(\varphi) \leq v(\psi) < 1$, and by inductive hypothesis and monotonicity of g, $w(\varphi) = g(v(\varphi)) \leq g(v(\psi)) = w(\psi)$, thus $w(\varphi \rightarrow \psi) = 1$. If $v(\psi) = 1$ then $w(\varphi \rightarrow \psi) \geq w(\psi) \geq \beta$ again by the induction hypothesis. Assume now $v(\varphi \rightarrow \psi) < 1$, then $v(\varphi) > v(\psi) < 1$ and thus $w(\psi) = g(v(\psi)) < \beta$. Moreover, $w(\varphi) > w(\psi)$: if $v(\varphi) = 1$ because $w(\varphi) \geq \beta$, if $v(\varphi) < 1$ because $w(\varphi) = g(v(\varphi)) > g(v(\psi))$. Therefore $w(\varphi \rightarrow \psi) = w(\psi) = g(v(\varphi \rightarrow \psi)) < \beta$.

3. $v(\varphi) \leq v(\psi) < 1$ implies by **2′**: $w(\varphi) = g(v(\varphi)) \leq g(v(\psi)) = w(\psi)$.

4. $v(\varphi) < v(\psi)$ implies $v(\psi \rightarrow \varphi) = v(\varphi) < 1$, hence $w(\psi \rightarrow \varphi) < \beta$ by '2, and thus $w(\varphi) < w(\psi)$.

5. If φ is shy then $\varphi = \varphi'(\rho_1, ..., \rho_n)$ where φ' is Gödel formula and $\rho_i \in Var \cup \Box\mathcal{L}$; therefore, $w(\varphi) = \varphi'(w(\rho_1), ..., w(\rho_n)) = \varphi'(g(v(\rho_1)), ..., g(v(\rho_n))) = g(v(\varphi'(\rho_1, ..., \rho_n)) = g(v(\varphi))$ because g is an endomorphism of the Heyting algebra $[0, 1]$. Therefore, $v(\varphi) = 1$ implies $w(\varphi) = g(1) = 1$. Moreover, $v(\theta \rightarrow \varphi) = 1$ implies trivially $w(\theta \rightarrow \varphi) = 1$ when $v(\varphi) = 1$, and $w(\theta) \leq w(\varphi)$ when $v(\varphi) < 1$ by property **4**.

6. The axioms of $\mathcal{G}$ give 1 under any Gödel valuation. The specific axioms of $\mathcal{G}_\Box$ are shy, $w(\text{R1}) = 1$ because w is the identity in Q, and R2, R3, R4.are of the form $\theta \rightarrow \varphi$ with φ shy. The other elements of $T\mathcal{G}_\Box(Q)$ are shy by construction. ■

Canonical model $M_\Box(Q)$:

- W^*: all normal valuations $v : Var \cup Q \cup \Box\mathcal{L}_{\Box Q} \rightarrow [0, 1]$ satisfying $v(T\mathcal{G}_\Box(Q)) = 1$ and such that there are δ_r^-, δ_r $(r \in Q)$ with $r^- < \delta_r^- < \delta_r < r$ and $\text{Im}(v) \subseteq Q \cup \bigcup_{r \in Q,\, r>0} [\delta_r^-, \delta_r)$.
- $S^*vw = \inf_{\theta \in \mathcal{L}_{\Box Q}} (v(\Box\theta) \Rightarrow w(\theta))$.
- $e^*(v, -) = v \restriction Var$.

LEMMA 9.3. *$e^*(v, \theta) = v(\theta)$ for any world v of $M_\Box(Q)$ and formula θ.*

PROOF. As in the case of $\mathcal{G}_\Box$, it is enough to check $\inf_{w \in W^*}(S^*vw \Rightarrow w(\varphi)) \leq v(\Box\varphi)$ whenever $v(\Box\varphi) < 1$. This is done in two stages:

Claim 1. *If $\alpha = v(\Box\varphi) < 1$ there exists a Gödel valuation $u : Var \cup Q \cup \Box\mathcal{L}_{\Box Q} \rightarrow [0, 1]$ such that $u(T\mathcal{G}_\Box(Q)) = 1$, $u(\varphi) < 1$ and for any θ and $r \in Q$*

1. $u(\theta) = 1$ *if* $v(\Box\theta) > \alpha$
2. $u(r) \leq u(\theta)$ *if* $r \leq v(\Box\theta)$
3. $u(r) < u(\theta)$ *if* $r < v(\Box\theta)$ *and* $r \leq \alpha$
4. $\alpha \in [\beta^-, \beta)$ *if* $\beta = \min\{r \in Q : u(r) = 1\}$.

PROOF. Let $T_{\varphi,v}$ be the theory

$$\begin{array}{rcl}\{\theta : v(\Box\theta) > \alpha\} & \cup & \{r \to \theta : r \in Q, r \leq v(\Box\theta)\} \\ & \cup & \{(\theta \to r) \to r : r \in Q, r \leq \alpha, r < v(\Box\theta)\}.\end{array}$$

Then $T_{\varphi,v} \nvdash_{\mathcal{G}_\Box(Q)} \varphi$. Otherwise, $\mu_1, ..., \mu_k \vdash_{\mathcal{G}_\Box(Q)} \varphi$ for some $\mu_i \in T_{\varphi,v}$ and thus

$$\Box\mu_1, ..., \Box\mu_k \vdash_{\mathcal{G}_\Box(Q)} \Box\varphi.$$

But $v(\Box\mu) > \alpha$ for any other $\mu \in T_{\varphi,v}$: for the first group of axioms, by construction; for the second, because $v(\Box\theta) \geq r$ implies $v(\Box(r \to \theta)) \geq v(r \to \Box\theta) = r \to v(\Box\theta) = 1$ by R3 and normality of v; for the third, because $v(\Box\theta) > r \leq \alpha$ implies by R4 and normality: $v(\Box((\theta \to r) \to r)) \geq (v(\Box\theta) \to r) \to r = (r \to r) = 1$. Hence, we obtain the contradiction

$$\alpha < \min\{\Box\mu_1, ..., \Box\mu_k\} \leq v(\Box\varphi).$$

Therefore, $T_{\varphi,v}, T\mathcal{G}_\Box(Q) \nvdash \varphi$, and we may use the strong completeness theorem of Gödel logic to get a valuation $u : Var \cup Q \cup \Box\mathcal{L} \to [0,1]$ such that $u(T_{\varphi,v} \cup T\mathcal{G}_\Box(Q)) = 1 > u(\varphi)$. Conditions **1**, **2** hold by construction, **3** is satisfied because $r \leq \alpha = v(\Box\varphi)$ implies $u(r) \leq u(\varphi) < 1$ by **2** and thus $u((\theta \to r) \to r) = 1$ implies $u(\theta) > u(r)$. To verify **4**, notice that $r \leq \alpha$ implies $u(r) < 1$ as just explained, thus $\beta > \alpha$. On the other hand $\alpha < r < \beta$ implies $v(\Box r) \geq v(r) = r > \alpha$ by R2, and thus $u(r) = 1$ by **1**, contradicting the definition of β. Therefore, $\beta^- \leq \alpha$.

Claim 2. *If* $\alpha = v(\Box\varphi) < 1$ *then for any* $\epsilon > 0$ *there exists a world* w *of* $\mathcal{M}_\Box(Q))$ *such that* $(Svw \Rightarrow w(\varphi)) < \alpha + \epsilon$.

PROOF. According to Lemma 9.2, the valuation u of the previous claim may be transformed in a valuation w in $M_\Box(Q)$ such that $w(\varphi) < \beta$ and the conditions on u become the following conditions on w:

1. $w(\theta) \geq \beta$ if $v(\Box\theta) > \alpha$.
2. $r \leq w(\theta)$ if $r \leq v(\Box\theta)$ and $r < \beta$ (that is, $r \leq \alpha$).
3. $r < w(\theta)$ if $r < v(\Box\theta)$ and $r \leq \alpha$.
4. $\alpha, w(\varphi) \in [\beta^-, \beta)$.

Only **4** needs some explanation: if $r < \beta$ then $r \leq \beta^- \leq v(\Box\varphi) = \alpha$ and thus $r \leq w(\varphi)$ by **3** above, hence $\beta^- \leq w(\varphi)$. Moreover, we may choose the parameters $\delta_r'^-, \delta_r'$, of w so that

$$r^- < \delta_r^- < \delta_r \leq \delta_r'^- < \delta_r' < r$$

for $0 < r < \beta$, where δ_r^-, δ_r are the pairs associated to v, and

$$\beta^- \leq \alpha \leq \delta_\beta'^- < \delta_\beta' < \alpha + \epsilon, \beta.$$

Then we have $w(\varphi) < \delta_\beta' < \alpha + \epsilon$. To conclude $(S^*vw \Rightarrow w(\varphi)) = w(\varphi) < \alpha + \epsilon$ it is enough to show that $v(\Box\theta) \Rightarrow w(\theta) \geq \beta$ for any θ. If $v(\Box\theta) > \alpha$ this follows from **1** above. If $v(\Box\theta) \leq \alpha < \beta$ we have the cases:

(i) $r^- < v(\Box\theta) < r \leq \beta$, then $r^- \leq w(\theta)$ by **2**. If $r^- \in Q$ we apply **3** to conclude that $r^- < w(\theta)$, and if $r^- \notin Q$ we may conclude similarly that $r^- < w(\theta)$ because r^- is not in the image of w. In any case, $v(\Box\theta) \leq \delta_r \leq \delta_r'^- \leq w(\theta)$ and $v(\Box\theta) \Rightarrow w(\theta) = 1$

(ii) $v(\Box\theta) = r < \beta$, then $r \leq w(\theta)$ by **2** above. ■

Before stating the completeness theorem, we introduce an alternative consequence relation:

Definition 9.2. *$T \models_{\leq GK(Q)} \varphi$ iff $\sup_{[0,1]}\{r \in Q : r \leq e(x,T)\} \leq e(x,\varphi)$ at any world x of any GK-model.*

Obviously, $T \models_{\leq GK} \varphi$ implies $T \models_{\leq GK(Q)} \varphi$. In fact, both notions coincide, and they coincide with $\models_{GK}$ under mild conditions on T.

Theorem 9.3. *The following are equivalent for T, φ in $\mathcal{L}_{\Box Q}$:*

1. $T \vdash_{\mathcal{G}_\Box(Q)} \varphi$
2. $T \models_{\leq GK} \varphi$
3. $T \models_{\leq GK(Q)} \varphi$

In case any formula in T is shy or has the form $\theta \to \psi$ with ψ shy:

4. $T \models_{GK} \varphi$.

Proof. Since $T \vdash_{\mathcal{G}_\Box(Q)} \varphi \Rightarrow T \models_{\leq GK} \varphi \Rightarrow T \models_{\leq GK(Q)} \varphi \Rightarrow T \models_{GK} \varphi$ we have to prove only $3 \Rightarrow 1$ and $4 \Rightarrow 1$ (for shy T). If $T \nvdash_{\mathcal{G}_\Box(Q)} \varphi$ then $T \cup T\mathcal{G}_\Box(Q) \nvdash \varphi$, and there is a Gödel valuation v such that $v(T \cup T\mathcal{G}_\Box(Q)) = 1$ and $v(\varphi) < 1$. By the Normalization Lemma 9.2 we may transform v to w in $M_\Box(Q)$ so that $\inf w(T) \geq \beta > w(\varphi)$ for certain $\beta \in Q$. By the previous lemma, this means $\inf e^*(w,T) \geq \beta > e^*(w,\varphi)$ in the canonical model, hence $T \not\models_{\leq GK(Q)} \varphi$. Moreover, if the formulas in T have the form ψ or $\theta \to \psi$, with ψ shy, then by Lemma 9.2-5 $w(T) = 1$ and thus $e^*(w,T) = 1$; hence $T \not\models_{GK} \varphi$. ■

Corollary. *For any Q and formula φ in $\mathcal{L}_{\Box Q}$: $\vdash_{\mathcal{G}_\Box(Q)} \varphi \Leftrightarrow \models_{GK} \varphi$.*

Proof. The condition in 4 of Theorem 9.3 is trivially satisfied by empty T, and no condition on Q is required since it is enough to consider the finitely many truth constants appearing in φ. ■

Note that GK-models with crisp accessibility relation are no enough for completeness of $\mathcal{G}_\Box(Q)$; for example, the formula $(\Box\frac{1}{2} \to \frac{1}{2}) \vee \Box 0$ is invalid but is valid in all a-crisp GK-models. However, if $\overline{Q}$ denotes the topological closure of Q in $[0,1]$ (still countable or Q itself if Q is finite) it may be shown that strong completeness of $\mathcal{G}_\Box(Q)$ holds with respect to GK-models where the accessibility relation takes values in $\overline{Q}$ only. For weak completeness only Q-valued accessibility has to be considered.

The proof of the completeness of $\mathcal{G}_\Diamond(Q)$ follows similar lines to that of $\mathcal{G}_\Box(Q)$, and again we have the finite model property for this logic. Also the results of Section 8 transfer without difficulty to the systems with truth constants.

10. Final comment

The main results of this paper were announced at the meeting on "Logic, Computability and Randomness", Cordoba, Argentina, Sept. 2004. Publication was delayed, aiming to axiomatize the full logic with both modal operators combined, which resulted elusive. It may be seen that the union of the systems $\mathcal{G}_\Box$ and $\mathcal{G}_\Diamond$ is not enough for that purpose. However, we have found recently that Fischer Servi [8] "connecting axioms":

$$\Diamond(\varphi \to \psi) \to (\Box\varphi \to \Diamond\psi)$$
$$(\Diamond\varphi \to \Box\psi) \to \Box(\varphi \to \psi)$$

together with $\mathcal{G}_\Box \cup \mathcal{G}_\Diamond$ constitute a strongly complete axiomatization. This will appear elsewhere.

Acknowledgements. The authors are indebted to F. Bou, F. Esteva and L. Godo for helpful comments on different drafts of this paper. The second author acknowledges partial support of Argentinean projects: PIP 112-200801-02543 2009-2011 and CyT-UBA X484.

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