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LOGICS AND PSEUDOGROUPS

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INTRODUCTION. Our purpose in this paper is to study the connection between "logics" on one side and "pseudogroups" of partial isomorphism on the other.

The method of extension of partial isomorphisms (back-and-forth) was first used by Cantor in the beginning of set theory. In 1954 R. Fraïssé introduced this method in the context of model theory and gave an algebraic characterization of elementary equivalence in first order logic (see [5]). His paper was followed by many others as those of Erenfeucht [4], Karp [7] and Lindström [9], making of partial isomorphism a fundamental tool in model theory. The first author of this paper gave in [1] a characterization by partial isomorphisms of elementary equivalence in logics generated by arbitrary families of quantifiers.

Pseudogroups were introduced by Lie in 1870 (under the name "continuous groups", even though they are not really groups), as families of locally defined diffeomorphisms in a manifold. He intended to extend to differential equations some results of Galois theory, and to study the transformation groups of some general geometrical structures (see [12], for more historical details and definitions). Today, this structure plays a central role in the study of differential geometry. In 1952, C. Ehresmann [3] led by the geometrical problem of finding a precise definition of local structure, introduced a general notion of pseudogroup in the context of category theory (see [2]).

Here, following an idea of Fraissē in [6], we will show how the formulae of a logic are invariants under the action of given pseudogroups of partial

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isomorphisms. In this way, logics may be seen as systems of invariants in which the usual closure conditions imposed on logics (negations, conjunctions, etc.) arise naturally.

In § 1, we consider the Galois' connection that associates to each logic L the class of partial isomorphisms $M(L)$ which leave invariant the formulae of L, and associates to each class M of partial isomorphisms the logic $\mathcal{L}(M)$ of formulae invariant under the elements of M. It turns out that a class of partial isomorphisms has the form $M(L)$ if and only if it is a pseudogroup (in the sense of Ehresmann), and a logic has the form $\mathcal{L}(M)$ if and only if it is closed under equality and atomic formulae, negations, arbitray conjunctions, and change of variables.

In § 2, we consider logics generated by inverse families of pseudogroups, showing that most natural logics can be so represented, but some familiar logics can not be generated by descending chains of pseudogroups.

In § 3, we introduce a multiplication between pseudogroups which is reflected in the logics as the operation of substituting atomic subformulae of a formula by other formulae. From this we obtain that finitary or infinitary extensions of first order logic by means of Lindström-Mostowski quantifiers are generated as invariants of chains of iterated powers of a fixed pseudogroup.

§ 1. FORMULAE AND PARTIAL ISOMORPHISMS.

Let τ be a purely relational type of structures. For each $n \in \omega$, a *n-ary formula of type* τ is a class of first order structures of the form $(\mathfrak{A}, a_1, \ldots, a_n)$, where $\mathfrak{A}$ is of type τ, closed under isomorphism. A 0-ary formula is called a *sentence*. From our point of view there is no difference between sentences and quantifiers in the sense of Lindström [8]. As usual, we will write:

$$\mathfrak{A} \models \varphi[a_1, \ldots, a_n] \quad \text{for} \quad (\mathfrak{A}, a_1, \ldots, a_n) \in \varphi.$$

A *logic of type* τ will be just a class of formulae of type τ. For definiteness, we could assume that the formulae of a given logic L are coded by a class of sets: $\mathrm{Form}(L) = \bigcup_{n \in \omega} L_n$, *the syntax* of L; together with a class of pairs:

$$\models \subseteq \bigcup_{n \in \omega} S_n^\tau \times L_n ,$$

where

$$S_n^\tau = \{(\mathfrak{A}, a_1, \ldots, a_n) \mid \mathfrak{A} \text{ is a structure of type } \tau\}.$$

In this way we would avoid talking about classes of classes. However, it is obvious that this syntactical-like approach, the usual one in abstract logic, is unnecessary if one allows a set theory with classes of classes. We will follow the syntax free approach.

A *partial isomorphism* from $\mathfrak{A}$ to $\mathfrak{B}$, two structures of type τ, is an isomorphism $f: \mathfrak{A}\restriction A' \longrightarrow \mathfrak{B}\restriction B'$ where $A' \subseteq A$, $B' \subseteq B$. This includes the empty partial isomorphism.

M_0^τ is the category having for objects all structures of type τ and where the set of morphisms $M_0^\tau(\mathfrak{A}, \mathfrak{B})$ consists of all partial isomorphisms from $\mathfrak{A}$ to $\mathfrak{B}$. Morphisms are composed as binary relations, this may yield the empty partial isomorphism. If $f \in M_0^\tau(\mathfrak{A}, \mathfrak{B})$, departing from ordinary usage for categories, we will reserve the notation *dom* f and *cod* f for the set theoretical domain and codomain of f, respectively, and no to denote $\mathfrak{A}$ and $\mathfrak{B}$.

Each partial isomorphism $f \in M_0^\tau(\mathfrak{A},\mathfrak{B})$ has an inverse $f^{-1} \in M_0^\tau(\mathfrak{B},\mathfrak{A})$, and for $S \subseteq A$, it has a restriction $f\restriction S \in M_0^\tau(\mathfrak{A},\mathfrak{B})$ obtained by restricting f to $(dom\ f) \cap S$. If $f_i \in M_0^\tau(\mathfrak{A}, \mathfrak{B})$, $i \in I$, is a family directed by inclusion, then its directed limit is $\bigcup_{i \in S} f_i \in M_0^\tau(\mathfrak{A}, \mathfrak{B})$.

DEFINITION 1. A *pseudogroup of partial isomorphisms* (of type τ) is a subcategory of M_0^τ closed under inverses, restrictions and directed limits, and containing the subcategory M_{iso}^τ of (total) isomorphisms between structures of type τ.

This is analogous to the definition of pseudogroup in differential geometry, where instead of structures we have differentiable manifolds, and instead of partial isomorphisms we have diffeomorphisms defined in open subsets (see Definition 1.1, in [12], or § 4 in [3]). The main differences being that instead of considering a single object (a manifold) we consider several objects (structures of type τ) and so the property of containing the identity in [12] becomes here the property of containing all the isomorphisms. This fits in the general definition of pseudogroup introduced by Ehresmann in [2].

DEFINITION 2. A partial isomorphism $f \in M_0^\tau(\mathfrak{A}, \mathfrak{B})$ *preserves* a n-ary formula φ, or φ is *invariant* for f, if for any $a_1,\dots,a_n \in dom\ f$ we have:

$$\mathfrak{A} \models \varphi[a_1,\dots,a_n] \iff \mathfrak{B} \models \varphi[f(a_1),\dots,f(a_n)].$$

DEFINITION 3. For any class of partial isomorphism M, let $\mathcal{L}(M)$ be the class of formulas invariant for all elements of M. We call $\mathcal{L}(M)$ the *primitive logic generated by* M.

DEFINITION 4. For any logic L of type τ, let $M(L)$ be the subcategory of M_0^τ given by $M(L)(\mathfrak{A},\mathfrak{B}) = \{f \in M_0(\mathfrak{A},\mathfrak{B}) \mid f$ preserves all formulae in $L\}$.

The assignment $M \mapsto \mathcal{L}(M)$, $L \mapsto M(L)$ is a *Galois connection* in the sense that it satisfies:

1) $M \subseteq M' \Rightarrow \mathcal{L}(M) \supseteq \mathcal{L}(M')$
2) $L \subseteq L' \Rightarrow M(L) \supseteq M(L')$
3) $M \subseteq M(\mathcal{L}(M))$
4) $L \subseteq \mathcal{L}(M(L))$.

From this it follows that L is a primitive logic if and only if $L = \mathcal{L}(M(L))$. Analogously, $M = M(L)$ for some logic L if and only if $M = M(\mathcal{L}(M))$. Our first objective will be to characterize these fixed points of the connection.

THEOREM 1. $M = M(L)$ *for some logic* L *if and only if* M *is a pseudogroup.*

PROOF. It is very easy to check that $M(L)$ is always a pseudogroup. For the converse, let M be a pseudogroup and for each $n \in \omega$ define in the class S_n^τ the following relation:

$$(\mathfrak{A},\overline{a}) \underset{M}{\sim} (\mathfrak{B},\overline{b}) \iff \text{there is } f \in M(\mathfrak{A},\mathfrak{B}) \text{ such that } \overline{a} \subseteq dom\, f \text{ and } f(\overline{a}) = \overline{b}.\ ^{(*)}$$

This is an equivalence relation with its classes closed under isomorphism. Reflexivity and transitivity follows from the fact that M is a subcategory of M_0^τ, symmetry from the closure of M under inverses, and closure of the equivalence clauses under isomorphism from the fact that $M \supseteq M_{iso}^\tau$. Let L be the logic having for n-ary formulae the equivalence classes of $\underset{M}{\sim}$ in S_n^τ. By construction, $L \subseteq \mathcal{L}(M)$ and so $M \subseteq M(\mathcal{L}(M)) \subseteq M(L)$. Now consider $f \in M(L)(\mathfrak{A},\mathfrak{B})$, then for any n-ary formula $\Psi \in L$ and $S = \{a_1,\dots,a_n\} \subseteq dom\, f$, we have $(\mathfrak{A},\overline{a}) \in \Psi \iff (\mathfrak{B}, f(\overline{a})) \in \Psi$. Hence, $(\mathfrak{A},\overline{a}) \underset{M}{\sim} (\mathfrak{B}, f(\overline{a}))$. This means that there is $g \in M(\mathfrak{A},\mathfrak{B})$ such that $g \supseteq f \restriction S$, and so $f \restriction S = g \restriction S \in M(\mathfrak{A},\mathfrak{B})$, by the closure of pseudogroups under restrictions. As this holds for all finite restrictions, then $f \in M(\mathfrak{A},\mathfrak{B})$ by the closure of pseudogroups under directed limits. We have shown $M(L) \subseteq M$ and so $M(L) = M$. □

(*) In the sequel we will use the notation $\overline{a}$ to denote sequences $(a_1,\dots,a_n)$. Given a partial isomorphism f, then $\overline{a} \subseteq dom\, f$ means $a_1,\dots,a_n \in dom\, f$, and $f(\overline{a}) = \overline{b}$ means $f(a_i) = b_i$ for $i = 1,\dots,n$.

Now we will characterize the primitive logics. First we need to define some operations on formulae:

1. Any n-ary symbol R of the type τ defines canonically a n-ary *atomic formula*: $\overline{R} = \{(\mathfrak{A}, a_1,\ldots,a_n) \mid (a_1,\ldots,a_n) \in R^{\mathfrak{A}}\}$.

2. The *equality* formula (of type τ) is $E = \{(\mathfrak{A}, a_1, a_2) \mid a_1 = a_2,\ \mathfrak{A}$ of type $\tau\}$.

3. The *negation*, $\neg\varphi$, of a n-ary formula φ is the class $\neg\varphi = \{(\mathfrak{A}, a_1,\ldots,a_n) \mid \mathfrak{A}$ of type τ, $(\mathfrak{A}, a_1,\ldots,a_n) \notin \varphi\}$. If φ_i, $i \in I$, is a class of formulae of the same arity n, then their *conjunction* $\bigwedge_{i \in I} \varphi_i$, is just their intersection.

4. Let φ be an n-ary formula and $\pi : \{1,2,\ldots,n\} \longrightarrow \{1,2,\ldots,k\}$, $k \in \omega$, a function (for $n = 0$, $\pi : \phi \longrightarrow \{1,2,\ldots,k\}$ is the empty function), then φ_π, called a *variant* of φ, is the k-ary formula defined by:

$$\varphi_\pi = \{(\mathfrak{A}, a_1,\ldots,a_k) \mid (\mathfrak{A}, a_{\pi(1)},\ldots,a_{\pi(n)}) \in \varphi\}.$$

If we write $\varphi(x_1,\ldots,x_n)$ to indicate that the formula is n-ary, then the definition of φ_π may be expressed in a syntactic-like manner by the equivalence: $\varphi_\pi(x_1,\ldots,x_k) \equiv \varphi(x_{\pi(1)},\ldots,x_{\pi(n)})$. Notice that if $\varphi_1,\ldots,\varphi_m$ are formulae of possibly different arities, they may be "brought" to the same arity by adequate variants, and so their conjunction may be taken in the common arity. Of course, this may not be possible for an infinite family.

The proof of the following lemma is straightforward.

LEMMA 1. *$\mathcal{L}(M)$ contains the atomic and the equality formulae, and it is closed under variants, negations, and conjunctions of arbitrary classes of formulae (of the same arity).*

We will show that these conditions are also sufficient to characterize primitive logics.

Given a logic L, let Var(L) be the class of all variants of formulae in L, and let $\mathrm{Bool}_\infty(L)$ be the closure of L under negations and arbitrary conjunctions of formulae of the same arity. Hence, if $L_n = \{\varphi \in L \mid \varphi$ n-ary$\}$,

$$\mathrm{Bool}_\infty(L) = \bigcup_{n \in \omega} \overline{L}_n$$

where $\overline{L}_n$ is the *complete* boolean algebra generated by L_n, in the class of all formulae. Finally, for a similarity type τ, let $At = At(\tau)$ denote the class of atomic and equality formulae.

LEMMA 2. *For any logic* L, $\mathcal{L}(M(L)) = \mathrm{Bool}_\infty(\mathrm{Var}(L \cup At(\tau)))$.

PROOF. By lemma 1, $\mathcal{L}(M(L)) \supseteq \mathrm{Bool}_\infty(\mathrm{Var}(L\cup At))$. For the other inclusion, let $M = M(L)$ and consider the equivalence relations: $(\mathfrak{A}, \bar{a}) \underset{M}{\sim} (\mathfrak{B},\bar{b})$ in S_n^τ, $n\in\omega$, as defined in the proof of Theorem 1. Let, for each $n\in\omega$. $V_n = \{\varphi\in \mathrm{Var}(L\cup At) \mid \varphi \text{ is } n\text{-ary}\}$, and for each $(\mathfrak{A}, \bar{a}) \in S_n$ define:

$$t_{(\mathfrak{A},\bar{a})} = \bigwedge_{\substack{\varphi\in V_n \\ (\mathfrak{A},\bar{a})\in\varphi}} \varphi \ \wedge \bigwedge_{\substack{\varphi\in V_n \\ (\mathfrak{A},\bar{a})\notin\varphi}} \neg\varphi$$

then we may see that $t_{(\mathfrak{A},\bar{a})} \in \mathrm{Bool}_\infty(\mathrm{Var}(L\cup At))$, and

(1) $$(\mathfrak{A}, \bar{a}) \underset{M}{\sim} (\mathfrak{B}, \bar{b}) \iff t_{(\mathfrak{A},\bar{a})} = t_{(\mathfrak{B},\bar{b})} .$$

From left to right the equivalence follows from the fact that if $f\in M(\mathfrak{A}, \mathfrak{B})$ and $f(\bar{a}) = \bar{b}$, then, as $M = M(L)$, f preserves the formulae in $L\cup At$ and by lemma 1, those in $\mathrm{Var}(L\cup At)$. In particular, the same n-ary formulas in this class hold for the tuple $\bar{a}$ than for the tuple $\bar{b}$. Conversely, if $t_{(\mathfrak{A},\bar{a})} = t_{(\mathfrak{B},\bar{b})}$ then $\mathfrak{A} \models \varphi[\bar{a}] \iff \mathfrak{B} \models \varphi[\bar{b}]$, for any $\varphi\in V_n$. Now, we may see that for any formula in $L\cup At$, say a k-ary formula ψ, and sequences $(a_{i_1},\dots,a_{i_k})$, $(b_{i_1},\dots,b_{i_k})$, extracted from $(a_1,\dots,a_n)$, $(b_1,\dots,b_n)$, respectively,

$$\mathfrak{A} \models \psi[a_{i_1},\dots,a_{i_k}] \iff \mathfrak{A} \models \psi_\pi[a_1,\dots,a_n]$$
$$\mathfrak{B} \models \psi[b_{i_1},\dots,b_{i_k}] \iff \mathfrak{B} \models \psi_\pi[b_1,\dots,b_n]$$

where $\pi : \{1,\dots,k\} \longrightarrow \{1,\dots,n\}$ is the function $\pi(r) = i_r$. Hence, as $\psi_\pi \in V_n$,

$$\mathfrak{A} \models \psi[a_{i_1},\dots,a_{i_k}] \iff \mathfrak{B} \models \psi[b_{i_1},\dots,b_{i_k}] ,$$

showing that the assignment $a_r \mapsto b_r$ preserves the formulae in $L \cup At$. Since At contains "=", $f(a_r) = b_r$ is a well defined partial isomorphism belonging to $M(L) = M$. Therefore $(\mathfrak{A}, \bar{a}) \underset{M}{\sim} (\mathfrak{B}, \bar{b})$, completing the proof of (1).

To finish the proof, consider a n-ary formula $\psi \in \mathcal{L}(M(L))$ then $(\mathfrak{A}, \bar{a}) \underset{M}{\sim} (\mathfrak{B}, \bar{b})$ implies the existence of $f \in M(L)(\mathfrak{A}, \mathfrak{B})$ such that $f(\bar{a}) = \bar{b}$; as f preserves ψ, then $\mathfrak{A} \models \psi[\bar{a}] \iff \mathfrak{B} \models \psi[\bar{b}]$. Therefore, ψ is a union of equivalence classes of M; then by (1)

$$\psi = \bigvee_{i\in I} t_{(\mathfrak{A}_i, \bar{a}_i)} \in \mathrm{Bool}_\infty(\mathrm{Var}(L\cup At)).$$

With this we finish the proof. □

From lemma 1, and the fact that if L satisfies the conditions of next theorem then $L = \mathrm{Bool}_\infty(\mathrm{Var}(L\cup At)) = \mathcal{L}(M(L))$, we have:

THEOREM 2. *A logic* L *is primitive if and only if it contains the atomic and equality formulae, and it is closed under variants, negations and conjunctions of arbitrary classes of formulae (of the same arity).*

EXAMPLE 1. $\mathcal{L}(M_0^\tau) = \mathcal{L}(M(At)) = Bool_\infty(Var(At))$, the quantifier free formulae of $L_{\infty\omega}(\tau)$ with finitely many free variables, is the smallest primitive logic (of type τ). If τ is finite, then $\mathcal{L}(M_0^\tau)$ consists of the quantifier free formulae of $L_{\omega\omega}(\tau)$.

EXAMPLE 2. $L_{\omega\omega}$ and $L_{\infty\omega}$ are not primitive logics. To see that $L_{\infty\omega}$ is not primitive, observe that for each ordinal α there is a sentence $\varphi_\alpha \in L_{\infty\omega}$ with the unique model $(\alpha,\in)$, but $\neg \bigwedge_{\alpha\in Ord} \neg\varphi_\alpha \notin L_{\infty\omega}$ because well order is not definable in this logic. In a similar way we show that no logic $L_{k\omega}$ is primitive.

EXAMPLE 3. The logic $L^\alpha_{\infty\omega}$ of formulae of $L_{\infty\omega}$ of quantifier rank less than α *is primitive if and only if* α *is a successor ordinal.* If τ is finite, we conclude that $L^k_{\omega\omega}$, for $k\in\omega$, is primitive.

§ 2. LOGICS GENERATED BY FAMILIES OF PSEUDOGROUPS.

As the examples show, most natural logics are not primitive. However, they are generated by inverse families of pseudogroups. This is, they are of the form $L = \bigcup_{i\in I} \mathcal{L}(M_i)$, where the M_i form a family inversely directed by inclusion, when the type is finite. To see this, we need the following observations. From Lemma 2,

$$\mathcal{L}(M(L)) = \bigcup_{n\in\omega} \overline{V}_n$$

where $V_n = \{\varphi \in Var(L\cup At) \mid \varphi \text{ is } n\text{-ary}\}$, and $\overline{V}_n$ denotes the *complete* boolean algebra generated by V_n. If L and τ are finite, then so is $L\cup At(\tau)$, and so is each V_n. Therefore, in this case, $\overline{V}_n$ is simply the finitary boolean algebra generated by V_n.

THEOREM 3. *A logic* L *of finite type is generated by an inverse family of pseudogroups if and only if* L *contains the atomic and equality formulae, and it is closed under variants negations and finite conjunctions.*

PROOF. If L is a directed union of primitive logics, the closure conditions are obvious. Conversely, suppose that L satisfies the hypothesis of closure and let $F = \{S\subseteq L \mid S \text{ finite}\}$ be directed by inclusion, then the family of pseudogroups $\{M(S) \mid S\in F\}$ is inversely directed, and $L = \bigcup\{S \mid S\in F\} \subseteq \bigcup_{S\in F} \mathcal{L}(M(S))$. Now, by the above remarks, each $\mathcal{L}(M(S))$ has the form

$\bigcup_{n\in\omega}\overline{V}_n$ where $\overline{V}_n$ is just the finitary boolean algebra generated by V_n. Since $V_n \subseteq L$, then $\overline{V}_n \subseteq L$ and so $\mathcal{L}(M(S)) \subseteq L$. Hence $L = \bigcup_{S\in F}\mathcal{L}(M(S))$. □

EXAMPLE 4. The equations: $L_{\omega\omega} = \bigcup_{k\in\omega} L^k_{\omega\omega}$ (for τ finite), $L^{\alpha}_{\infty\omega} = \bigcup_{\lambda<\alpha} L^{\lambda+1}_{\infty\omega}$ (α limit), and $L_{\infty\omega} = \bigcup_{\lambda\in \mathrm{Ord}} L^{\lambda+1}_{\infty\omega}$, give representations of these non-primitive logics as directed unions of *chains* of primitive logics (cf. example 3). Therefore, these logics are generated by descending chains of pseudogroups. It is easy to see that these pseudogroups are generated in turn (in the obvious sense) by the families of partial isomorphisms introduced by Fraïssé [5] and Karp [7] to characterize elementary equivalence in the corresponding logics. However, the non-primitive logics $L_{k\omega}$, which are generated by inverse families of pseudogroups after Theorem 2, are generated by descending chains only in exceptional cases. It may be shown:

THEOREM 4. *For* $|\tau| < k$, $L_{k\omega}$ *is generated by a descending chain of pseudogroups if and only if* k *is strongly inaccessible (or* $k = \omega, \infty$*).*

PROOF. We may assume k to be regular, because for singular k, $L_{k\omega} = L_{k^+\omega}$. Moreover, we will use the following estimate for regular k:

(1) $$|L_{k\omega}| \leqslant \sup_{\delta<k} 2^{\delta} .$$

The "if" part of the theorem follows from the fact that for inaccessible k, $L_{k\omega} = L^k_{\infty\omega}$, a union of primitive logics (Example 4). To prove the "only if" part of the theorem, let $L_{k\omega} = \bigcup_{i\in I} L_i$, where $\{L_i \mid i \in I\}$ is a chain of primitive logics, ordered by inclusion. Then we may extract a well ordered subchain $\{L_{i_\beta} \mid \beta < \mu\}$, with μ a regular cardinal, such that $L_{k\omega} = \bigcup_{\beta<\mu} L_{i_\beta}$.

First, we show that $\mu \geqslant k$. Let $\varphi \in L_{\infty\omega} - L_{k\omega}$ be of minimal complexity (it exists because $L_{\infty\omega}$ is a proper class and $L_{k\omega}$ is a set). As $L_{k\omega}$ is closed under $\neg$ and $\exists$, we may assume that $\varphi = \bigwedge_{j\in J}\varphi_j$, with $\varphi_j \in L_{k\omega}$. For each $\beta < \mu$, let $\theta_\beta = \bigwedge\{\varphi_j \mid \varphi_j \in L_{i_\beta}\}$, or a tautology if there is no φ_j in L_{i_β}. Then $\theta_\beta \in L_{i_\beta} \subseteq L_{k\omega}$ because L_{i_β} is primitive, and obviously $\varphi = \bigwedge_{\beta<\mu}\theta_\beta$. If $\mu < k$, we would have $\varphi \in L_{k\omega}$, a contradiction. Hence $\mu \geqslant k$.

Next, we show that for any cardinal $\delta < k$, $|L_{k\omega}| \geqslant 2^{2^{\delta}}$. Fix $\delta < k$. For any ordinal $\alpha \leqslant \delta$ there is a "Scott sentence" φ_α belonging to $L_{k\omega}$ with the unique model $(\alpha, \in)$. Let $\varphi^{<x}_\alpha$ be the formula with a free variable x which relativizes φ_α to the predecessors of x, and consider the family of sentences:

$$\mathbb{F} = \{\varphi_\delta\} \cup \{\forall x\,(\varphi^{<x}_\alpha \rightarrow P(x)),\ \forall x\,(\varphi^{<x}_\alpha \rightarrow \neg P(x) \mid \alpha < \delta\}.$$

Then $\mathbb{F} \subseteq L_{k\omega}$ and $|\mathbb{F}| \leqslant \delta < k \leqslant \mu$. As μ is regular, there exists $\gamma < \mu$ such that $\mathbb{F} \subseteq L_{i_\gamma}$. Now, for each $S \subseteq \delta$, let $\Psi_S \in L_{i_\gamma}$ be the sentence

$$\varphi_\delta \wedge \bigwedge_{\alpha\in S} \forall x(\varphi_\alpha^{<x} \rightarrow P(x)) \wedge \bigwedge_{\beta\notin S} \forall x(\varphi_\beta^{<x} \rightarrow \neg P(x)).$$

This is a "Scott sentence" for the structure $(\delta,\in,S)$. Finally, for each $I \subseteq P(\delta)$, let $\sigma_I = \bigvee_{S\in I} \Psi_S$. Then $\sigma_I \in L_{i_\gamma}$ and it has for models the class $\{(\delta,\in,S) \mid S\in I\}$. This means that distinct $I \subseteq P(\delta)$ yield non-equivalent σ_I, all of them belonging to L_{i_γ}. Hence, $|L_{k\omega}| \geqslant |L_{i_\gamma}| \geqslant |P(P(\delta))| = 2^{2^\delta}$.

To finish the proof, notice that by the last fact and (1) above,

$$2^\delta < 2^{2^\delta} \leqslant \sup_{\delta<k} 2^\delta \leqslant 2^k \quad \text{for all } \delta<k.$$

By König's Theorem:

$$2^{2^\delta} \leqslant \sup_{\delta<k} 2^\delta < \prod_{\delta<k} 2^k = 2^k,$$

which implies $2^\delta < k$ for all $\delta<k$. □

The above theorem shows that in a very general sense there is not a characterization of elementary equivalence in $L_{\omega_1\omega}$ by descending chains of sets of partial isomorphisms, as is the case with $L_{\omega\omega}$ (Fraïssé) or $L_{\infty\omega}$ (Karp).

§ 3. SUBSTITUTION OF FORMULAE AND PRODUCT OF PSEUDOGROUPS.

Primitive logics do not include full first order logic, even for τ finite. This is due to the fact that primitive logics are not closed under their own "logical operators". Each formula may be seen as a logical operator acting on its "atomic components", and one should be allowed to iterate this operators. This would correspond, semantically, to compose certain functors, and syntactically to substitute atomic formulae for more complex formulae in a given formula. We define more precisely this operation.

Let ψ be a r+n-ary formula then for any structure $\mathfrak{A}$ and $\bar{a}\in A^r$ introduce

$$\psi^{\mathfrak{A}\bar{a}} = \{\bar{x}\in A^n \mid \mathfrak{A} \models \psi[\bar{a},\bar{x}]\}.$$

(If $r=0$, we write $\psi^{\mathfrak{A}}$).

DEFINITION. Let φ be a n-ary formula of type $\langle n_1,\dots,n_k\rangle$ and for each $i=1,\dots,k$ a $r+n_i$-ary formulae ψ_i, then the r+n-ary formula,

$$\varphi(\psi_1,\dots,\psi_k) = \{(\mathfrak{A},\bar{a}\,\bar{b}) \mid \bar{a}\in A^r,\ \bar{b}\in A^n,\ (A,\psi_1^{\mathfrak{A}\bar{a}},\dots,\psi_k^{\mathfrak{A}\bar{a}},\bar{b}) \in \varphi\}$$

is called the *substitution of* $\psi_1,\dots,\psi_k$ *in* φ.

The type of $\varphi(\psi_1,\ldots,\psi_k)$ is the union of the types of the ψ_i's. However, we will assume all formulae of the same type for simplicity.

In the usual notation this may be written

$$\mathfrak{A} \models \varphi(\psi_1,\ldots,\psi_k)[\overline{a}\ \overline{b}] \iff (A,\psi_1^{\mathfrak{A}\,\overline{a}},\ldots,\psi_k^{\mathfrak{A}\,\overline{a}}) \models \varphi[b].$$

For the case $n=0$, it corresponds to the application of the generalized quantifier defined by the sentence φ to the formulae $\psi_1,\ldots,\psi_k$. All logics ordinarily studied are closed under substitution of formulae. An exception would be $L^{\alpha}_{\infty\omega}$ for successor ordinal α.

Now, we will introduce a *multiplication* between pseudogroups. Given fixed structures $(\mathfrak{A}_1,\overline{a}_1)$, $(\mathfrak{A}_2,\overline{a}_2)$ in S^{τ}_r with $A_1\cap A_2=\phi$ we define first for each $n\in\omega$ the following equivalence relation in $A_1^n\cup A_2^n$. For $\overline{x}\in A_i^n$, $\overline{y}\in A_j^n$:

$$\overline{x} \underset{M\overline{a}_1\overline{a}_2}{\sim} \overline{y} \iff [(\mathfrak{A}_i,\overline{a}_i\overline{x}) \underset{M}{\sim} (\mathfrak{A}_j,\overline{a}_j\overline{y})]$$

where M is a given pseudogroup. The equivalence class of $\overline{x}$ will be denoted $[\overline{x}]^M_{\overline{a}_1\overline{a}_2}$. If $X\subseteq A^n\cup B^n$ is a union of these classes, we will say that X is *closed* under $M\overline{a}_1\overline{a}_2$. We will need the following lemma.

LEMMA 3. *If* $X\subseteq A_1^n\cup A_2^n$ *runs through the* $M\overline{a}_1\overline{a}_2$*-closed sets then the pairs* $X\cap A^n$, $X\cap B^n$ *coincide with the pairs* $\psi^{\mathfrak{A}_1\overline{a}_1}$, $\psi^{\mathfrak{A}_2\overline{a}_2}$, *where* ψ *runs through the* $n+r$*-ary formulae of* $\mathcal{L}(M)$.

PROOF. If $X\subseteq A_1^n\cup A_2^n$ is $M\overline{a}_1\overline{a}_2$ closed then $X\cap A_1^n = \bigcup_{j\in J}\{[\overline{x}_j]^M_{\overline{a}_1\overline{a}_2}\cap A_1^n\} = \{x\in A_1^n \mid (\mathfrak{A}_1,\overline{a}_1\overline{x}) \in \bigcup_{j\in J}[(\mathfrak{A}_{i_j},a_{i_j}x_{i_j})]_M\}$, where $i_j\in\{1,2\}$.

But we know that a union of $\tilde{M}$-equivalence classes is a formula of $\mathcal{L}(M)$, call it ψ, then

$$X\cap A_1^n = \{\overline{x}\in A_1^n \mid (\mathfrak{A}_1,\overline{a}_1\overline{x})\in\psi\} = \psi^{\mathfrak{A}_1\overline{a}_1}.$$

Now, the family $(\mathfrak{A}_{i_j},\overline{a}_{i_j},\overline{x}_{i_j})$, $j\in J$, depends on X only; then we have, similarly, for the same formula, $X\cap A_2^n = \psi^{\mathfrak{A}_2\overline{a}_2}$.

Conversely, given $\psi\in\mathcal{L}(M)$, $r+n$-ary, take $\overline{x}\in A_i^n$, $\overline{y}\in A_j^n$, $i,j\in\{1,2\}$, then

$$\begin{aligned} \overline{x} \underset{M\overline{a}_1\overline{a}_2}{\sim} \overline{y} &\Rightarrow (\mathfrak{A}_i,\overline{a}_i\overline{x}) \underset{M}{\sim} (\mathfrak{A}_j,\overline{a}_j\overline{y}) \\ &\Rightarrow [(\mathfrak{A}_i,\overline{a}_i\overline{x})\in\psi \iff (\mathfrak{A}_j,\overline{a}_j\overline{y})\in\psi] \\ &\Rightarrow [\overline{x}\in\psi^{\mathfrak{A}_i\overline{a}_i} \iff \overline{y}\in\psi^{\mathfrak{A}_j\overline{a}_j}]. \end{aligned}$$

Then $\psi^{\mathfrak{A}_1\overline{a}_1}\cup\psi^{\mathfrak{A}_2\overline{a}_2}$ must be a union of $M\overline{a}_1\overline{a}_2$-equivalence classes, i.e. a $M\overline{a}_1\overline{a}_2$-closed subset X of $A_1^n\cup A_2^n$. Therefore, we have

$$\psi^{\mathfrak{A}_1\overline{a}_1} = X\cap A_1^n, \qquad \psi^{\mathfrak{A}_2\overline{a}_2} = X\cap A_2^n. \qquad \square$$

$\mathfrak{A}$ is a *weak substructure* of $\mathfrak{B}$ if $A \subseteq B$ and $R^{\mathfrak{A}} \subseteq R^{\mathfrak{B}}$ for all symbols $R \in \tau$. The *disjoint union* of two structures $\mathfrak{A}$ and $\mathfrak{B}$ is the structure $\mathfrak{A} \amalg \mathfrak{B} = (A \amalg B, \ldots, R^{\mathfrak{A}} \amalg R^{\mathfrak{B}}, \ldots)_{R\in\tau}$. Given $\bar{a} \in A^n$, $\bar{b} \in B^n$, a weak substructure C of $\mathfrak{A} \amalg \mathfrak{B}$ is said to be $M\bar{a}\bar{b}$-*closed* if R^C is $M\bar{a}\bar{b}$-closed for all $R \in \tau$.

DEFINITION. Let N and M be pseudogroups then their *product* NM is defined by the following condition:

$$f \in NM(\mathfrak{A}, \mathfrak{B}) \iff \text{for all } \bar{a} \subseteq \text{dom } f \text{ and all } Maf(a)\text{-closed weak substructure } C \text{ of } \mathfrak{A}\amalg\mathfrak{B},\ (C \restriction A, \bar{a}) \underset{N}{\simeq} (C \restriction B, f(\bar{a})).$$

The following lemma shows that NM is in fact a pseudogroup.

Given classes of formulas of type τ, L_1, L_2 let

$$L_1 L_2 = \{\varphi(\psi_1, \ldots, \psi_k) \mid \varphi \in L_1,\ \psi_i \in L_2,\ i = 1, \ldots, k\}.$$

THEOREM 5. $NM = M(\mathcal{L}(N)\mathcal{L}(M))$.

PROOF. Let $f \in NM(\mathfrak{A}, \mathfrak{B})$ and $\varphi \in \mathcal{L}(N)$ n-ary, $\psi_i \in \mathcal{L}(M)$, $r+n_i$-ary, $i = 1, \ldots, k$. Given $\bar{a} \in A^r$, $\bar{b} \in A^n$, $\bar{a}\bar{b} \subseteq \text{dom } f$, find variants $\bar{\varphi}$, $\bar{\psi}_i$, of φ, ψ_i, respectively, such that

$$\bar{\varphi}[x_1, \ldots, x_r;\ x_{r+1}, \ldots, x_{r+n}] \equiv \varphi[x_{r+1} \ldots x_{r+n}],$$
$$\bar{\psi}_i[x_1 \ldots x_r;\ x_{r+1} \ldots x_{r+n};\ x_{r+n+1} \ldots x_{r+n+n_i}] \equiv \psi_i[x_1 \ldots x_r;\ x_{r+n+1} \ldots x_{r+n+n_i}].$$

Then $\bar{\psi}_i^{\mathfrak{A}\bar{a}\bar{b}} = \psi_i^{\mathfrak{A}\bar{a}}$. By lemma 3 and the definition of NM,

$$(A, \ldots, \bar{\psi}_i^{\mathfrak{A}\bar{a}\bar{b}}, \ldots, \bar{a}\bar{b}) \underset{N}{\simeq} (B, \ldots, \bar{\psi}_i^{\mathfrak{B}f(\bar{a})f(\bar{b})}, \ldots, f(\bar{a})f(\bar{b})).$$

Since $\bar{\varphi} \in \mathcal{L}(N)$, this implies

$$(A, \ldots, \psi_i^{\mathfrak{A}\bar{a}\bar{b}}, \ldots) \models \bar{\varphi}[\bar{a}\bar{b}] \iff (B, \ldots, \psi_i^{\mathfrak{B}f(a)f(b)}, \ldots) \models \bar{\varphi}[f(\bar{a}\bar{b})].$$

Hence, returning to the original formulae

$$(A, \ldots, \psi_i^{\mathfrak{A}\bar{a}}, \ldots) \models \varphi[\bar{b}] \iff (B, \ldots, \psi_i^{\mathfrak{B}f(\bar{a})}, \ldots) \models \varphi[f(\bar{b})]$$

which means, by definition

$$\mathfrak{A} \models \varphi(\psi_1, \ldots, \psi_k)[\bar{a}\bar{b}] \iff \mathfrak{B} \models \varphi(\psi_1, \ldots, \psi_k)[f(\bar{a}\bar{b})].$$

Hence, $f \in M(\mathcal{L}(N)\mathcal{L}(M))(\mathfrak{A}, \mathfrak{B})$.

To show the converse, assume $f \in M(\mathcal{L}(N)\mathcal{L}(M))(\mathfrak{A}, \mathfrak{B})$ and $C \subseteq \mathfrak{A} \amalg \mathfrak{B}$, $\bar{a} \subseteq \text{dom } f$, as in the definition of NM. By lemma 3 the structures $(C \restriction A, \bar{a})$ and $(C \restriction B, f(\bar{a}))$ are of the form $\mathfrak{A}' = (A, \ldots, \psi_i^{\mathfrak{A}\bar{a}}, \ldots)$, $\mathfrak{B}' = (B, \ldots, \psi_i^{\mathfrak{B}f(a)}, \ldots)$ for some formulae $\psi_i \in \mathcal{L}(M)$. Hence, for any $\varphi \in \mathcal{L}(N)$, as we have $\mathfrak{A} \models \varphi(\psi_1, \ldots, \psi_k)[\bar{a}] \iff \mathfrak{B} \models \varphi(\psi_1, \ldots, \psi_k)[f(\bar{a})]$, by hypothesis; then $\mathfrak{A}' \models \varphi[\bar{a}] \iff \mathfrak{B}' \models \varphi[f(a)]$. As this holds for all

$\varphi \in \mathcal{L}(N)$ then $f \upharpoonright \bar{a} \in M(\mathcal{L}(N))(\mathfrak{A}, \mathfrak{b}) = N(\mathfrak{A}, \mathfrak{b})$, which implies $(C \upharpoonright A, \bar{a}) \underset{N}{\sim} (C \upharpoonright B, f(\bar{a}))$. □

COROLLARY 1. $NM \subseteq N \cap M$.

PROOF. It is clear that $\mathcal{L}(N)\mathcal{L}(M) \supseteq \mathcal{L}(N)$, substituting the atomic formulae $\varphi \in \mathcal{L}(N)$ by themselves; then $NM = M(\mathcal{L}(N)\mathcal{L}(M)) \subseteq M(\mathcal{L}(N)) = N$. On the other hand, it may be shown that $\mathrm{Var}[\mathcal{L}(N)\mathcal{L}(M)] \supseteq \mathcal{L}(M)$. To see this, let $\psi \in \mathcal{L}(M)$ be n-ary and R any atomic formula, say r-ary and let $\bar{\psi}$ be a variant of ψ such that

$$\bar{\psi}[x_1,\ldots,x_n,x_{n+1},\ldots,x_{n+r}] \equiv \psi[x_1,\ldots,x_n].$$

As $R \in \mathcal{L}(N)$ then $R(\bar{\psi}) \in \mathcal{L}(N)\mathcal{L}(M)$. Now, for $\bar{a} \in A^n$, $\bar{b} \in A^r$:

$$\mathfrak{A} \models R(\bar{\psi})[\bar{a}\bar{b}] \iff \mathfrak{A}' = (A,\ldots,\bar{\psi}^{\mathfrak{A}\bar{a}},\ldots) \models R[\bar{b}] \iff$$

$$\iff \bar{b} \in R^{\mathfrak{A}'} = \bar{\psi}^{\mathfrak{A}\bar{a}} \iff \mathfrak{A} \models \bar{\psi}[\bar{a}\bar{b}] \iff \mathfrak{A} \models \psi[\bar{a}]$$

taking the variant of $R(\bar{\psi})(x_1,\ldots,x_n,x_{n+1},\ldots,x_{n+r})$ which identifies the variables $x_{n+1},\ldots,x_{n+r}$ with x_1 say, then this variant is identical to ψ, showing that $\psi \in \mathrm{Var}(\mathcal{L}(N)\mathcal{L}(M))$. Now,

$$NM = M(\mathcal{L}(N)\mathcal{L}(M)) = M(\mathrm{Var}(\mathcal{L}(N)\mathcal{L}(M))) \subseteq M(\mathcal{L}(M)) = M. \qquad \square$$

COROLLARY 2. $M_1 \subseteq M_2$, $N_1 \subseteq N_2$ *implies* $N_1 M_1 \subseteq N_2 M_2$.

PROOF. Left to the reader.

COROLLARY 3. *Given a pseudogroup* M, *then* $\mathcal{L}(M)$ *is closed under substitution iff* $MM = M$.

PROOF. We already have $MM \subseteq M$. If $\mathcal{L}(M)$ is closed under substitution then $\mathcal{L}(M)\mathcal{L}(M) \subseteq \mathcal{L}(M)$ which means $MM = M(\mathcal{L}(M)\mathcal{L}(M)) \supseteq M(\mathcal{L}(M)) = M$; hence, $MM = M$. Conversely, if $MM = M$ then $\mathcal{L}(M) \supseteq \mathcal{L}(MM) = \mathcal{L}(M(\mathcal{L}(M)\mathcal{L}(M))) \supseteq \mathcal{L}(M)\mathcal{L}(M)$. □

Not any primitive logic is closed under substitutions, cf. $L^{\alpha+1}_{\infty\omega}$, and not any logic closed under substitution is primitive, cf. $L_{\omega\omega}$. We will relate now these two types of logics.

For a pseudogroups M define by transfinite induction:

$$M^1 = M, \qquad M^{\alpha+1} = M(M^{\alpha}), \qquad M^{\alpha} = \bigcap_{\delta>\alpha} M^{\alpha} \quad (\alpha \text{ limit}).$$

Obviously, all the M^{α} are pseudogroups (theorem 5, and induction in ordinals), and by corollary 1 we have a chain:

$$M = M' \supseteq \ldots \supseteq M^{\alpha} \supseteq \ldots \supseteq M^{\beta} \supseteq \ldots, \qquad \alpha < \beta$$

hence,

$$\mathcal{L}(M) \subseteq \mathcal{L}(M^2) \subseteq \ldots \subseteq \mathcal{L}(M^{\alpha}) \subseteq \ldots$$

where it can be seen from lemma 2, corollary 1, and the fact that $\mathrm{Var}(L_1L_2) \subseteq (\mathrm{Var}\ L_1)(\mathrm{Var}\ L_2)$ that

(1) $\qquad \mathcal{L}(M^{\alpha+1}) = \mathrm{Bool}_\infty(\mathcal{L}(M)\mathcal{L}(M^\alpha))$,

and from the properties of the Galois connection:

(2) $\qquad \mathcal{L}(M^\alpha) = \mathrm{Bool}_\infty(\bigcup_{\delta<\alpha} \mathcal{L}(M^\delta))$ for α limit.

LEMMA 4. *For any ordinals* α, β, $\mathcal{L}(M^\alpha)\mathcal{L}(M^\beta) \subseteq \mathcal{L}(M^{\beta+\alpha})$.

PROOF. By induction on α. For $\alpha = 0$ it is trivial. Consider, now $\varphi \in \mathcal{L}(M^{\alpha+1})$ and $\psi_1,\ldots,\psi_k \in \mathcal{L}(M^\beta)$; then by (1), $\varphi = \bigvee_i \bigwedge \theta^i(\mu_1^i,\ldots,\mu_k^i)$ where $\theta^i \in \mathcal{L}(M)$ and $\mu_1^i,\ldots,\mu_k^i \in \mathcal{L}(M^\alpha)$. By induction hypothesis

$$\mu_j^i(\psi_1,\ldots,\psi_k) \in \mathcal{L}(M^{\beta+\alpha}) \quad \text{for all } i,j.$$

Hence

$$\varphi(\psi_1,\ldots,\psi_k) = \bigvee_i \bigwedge \theta^i(\mu_1^i(\psi_1,\ldots,\psi_k),\ldots,\mu_k^i(\psi_1,\ldots,\psi_k))$$

belongs to $\mathrm{Bool}_\infty(\mathcal{L}(M)\mathcal{L}(M^{\beta+\alpha})) = \mathcal{L}(M^{\beta+(\alpha+1)})$. If α is a limit ordinal and we have $\mathcal{L}(M^\delta)\mathcal{L}(M^\beta) \subseteq \mathcal{L}(M^{\beta+\delta})$ for all $\delta<\alpha$, then

$$\mathcal{L}(M^\alpha)\mathcal{L}(M^\beta) \subseteq \mathrm{Bool}_\infty(\bigcup_{\delta<\alpha} \mathcal{L}(M^\delta)\mathcal{L}(M^\beta)) \subseteq \mathrm{Bool}_\infty(\bigcup_{\delta<\alpha} \mathcal{L}(M^{\beta+\delta})) = \mathcal{L}(M^{\beta+\alpha}). \qquad \square$$

Now, given a *set* of formulae F, define $L_\infty[F]$ as the smallest logic which:

(i) contains F, the atomic and the equality formulae;
(ii) is closed under variants;
(iii) is closed under negations;
(iv) is closed under conjunctions of *sets* of n-ary formulae;
(v) is closed under *quantification* by formulae of F (that is, $Q \in F$, $\psi_1,\ldots,\psi_k \in L$, then $Q(\psi_1,\ldots,\psi_k) \in L$ whenever the substitution may be performed).

We call this the *infinitary logic generated by* F. Analogously, we define $L_\omega[F]$, the *finitary logic generated by* F, changing clause (iv) to:

(iv') is closed under finite conjunctions of n-ary formulae.

For example $L_{\infty\omega} = L_\infty[F]$, and $L_{\omega\omega} = L_\omega[F]$, where $F = \{\exists\}$. In general, if F is a set of Lindström-Mostowski quantifiers, then $L_{\infty\omega}(Q \mid Q \in F) = L_\infty[F \cup \{\exists\}]$ and for F finite, $L_{\omega\omega}(Q \mid Q \in F) = L_\omega[F \cup \{\exists\}]$.

THEOREM 6. *Let* $M = \mathcal{M}(F)$, F *a set of formulae, then*

(a) $L_\infty[F] = \bigcup_{\alpha\in\mathrm{Ord}} \mathcal{L}(M^\alpha)$.

(b) $L_\omega[F] = \bigcup_{n\in\omega} \mathcal{L}(M^n)$ *if* F *is finite (and* τ *finite).*

PROOF. The logics in the right hand side are closed under substitution by Lemma 4, the other properties (i) - (iv) are obvious. Now calling $F' = F \cup At$, we have

(3) $\mathcal{L}(M) = \mathrm{Bool}_\infty(\mathrm{Var}\ F')$

and so

(4) $\mathcal{L}(M^{\alpha+1}) = \mathrm{Bool}_\infty((\mathrm{Bool}_\infty\ \mathrm{Var}\ F')\mathcal{L}(M^\alpha)) = \mathrm{Bool}_\infty((\mathrm{Var}\ F')\mathcal{L}(M^\alpha))$.

If F' is a set then by (2), (3), (4) and induction it is shown simultaneously that each $\mathcal{L}(M^\alpha)$ is a set and, using (i) - (v), $\mathcal{L}(M^\alpha) \subseteq L_\infty[F]$. This shows part (a) in the theorem. Also, if F and the type τ are finite then $\mathrm{Var}\ F'$ is finite in each arity $n \in \omega$. Using (3), (4), and the fact that a finitely generated boolean algebra is finite and complete, one shows by induction on $k \in \omega$ that each $\mathcal{L}(M^k)$ has finitely many n-ary formulae for each $n \in \omega$ and so $\mathcal{L}(M^k) \subseteq L_\omega[F]$. □

REMARK. If we define $qr_F(\varphi)$ as the minimum number of "applications" of rule (v) necessary to obtain $\varphi \in L_\infty[F]$, and $L^\alpha_\infty[F] = \{\varphi \in L_\infty[F] \mid qr_F(\varphi) < \alpha\}$, then we obviously have also

$$L^\alpha_\infty[F] = \bigcup_{\delta<\alpha} \mathcal{L}(M^\delta).$$

This last theorem generalizes the characterization of formulae of $L\omega\omega$ as invariants by Fraïssés families of partial isomorphisms in [11]. From it one can obtain also the characterization of elementary equivalence for arbitrary quantifiers given in [1]. In fact, it improves said characterization, since we use here arbitrary formulae instead of just sentences (quantifiers) to generate the logics, and we do not ask the existential quantifier to be among the generators.

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