MAXIMALITY OF CONTINUOUS LOGIC

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Abstract. The analogue of Lindström´s characterization of elementary logic in terms of compactness and the downward Löwenheim-Skolem theorem is shown for continuous logic, introduced originally for Banach spaces by Henson and Iovino as a logic of approximate satisfaction and generalized later to metric spaces by Ben Yaacov and Usvyatsov. For this purpose we characterize equivalence of models in this logic by means of partial approximations and identify some useful properties of its compact extensions.

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1. Introduction

Continuous logic has its prehistory in Chang and Keisler´s monograph on logic with values in compact Hausdorff spaces [13], see also [12]. It had an independent revival in Krivine's successful use of model theoretic methods in Banach spaces [22, 23], work continued by Henson and Iovino [17, 18], and recently generalized to metric spaces by Ben Yaacov and Usvyatsov [5, 6] and [3]. Appropriate versions of most techniques and properties of classical model theory generalize to continuous logic making it well suited for exploiting model theoretic methods in analysis: ultraproducts, compactness, Löwenheim-Skolem theorems, omitting types, and large portions of stability theory. Noteworthy are Keisler-Shelah's theorem on ultraproducts and Morely's categoricity theorem.

The models of this logic are bounded complete metric structures, equipped with uniformly continuous maps and [0,1]-valued predicates. The language has a connective for each continuous truth table $c : [0,1]^n \to [0,1]$ and the quantifiers are interpreted as infima and suprema. The metric plays the role of an identity predicate.[1] We will depart of the usual presentations of continuous logic in that we take 1 as the distinguished truth value and, more significantly, we allow infinitary continuous connectives $c : [0,1]^\omega \to [0,1]$. The aim of this paper is to prove the following Lindström style result which shows this logic has maximal expressive strength if we wish to maintain the two model theoretic properties mentioned in the theorem.

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[1]A more general setting allows unbounded pointed metric spaces, but restricts quantification to closed balls around a distinguished point, see Chapter 5 paper by Dueñez and Iovino in this volume. Our maximality result holds in this setting.

(Theorem 6.5) *Let L be an extension of continuous logic closed under Łukasiewicz connectives $\to, \neg$ and satisfying compactness and the separable downward Löwenheim-Skolem property. Then any sentence of L is equivalent to a sentence of continuous logic.*

The compactness and Löwenheim-Skolem properties are here natural versions of the analogous classical properties, compactness being asked only for equicontinuous classes of structures, those having a common uniform continuity modulus for each basic function and predicate, and separability being the adequate countable power condition for complete metric structures. Two sentences are said to be equivalent if they attain the same value in all suitable structure. Our main tools are a natural notion of approximation which behaves well in compact extensions, and a characterization of equivalence of models in continuous logic in the style of Fraissé by means of partial approximations. Some topological ideas and the close relation of continuous logic to Łukasiewicz logic will be useful also. We obtain, in fact, a weaker maximality result for Łukasiewicz logic:

(Theorem 6.3) *Let L be an extension of Łukasiewicz logic closed under Łukasiewicz connectives $\to, \neg$ and satisfying compactness and the separable downward Löwenheim-Skolem property. Then any sentence of L has the same models as a countable theory of Łukasiewicz logic in complete structures.*

It follows that continuous and Łukasiewicz logic have the same axiomatizability strength.

This paper may be seen as a sequel of Iovino's maximality result for the approximation logic of Banach spaces with respect to compactness and the elementary chain property [19], and the characterization by Iovino and the author in terms of uncountable omitting types in [9].

Versions of our results have been announced in [9] and in several talks as early as 2008. The paper intends to be self-contained but we refer the reader to [6], [3] and [20] for continuous logic, and [16] for Lukasiewicz logic.

2. Preliminaries

We start considering the language of continuous logic interpreted in non-metric $[0,1]$-valued structures. A *(first order) many-sorted signature* is a sequence

$$\tau = \{\mathcal{S}, (R, \overline{s}_R), ..; (f, \overline{s}_f), .., c_{s_c}, ..\},$$

were $\mathcal{S}$ is a nonempty set of sorts, R, f, and c are predicate, function, and constant symbols, respectively, and $\overline{s}_R$, $\overline{s}_f, s_c$, are finite sequences in $\mathcal{S}$ associated to each symbol[2]. In the one-sorted case these sort sequences may be identified with natural numbers.

A $[0,1]$-*valued τ-structure* is a sequence

$$\mathfrak{A} = (\{A_s : s \in \mathcal{S}\}, R^{\mathfrak{A}}, ..; f^{\mathfrak{A}} ..; c^{\mathfrak{A}}, ..),$$

where A_s is a nonempty universe for each sort and

$$R^{\mathfrak{A}} : \Pi_{i=1}^{n} A_{s_{R,i}} \to [0,1], \quad f^{\mathfrak{A}} : \Pi_{j=1}^{m} A_{s_{f,j}} \to A_{s_{f,m+1}}, \quad c^{\mathfrak{A}} \in A_{s_c}.$$

St_τ will denote the class of all $[0,1]$-valued τ-structures. This class has natural notions of isomorphism and substructure. An *isomorphism* $h : \mathfrak{A} \to \mathfrak{B}$ is an

[2]Through the paper $\overline{x}$, $\overline{a}, ...$ will denote finite sequences and their components will be denoted, respectively, x_i, $a_i, ...$

ordinary isomorphism of the many-sorted algebraic reducts of $\mathfrak{A}$ and $\mathfrak{B}$ satisfying, moreover, $R^{\mathfrak{B}}[h(a_1),..,h(a_n)] = R^{\mathfrak{A}}[a_1,..,a_n]$ for any predicate symbol R in τ and a_i in $A_{s_{R,i}}$. Its existence will be denoted $\mathfrak{A} \simeq \mathfrak{B}$. Similarly, $\mathfrak{A}$ is a *substructure* of $\mathfrak{B}$, denoted $\mathfrak{A} \leq \mathfrak{B}$, if it is a substructure of $\mathfrak{B}$ in the classical sense with respect to the operations in τ, and $R^{\mathfrak{A}}[\overline{a}] = R[\overline{a}]^{\mathfrak{B}}$ for the predicate symbols.

The language of *Lukasiewicz predicate logic,* denoted Ł∀, is built in the same way as the classical one based on the primitive connective symbols →, ¬, and quantifier symbols $\exists^t$, $\forall^t$ for each sort t. The set of formulas built from a signature τ will be denoted Ł$\forall_\tau$. Terms, evaluated as in classical logic, give rise to maps $t^A : A_{\overline{s}} \to A_t$ in the usual way, while formulas $\varphi(x_1, ..., x_n)$ give rise to maps $\varphi^{\mathfrak{A}} : A_{\overline{s}} \to [0, 1]$ and sentences determine truth values $\varphi^{\mathfrak{A}} \in [0, 1]$, defined inductively by composing with Łukasiewicz's functional interpretation of the connectives in [0,1]:

$$\begin{aligned} p \to q &= \min(1 - p + q, 1) \\ \neg p &= 1 - p, \end{aligned}$$

(we utilize the same name for the symbol and its interpretation), and interpreting quantifiers as suprema and infima. That is, for $\overline{a}$ in $A_{\overline{s}}$:

$$\exists^t x\varphi(x)^{\mathfrak{A}}[\overline{a}] \quad := \quad \sup\nolimits_{b\in A_t} \varphi^{\mathfrak{A}}[b, \overline{a}]$$

and $\forall x\varphi(x)^{\mathfrak{A}}[\overline{a}] = \neg\exists^t\neg x\varphi(x)^{\mathfrak{A}}[\overline{a}] = \inf_{b\in A_t} \varphi^{\mathfrak{A}}[b, \overline{a}]$.

The language of *restricted continuous logic*, denoted CL°, is obtained by adding a connective symbol c for each continuous map $c : [0, 1]^n \to [0, 1]$ and closing under these operators: $c(\varphi_1[\overline{x}], .., \varphi_n[\overline{x}])$. The language of *continuous logic*, CL, is obtained similarly by adding all infinitary continuous connectives $c : [0, 1]^\omega \to [0, 1]$ for the Thychonov topology and closing the language under their application to sequences of formulas in the same finite set of variables: $c(\varphi_1[\overline{x}], \varphi_2[\overline{x}], \varphi_3[\overline{x}]...)$. Notice that a formula of CL may carry a countable infinite vocabulary but have a finite number of free variables. Since Łukasiewicz connectives are clearly continuous, Ł∀ becomes a sublogic of CL°, in turn a sublogic of CL. The additional strength provided by infinitary continuous connectives amounts to allowing "definable predicates" (see Lemma 2.3). For technical reasons and because the literature is ambiguous about allowing the latter in the syntax of continuous logic we distinguish CL° and CL.

Clearly, if $h : \mathfrak{A} \to \mathfrak{B}$ is an isomorphism, then $\varphi^{\mathfrak{B}}[h(a_1), .., h(a_n)] = \varphi^{\mathfrak{A}}[a_1, .., a_n]$ for any $\overline{a}$ in A^n and formula φ of CL and a_i in $\mathfrak{A}$. If $\mathfrak{A} \leq \mathfrak{B}$ then $\varphi^{\mathfrak{B}}[a_1, .., a_n] = \varphi^{\mathfrak{A}}[a_1, .., a_n]$ for any quantifier free φ.

Satisfaction of formulas of CL in [0,1]-valued structures is defined in terms of the designed truth value 1:

$$\mathfrak{A} \models \varphi[\overline{a}] \;\; (\mathfrak{A} \textit{ satisfies } \varphi \textit{ at } \overline{a}) \;\; \text{if and only if} \;\; \varphi^{\mathfrak{A}}[\overline{a}] = 1.$$

In the literature of continuous logic, 0 is utilized instead and the meaning of the quantifiers is switched, an inessential difference.

If $\mathfrak{A} \models \varphi$ for a sentence $\varphi \in$ CL, we say that $\mathfrak{A}$ is *a model of* φ. This nomenclature extends to theories, that is sets of sentences $T \subseteq$ CL, and $Mod_\tau(\varphi)$, $Mod_\tau(T)$ will denote the class of $[0, 1]$-valued τ-models of φ or T, respectively. It should be clear that $Mod_\tau(\varphi)$ does not describe the full meaning of φ, as it says nothing of the structures $\mathfrak{A}$ where $\varphi^{\mathfrak{A}} = \frac{1}{2}$, for example.

A structure $\mathfrak{A} \in St_\tau$ is *crisp* (two-valued, classical) if its basic relations take values in $\{0, 1\}$. A simple induction in formulas yields that in crisp structures any formula $\varphi \in \text{Ł}\forall_\tau$ is two-valued and $\mathfrak{A} \models \varphi[\bar{a}]$ coincides with classical satisfaction. This is clearly not the case for CL.

For simplicity, we will usually refer to the one-sorted case through the rest of the paper, resorting to the many-sorted case when strictly necessary only.

2.1. **Language reductions.** Lukasiewicz connectives $\rightarrow$ and $\neg$ have a high expressive power which permits to recover the lattice connectives:

$$\begin{aligned} p \vee q &:= \max\{p, q\} &&= (p \rightarrow q) \rightarrow q \\ p \wedge q &:= \min\{p, q\} &&= \neg(\neg p \vee \neg q) \end{aligned}$$

and a host of arithmetical connectives, as:

$$\begin{array}{llll} p \oplus q & := \min\{p+q, 1\} & = \neg p \rightarrow q & \textit{truncated addition} \\ p \searrow q & := \max\{p-q, 0\} & = \neg(p \rightarrow q) & \textit{truncated substraction} \\ |p-q| & := (p \searrow q) \oplus (q \searrow p) & = (p \searrow q) \vee (q \searrow p) & \textit{truth value distance.} \end{array}$$

Equivalence becomes the negation of truth distance:

$$p \leftrightarrow q \quad := (p \rightarrow q) \wedge (q \rightarrow p) \quad \equiv (p \rightarrow q) \odot (q \rightarrow p) \quad \equiv \neg|p-q| \ .$$

Any of $\{\oplus, \neg\}$ or $\{\searrow, \neg\}$ may be taken as complete set of connectives for Ł$\forall$ (this is not the case of $\{\wedge, \neg\}$ or $\{\vee, \neg\}$). The first pair is utilized in Chang's algebraization of Lukasiewicz propositional logic in terms of MV-algebras [10], while the latter is preferred in the literature of continuous logic. McNaughton's theorem [25] characterizes the Łukasiewicz definable connectives as the continuous piecewise linear maps $c : [0,1]^n \rightarrow [0,1]$ with integer coefficients. It follows that reflexive rational bounds on truth values of formulas are expressible in Łukasiewicz logic. Irreflexive bounds as $\varphi^{\mathfrak{A}} > r$ are not similarly expressible due to continuity of the connectives.

Lemma 2.1. (Chang [11], Belluce [1], Mundici [26]) *For each rational* $r = (0,1)$ *there are Łukasiewicz connectives* β_r *and* γ_r *such that*

$$\mathfrak{A} \models \beta_r(\varphi) \textit{ iff } \varphi^{\mathfrak{A}} \geq r, \qquad \mathfrak{A} \models \gamma_r(\varphi) \textit{ iff } \varphi^{\mathfrak{A}} \leq r.$$

Moreover, $\beta_r(\varphi^{\mathfrak{A}}) \geq 1-\varepsilon$ *implies* $\varphi^{\mathfrak{A}} \geq r-\varepsilon$, *and* $\gamma_r(\varphi^{\mathfrak{A}}) \geq 1-\varepsilon$ *implies* $\varphi^{\mathfrak{A}} \leq r+\varepsilon$, *for any* $\varepsilon \in [0,1)$.

Proof. The map $\beta_{\frac{n}{m}} : [0,1] \rightarrow [0,1]$ which is 0 for $x \leq \frac{n-1}{m}$, $mx - n + 1$ in $[\frac{n-1}{m}, \frac{n}{m}]$, and 1 for $x \geq \frac{n}{m}$, satisfies McNaughton's conditions and the claimed properties. And so does $\gamma_{\frac{n}{m}}(x) := 1 - \beta_{\frac{n+1}{m}}(x)$. □

Notation. $\varphi_{\geq r}$ and $\varphi_{\leq r}$ will abbreviate $\beta_r(\varphi)$ and $\gamma_r(\varphi)$, respectively. As these connectives depend on the actual fraction $\frac{n}{m}$ representing r, the generic notation $\varphi_{\geq r}$, $\varphi_{\leq r}$ will refer to the reduced fraction of r.

It follows from Corollary 1.5 in [6] that any continuous map $c : [0,1]^n \rightarrow [0,1]$ may be approximated uniformly by combinations of Łukasiewicz connectives and rational constants, thanks to the Stone-Weirstrass theorem and compactness of $[0,1]^n$ (see also Proposition 1.18 in [9]). This fact extends readily to infinitary continuous connectives if we add projections $[0,1]^\omega \stackrel{\pi_n}{\rightarrow} [0,1]^n$, and it implies that CL may be approximated by the sublanguage Ł$\forall(\mathbb{Q})$ which results of adding to Ł$\forall$ a constant connective of value r for each rational $r \in (0,1)$.[3]

[3]Called rational Pavelka-Łukasiewicz logic in [16].

Lemma 2.2. *For any formula $\varphi(\overline{x})$ of CL there is a sequence of formulas $\varphi_n(\overline{x})$ in Ł$\forall(\mathbb{Q})_\tau$ such that $\varphi^{\mathfrak{A}}[\overline{a}]$ converges to $\varphi_n^{\mathfrak{A}}[\overline{a}]$, uniformly on $(\mathfrak{A},\overline{a}) \in St_{\tau\cup\{\overline{c}\}}$.*

Proof. By Induction on complexity of $\varphi(\overline{x})$ and utilizing the fact that the connectives are uniformly continuous, one proves the existence for all n of $\varphi_n(\overline{x})$ such that $|\varphi_n^{\mathfrak{A}}[\overline{a}] - F(\mathfrak{A},\overline{a})| < \frac{1}{n}$ for all $(\mathfrak{A},\overline{a})$. □

The approximation in the lemma cannot be achieved with formulas taken from Ł$\forall$, since in crisp structures these formulas take values in $\{0,1\}$ and so must do their limits. As consequence of the lemma the countable theory $T_\varphi\{(\varphi_n)_{\geq 1-\frac{1}{n}} : n \in \omega\}$ of Ł$\forall(\mathbb{Q})_\tau$ has the same models as φ. This will be improved for continuous structures in Corollary 6.4.

Given a sequence of formulas $\varphi_n(\overline{x})$ of CL converging uniformly on $St_{\tau\cup\{\overline{c}\}}$ to a map $F : St_{\tau\cup\{\overline{c}\}} \to [0,1]$ then F is called a (global) *definable predicate* in the sense of [6, 3]. The next lemma shows that CL is closed under these predicates.

Lemma 2.3. *If $\varphi_n(\overline{x}) \to F$ uniformly in $St_{\tau\cup\{\overline{c}\}}$, there is a formula $\varphi(\overline{x})$ of CL such that $\varphi^{\mathfrak{A}}[\overline{a}] = F(\mathfrak{A},\overline{a})$ for any $(\mathfrak{A},\overline{a})$.*

Proof. Find a subsequence of φ_{i_n} of $\{\varphi_n\}$ such that $|\varphi_{i_n}^{\mathfrak{A}}[\overline{a}] - \varphi_{i_{n+1}}^{\mathfrak{A}}[\overline{a}]| < 2^{-n}$ for all $(\mathfrak{A},\overline{a})$, then Lemma 3.7 in [6] grants that there is a continuous connective $c : [0,1]^\omega \to [0,1]$ such that $c(\varphi_{i_1}^{\mathfrak{A}}[\overline{a}], \varphi_{i_2}^{\mathfrak{A}}[\overline{a}], ...) = \lim \varphi_{i_n}^{\mathfrak{A}}[\overline{a}] = \lim \varphi_n^{\mathfrak{A}}[\overline{a}]$ for all $(\mathfrak{A},\overline{a})$. □

Lemmas 2.2 and 2.3 put together say that, functionally speaking, CL is the uniform closure of Ł$\forall(\mathbb{Q})$ in [0,1]-valued structures.

2.2. **The Löwenheim-Skolem property.** $\mathfrak{A}$ is a CL-*substructure* of $\mathfrak{B}$, denoted $\mathfrak{A} \prec_{\mathrm{CL}} \mathfrak{B}$, if and only if $\mathfrak{A} \leq \mathfrak{B}$ and $\varphi^{\mathfrak{A}}[\overline{a}] = \varphi^{\mathfrak{B}}[\overline{a}]$ for all $\varphi \in CL_\tau$ and $\overline{a}$ in $\mathfrak{A}$. The following properties of classical elementary embedding hold. In fact, they hold for any language L between Ł$\forall$ and CL generated by the quantifiers and any family of continuous connectives containing $\to$, $\neg$.

Lemma 2.4. a) *Given $\mathfrak{A} \leq \mathfrak{B}$, then $\mathfrak{A} \prec_{CL} \mathfrak{B}$ if and only if for any formula $\varphi(x,\overline{y})$ in CL, $\overline{a}$ in A, b in B, and $r < 1$, $\mathfrak{B} \models \varphi[b,\overline{a}]$ implies there is $a \in A$ such that $\mathfrak{B} \models \varphi_{\geq r}[a,\overline{a}]$.*

b) *Given $\mathfrak{B}$ and $C \subseteq B$ there is $\mathfrak{A} \prec_{CL} \mathfrak{B}$ with $C \subseteq A$ and $|A| \leq |C| + |\tau| + \omega$.*

Proof. a) One direction is trivial. For the other, show $\varphi^{\mathfrak{A}}[\overline{a}] = \varphi^{\mathfrak{B}}[\overline{a}]$ by induction in the complexity of formulas, the atomic case and the step for connectives are automatic. Assume $\exists x\varphi(x,\overline{a})^{\mathfrak{A}} > \beta$ then there is a rational $r > \beta$ and a in A such that $\varphi^{\mathfrak{A}}[a,\overline{a}] \geq r$, equivalent by induction hypothesis to $\varphi^{\mathfrak{B}}[a,\overline{a}] \geq r$, which implies $\exists x\varphi(x,\overline{a})^{\mathfrak{B}} > \beta$. Reciprocally, $\exists x\varphi(x,\overline{a})^{\mathfrak{B}} > \beta$ implies $\varphi^{\mathfrak{B}}[b,\overline{a}] \geq r$ for some $r > \beta$ which by the hypothesis of the lemma implies $\mathfrak{B} \models (\varphi_{\geq r})_{1-t}[a,\overline{a}]$ for some a in A and t arbitrarily small. Hence, By Lemma 2.1 $\mathfrak{B} \models (\varphi_{\geq r})_{1-t}[a,\overline{a}] \geq r - t > \alpha$ if t is small enough.

b) A τ-theory of power κ in CL is equivalent to a theory in Ł$\forall(\mathbb{Q})_\tau$ of power $\kappa+\omega$ by the remark after Lemma 2.2. For each φ in Ł$\forall(\mathbb{Q})_\tau$ and rational $r \in (0,1)$ choose a Skolem function $f_{\varphi,r}(\overline{y})$ such that $\mathfrak{B} \models \exists x\varphi(x,\overline{a})$ implies $\varphi^{\mathfrak{B}}[f_{\varphi,r}(\overline{a}),\overline{a}] \geq r$. Let A be the closure of C under these maps and apply (a) to the induced substructure. As the number of Skolem functions is bounded by $\kappa + \omega$, then the bound of $|A|$ is attained. □

Taking countable τ and $C = \emptyset$ in part (b) of the Lemma we obtain:

Proposition 2.5. (Löwenheim-Skolem property) *Any countable theory of CL having* $[0,1]$*-valued models has a finite or countable model.*

2.3. **Continuous structures.** The natural notion of identity in a $[0,1]$-valued structure $\mathfrak{A}$ is a distinguished binary predicate $\approx^{\mathfrak{A}}$: $A^2 \to [0,1]$ such that $\approx^{\mathfrak{A}}$ $(a,b) = 1$ implies $a = b$ and satisfying the usual congruence axioms. As noticed first by Katz [21] and independently in [16] and [5], these axioms become the axioms of a metric for the predicate $d(x,y) := \neg x \approx y$ under which the basic relations and functions result 1-Lipschitz continuous. Continuous logic takes d as the primitive notion and generalizes the Lipschitz conditions to arbitrary uniform continuity.

A [0,1]-valued structures is a *continuous structure* if it has a distinguished relation symbol $d_s \in \tau$ for each sort s, interpreted by a metric $d_s^{\mathfrak{A}} : A_s^2 \to [0,1]$, under which the relations $R^{\mathfrak{A}}$ and maps $f^{\mathfrak{A}}$ $(R,\ f \in \tau)$ are uniformly continuous with respect to the sup metric induced in $\Pi_i A_{s_{R,i}}$ and $\Pi_i A_{s_{f,i}}$, respectively.

This means that there is a system $S = \{m_\alpha\}_{\alpha\in\tau}$ of uniform continuity moduli $m_\alpha : (0,1) \to (0,1)$ such that for any $\varepsilon > 0$:

$$d(\overline{a}, \overline{b}) < m_R(\varepsilon) \text{ implies } |R(\overline{a}) - R(\overline{b})| \leq \varepsilon$$

$$d(\overline{a}, \overline{b}) < m_f(\varepsilon) \text{ implies } d(f(\overline{a}), f(\overline{b})) \leq \varepsilon,$$

where $d(\overline{a}, \overline{a}') = \max_i d(a_i, a_i')$.

It is not necessary to specify m for the metrics d_s or the constant symbols since a metric is always 1-Lipschitz and the constants satisfy any moduli. Without loss of generality, we may assume that all moduli satisfy $m(\varepsilon) \leq \varepsilon$, and they are given for $\varepsilon = \frac{1}{n}$ only. Therefore, thanks to Lemma 2.1, the above conditions are expressible in $L\forall$ by a theory U_S which has the following countable schema for each relation and function symbol in τ:

$$U_S(R) : \forall\overline{xy}(d_s(\overline{x},\overline{y})_{\geq r} \vee |R(\overline{x}) - R(\overline{y})|_{\leq \frac{1}{n}}),\ \ n \in \omega^+,\ r < m_R(\tfrac{1}{n}),\ r \in \mathbb{Q}^+$$

$$U_S(f) : \forall\overline{xy}(d_s(\overline{x},\overline{y})_{\geq r} \vee d(f(\overline{x}), f(\overline{y}))_{\leq \frac{1}{n}}),\ \ n \in \omega^+,\ r < m_f(\tfrac{1}{n}),\ r \in \mathbb{Q}^+.$$

$St_{\tau,S}$ will denote the class of metric τ-structures satisfying the moduli in S that we will call *S-models*. These classes may overlap; for example, crisp τ-structures with the discrete metric belong to all $St_{\tau,S}$. We will write $Mod_{\tau,S}(T)$ for the S-models of T.

Call a class $K \subseteq St_\tau$ *equicontinuous* if all structures in K share a common moduli system, that is, $K \subseteq St_{\tau,S}$ for some S. Perhaps the most important property of CL in continuous structures is the validity of a Łoś theorem for metric ultraproducts of equicontinuous families which yields compactness for satisfaction in each class $St_{\tau,S}$ (see [3]).

Proposition 2.6. (Compactness) *Given a moduli system S for τ, if any finite part of a theory $T \subseteq CL$ has S-models, then the theory has S-models.*

2.4. **Complete structures.** Our maximality results will depend strongly on having complete continuous models.

A continuous structure is *complete* if each metric space $(A_s, d_s^{\mathfrak{A}})$ is complete. $St_{\tau,S}^c$ will denote the class of complete S-structures and $Mod_{\tau,S}^c(T)$ will denote the class of complete S-models of T.

Any continuous structure $\mathfrak{A} \in St_{\tau,S}$ has a *(metric) completion* $\widehat{\mathfrak{A}}$ in $St_{\tau,S}^c$ which is well defined because uniformly continuous functions between metric spaces extend univocally to their completions with the same uniform convergence modulus. Moreover, the uniqueness of the extensions of the maps $t^A : A^n \to A,\ \varphi^{\mathfrak{A}} : A^n \to [0,1]$ to $\widehat{A}$ grants

$$\mathfrak{A} \prec_{CL} \widehat{\mathfrak{A}}.$$

Therefore, taking the completion of the model provided by Proposition 2.6, it may be seen that the compactness property holds in the class of complete structures. This may be obtained also from the fact that metric ultraproducts may be chosen ω_1-saturated. Similarly, taking the completion of the model provided by Proposition 2.5, the Löwenheim-Skolem property becomes in complete structures:

Proposition 2.7. (Separable Löwenheim-Skolem property) *Any countable theory of CL having S-models has a separable S-model.*

3. Characterizing equivalence

In this section, we characterize Ł∀-equivalence of (not necessarily complete) continuous structures by means of ranked families of partial approximations. Equivalence of structures is defined in the obvious way for any language L between Ł∀ and CL: $\mathfrak{A} \equiv_{\mathrm{L}} \mathfrak{B}$ if and only if $\varphi^{\mathfrak{A}} = \varphi^{\mathfrak{B}}$ for all suitable sentences $\varphi \in L$. Whenever L is closed under Łukasiewicz connectives this reduces by Lemma 2.1 to

$$\mathfrak{A} \equiv_{\mathrm{L}} \mathfrak{B} \ \text{ if and only if } Th_L(\mathfrak{A}) = Th_{\mathrm{L}}(\mathfrak{B}).$$

Our characterization of Ł∀-equivalence below will imply that CL and Ł∀ share the same equivalence in continuous structures (Corollary 3.4):

$$\mathfrak{A} \equiv_{\mathrm{CL}} \mathfrak{B} \ \text{ if and only if } \ \mathfrak{A} \equiv_{\mathrm{Ł}\forall} \mathfrak{B}.$$

For the proof of the next lemma it will be convenient to consider the Lukasiewicz connectives $\wedge$, $(\ )_{\geq \frac{k}{n}}$, and $(\ )_{\leq \frac{k}{n}}$, as primitive and define a complexity rank on formulas of Ł∀ as follows:

For terms, $\rho(t)$ is the usual syntactic depth (0 for variables and constants)
$\rho(\alpha(t_1, .., t_k)) = \Sigma_i \rho(t_i)$, for a n-ary predicate symbol $\alpha \in \tau$
$\rho(\varphi \to \psi) = \max\{\rho(\varphi), \rho(\psi)\} + 1$
$\rho(\neg\varphi) = \rho(\varphi) + 1$
$\rho(\varphi \wedge \psi) = \max\{\rho(\varphi), \rho(\psi)\}$
$\rho(\varphi_{\geq \frac{k}{n}}) = \rho(\varphi_{\leq \frac{k}{n}}) = \rho(\varphi) + n$
$\rho(\exists v\varphi) = \rho(\varphi) + 1$.

Denote by $\text{Ł}\forall^{\ell}_{\mu,\overline{v}}$ the set of Lukasiewicz μ-formulas of rank $\leq \ell$ with free variables in $\overline{v}$. It is easy to verify by induction on ℓ that for any finite signature μ and finite list of variables $\overline{v}$, there are finitely many terms and finitely many nonequivalent formulas in $\text{Ł}\forall^{\ell}_{\mu,\overline{v}}$. For terms this is clear and, a fortiori, for atomic formulas. Moreover, any nonatomic formula of rank $\leq \ell+1$ has one of the forms: $\varphi \to \psi$, $\neg\varphi$, $\varphi_{\geq \frac{k}{n}}$, $\varphi_{\leq \frac{k}{n}}$, $\exists v\varphi$, where φ and ψ, have rank $\leq \ell$ and $n \leq \ell + 1$, or it is equivalent to a $\wedge$-combination of formulas of these forms. By induction hypothesis there are finitely many representatives for the displayed forms and the result follows because a finitely generated $\wedge$-semi-lattice is finite.

Proviso. For a signature τ, τ^* will denote in the rest of this paper the set of atomic formulas of rank at most 1; that is, formulas $\alpha(t_1, .., t_k)$ where $\alpha \in \tau \cup \{d\}$ and at most one term t_i has the form $f(\overline{x})$ for a function symbol $f \in \tau$, the other being variables or constant symbols.

Definition 3.1. *Given* a finite $\mu \subseteq \tau$, and $\varepsilon > 0$, a *partial $\mu\varepsilon$-approximation* between two τ-structures $\mathfrak{A}$ and $\mathfrak{B}$ is a relation $R \subseteq A \times B$ such that $|\varphi^{\mathfrak{A}}[\overline{a}] - \varphi^{\mathfrak{B}}[\overline{b}]| \leq \varepsilon$ for all $\varphi \in \mu^*$ and all suitable $\overline{a}, \overline{b}$ such that $(a_i, b_i) \in R$.

Notice that the empty relation is a partial $\mu\varepsilon$-approximation if and only $|\varphi^{\mathfrak{A}} - \varphi^{\mathfrak{B}}| \leq \varepsilon$ for any sentence $\varphi \in \mu^*$. We may identify a finite partial approximation $R = \{(a_i, b_i)\}_{i=1,..t} \subseteq A \times B$ with the ordered pair $(\overline{a}, \overline{b}) = ((a_1, ...a_t), (b_1, ...b_t)) \in A^t \times B^t$, the empty relation being identified with the pair $(\Lambda, \Lambda) \in A^0 \times B^0$, where $\Lambda = \emptyset$ stands for the empty sequence.

Lemma 3.2. *If* $\mathfrak{A} \equiv_{L\forall(\tau)} \mathfrak{B}$*, then for each finite* $\mu \subseteq \tau$, $n \in \omega^+$, *and* $\varepsilon > 0$, *there is a sequence* $I_0, ..., I_n$ *of nonempty sets of (finite) partial* $\mu\varepsilon$*-approximations between* A *and* B *with the following extension property:* $R \in I_{j+1}$ *and* $a \in A$ $(b \in B)$ *imply there is* $R' \in I_j$ *such that* $R' \supseteq R$ *and* $a \in domR'$ $(b \in ranR')$.

Proof. By the previous remarks, to prove the claim it is enough (given μ, ε and $n \in \omega$) to define nonempty sets $I_j \subseteq A^{n-j} \times B^{n-j}$, $j = 0, .., n$, satisfying:

(1) If $(\overline{a}, \overline{b}) \in I_j$, then $|\varphi^{\mathfrak{A}}[\overline{a}] - \varphi^{\mathfrak{B}}[\overline{b}]| \leq \varepsilon$ for all atomic $\varphi(\overline{x}) \in \mu^*$ with length$(\overline{x}) = n - j$.
(2) If $(\overline{a}, \overline{b}) \in I_{j+1}$, and $a \in A$ $(b \in B)$ then there is $b \in B$ $(a \in A)$ such that $(\overline{a}a, \overline{b}b) \in I_j$.

In the next definition let $\overline{v}_j$ denote a sequence of j many variables and $\overline{v}_0$ the empty sequence of variables. Fix N such that $\frac{1}{3^N} \leq \varepsilon$ and define for $j \leq n$

$$I_j = \{(\overline{a}, \overline{b}) \in A^{n-j} \times B^{n-j} : |\varphi^{\mathfrak{A}}[\overline{a}] - \varphi^{\mathfrak{B}}[\overline{b}]| \leq \frac{1}{3^{N+j}} \text{ for all } \varphi \in \text{Ł}\forall^{3^{N+j+1}}_{\mu,\overline{v}_{n-j}}\}.$$

Then (1) holds by definition. To prove the extension property (2), assume $(\overline{a}, \overline{b}) \in I_{j+1}$, $j < n$, and given $a \in A$ consider the formula

$$Th_j(\overline{x}, v) := \bigwedge_i (\varphi_i(\overline{x}v)_{\geq \frac{k_i}{3^{N+j+1}}} \wedge \varphi_i(\overline{x}v)_{\leq \frac{k_i+1}{3^{N+j+1}}})$$

where $\{\varphi_i(\overline{x}v)\}_i$ is a finite list of equivalence representatives of the formulas in $\text{Ł}\forall^{3^{N+j+1}}_{\mu^*,\overline{v}_{n-j}}$, and k_i is chosen so that $\varphi_i^{\mathfrak{A}}(\overline{a}a) \in [\frac{k_i}{3^{N+j+1}}, \frac{k_i+1}{3^{N+j+1}}]$. Clearly $\exists v Th_j(\overline{x}v)$ has rank $2 \cdot 3^{N+j+1} + 1 \leq 3^{N+j+2}$, (belongs to $\text{Ł}\forall^{3^{N+j+2}}_{\mu^*,\overline{v}_{n-(j+1)}}$) and since $(\overline{a}, \overline{b}) \in I_{j+1}$ then

$$|\exists v Th_j(\overline{a}v)^{\mathfrak{A}} - \exists v Th_j(\overline{b}v)^{\mathfrak{B}}| \leq \frac{1}{3^{N+j+1}}.$$

As $A \models \exists v Th_j(\overline{a}v)$ by construction, then $\exists v Th_j(\overline{b}v)^B \geq 1 - \frac{1}{3^{N+j+1}}$ and we may find $b \in B$ such that $Th_j^{\mathfrak{B}}[\overline{b}b] \geq 1 - \frac{2}{3^{N+j+1}}$, thus for each φ_i

$$[\varphi_i[\overline{b}b]_{\geq \frac{k_i}{3^{N+j+1}}}], \ [\varphi_i[\overline{b}b]_{\leq \frac{k_i+1}{3^{N+j+1}}}]^{\mathfrak{B}} \geq Th_j^{\mathfrak{B}}[\overline{b}b] \geq 1 - \frac{2}{3^{N+j+1}},$$

and by Lemma 2.1: $\frac{k_i}{3^{N+j+1}} - \frac{2}{3^{N+j+1}} \leq \varphi_i^{\mathfrak{B}}[\overline{b}b] \leq \frac{k_i+1}{3^{N+j+1}} + \frac{2}{3^{N+j+1}}$. As $\varphi_i^{\mathfrak{A}}[\overline{a}a] \in [\frac{k_i}{3^{N+j+1}}, \frac{k_i+1}{3^{N+j+1}}]$ by construction, this means $|\varphi_i^{\mathfrak{A}}[\overline{a}a] - \varphi_i^{\mathfrak{B}}[\overline{b}b]| \leq 3\frac{1}{3^{N+j+1}} = \frac{1}{3^{N+j}}$. Since this is true for each $\varphi \in \text{Ł}\forall^{3^{N+j+1}}_{\mu,\overline{v}}$ we conclude that $(\overline{a}a, \overline{b}b) \in I_j$. The other direction is similar. To finish, notice that $(\Lambda, \Lambda) \in I_n = A^0 \times B^0$ because $\mathfrak{A} \equiv^{3^{N+n}}_{\text{Ł}\forall(\mu)} \mathfrak{B}$. Hence, by the extension property all I_j are nonempty. □

We prove the reciprocal for CL$^\circ$ utilizing the simpler rank notion $r(c(\theta_1,..,\theta_k)) = 1 + \max_i r(\theta_i)$ for all the continuous connectives. $\mathfrak{A} \equiv^n_{\mathrm{CL}^\circ(\mu)} \mathfrak{B}$ will denote equivalence of A and B with respect to μ-sentences of CL$^\circ$ of rank r at most n.

Lemma 3.3. *Let $n \in \omega$ and $\mu \subseteq \tau$ be finite. If there is for each $\varepsilon > 0$ a sequence $I^\varepsilon_0, ..., I^\varepsilon_n$ of nonempty sets of partial $\mu\varepsilon$-approximations with the extension property between continuous structures $\mathfrak{A}$ and $\mathfrak{B}$ as in Lemma 3.2, then $\mathfrak{A} \equiv^n_{CL^\circ(\mu)} \mathfrak{B}$.*

Proof. Fix μ, n and let $I^\varepsilon_0, ..., I^\varepsilon_n$ be given for each $\varepsilon > 0$ with the described properties. Write $(\overline{a}, \overline{b}) \in^* I^\varepsilon_j$ to mean that there is $R \in I^\varepsilon_j$ such that $(a_i, b_i) \in R$ for $i = 1, .., s$. Since $I^\varepsilon_j \neq \emptyset$, we have always $(\Lambda, \Lambda) \in^* I^\varepsilon_j$. We will show by induction in the rank $r(\varphi)$ of $\varphi(\overline{x}) \in \mathrm{CL}^{\circ n}_\mu$ that for any $\varepsilon > 0$ there is $\delta \leq \varepsilon$ such that for all $\overline{a}, \overline{b}$ of the same length $\geq$ length$(\overline{x})$:

$$r(\varphi) \leq j,\ \rho \leq \delta \text{ and } (\overline{a}, \overline{b}) \in^* I^\rho_j \text{ imply } |\varphi^{\mathfrak{A}}[\overline{a}] - \varphi^{\mathfrak{B}}[\overline{b}]| \leq \varepsilon.$$

- For φ atomic of rank ≤ 1 take $\delta = \varepsilon$, the result follows by definition of partial $\mu\rho$-approximation because $\varphi \in \mu^*$.

- For φ atomic of rank ≥ 2, $\varphi := \psi(t_1(\overline{x}), ..t_k(\overline{x}), \overline{x})$ where ψ is atomic of rank 1 and at least one t_i has positive rank; thus $r(\varphi) = \Sigma_i r(t_i) + 1 \geq k+1$. As the atomic formula $\gamma_i := d(t_i(x), y)$ has rank $r(t_i) < r(\varphi)$, there is δ_i satisfying the induction hypothesis for γ_i and $m_\psi(\frac{\varepsilon}{2})$ instead of ε, where m_ψ is the continuity modulus of ψ. Define $\delta = \min_i \delta_i$ and assume: $r(\varphi) \leq j$, $\rho \leq \delta$, and $(\overline{a}, \overline{b}) \in^* I^\rho_j$. Then $k < r(\varphi) \leq j$ and applying the extension property k times we may find $b_1, ...b_k$ such that $(\overline{a} t_1[\overline{a}]...t_k[\overline{a}], \overline{b} b_1...b_k) \in^* I^\rho_{j-k}$. Since $\psi \in \mu^*$, this implies

$$|\psi^{\mathfrak{A}}(t_1[\overline{a}], .., t_k[\overline{a}]) - \psi^{\mathfrak{B}}(b_1, .., b_k)| \leq \rho \leq \delta \leq \delta_{\gamma_1} \leq m_\psi(\frac{\varepsilon}{2}) \leq \frac{\varepsilon}{2}.$$

On the other hand, as $r(\gamma_i) = r(t_i) \leq \Sigma_i(r(t_i) - 1) + 1 \leq j - k$, $\rho \leq \delta \leq \delta_i$ and $(\overline{a}...t_i[\overline{a}].., \overline{b}...b_i..) \in^* I^\rho_{j-k}$, we may conclude by the properties of δ_i that $|d(t_i[\overline{a}], t_i[\overline{a}]) - d(t_i[\overline{b}], b_i)| \leq m_\psi(\frac{\varepsilon}{2})$; hence, $d(t_i[\overline{b}], b_i) \leq m_\psi(\frac{\varepsilon}{2})$ for all i and thus:

$$|\psi^{\mathfrak{B}}(t_1[\overline{b}], .., t_k[\overline{a}]) - \psi^{\mathfrak{B}}(b_1, .., b_k)| \leq \frac{\varepsilon}{2}.$$

Putting all together, $|\psi^{\mathfrak{A}}(t_1[\overline{a}], ..) - \psi^{\mathfrak{B}}(t_1[\overline{b}], .., t_k[\overline{a}])| \leq \varepsilon$.

- If φ has the form $\exists v\theta$ define $\delta_\varphi(\varepsilon) = \delta_\theta(\varepsilon)$. If $(\overline{a}, \overline{b}) \in^* I^\rho_j$ with $j \geq r(\varphi) \geq 1$, $\rho \leq \delta_\varphi(\varepsilon)$ then for each $a \in A$ there is $b \in B$ such that $(\overline{a}a, \overline{b}b) \in I^{n\rho}_{j-1}$; hence $|\theta^{\mathfrak{A}}[\overline{a}a] - \theta^{\mathfrak{B}}[\overline{b}b]| \leq \varepsilon$ by induction hypothesis. Taking suprema, first over a and then over b, $|(\exists v\rho)^{\mathfrak{A}}[\overline{a}] - (\exists v\rho)^{\mathfrak{B}}[\overline{b}]| \leq \varepsilon$.

- If φ has the form $c(\theta_1, .., \theta_k)$ where c is a continuous connective, each θ_i has rank $< r(\varphi)$ and thus $\delta_{\theta_i}(\varepsilon)$ exists. Let m_c be the uniform continuity modulus of c and define $\delta_\varphi(\varepsilon) = \min_i \delta_{\theta_i}(m_c(\varepsilon))$. If $(\overline{a}, \overline{b}) \in^* I^{n\rho}_j$ with $j \geq r(\varphi)$, $\rho \leq \delta_\varphi(\varepsilon)$ then $r(\theta_i) \leq j$ and $\rho \leq \delta_{\theta_i}(m_c(\varepsilon))$; hence, $|{\theta_i}^{\mathfrak{A}}[\overline{a}] - {\theta_i}^{\mathfrak{B}}[\overline{b}]| \leq m_c(\varepsilon)$ by induction hypothesis, and thus $|c(\theta_1, .., \theta_n)^{\mathfrak{A}}[\overline{a}] - c(\theta_1, .., \theta_n)^{\mathfrak{B}}[\overline{b}]| \leq \varepsilon$.

- Finally, given a sentence $\varphi \in \mathrm{CL}^{\circ n}_\mu$, pick any $(\overline{a}, \overline{b}) \in^* I^{\delta_\varphi(\varepsilon)}_n$, then $|\varphi^{\mathfrak{A}} - \varphi^{\mathfrak{B}}| = |\varphi^{\mathfrak{A}}[\overline{a}] - \varphi^{\mathfrak{B}}[\overline{b}]| \leq \varepsilon$. As ε is arbitrary, $|\varphi^{\mathfrak{A}} - \varphi^{\mathfrak{B}}| = 0$. □

Remark. The complexity of the proof of Lemma 3.3 in the atomic case is due to the presence of function symbols. For purely relational signatures, the proof would be simpler.

Corollary 3.4. *If* $\mathfrak{A}$ *and* $\mathfrak{B}$ *are continuous structures, then* $\mathfrak{A} \equiv_{Ł\forall} \mathfrak{B}$ *implies* $\mathfrak{A} \equiv_{CL} \mathfrak{B}$.

Proof. From lemmas 3.2 and 3.3 we have that $\mathfrak{A} \equiv_{Ł\forall} \mathfrak{B}$ implies $\mathfrak{A} \equiv_{\mathrm{CL}^\circ} \mathfrak{B}$. Since any φ sentence of CL may be approximated uniformly by sentences φ_n of $Ł\forall(\mathbb{Q}) \subseteq \mathrm{CL}^\circ$ we then have $|\varphi^{\mathfrak{A}} - \varphi^{\mathfrak{B}}| = |\lim \varphi_n^{\mathfrak{A}} - \lim \varphi_n^{\mathfrak{B}}| = \lim |\varphi_n^{\mathfrak{A}} - \varphi_n^{\mathfrak{B}}| = 0$. □

From this corollary and Proposition 2.6 it follows by a topological argument (Proposition 2.4 in [9], Proposition 4.4 in the next section) that Ł∀ and CL have the same axiomatizability strength in any equicontinuous class of structures.

4. General [0, 1]-valued logics

A [0,1]-*valued logic* is a triple $L = (L, \overline{St}, V)$ where L and $\overline{St}$ are applications assigning to each signature τ a set of sentences L_τ, and a subclass $\overline{St}_\tau \subseteq St_\tau$, respectively, and $V : L_\tau \times \overline{St}_\tau \to [0, 1]$ is a semantical map subject to the following axioms where we write $\varphi^{\mathfrak{A}}$ for $V(\varphi, \mathfrak{A})$:

- *Isomorphism.* $\mathfrak{A} \simeq \mathfrak{B}$ implies $\varphi^{\mathfrak{A}} = \varphi^{\mathfrak{B}}$
- *Reducts.* If $\tau \subseteq \mu$ then $L_\tau \subseteq L_\mu$ and $\varphi^{\mathfrak{A}} = \varphi^{\mathfrak{A}}$ for any $\varphi \in L_\tau$, $\mathfrak{A} \in \overline{St}_\mu$.
- *Renaming.* A bijection $\alpha : \tau \to \mu$ preserving type of symbols induces a translation $\widehat{\alpha} : L_\tau \to L_\mu$ such that $(\widehat{\alpha}\varphi)^{\mathfrak{A}} = \varphi^{\mathfrak{A}^\alpha}$ for any $\mathfrak{A} \in \overline{St}(L)_\mu$, and $\varphi \in L_\tau$, where $\mathfrak{A}^\alpha \in \overline{St}(L)_\tau$ is the renaming of $\mathfrak{A}$ via α.

This is a slight reformulation of Definition 1.10 of a [0,1]-valued logic in [9] which paraphrases Lindström definition of model theoretic logic in [24]. In the translation axiom α may permute sorts, which means sentences may be duplicated in disjoint sort systems.

For our purposes $\overline{St}_\tau = \cup_S St_{\tau,S}$ (or $\overline{St}_\tau = \cup_S St^c_{\tau,S}$) the class of continuous (complete) structures, if τ has distinguished distance predicates, and $\overline{St}_\tau$ is empty otherwise.

All the model theoretic concepts and properties we have considered so far for CL on these classes may be applied to L, particularly compactness in the classes $St_{\tau,S}$ and the (separable) Löwenheim-Skolem property. In addition, we will utilize the following closure property:

Definition 4.1. L will be Ł-*regular* if the collection of maps $F_\varphi : \overline{St} \to [0, 1]$, $\mathfrak{A} \mapsto \varphi^{\mathfrak{A}}$, for all $\varphi \in L_\tau$, is closed under composition with Łukasiewicz connectives $\to$, $\neg$.

There are at least two ways to compare [0,1]-valued logics:

- *Full strength*: $L \leq L'$ if for each sentence $\varphi \in L_\tau$ there is a sentence $\psi \in L'_\tau$ such that $\varphi^{\mathfrak{A}} = \psi^{\mathfrak{A}}$ for any $\mathfrak{A} \in \overline{St}_\tau$.
- *Axiomatic strength*: $L \leq_{ax} L'$ if for each sentence $\varphi \in L_\tau$ there is a theory $T \subseteq L'_\tau$ of power such that $Mod_\tau(\varphi) = Mod_\tau(T)$.

The second relation is weaker than the first even if T is a sentence. Natural (full) extensions of CL are:

By connectives. These must be discontinuous which necessarily destroy compactness. Indeed, if $c : [0, 1] \to [0, 1]$ and there is $x_n \to x$ in $[0, 1]$ with $\liminf_n c(x_n) < c(x)$, there are rationals r, s such that $c(x_n) \leq r < s \leq c(x)$ for infinitely many

$n \in \omega$, then the theory $T = \{c(p)_{\leq r},\ c(q)_{\geq s},\ (p \leftrightarrow q)_{\geq n/n+1} : n \in \omega\}$ is finitely satisfiable but unsatisfiable.

Infinitary continuous logic, $\mathrm{CL}_{\omega_1\omega}$, introduced by Ben-Yaacov and Iovino in [4], allows countable conjunctions $\bigwedge \varphi_n(\overline{x})$ when the φ_n share the same uniform convergence modulus. Compactness is lost but formulas stay continuous with respect to the metric and it shares the good behavior of classical $L_{\omega_1\omega}$, see [7]. Infinitary logic on continuous structures without restriction on the countable conjunctions has been considered by Eagle [14].

Second order continuous logic, CL^{II}. Allow formulas $\exists X_m \varphi(X)$, $\forall X_m \varphi(X)$ where X_m is an n-ary predicate variable X with a specified moduli of continuity m, with the interpretation:

$$\exists X_m \varphi(X)^{\mathfrak{A}} = \sup_{f \in U\mathcal{C}_m(A^n,[0,1])} \varphi(X)^{(\mathfrak{A},f)},$$

where $U\mathcal{C}_m(A^n,[0,1])$ is the set of functions $f : A^n \to [0,1]$ with uniform continuity modulus m.

Second order existential continuous logic $\Sigma^1_1(\mathrm{CL})$. The existential fragment of CL^{II}. It is not difficult to show that this logic inherits from CL compactness, the (separable) downward Löwenheim-Skolem property, the witnessed model property, and the complete model property.

By quantifiers. Further extensions may be obtained adding generalized quantifiers. For example, if κ is a cardinal:

$W_\kappa x \varphi(x)^{\mathfrak{A}} = \sup\{\varepsilon : (\varphi^{\mathfrak{A}})^{-1}([0,\varepsilon]) \text{ has weight } \geq \kappa\}$.
$I x \varphi(x)^{\mathfrak{A}} = \sup\{\varepsilon : \varphi^{\mathfrak{A}}(A) \supseteq [0,\varepsilon]\}$.

4.1. **A topological view.** The *L-topology* of a [0,1]-valued logic L is the one generated in each $\overline{St}_\tau$ by the classes $Mod(\theta)$, $\theta \in L_\tau$, as a sub-basis of closed classes. We denote this topology $\Gamma_\tau(L)$. The spaces $(\overline{St}_\tau, \Gamma_\tau(L))$ are proper classes but $\Gamma_\tau(L)$ is parametrized by sets (of sets of sentences) which permits to use safely all topological concepts (cf. [9], [8]).

Lemma 4.2. (i) *$\Gamma_\tau(L)$ is invariant under isomorphisms.*

(ii) *If the logic is closed under disjunctions, then the closed sub-basis is a basis and the closed classes are exactly the axiomatizable classes: $Mod(T)$, $T \subseteq L_\tau$.*

(iii) *A Ł-regular logic is topologically regular; that is, points and closed classes are separable by open classes in $(St_\tau, \Gamma_\tau(L))$. These spaces are in fact completely regular, and thus uniformizable, having for uniformity basis the relations*

$$U_{\varepsilon,\varphi}(A,B) \text{ iff } |\varphi^{\mathfrak{A}} - \varphi^{\mathfrak{B}}| \leq \varepsilon, \quad \varphi \in L, \varepsilon > 0.$$

(iv) *Model theoretic compactness of L is equivalent to topological compactness of the spaces $(St_\tau, \Gamma_\tau(L))$.*

(v) $L \leq_{ax} L'$ if and only if $\Gamma_\tau(L')$ is finer than $\Gamma_\tau(L)$ for any τ.

Proof. (i), (ii), and (v) are immediate, (iv) follows by Alexander sub-basis lemma. For (iii) notice that a Ł-regular logic must be closed under disjunctions and the connectives $(\ )_{\leq r}$, $(\ )_{\geq r}$. Moreover, $\mathfrak{A} \notin C = Mod(T)$ implies $\varphi^{\mathfrak{A}} < r < s < 1$ for some rationals r, s. and thus the complements in St_τ of the class $Mod(\varphi_{\geq r})$ and $Mod(\varphi_{\leq s})$ separate $\mathfrak{A}$ and C. □

Define in any topological space X the equivalence relation $x \equiv y$ if and only if $cl\{x\} = cl\{y\}$, where cl denotes topological adherence. Thus, $x \equiv y$ if and only if x and y belong to the same closed (open) subsets of X. It is a topology exercise to show that if X is a regular space. then the quotient space $X_{/\equiv}$ is Hausdorff. The next lemma will be useful in the proof of the maximality theorems.

Lemma 4.3. *If K_1 and K_2 are disjoint compact subclasses of a regular topological space X, which cannot be separated by a finite intersection of basic closed sets of a given basis, then there exist $x_i \in K_i$, $i = 1, 2$, such that $x_1 \equiv x_2$.*

Proof. Let $\eta : X \to X_{/\equiv}$ be the quotient map. By continuity, the images ηK_1 and ηK_2 are compact in $X_{/\equiv}$ and thus closed. They cannot be disjoint; otherwise, their inverse images would be disjoint closed sets separating K_1 and K_2, and by a compactness argument again the separation could be achieved by a finite intersection of basic sets. Pick $x_i \in K_i$, $i = 1, 2$, with $\eta x_1 = \eta x_2 \in \eta K_1 \cap \eta K_2$, then $x_1 \equiv x_2$. □

It should be clear that the topological relation $\equiv$ in $(St_\tau, \Gamma_\tau(L))$ coincides with the logical relation $\equiv_L$. We have then,

Proposition 4.4. *Let L and L' be two Ł-regular [0,1]-valued logics on continuous structures with $L \leq L'$ and L' compact in each $St_{\tau,S}$. If $\mathfrak{A} \equiv_L \mathfrak{B}$ implies $\mathfrak{A} \equiv_{L'} \mathfrak{B}$ for any pair of models, then $L \sim_{ax} L'$ in any $St_{\tau,S}$. More precisely, any $\varphi \in L'_\tau$ has the same models in $St_{\tau,S}$ as a countable theory of L.*

Proof. Let $\varphi \in L'$ and consider $\varphi_{\leq r}$ for rational $r \in (0, 1)$, then $Mod(\varphi)$ and $Mod(\varphi_{\leq r})$ are disjoint compact subclasses of $St_{\tau,S}$ in the L'-topology, a fortiori in the L-topology. Then they must be separated by an intersection of basic closed sets of the later topology, otherwise by Lemma 4.3 there should exist S-models $\mathfrak{A} \models \varphi$, $\mathfrak{B} \models \varphi_{\leq r}$ with $\mathfrak{A} \equiv_L \mathfrak{B}$ and thus $\mathfrak{A} \equiv_{L'} \mathfrak{B}$, but the last equivalence is impossible because φ, $\varphi_{\leq r} \in L'$. As L is closed under disjunctions, the separating class has the form $C_r = Mod(F_r)$ for a finite theory F_r. As $Mod(\varphi) \subseteq C_r$ and $Mod(\varphi_{\leq r}) \cap C_r = \emptyset$ for each $r < 1$, then $Mod(\varphi) = \cap_r C_r = Mod(\cup_r F_r)$. □

5. Compact extensions and approximations

We prove in this section several properties of compact extensions of Ł∀ on continuous complete structures utilizing a simple and robust notion of approximation. We prove in particular that arbitrarily approximable model-classes of the logic are jointly satisfiable, approximable complete structures are equivalent in the logic, and each sentence of the logic has countable dependence number in any equicontinuous class of complete structures.

Definition 5.1. Given τ-structures $\mathfrak{A}$ and $\mathfrak{B}$, a subsignature $\mu \subseteq \tau$, and $\varepsilon \in (0, 1)$, a relation $R \subseteq A \times B$ is a *$\mu\varepsilon$-approximation* between $\mathfrak{A}$ and $\mathfrak{B}$ if

i) $|\varphi^{\mathfrak{A}}[\overline{a}] - \varphi^{\mathfrak{B}}[\overline{b}]| \leq \varepsilon$ for all $\varphi \in \mu^*$ and all suitable $\overline{a}, \overline{b}$ such that $(a_i, b_i) \in R$.
ii) $d^{\mathfrak{A}}(a, domR) \leq \varepsilon$ and $d^{\mathfrak{B}}(b, rangR) \leq \varepsilon$ for all $a \in A$, and $b \in B$.

We will write $\mathfrak{A} \simeq_{\mu\varepsilon} \mathfrak{B}$ ($R : \mathfrak{A} \simeq_{\mu\varepsilon} \mathfrak{B}$) to indicate that there is (R is) such an approximation. Notice that a $\mu\varepsilon$-approximation between crisp structures is an ordinary μ-isomorphism. The next lemma shows that if we have $\mu\varepsilon$-approximations for arbitrarily small ε, then we may assume that these are total functions having an inverse "up to ε".

Lemma 5.2. *Given τ-structures $\mathfrak{A}$, $\mathfrak{B}$, a finite $\mu \subseteq \tau$, and $\varepsilon > 0$, there is $\rho = \rho(\mu, \varepsilon) > 0$ such that if $\mathfrak{A} \approx_{\mu,\rho} \mathfrak{B}$, then there are maps $h : \mathfrak{A} \to \mathfrak{B}$ and $g : \mathfrak{B} \to \mathfrak{A}$ such that*

(1) $|\varphi^{\mathfrak{A}}[\overline{a}] - \varphi^{\mathfrak{B}}[h(a_1), .., h(a_k)]| \leq \varepsilon$ for all $\varphi \in \mu$ and $\overline{a}$ in A
(2) $d^{\mathfrak{B}}[h(f^{\mathfrak{A}}(\overline{a})), f^{\mathfrak{B}}(h(a_1), .., h(a_k)))] \leq \varepsilon$ for all $f \in \mu$ and $\overline{a}$ in A,
and similarly for g. Moreover,
(3) $d^{\mathfrak{A}}(a, gh(a)) \leq \varepsilon$ and $d^{\mathfrak{B}}(hg(b), b) \leq \varepsilon$ for any $a \in A$, $b \in B$.

Proof. Let $\rho = \min\{m_{\varphi^{\mathfrak{A}}}(\varepsilon/3), m_{\varphi^{\mathfrak{B}}}(\varepsilon/3) : \varphi \in \mu^*\}$ and assume $R : A \approx_{\mu,\rho} B$. We show first that there is $R' : A \simeq_{\mu,\varepsilon} B$ with $domR' = A$ and $rangR' = B$. Define $(a', b') \in R'$ if and only if there is $(a, b) \in R$ such that $d(a', a)$, $d(b', b) \leq \rho(\mu, \varepsilon)$. Then $(a_i', b_i') \in R'$ implies for any $\varphi \in \mu^* : |\varphi^{\mathfrak{A}}\overline{a}' - \varphi^{\mathfrak{B}}\overline{b}'| \leq |\varphi^{\mathfrak{A}}\overline{a}' - \varphi^{\mathfrak{A}}\overline{a}|$ + $|\varphi^{\mathfrak{A}}\overline{a} - \varphi^{\mathfrak{B}}\overline{b}|$ + $|\varphi^{\mathfrak{B}}\overline{b} - \varphi^{\mathfrak{B}}\overline{b}'| \leq \varepsilon/3 + \rho + \varepsilon/3 \leq \varepsilon$; the first and last bounds follow by uniform continuity of φ in A and B, the second the defining properties of R. Moreover, for any $a' \in A$ there is $(a, b) \in R_{\mu,\rho}$ such that $d(a', a) \leq \rho$ by condition (ii), then $(a', b) \in R'$, showing that $domR'$ is all of A, similarly, its image is all of B. Now let $h : A \to B$, $g : B \to A$ be choice functions for R' and R'^{-1}, respectively. Then (1) follows by construction, and (2) follows applying (1) to the formula $d(f(\overline{x}), y) \in \mu^*$ to obtain $|d^{\mathfrak{A}}(f(\overline{a}), a) - d^{\mathfrak{B}}(f(h(a_1), .., h(a_k)), h(a))| \leq \varepsilon$, and then making $a := f(\overline{a})$. For (3) observe that $(a, h(a))$, $(g(b), b) \in R'$, then $|d^{\mathfrak{A}}(a, g(b)) - d^{\mathfrak{B}}(h(a), b)| \leq \varepsilon$. Making $b := h(a)$ we obtain $d^{\mathfrak{A}}(a, g(h(a))) \leq \varepsilon$ and making $a := g(b)$yields $d^{\mathfrak{B}}(h(g(b)), b) \leq \varepsilon$. □

Proposition 5.3. ***(joint consistency)*** *Let T_1, T_2 be τ-theories of a compact extension of Ł∀ on complete continuous structures. If for each finite $\mu \subseteq \tau$ and $\varepsilon > 0$ there are equicontinuous families $\{\mathfrak{A}_{\mu,\varepsilon}\}_{\mu,\varepsilon}$ and $\{\mathfrak{B}_{\mu,\varepsilon}\}_{\mu,\varepsilon}$ of complete structures such that $\mathfrak{A}_{\mu,\varepsilon} \simeq_{\mu,\varepsilon} \mathfrak{B}_{\mu,\varepsilon}$ and $\mathfrak{A}_{\mu,\varepsilon} \models T_1$, $\mathfrak{B}_{\mu,\varepsilon} \models T_2$, then $T_1 \cup T_2$ is satisfiable.*

Proof. By Lemma 5.2, we may assume the approximations are of the form $h_{\varepsilon,\mu} : A_{\mu,\varepsilon} \to B_{\mu,\varepsilon}$ with a right ε-inverse $g_{\varepsilon,\mu} : B_{\mu,\varepsilon} \to A_{\mu,\varepsilon}$ The obvious idea of using compactness to transform them in an isomorphism $h : \mathfrak{A} \to \mathfrak{B}$ with $\mathfrak{A} \models T_1$, $\mathfrak{B} \models T_2$ does not work directly because the $h_{\varepsilon,\mu}$, $g_{\varepsilon,\mu}$ are not necessarily uniformly continuous. To circumvent this problem, we consider diagrams of continuous structures of the form:

$$\begin{array}{ccc} (A^\circ, \Delta_{A^\circ}, D_{A^\circ}, ...) & \underset{g}{\overset{h}{\rightleftarrows}} & (B^\circ, \Delta_{B^\circ}, D_{B^\circ}, ...) \\ \downarrow \pi_1 & & \downarrow \pi_2 \\ (A, d_A, ..) & & (B, d_B, ..) \end{array}$$

where

- A and B have signature τ and distinguished metrics d_A, d_B; moreover, and $A \models T_1$, $B \models T_2$.
- A° and B° have type $\tau \cup \{D\}$, where D_{A°, D_{B° are pseudometrics, and the discrete metrics Δ_{A°, Δ_{B° as distinguished metrics.
- π_1 is a map with dense image in A, preserving the truth value of the predicates in τ (that is, $\varphi^A(\pi_1 a_1, .., \pi_k a_k) = \varphi^{A_\circ}(\pi_1, .., \pi_k))$ and the operations in τ (that is, $\pi_1 f^{A_\circ}(a_1, .., a_k) = f^A(\pi_1 a_1, .., \pi_k a_k))$, but not preserving the distinguished metrics and sending instead the pseudometric D_{A° to the metric d_A (that is, $d_A(\pi_1 a, \pi_2 b) = D_{A^\circ}(a, b))$. Similarly for π_2.

- h preserves the value of the predicates in $\tau\cup\{D\}$ but it is not asked to preserve Δ (that is, it is not asked to be one to one) and it preserves the operations in τ only up to D; that is, $D(hf^{A\circ}(a_1,..,a_k), f^{B^\circ}(ha_1,..,ha_k)) = 0$.

- g is a left inverse of h up to D; that is, it preserves the value of D and $D(gh(a)), a) = 0$.

These diagrams may be taken as many-sorted structures or coded as single-sorted structures $([A^\circ, B^\circ, A, B], ...)$. In any case, the properties described above may be axiomatized by the following theory, where the predicate superscripts P_1, P_2, P_1°, P_2° specify the sorts of the variables and formulas in the many-sorted version, or denote relativizations in the single sorted version. For convenience, the properties of h are not stated sharply but by a sequence of approximations. For the sake of clearness, we use sometimes the identity $\approx$ associated to the distinguished metric in the sorts P_i:

For $i = 1, 2$:
A1 γ^{P_i} for all $\gamma \in T_i$
A2 $\forall x^{P_i^\circ} y^{P_i^\circ} z^{P_i^\circ} (\neg D(x,x)\wedge(D(x,y) \to D(y,x))\wedge(D(x,y)\odot D(y,z) \to D(x,z)))$
A3 $\forall y^{P_i} \exists x^{P_i^\circ} (\pi_i(x) \approx y)$
A4 $\forall \overline{x}^{P_i^\circ} (\varphi^{P_i}(\pi_i(x_1)...\pi_i(x_k)) \leftrightarrow \varphi^{P_i^\circ}(\overline{x}))$ for each predicate symbol $\varphi \in \tau$
A5 $\forall \overline{x}^{P_i^\circ} (f^{P_i}(\pi_i(x_1)...\pi_i(x_k)) \approx \pi_i(f^{P_i^\circ}(\overline{x})))$ for each function symbol $f \in \tau$
A6 $\forall x^{P_i^\circ} y^{P_i^\circ} (d^{P_i}(\pi_i(x), \pi_i(y))) \leftrightarrow D^{P_i^\circ}(x,y))$
For each n :
A7 $\forall \overline{x}^{P_1^\circ} [\varphi^{P_2^\circ}(h(x_1),..,h(x_k)) \leftrightarrow \varphi^{P_1^\circ}(\overline{x})]_{\geq 1-\frac{1}{n}}$ for each predicate symbol $\varphi \in \tau \cup \{D\}$
A8 $\forall \overline{x}^{P_1^\circ} [D^{P_2^\circ}(h(f^{P_1^\circ}(\overline{x})), f^{P_2^\circ}(h(\overline{x})))]_{\leq \frac{1}{n}}$ for each function symbol $f \in \tau$
A9 $\forall x^{P_2^\circ} y^{P_2^\circ} [D^{P_1^\circ}(g(x), g(y)) \leftrightarrow D^{P_2^\circ}(x,y)]_{\geq 1-\frac{1}{n}}$
A10 $\forall y^{P_2^\circ} [D(h(g(y)), y)]_{\leq \frac{1}{n}}$.

Given a finite part Σ of this theory, where μ is the subsignature of τ occurring in Σ and ε the minimum $\frac{1}{n}$ occurring, then the structure $C_{\mu,\varepsilon}$

$$\begin{array}{ccc} A^\circ_{\mu,\varepsilon} & \overset{h_{\mu,\varepsilon}}{\underset{g_{\mu,\varepsilon}}{\rightleftarrows}} & B^\circ_{\mu,\varepsilon} \\ \downarrow \pi_1 & & \downarrow \pi_2 \\ A_{\mu,\varepsilon} & & B_{\mu,\varepsilon} \end{array}$$

where $A^\circ_{\mu,\varepsilon}$ $(B^\circ_{\mu,\varepsilon})$ is a copy of $A_{\mu,\varepsilon}$ $(B_{\mu,\varepsilon})$ with the discrete metric Δ as distinguished metric, D interpreted by d, and π_1, π_2 the identity functions, is a model of Σ. Axioms A1-A6 are satisfied trivially, axioms A7, A8 are satisfied by properties of functional $\mu\varepsilon$-approximations, and A9, A10 by Lemma 5.2.

The predicates and operations in $A^\circ_{\mu,\varepsilon}$, $B^\circ_{\mu,\varepsilon}$, and the maps $h_{\varepsilon,\mu}$, $g_{\mu,\varepsilon}$, π_1, π_2 are trivially uniformly continuous with respect to Δ, say with Lipschitz constant $\frac{1}{2}$, and the structures $A_{\mu,\varepsilon}$, $B_{\mu,\varepsilon}$ are equicontinuous by hypothesis. Hence, the $C_{\mu,\varepsilon}$ are equicontinuous and thus we may apply compactness to obtain a model C of T. Given such a model, and referring to the first diagram, Δ_{A°, Δ_{B°, d_A and d_B are metrics, D_{A° and D_{B° are pseudometrics by A2 but might not be metrics, the images of π_1 and π_2 are dense substructures of A and B, respectively, by axiom A3.

Let $x \sim_D y$ iff $D(x,y) = 0$ be the similarity induced by the pseudometric D in A° and B°, then we have $d(\pi_i(x), \pi_i(y)) = D(x,y)$ by A6. In particular, since d is a metric,

$$\pi_i(x) = \pi_i(y) \text{ if and only if } x \sim_D y.$$

Moreover, we have $D^{A^\circ}(x, y) = D^{B^\circ}(h(x), h(y))$ by A7, and thus

$$x \sim_D y \text{ if and only if } h(x) \sim_D h(y).$$

This means that the map $\widehat{h} : A \to B$ defined as $\widehat{h}(\pi_1(x)) := \pi_2 h(x)$ is a well-defined bijection between the images of π_1 and π_2, since

$$\pi_1(x) = \pi_1(y) \Leftrightarrow x \sim_D y \Leftrightarrow h(x) \sim_D h(y) \Leftrightarrow \pi_2 h(x)) = \pi_2 h(y).$$

It is actually an isomorphism because by A6 and A7 applied to D: $d^A(\pi_1(x), \pi_1(y)) \overset{(A6)}{=} D^{A^\circ}(x, y) \overset{(A7)}{=} D^{B^\circ}(h(x), h(y)) \overset{(A6)}{=} d^B(\pi_2 h(x), \pi_2 h(y)) \overset{def}{=} d_B(\widehat{h}(\pi_1(x)), \widehat{h}(\pi_1(y)))$. In particular, h is uniformly continuous. Moreover, by A4 and A7, for any relation symbol $\varphi \in \tau$: $\varphi^{\mathfrak{A}}(p_1(x), ..) \overset{(A4)}{=} \varphi^{A^\circ}(x, ..) \overset{(A7)}{=} \varphi^{B^\circ}(h(x), ..) \overset{(A4)}{=} \varphi^{\mathfrak{B}}(p_2(h(x), ..) \overset{def}{=} \varphi^{\mathfrak{B}}(\widehat{h}(p_1(x)), ..)$; and by A8, $h(f^{A^\circ}(x, ..))) \sim_D f^{B^\circ}(h(x), ..)$ for any function symbol $f \in \tau$; thus $\widehat{h}(f^A(\pi_1(x), ..)) \overset{(A5)}{=} \widehat{h}(\pi_1(f^{A^\circ}(x, ..)) \overset{def}{=} \pi_2 h(f^{A^\circ}(x, ..)) = \pi_2(f^{B^\circ}(h(x), ..)) \overset{(A5)}{=} f^B(\pi_2 h(x)), ..) \overset{def}{=} f^B(\widehat{h}(\pi_1(x)), ..)$.

Moreover, $\widehat{h} : \pi_1(A^\circ) \to \pi_2(B^\circ)$ is surjective because g may be lifted similarly to $\widehat{g} : \pi_2(B^\circ) \to \pi_1(A^\circ)$ as $\widehat{g}(\pi_2(x)) = \pi_1 g(x)$ by A9, and A10 plus A6 imply $\pi_2(y) = \pi_2(h(g(y)) = \widehat{h}(\pi_1(g(x))) = \widehat{h}(\widehat{g}(\pi_2(x)))$. As A and B are complete, the uniformly continuous maps and $\widehat{h}$, $\widehat{g}$ may be extended to an isomorphism between A and B. □

The previous proposition does not require closure of the logic L under Łukasiewicz connectives, but the following corollaries do.

Corollary 5.4. *For any L-regular compact extension $L \geq Ł\forall$ and complete τ-structures $\mathfrak{A}$, $\mathfrak{B}$: $\mathfrak{A} \simeq_{\mu\varepsilon} \mathfrak{B}$ for all finite $\mu \subseteq \tau$ and $\varepsilon > 0$ implies $\mathfrak{A} \equiv_L \mathfrak{B}$.*

Proof. Assume $\varphi^{\mathfrak{A}} \leq r < s \leq \varphi^{\mathfrak{B}}$ for some φ and rationals $r < s$. Then $\mathfrak{A} \models \varphi_{\leq r}$, $\mathfrak{B} \models \varphi_{\geq s}$ and by Proposition 5.3 $\{\varphi_{\leq r}\} \cup \{\varphi_{\geq s}\}$ would be satisfiable, a contradiction. □

We may refine the previous result. It is easy to verify that the relations $\approx_{\mu,\varepsilon}$for $\mu \subseteq_{fin} \tau$, $\varepsilon > 0$, form a uniformity basis in $St_{\tau,S}$ which generates what we call the *approximation uniformity.* The following corollary says that the approximation uniformity is finer than the canonical uniformity of the L-topology (see Lemma 4.2 (iii)).

Corollary 5.5. *Let $L \geq Ł\forall$ be a compact L-regular logic, then for each sentence $\varphi \in L_\tau$ and $\varepsilon > 0$ there are a finite $\mu \subseteq \tau$ and $\delta > 0$ such that $\mathfrak{A} \approx_{\mu,\delta} \mathfrak{B}$ implies $|\varphi^{\mathfrak{A}} - \varphi^{\mathfrak{B}}| \leq \varepsilon$ for all complete τ-structures $\mathfrak{A}$, $\mathfrak{B}$.*

Proof. Suppose no, then for some $\varepsilon > 0$ and each μ, δ, there are $A_{\mu,\delta} \approx_{\mu,\delta} B_{\mu,\delta}$ such that $|\varphi^{A_{\mu,\delta}} - \varphi^{B_{\mu,\delta}}| \geq \varepsilon$. Then $[A_{\mu,\delta}, B_{\mu,\delta}] \models (\varphi^P \leftrightarrow \varphi^{P'})_{\leq 1-\varepsilon}$ while $[A_{\mu,\delta}, A_{\mu,\delta}] \models \varphi^P \leftrightarrow \varphi^{P'}$ (trivially). But $[A_{\mu,\delta}, A_{\mu,\delta}] \approx_{\mu,\delta} [A_{\mu,\delta}, B_{\mu,\delta}]$ for all μ, δ by construction which contradicts Proposition 5.3. □

A sentence $\varphi \in L_\tau$ is said to have *dependence number* κ if there is a $\mu \subseteq \tau$ of power less or equal than κ such that for any pair of models $\mathfrak{A}$, $\mathfrak{B} \in St_\tau$, $\mathfrak{A} \restriction \mu \simeq \mathfrak{B} \restriction \mu$ implies $\varphi^{\mathfrak{A}} = \varphi^{\mathfrak{B}}$. Any sentence of a compact extension closed under Boolean connectives of classical logic has finite dependence (Lindström [24]). This is not the case of CL. However,

Corollary 5.6. (Countable dependence) *Any sentence of a L-regular compact extension L of Ł∀ has countable dependence on each equicontinuous class of complete structures.*

Proof. Given φ, and $n \in \omega^+$ find finite μ_n, $\delta_n > 0$ such that $\mathfrak{A} \approx_{\mu_n,\delta_n} \mathfrak{B}$ implies $|\varphi^{\mathfrak{A}} - \varphi^{\mathfrak{B}}| \leq \frac{1}{n}$, for any complete $\mathfrak{A}$, $\mathfrak{B}$. Let $\mu = \cup_n \mu_n$ then, trivially, $\mathfrak{A} \restriction \mu \simeq \mathfrak{B} \restriction \mu$ implies $\mathfrak{A} \approx_{\mu_n,\delta_n} \mathfrak{B}$ and thus $|\varphi^{\mathfrak{A}} - \varphi^{\mathfrak{B}}| \leq \frac{1}{n}$ for all n. □

6. Lindström´s theorem

The following property generalizes the separable Löwenheim-Skolem property and will be enough to obtain our maximality results.

Definition 6.1. A logic L has the *approximate Löwenheim-Skolem property* if any satisfiable countable theory has for each $\varepsilon > 0$ a model A which may be covered with countably many ε-balls. We call this a ε-separable model, and the set of balls centers we will call a ε-dense subset.

Our main result hinges in the next separation lemma which does not assume L-regularity of L.

Proposition 6.2. (Separation) *Let $L \geq$ Ł∀ be a compact logic satisfying the approximate Löwenheim-Skolem property, S a moduli system S for τ, and $T_1 \cup T_2 \subseteq L_\tau$ unsatisfiable in $St^c_{\tau,S}$. Then T_1 and T_2 are separable in $St^c_{\tau,S}$ by a sentence $\varphi \in$ Ł$\forall_\tau$. That is,*

$$Mod^c_{\tau,S}(T_1) \subseteq Mod^c_{\tau,S}(\varphi), \quad Mod\psi_{\tau,S}(T_2) \cap Mod^c_{\tau,S}(\varphi) = \emptyset.$$

Proof. If $T_1 \cup T_2$ is unsatisfiable by S-models, then by compactness of L there are finite subsets $\Delta_i \subseteq T_i$ such that $\Delta_1 \cup \Delta_2$ is similarly unsatisfiable and it is enough to separate Δ_1 and Δ_2. The closed classes $Mod^c_{\tau,S}(\Delta_i)$ are compact in the L-topology of $St^c_{\tau,S}$, a fortiori in the weaker Ł∀-topology. Assume the claimed separation is not possible, since the closed basics of the Ł∀-topology are closed under intersections, then by regularity of the Ł∀-topology and Lemma 4.3 there exist $\mathfrak{A}, \mathfrak{B} \in St_{\tau,S}$ such that $\mathfrak{A} \models \Delta_1$, $\mathfrak{B} \models \Delta_2$ and $\mathfrak{A} \equiv_{\text{Ł}\forall} \mathfrak{B}$. By Lemma 3.2 there is for each finite $\mu \subseteq \tau$, $\varepsilon > 0$ and each $n \in \omega$ a family of relations $I_0^n, ..., I_n^n$ coding sets of partial $\mu\frac{\varepsilon}{6}$-approximations with the extension property between $\mathfrak{A}$ and $\mathfrak{B}$ (the choice $\varepsilon/6$ will be useful later).[4] For the rest of the proof we fix μ, ε, and show how to transform, using compactness of L, the various families $I_0^n, ..., I_n^n$ (depending on n) in a single $\mu\varepsilon$-approximation $R_{\mu,\varepsilon} : \mathfrak{A}' \approx \mathfrak{B}'$, where $\mathfrak{A}' \models \Delta_1$, $\mathfrak{B}' \models \Delta_2$ and $\mathfrak{A}'$, $\mathfrak{B}'$ share the convergence moduli S (this is extremely important for the proof). Since μ, ε are arbitrary, this will imply by Proposition 5.3 that $\Delta_1 \cup \Delta_2$ is satisfiable, a contradiction.

To obtain the claimed μ, ε-approximation, consider the following countable theory in the signature $\{P, P'\} \cup \mu \cup \mu' \cup \{I_n\}_{n\in\omega}$, where P, P' are two sorts, u is considered of sort P, μ' is a copy of μ in sort P', and for each μ-sentence φ the corresponding sentences in sorts P, P' are denoted φ^P, $\varphi^{P'}$, respectively.

[4]For convenience we will invert the order of the I_i^n so that the extension property goes from I_i^n to I_{i+1}^n.

A1. Δ_1^P, $\Delta_2^{P'}$
A2. $\forall \overline{xy}(I_j\overline{xy} \to \wedge_i(P(x_i) \wedge P'(y_i))$, for each $j \in \omega$
A3. $\exists xyI_1xy$
A4. $\forall \overline{xy}xy(I_j\overline{xy} \to \exists wI_{j+1}\overline{x}x\overline{y}w \wedge \exists wI_{j+1}\overline{x}w\overline{y}y)$, for each $j \geq 1$
A5. $\forall \overline{xy}(I_j\overline{xy} \to (|\varphi^P[\overline{x}] - \varphi^{P'}[\overline{x}]|_{\leq \varepsilon/2}))$, for each atomic $\varphi \in \mu^*$ and $j \geq 1$.

Recall that μ and ε will remain fixed through the argument. Each finite part Σ of this theory where $N = n_\Sigma$ is the largest subindex of the I_j occurring in Σ has the following noncontinuous model

$$M_N = ([\mathfrak{A}, \mathfrak{B}], I_1^N, ..., I_N^N, \emptyset, \emptyset, ...).$$

Since the I_j^N are not necessarily equicontinuous in $[A, B]$ for varying N, we cannot apply compactness to obtain a model of the full theory. To overcome this problem, we make them equicontinuous taking convenient "distance" predicates (cf. [3]), the finiteness of μ is crucial for this purpose. Define:

$$\widehat{I}_j^N\overline{a}\overline{b} := 1 - kd(\overline{a}\overline{b}, I_j^N) \text{ for } j \leq N, \quad \widehat{I}_j^N\overline{a}\overline{b} = 0 \text{ for } j > N,$$

where k is an integer such that $\frac{1}{k} \leq m_\varphi(\varepsilon/6)$ for all $\varphi \in \mu^*$. By construction, all the predicates $\widehat{I}_j^N$ are k-Lipschitz:

$$|\widehat{I}_j^N\overline{a}\overline{b} - \widehat{I}_j^N\overline{a}'\overline{b}'| \leq |kd(\overline{a}\overline{b}, I_j^N) - kd(\overline{a}'\overline{b}', I_j^N)| \leq kd(\overline{a}\overline{b}, \overline{a}'\overline{b}').$$

and this constant is independent of N. We must verify again axioms A1-A5 for $j \leq N$ in the now continuous structure

$$\widehat{M}_N = ([\mathfrak{A}, \mathfrak{B}], \widehat{I}_0^N, ..., \widehat{I}_j^N, \widehat{I}_{j+1}^N, ...).$$

A1 is obvious.
- A2. If $P(a_i) < 1$, then $P(a_i) = 0$ and thus $a_i \in B$. This implies $d(\overline{a}\overline{b}, I_j^N) = 1$ and thus $\widehat{I}_j^N\overline{a}\overline{b} = 0 \leq P(a_i)$, etc.
- A3. Given $a \in A$, there is b such that $(a, b) \in I_1^N$ hence $\widehat{I}_1^N(a, b) = 1 = (\exists xyI_1xy)^{\widehat{M}_N}$.
- A4. $\widehat{I}_j^N\overline{a}\overline{b} \geq r > 0$ implies for small enough $\delta > 0$ the existence of $(\overline{a}'\overline{b}') \in I_j^N$ such that $d(\overline{a}\overline{b}, \overline{a}'\overline{b}') \leq \frac{1-r+\delta}{k} < 1$. Moreover, given $a \in A$ there is c such that $(\overline{a}'a\overline{b}'c) \in I_{j+1}^N$. Then $d(\overline{a}a\overline{b}c, I_{j+1}^N) \leq d(\overline{a}a\overline{b}c, \overline{a}'a\overline{b}'c) = d(\overline{a}\overline{b}, \overline{a}'\overline{b}') \leq \frac{1-r+\delta}{k}$, and thus $(\exists wI_{j+1}\overline{a}a\overline{b}w')^{\widehat{M}_N} \geq 1 - kd(\overline{a}a\overline{b}c, I_{j+1}^N) \geq r - \delta$. As δ is arbitrarily small, $(\exists wI_{j+1}\overline{a}a\overline{b}w')^{\widehat{M}_N} \geq r$, showing that $\widehat{I}_j^N\overline{a}\overline{b} \leq (\exists wI_{j+1}\overline{a}a\overline{b}w')^{\widehat{M}_N}$. The other case is similar.
- A5. As in A4, $\widehat{I}_j^N\overline{a}\overline{b} \geq r > 0$ means that $d(\overline{a}\overline{b}, \overline{a}'\overline{b}') \leq \frac{1-r+\delta}{k} < \frac{1}{k}$ for some $(\overline{a}'\overline{b}') \in I_j^N$ and small enough δ; hence, $d(a_i, a_i'), d(b_i, b_i') < \frac{1}{k} < m_\varphi(\varepsilon/6)$ and thus

$$\begin{aligned} |\varphi^P[\overline{a}] - \varphi'^{P'}[\overline{b}]| &\leq |\varphi^P[\overline{a}] - \varphi^{P'}[\overline{a}']| + |\varphi^P[\overline{a}'] - \varphi^{P'}[\overline{b}']| + |\varphi^{P'}[\overline{b}'] - \varphi^{P'}[\overline{b}]| \\ &\leq \frac{\varepsilon}{6} + \frac{\varepsilon}{6} + \frac{\varepsilon}{6} = \varepsilon/2, \end{aligned}$$

the first and last bounds $\frac{\varepsilon}{6}$ due to uniform continuity of φ and the definition of k, and the middle one because $(\overline{a}'\overline{b}') \in I_j^N$. Thus $\widehat{I}_j^N\overline{a}\overline{b} \leq (|\varphi^P[\overline{x}] - \varphi'^{P'}[\overline{x}]|_{\leq \varepsilon/2}) = 1$.

By construction, the continuity moduli of each predicate in $\widehat{M}_N$ is independent of N since $[\mathfrak{A}, \mathfrak{B}]$ remains fixed. Then we may apply compactness to obtain a continuous S-model of the full theory A1-A5:

$$M_{\mu\varepsilon} = ([\mathfrak{A}^\circ, \mathfrak{B}^\circ], \widehat{I}_1, ..., \widehat{I}_j, \widehat{I}_{j+1}, ...),$$

which by the (approximate) downward Löwenheim-Skolem theorem for L we may assume is ε-separable. Moreover, we have

1. By A3, there is (a_1^*, b_1^*) such that $\widehat{I}_1(a_1^*, b_1^*) \geq 1 - \frac{\varepsilon}{2^2}$.

2. By A4, if $\widehat{I}_j(\overline{a}, \overline{b}) \geq 1 - \rho$ and $a \in A$ ($b \in B$) there is $b \in B$ ($a \in A$) such that $\widehat{I}_{j+1}(\overline{a}a, \overline{b}b) \geq 1 - \rho - \frac{\varepsilon}{2^{n+2}}$.

3. By A5, if $\widehat{I}_j(\overline{a}, \overline{b}) \geq 1 - \rho$ then $|\varphi^{\mathfrak{A}^\circ}[\overline{a}] - \varphi^{\mathfrak{B}^\circ}[\overline{b}]| \leq \frac{\varepsilon}{2} + \rho$ for all atomic $\varphi \in \mu^*$.

Choose listings $\{a_1, a_2...\}$, $\{b_1, b_2...\}$ of an ε-dense subsets of A° and B°, respectively, and construct a relation

$$R = \{(a_i^*, b_i^*) : i \in \omega\} \subseteq A^\circ \times B^\circ$$

starting with (1) and utilizing (2) in a back-and-forth manner to choose inductively (a_n^*, b_n^*) so that the domain and range of R contain $\{a_i\}_i$ and $\{b_i\}$, respectively, and

$$\widehat{I}_n(a_1^*...a_n^*, b_1^*...b_n^*) \geq 1 - (\frac{\varepsilon}{2^2} + ... + \frac{\varepsilon}{2^{n+1}})$$

for all $n \geq 1$. By (3), $|\varphi^{\mathfrak{A}^\circ} a_1^*...a_n^* - \varphi^{\mathfrak{B}^\circ} b_1^*...b_n^*| \leq \frac{\varepsilon}{2} + (\frac{\varepsilon}{4} + ... + \frac{\varepsilon}{2^n}) \leq \varepsilon$ for all atomic formulas in μ^*. Hence, R is a $\mu\varepsilon$-approximation from $\mathfrak{A}^\circ$ to $\mathfrak{B}^\circ$, which finishes the proof. □

Theorem 6.3. (**Maximality of Łukasiewicz logic**) *Let $L \geq Ł\forall$ be a Ł-regular compact logic with the approximate Löwenheim-Skolem property. Then any $\varphi \in L_\tau$ is equivalent with respect to satisfaction to a countable theory of Ł∀, in any equicontinuous class of complete structures.*

Proof. Consider $\varphi \in L_\tau$ and $\mathfrak{A} \notin Mod_\tau(\varphi)$ then $\mathfrak{A} \in C = Mod_\tau(\varphi_{\leq \frac{n}{n+1}})$ for some n. By Proposition 6.2, given S, there is a sentence $\varphi_n \in Ł\forall_\tau$ such that $Mod_{\tau,S}(\varphi) \subseteq Mod_{\tau,S}(\varphi_n)$ and $C \cap Mod_{\tau,S}(\varphi_n) = \emptyset$. Hence, $\mathfrak{A} \notin Mod_{\tau,S}(\varphi_n)$ and thus $Mod_{\tau,S}(\varphi) = \cap_n Mod_{\tau,S}(\varphi_n) = Mod_{\tau,S}(\{\varphi_n : n \in \omega\})$. □

Corollary 6.4. *CL and Ł∀ have the same axiomatizability strength in any equicontinuous class of (not necessarily complete) structures.*

Proof. Any continuous structure is CL-equivalent to its completion, thus two CL–theories have the same continuous models if and only if they have the same complete models. □

Theorem 6.5. (**Maximality of continuous logic**) *Let $L \geq Ł\forall$ be a Ł-regular compact logic with the approximate Löwenheim-Skolem property.Then any $\varphi \in L_\tau$ is equivalent to a sentence of CL, in any equicontinuous class of complete structures.*

Proof. We utilize the following version of the Stone-Weirstrass theorem which follows from Lemma 16.3 in [15]:

If X is a compact Hausdorff space and L is a sublattice of $C(X, [0,1])$ such that for any two distinct elements $x, y \in X$, and $a, b \in [0,1]$ there exists $f \in L$ with $f(x) = a$ and $f(y) = b$. Then L is dense in $C(X, [0,1])$.

For $\varphi \in L_\tau$ the map $F_\varphi : St_{\tau,S} \to [0,1]$, $A \longmapsto \varphi^{\mathfrak{A}}$, is continuous in the Ł∀-topology because the inverse image of each closed subbasic: $F_\varphi^{-1}([r,s]) = Mod(\varphi_{\geq r} \wedge \varphi_{\leq s})$ is L-closed and thus Ł∀-closed by Theorem 6.3. By continuity, F_φ factors through the quotient $St_{\tau/\equiv Ł\forall}$, which is compact and Hausdorff by regularity of the space $St_\tau(Ł\forall)$:

$$\begin{array}{ccc} St_\tau & \xrightarrow{F_\varphi} & [0,1] \\ {\scriptstyle \eta}\searrow & & \nearrow{\scriptstyle \widehat{F}_\varphi} \\ & St_{\tau/\equiv_{Ł\forall}} & \end{array}$$

Now, this is true also for the maps F_θ with $\theta \in$ CL. Let $\mathcal{F} = \{\widehat{F}_\theta : \theta \in \mathrm{CL}\} \subseteq \mathcal{C}(St_{\tau/\equiv_{Ł\forall}}, [0,1])$. Obviously, $\mathcal{F}$ is closed under composition with all continuous connectives and is a lattice thanks to the presence of $\wedge, \vee$. Given distinct points $M_{/\equiv_{Ł\forall}}$, $N_{/\equiv_{Ł\forall}}$ in $St_{\tau/\equiv_{Ł\forall}}$ there must exist a Ł∀-sentence θ such that $\theta^M < \theta^N$ and given values $a, b \in [0,1]$ we may find a continuous connective c (actually in $Ł\forall(\mathbb{Q})$) such that $c(\theta^M) = a$, $c(\theta^N) = b$, then the map $c \circ \widehat{F}_\theta = \widehat{F}_{c\theta}$ satisfies the hypothesis of the Stone-Weirstrass theorem granting that the uniform closure of $\mathcal{F}$ is $\mathcal{C}$. Therefore, $\widehat{F}_\varphi$ is the uniform limit of a sequence $\{\widehat{F}_{\theta_n}\}_n$ with $\theta_n \in \mathcal{F}$. Thus $F_\varphi = \widehat{F}_\varphi \circ \eta$ is the uniform limit of $F_{\theta_n} = \widehat{F}_{\theta_n} \circ \eta$ and by Lemma 2.3 $F_\varphi = F_\theta$ with $\theta \in$ CL.

Example $L = \Sigma^1_1(\mathrm{CL})$ inherits from CL compactness, and the downward Löwenheim-Skolem property and thus it satisfies Proposition 6.2; however, it does not satisfy the conclusion of Theorems 6.3 and 6.5, because the crisp structures $(\mathbb{Q},<)$ and $(\mathbb{R},<)$ are complete for the discrete metric and equivalent in Ł∀ (which collapses to classical logic in these structures) and thus in CL, but the order incompleteness of the first structure is expressible in $\Sigma^1_1(Ł\forall)$. We conclude that $\Sigma^1_1(\mathrm{CL})$ cannot be closed under both $\to$ and $\neg$, and this closure hypothesis cannot be eliminated from Theorem 6.3. Actually, $\Sigma^1_1(\mathrm{CL})$ is closed under $\wedge, \vee, \oplus, \odot$ and $(\ )_{\geq r}$ but not under, $\to, \neg$ or $(\ )_{\leq r}$.

7. Comments

The countable theory given in Theorem 6.3 and its corollary and the sentence obtained in Theorem 6.5 depend on the uniform continuity moduli system S. Is this necessarily so?

Do theorems 6.3 and 6.5 hold for arbitrary continuous structures? Otherwise, under which conditions on the extension L, they hold if we allow incomplete continuous structures? An obvious condition is asking L to have the property that any satisfiable theory has a complete model (as in [9]) because in such case two theories or sentences are equivalent in continuous structures if and only if they are equivalent in complete ones, thus we may restrict the logic to the latter and apply the maximality results.

Lemmas 2.2 and 2.3 say that the uniform closure of $Ł\forall(\mathbb{Q})$ in [0,1]-valued structures is CL. Which is the uniform closure of Ł∀?

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