

# On a characterization of path connected topological fields

Xavier Caicedo

Guillermo Mantilla-Soler

## Abstract

The aim of this paper is to give a characterization of path connected topological fields, inspired by the classical Gelfand correspondence between a compact Hausdorff topological space  $X$  and the space of maximal ideals of the ring of real valued continuous functions  $C(X, \mathbb{R})$ . More explicitly, our motivation is the following question: What is the essential property of the topological field  $F = \mathbb{R}$  that makes such a correspondence valid for all compact Hausdorff spaces? It turns out that such a perfect correspondence exists if and only if  $F$  is a path connected topological field.

## §1 Introduction

Let us recall briefly what happens with the real field. Let  $X \neq \emptyset$  be a compact Hausdorff topological space, and let  $C(X, \mathbb{R})$  be the set of continuous functions from  $X$  to  $\mathbb{R}$ . Since evaluating at a point  $x_0 \in X$  is a surjective ring homomorphism  $\text{ev}_{x_0} : C(X, \mathbb{R}) \rightarrow \mathbb{R}; f \mapsto f(x_0)$ , the kernel of such morphism,  $I_{\mathbb{R}}(x_0) := \text{Ker}(\text{ev}_{x_0})$ , is a maximal ideal of  $C(X, \mathbb{R})$ . By endowing the set of maximal ideals  $\text{Max}(C(X, \mathbb{R}))$  with the Zariski topology, Gelfand's theorem tells us that this correspondence is actually a homeomorphism, explicitly:

**Theorem.** *Let  $X$  be a non-empty compact Hausdorff topological space. Then the map*

$$\begin{aligned} I_{\mathbb{R}} : X &\rightarrow \text{Max}(C(X, \mathbb{R})) \\ x_0 &\mapsto I_{\mathbb{R}}(x_0) \end{aligned}$$

*is a homeomorphism.*

In other words the algebraic structure of the ring  $C(X, \mathbb{R})$  completely characterizes the space  $X$ . What is so special about  $\mathbb{R}$  that makes such a powerful characterization possible? Could we ensure the same kind of result if we replace  $\mathbb{R}$  by another topological field  $F$ ?

Let  $F$  be a topological field. Let  $X \neq \emptyset$  be a compact Hausdorff topological space, and let  $C(X, F)$  be the set of continuous functions from  $X$  to  $F$ . As in the real case given  $x_0 \in X$  if  $I_F(x_0)$  is the kernel of the evaluation morphism at the point  $x_0$  then  $I_F(x_0) \in \text{Max}(C(X, F))$ . The question we address is the following:

What conditions on a topological field  $F$  are sufficient and necessary such that for every compact Hausdorff topological space  $X$  the function

$$\begin{aligned} I_F : X &\rightarrow \text{Max}(C(X, F)) \\ x_0 &\mapsto I_F(x_0) \end{aligned}$$

is a homeomorphism? The answer is given by the main result of the paper:

**Theorem** (cf. Theorem 4.1). *Let  $F$  be topological field. Then the Gelfand map  $I_F$  is a homeomorphism for every compact Hausdorff topological space  $X$  if and only if  $F$  is path connected.*

The continuity of the Gelfand map holds more generally for any topological field. Injectivity and bicontinuity will be shown to be equivalent to path connectedness. And, remarkably, surjectivity will be seen to hold for any topological field.

Our proof of the later fact relies on the famous classification of non-discrete locally compact fields ([5], [10], or [2]), and the observation (Proposition 3.1) that for a non algebraically closed field  $F$  there is a polynomial function  $\psi : F^2 \rightarrow F$  such that

$$\psi^{-1}(\{0_F\}) = (0_F, 0_F).$$

This polynomial must have the form  $\psi(x, y) = x\phi_1(x, y) + y\phi_2(x, y)$ , and it generalizes the function  $\psi(x, y) = x^2 + y^2$  for  $\mathbb{R}$ . In the algebraically closed case such polynomial can not exist, but surjectivity of  $I_{\mathbb{C}}$  may be shown utilizing the function  $\psi(x, y) = x\bar{x} + y\bar{y}$  which has the displayed property. By analyzing the cases we have in hand, we suspect that there is for any path connected topological field  $F$  a similar function  $\psi(x, y) = x\phi_1(x, y) + y\phi_2(x, y)$  with  $\phi_1, \phi_2 : F^2 \rightarrow F$  continuous, but have not been able to prove this.

The subject of rings of continuous functions with values in a field is classic and has been studied by several authors, see for instance [1], [3], [6]. A survey article on the subject containing a great deal of results and many references is [9]. Although our characterization of path connected fields given by Theorem 4.1 is quite natural, it seems to have been missed in previous characterizations, see [6, Theorem 8].

## 1.1 Path connected fields

In this paper a *topological field* is a field  $F$  which is also a topological space in which the operations are continuous and the points are closed. Since the topology must be regular by general properties of topological groups,  $F$  must be Hausdorff. We denote by  $0_F$  and  $1_F$  the zero and identity of  $F$  respectively. Recall that a topological space  $X$  is *path connected* if for every two points  $x, y \in X$  there is a continuous path  $\gamma : [0, 1] \rightarrow X$  joining  $x$  and  $y$ , and it is *arcwise connected* if  $\gamma$  may be always chosen as a homeomorphism between  $[0, 1]$  and its image. The following observation shows that the various forms of path connectivity are equivalent in fields.

**Lemma 1.1.** *Let  $F$  be a topological field. The following are equivalent:*

- (i) *There is a continuous path  $\gamma : [0, 1] \rightarrow F$  joining  $0_F$  and  $1_F$ .*
- (ii)  *$F$  is contractible.*
- (iii)  *$F$  is path connected.*
- (iv)  *$F$  is arcwise connected.*

*Proof.* Path and arcwise connectendness are known to be equivalent in Hausdorff spaces ([11], Corollary 31.6). It is enough then to show the implications (i)  $\implies$  (ii) and (iii)  $\implies$  (i). Given a path  $\gamma$  joining  $0_F$  and  $1_F$ , the function  $H : [0, 1] \times F \rightarrow F; (t, \lambda) \mapsto (1 - \gamma(t))\lambda$  gives the result. For the later case, given a path  $\gamma_1$  joining two different points  $a, b \in F$  the function  $\gamma(t) = \frac{\gamma_1(t) - a}{b - a}$  gives a path between  $0_F$  and  $1_F$ .  $\square$

## §2 A first characterization

We prove in this section a first approximation to our main result (Theorem 2.4 below) by following essentially the lines of the proof that the map  $I_{\mathbb{R}}$  is injective and bicontinuous for any compact Hausdorff  $X$ .

Continuity of  $I_{\mathbb{R}}$  follows from definition of the Zariski topology, and from the fact that  $0$  is closed in  $\mathbb{R}$ . This holds for every topological field  $F$  and every space  $X$  (not necessarily compact or Hausdorff).

**Lemma 2.1.** *For any space  $X$  and any topological field the Gelfand map  $I_F : X \rightarrow \text{Max}(C(X, F))$  is continuous.*

*Proof.* By definition of the Zariski topology a closed subset of  $\text{Max}(C(X, F))$  is of the form

$$C_S := \{M \in \text{Max}(C(X, F)) \mid S \subseteq M\}$$

for some  $S \subseteq C(X, F)$ . Continuity of  $I_F$  follows since  $\{0_F\}$  is closed and

$$I_F^{-1}(C_S) = \{x : S \subseteq I_F(x)\} = \bigcap_{f \in S} f^{-1}(0_F).$$

$\square$

Injectivity of  $I_{\mathbb{R}}$  follows from the fact that  $\mathbb{R}$  is path connected and compact Hausdorff spaces are completely regular. These ideas can be generalized as follows:

**Lemma 2.2.** *Let  $F$  be a topological field. Then  $F$  is path connected if and only if for any compact Hausdorff space  $X$  the Gelfand map  $I_F : X \rightarrow \text{Max}(C(X, F))$  is injective.*

*Proof.* Assume  $I_F$  is injective for the space  $X = [0, 1]$  then  $I_F(0) \neq I_F(1)$ , pick  $f \in I_F(0) \setminus I_F(1)$  then  $f(0) = 0_F$  and  $f(1) \neq 0_F$ . Hence,  $\gamma = f(1)^{-1}f$  defines a path connecting  $0_F$  and  $1_F$ . Reciprocally, assume there is a continuous path  $\gamma$  as described. Let  $x \neq y \in X$ . Since  $X$  is completely regular it follows that there is a continuous function  $f : X \rightarrow [0, 1]$  such that  $f(x) = 0$  and  $f(y) = 1$ . Notice that  $\gamma \circ f(x) = 0_F$  and  $\gamma \circ f(y) = 1_F$ . In particular,  $\gamma \circ f \in I_F(x) \setminus I_F(y)$  hence  $I_F$  is injective.  $\square$

Path connectedness of  $F$  implies also that the image of the embedding is Hausdorff for any completely regular space. Recall that an open basis for  $\text{Max}(C(X, F))$  is given by the sets  $D(f) = \{M : f \notin M\}$ .

**Lemma 2.3.** *Let  $F$  be a path connected topological field. Then the image of  $I_F$  is Hausdorff for every compact Hausdorff space  $X$ .*

*Proof.* To separate  $I_F(x_0) \neq I_F(x_1)$  in  $\text{Max}(C(X, F))$  it is enough to show that there are  $f, g \in C(X, F)$  such that  $f(x_0)g(x_1) \neq 0_F$  (thus,  $I_F(x_0) \in D(f)$ ,  $I_F(x_1) \in D(g)$ ) and  $f, g \equiv 0_F$  (thus  $D(f) \cap D(g) = \emptyset$ ) Since  $x_0 \neq x_1$  and  $X$  is Hausdorff there are disjoint open sets  $U_i$ ,  $i = 0, 1$ , such that  $x_i \in U_i$ . Letting  $C_i := X \setminus U_i$  we see that the  $C_i$  are closed subsets of  $X$ . Therefore, by complete regularity of  $X$  there are  $\tilde{f}, \tilde{g} \in C(X, [0, 1])$  such that

$$\tilde{g}(x_1) = 1 = \tilde{f}(x_0) \text{ and } \tilde{g}(C_0) = 0 = \tilde{f}(C_1).$$

Fix a continuous path  $\gamma : [0, 1] \rightarrow F$  joining  $0_F$  and  $1_F$ , and let  $f := \gamma \circ \tilde{f}$  and  $g := \gamma \circ \tilde{g}$ . By the above  $f(x_0) = 1_F$  and  $g(x_1) = 1_F$ . Let  $x \in X$ , then  $x \in C_0$  or  $x \in C_1$ , which by construction implies that  $f(x)g(x) = 0_F$ .  $\square$

By standard topology a continuous injection from a compact space in a Hausdorff space is bicontinuous. Hence, combining lemmas 2.1, 2.1 and 2.3, we obtain a first characterization.

**Theorem 2.4.** *Let  $F$  be a topological field. Then  $F$  is path connected if and only if for every compact Hausdorff space  $X$  the Gelfand map  $I_F$  induces a homeomorphism between  $X$  and its image.*

### §3 Surjectivity

We will show in this section that the Gelfand map is surjective for any compact Hausdorff space  $X$  and any topological field  $F$ . In this case the techniques for  $I_{\mathbb{R}}$  do not generalize fully. Surjectivity of  $I_{\mathbb{R}}$  follows basically from the fact that for every positive  $n$  the vanishing locus of the polynomial  $f_n := x_1^2 + \dots + x_n^2$  in  $\mathbb{R}^n$  is the single point  $(0_F, \dots, 0_F)$ . Such a polynomial function can not exist in algebraically closed fields, but we notice that it exists for any non algebraically closed field  $F$ , which will lead us to the proof of surjectivity in this case.

**Proposition 3.1.** *Let  $F$  be a non algebraically closed field. Then for each positive integer  $n$  there is a polynomial map  $f_n : F^n \rightarrow F$  such that  $f_n^{-1}(0_F) = \{(0_F, \dots, 0_F)\}$ .*

*Proof.* Since  $F$  is not algebraically closed there is a monic polynomial  $f(x) \in F[x]$ , of positive degree  $m$ , which has no zeros in  $F$ . Let  $f_2(x, y) := y^m f(\frac{x}{y}) \in F[x, y]$  be the homogenization of  $f$ . Since  $f(x)$  has no roots in  $F$  then  $f_2^{-1}(0_F) = \{(0_F, 0_F)\}$ . Now proceed by induction. Let  $f_1 : F \rightarrow F$  be the identity function, and suppose that for  $n \geq 1$  we have defined  $f_n$ . Then  $f_{n+1}(x_1, \dots, x_{n+1}) := f_2(f_n(x_1, \dots, x_n), x_{n+1})$  is also polynomial and

$$f_{n+1}^{-1}(0_F) = (f_n \times f_1)^{-1}(\{(0_F, 0_F)\}) = \{(0_F, \dots, 0_F)\} \times \{0_F\} = \{(0_F, \dots, 0_F)\}.$$

□

For the complex numbers surjectivity of  $I_{\mathbb{C}}$  may be shown as for  $I_{\mathbb{R}}$  by taking the continuous function  $f_n := x_1 \bar{x}_1 + \dots + x_n \bar{x}_n$ . It is possible that for any field there is continuous function vanishing only at  $(0_F, \dots, 0_F)$  of the form  $f = x_1 \phi_1 + \dots + x_n \phi_n$ , where  $x_i : F^n \rightarrow F$  denotes the  $i$ -th coordinate function and  $\phi_i : F^n \rightarrow F$  is continuous. However, we have not been able to prove this. Fortunately, a weaker version of this idea gives an alternative argument that works for all fields.

**Theorem 3.2.** *Let  $F$  be a topological field. Then for any compact Hausdorff space  $X$  the Gelfand map  $I_F : X \rightarrow \text{Max}(C(X, F))$  is surjective.*

*Proof.* Let  $M \in \text{Max}(C(X, F))$  and for  $\psi \in M$  let  $D(\psi) := \psi^{-1}(F \setminus \{0_F\})$ . Notice that  $I_F^{-1}(M) = \{x : I_F(x) \supseteq M\} = \bigcap_{\psi \in M} \psi^{-1}(0_F)$ . Hence,

$$I_F^{-1}(M) = X \setminus \bigcup_{\psi \in M} D(\psi).$$

Suppose that  $M$  is not in the image of  $I_F$ . It follows from the compactness of  $X$  that there are finitely many  $\psi_1, \dots, \psi_n \in M$  such that  $X = D(\psi_1) \cup \dots \cup D(\psi_n)$ . Hence, any element is not zero by some  $\psi_i$ . We will show that there is a continuous function  $f_n : F^n \rightarrow F$  such that  $f_n(\psi_1, \dots, \psi_n) \in M$  and  $f_n(\psi_1(x), \dots, \psi_n(x))$  is never 0 in  $X$ . This contradicts the fact that  $M$  is a proper ideal. To achieve this we consider several cases.

**Case I.**  $F$  is non-discrete locally compact then  $F$  must be  $\mathbb{C}$ ,  $\mathbb{R}$ , a finite extension of  $\mathbb{Q}_p$  or a finite extension of  $\mathbb{F}_p((t))$ . For  $\mathbb{C}$  take  $f_n := x_1 \bar{x}_1 + \dots + x_n \bar{x}_n$ . In the remaining cases  $F$  is not algebraically closed. Therefore, the polynomial  $f_n$  provided by Proposition 3.1 will do:  $f_n(\psi_1(x), \dots, \psi_n(x))$  never vanishes because  $(\psi_1(x), \dots, \psi_n(x))$  is never  $(0_F, \dots, 0_F)$  and  $f_n(\psi_1, \dots, \psi_n) \in M$  since  $f_n$  does not have constant term.

**Case II.**  $F$  is discrete. Consider the continuous map  $x \mapsto (\psi_1(x), \dots, \psi_n(x))$  from  $X$  into  $F^n$ . Its image  $J = \{(\psi_1(x), \dots, \psi_n(x)) : x \in X\}$  is compact and thus necessarily finite since  $F^n$  is discrete, moreover, it does not include  $(0_F, \dots, 0_F)$  by hypothesis. Find a polynomial  $f_n(x_1, \dots, x_n)$  identically  $1_F$  in  $J$  and  $0_F$  in  $(0_F, \dots, 0_F)$  (Lagrange interpolation). Then  $f_n(\psi_1(x), \dots, \psi_n(x)) = 1_F$  for any  $x \in X$  and belongs to  $M$  because  $f_n$  does not have constant term.

**Case III.**  $F$  is not locally compact. Consider first  $D(\psi_1) \cup D(\psi_2)$  restricted to the compact space  $Y = X \setminus D(\psi_3) \cup \dots \cup D(\psi_n)$ , and consider the map  $x \mapsto [\psi_1(x), \psi_2(x)]$  from  $Y$  into  $P_1(F)$ , the projective  $F$ -line with the quotient topology, and let  $I$  be the image of this map. Recall that  $P_1(F) = F \cup \{[1, 0]\}$  where  $F$  is topologically embedded in  $P_1(F)$  via the identification  $F \sim \{[a, 1] : a \in F\}$ . As  $I$  is compact, the set  $P_1(F) \setminus I$  must be infinite, as otherwise  $P_1(F)$  would be compact and thus  $F$  would be locally compact. Then we may find  $[a, 1] \in F$  such that  $[\psi_1(x), \psi_2(x)] \neq [a, 1]$  for each  $x \in Y$ . If  $\psi_2(x) = 0$  then  $\psi_1(x) \neq 0$  and thus  $\psi_1(x) - a\psi_2(x) \neq 0$ . If  $\psi_2(x) \neq 0$  then  $\psi_1(x)/\psi_2(x) \neq a$  and also  $\psi_1(x) - a\psi_2(x) \neq 0$ . Therefore  $\psi_1 - a\psi_2$  does not vanish in  $Y$  and we have

$$X = D(\psi_1 - a\psi_2) \cup D(\psi_3) \cup \dots \cup D(\psi_n)$$

Repeating the procedure with the pair  $[\psi_1 - a\psi_2, \psi_3]$ , and continuing inductively, we obtain finally  $X = D(a_1\psi_1 + a_2\psi_2 + \dots + a_n\psi_n)$  where evidently  $a_1\psi_1 + a_2\psi_2 + \dots + a_n\psi_n \in M$ .  $\square$

## §4 Conclusion

Thanks to theorems 2.4 and 3.2, we have our main result:

**Theorem 4.1.** *A topological field  $F$  is path connected if and only if for every compact Hausdorff topological space  $X$  the Gelfand map  $I_F : X \rightarrow \text{Max}(C(X, F))$  is a homeomorphism.*

An important question to ask at this point is: are there other examples besides  $\mathbb{R}$  and  $\mathbb{C}$  that make Gelfand correspondence work? As it turns out there are many other path connected topological fields of 0 characteristic not isomorphic to either  $\mathbb{R}$  or  $\mathbb{C}$ . There are even examples in positive characteristic; see for instance [8], where it is shown that any discrete field may be embedded in a path connected field. However, by Pontryagin's classification theorem [7] the only locally compact ones are  $\mathbb{R}$  and  $\mathbb{C}$ .

An inspection of the proof of Theorem 3.2 shows that for any topological field  $F$ , given functions  $\psi_1, \dots, \psi_n \in C(X, F)$  such that  $X = D(\psi_1) \cup \dots \cup D(\psi_n)$  there exist continuous functions  $\phi_1, \dots, \phi_m : F^n \rightarrow F$  such that  $\sum_{i=1}^n \psi_i \phi_i(\psi_1, \dots, \psi_n)$  does not vanish in  $X$ . But the  $\phi_i$  depend on the  $\psi_i$ . One may wonder if the  $\phi_i$  may be chosen independently; that is, whether there exists  $f = x_1\phi_1 + \dots + x_n\phi_n$  such that  $f^{-1}(0_F) = (0_F, \dots, 0_F)$ . We call this *polynomially generated* functions. We are able to show:

**Proposition 4.2.** *Let  $F$  be a path connected metrizable topological field. Then for every positive integer  $n$  there is a continuous function  $f_n : F^n \rightarrow F$  such that  $f_n^{-1}(0_F) = \{(0_F, \dots, 0_F)\}$ .*

*Proof.* Let  $d$  be a metric on  $F^n$  and let  $\bar{d} := \min\{1, d\}$  be the standard bounded metric on  $F^n$ . Define  $\phi : F^n \rightarrow [0, 1]$  as the bounded distance to  $(0_F, \dots, 0_F)$ . As  $F$  is arcwise connected (Lemma 1.1), there is an embedding  $[0, 1] \xrightarrow{\gamma} F$  sending 0 to  $0_F$ . Take  $f_n = \gamma \circ \phi$ .  $\square$

We finish with the following question, that appeared naturally during the progress of the paper, for which we don't have an answer.

**Question 4.3.** *Let  $F$  be a path connected topological field. Is there is a polynomially generated function  $f_2 : F^2 \rightarrow F$  such that  $f_2^{-1}(\{0_F\}) = (0_F, 0_F)$ ?*

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Xavier Caicedo, Department of Mathematics, Universidad de los Andes, Bogotá, Colombia (xcaicedo@uniandes.edu.co)  
Guillermo Mantilla-Soler, Department of Mathematics, Universidad Konrad Lorenz, Bogotá, Colombia (gmantelia@gmail.com)