

# Lindström's theorem for positive logics, a topological view \*

Xavier Caicedo

Departamento de Matemáticas, Universidad de los Andes  
AA 4976, Bogotá, Colombia

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## Abstract

The topology induced on classes of first order structures by the elementary classes is characterized among finer topologies by intrinsic non-boolean properties. This yields a topological version of Lindström's first theorem which applies to certain extensions of first order logic lacking classical negation. In this setting, we examine the possibility of extending to infinite models the equilibrium game semantics of imperfect information logic on finite models, and show that under reasonable conditions this is possible only for essentially first order fragments of the later logic.

## 1 Introduction

Closure under classical negation is an essential feature in the usual proofs of the celebrated Lindström's theorems characterizing first order logic [21] and many other results in abstract model theory. We provide in this paper an intrinsic characterization of the elementary topology, actually a topological

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version of Lindström’s first theorem, from which negationless forms of this result follow. It is well known that for each first-order vocabulary  $\tau$  the class of first-order structures of type  $\tau$  becomes a topological space by taking as a basis the elementary classes  $Mod(\varphi)$ ,  $\varphi$  a first-order sentence, so that model theoretic properties as compactness correspond to genuine topological properties. We show that among finer topologies this is the unique regular compact topology for which the class of countable structures is dense, and the forgetful and renaming functors relating distinct vocabularies are continuous. Since any model theoretic logic gives rise similarly to a natural topology, we obtain versions of Lindström’s theorem which apply to certain extensions of first-order logic lacking classical negation.

One of the most interesting such extensions is imperfect information logic in its various versions (cf. [23], [30]). In this context, we discuss the problem of extending to infinite models the  $[0,1]$ -valued equilibrium semantics on finite models introduced in [28], [15] for these logics. We show that under natural desirable conditions this is possible only for essentially first order fragments of imperfect information logic.

Model theoretic logics without negation have been considered by García-Matos in [17]. Topological ideas have been present in classical model theory from its beginnings, mainly via the study of spaces of enumerated countable models [11] or spaces of types [24], [25]. Working directly with spaces of models appears first in Fraïssé’s beautiful proof of countable compactness of first order logic [13]. This approach has been exploited in the context of model theoretic logics by Mundici in [26] and the author in [5], [6], [7].

We refer the reader to [10], [2], [32] for unexplained concepts in model theory, abstract model theory or topology, respectively.

## 2 The elementary topology

Given a first-order vocabulary  $\tau$ ,  $L_{\omega\omega}(\tau)$  is the set of first-order sentences of type  $\tau$ . The *elementary topology* on the class  $St_\tau$  of first-order structures type  $\tau$  is obtained by taking the family of *elementary classes*

$$Mod(\varphi) = \{M : M \models \varphi\}, \quad \varphi \in L_{\omega\omega}(\tau)$$

as an open basis. Due to the presence of classical negation, this family is also a closed basis and thus the closed classes of  $St_\tau$  are the first-order axiomatizable classes  $Mod(T)$ ,  $T \subseteq L_{\omega\omega}(\tau)$ . Possible foundational problems

due to the fact that the topology is a class of proper classes may be settled observing that it is indexed by a set, namely the set of theories of type  $\tau$ .

The main facts of model theory are reflected by the topological properties of these spaces. Thus, the *downward Löwenheim-Skolem theorem* for sentences amounts to topological density of the subclass of countable structures. *Loś theorem* on ultraproducts grants that *U-limits* exist for any ultrafilter  $U$ , condition well known to be equivalent to topological compactness, which amounts in turn to model theoretic compactness.

These spaces are not Hausdorff or  $T_1$ , but having a clopen basis they are *regular*; that is, closed classes and exterior points may be separated by disjoint open classes. All properties of regular compact spaces are then available: normality, complete regularity, uniformizability, the Baire property, etc.

Many model theoretic properties are related to the continuity of natural operations between classes of structures. The following operations are readily seen to be continuous and play an important role in abstract model theory:

- *Reducts*,  $\pi_{\sigma\tau} : St_\sigma \rightarrow St_\tau$ ,  $\pi_{\sigma\tau}(M) = M \upharpoonright \tau$ , which forgets the interpretation of the symbols in  $\sigma \setminus \tau$  when  $\tau \subseteq \sigma$ .

- *Renamings*,  $\hat{\alpha} : St_\sigma \rightarrow St_\tau$  that change the names of the interpretations according to a bijection of vocabularies  $\alpha : \sigma \rightarrow \tau$  respecting kind and arity of the symbols. More precisely,  $(\alpha R)^{\hat{\alpha}(M)} = R^M$  for any  $R \in \sigma$ .

Not only these, but a host of operations are continuous for the elementary topology: first-order interpretations, cartesian products (as operations on several arguments), any operation with  $\Sigma_1^1$  graph (cf. [6]), any operation enjoying the *uniform reduction property* in the sense of Feferman and Vaught [12]. Consider, for example:

- *Restrictions*,  $r_{\tau P} : St_\tau \rightarrow St_\tau$ , where  $P \in \tau$  is a monadic predicate symbol, sending  $M$  to the substructure  $M \upharpoonright P^M$  with universe  $P^M$ , when this set is a subuniverse of  $M$ . If  $\tau$  has function symbols, this is a partial operation with closed domain. It is continuous because  $r_P^{-1}(Mod_\tau(\varphi)) = Mod_\tau(\varphi^P)$  where  $\varphi^P$  results of relativizing the quantifiers of  $\varphi$  to  $P$ .

- *Disjoint sums*, which assign to a family of structures  $(M_i)_{i \in I} \in \prod_i St_{\tau_i}$  the structure of type  $\oplus_i \tau_i := \cup_i \{P_i\} \cup \underline{\tau}_i$ :

$$\Sigma_{i \in I} M_i = (\cup_i \underline{A}_i, \underline{M}_i)_{i \in I},$$

where the  $\underline{M}_i$  are mutually disjoint renamings of the  $M_i$  by disjoint vocabularies  $\underline{\tau}_i$ , and have universes  $\underline{A}_i$  which interpret the predicates  $P_i$ . The resulting map  $S : \prod_i St_{\tau_i} \rightarrow St_{\oplus_i \tau_i}$  is continuous due to Feferman-Vaught theorem

[12] which grants for each sentence  $\varphi \in L_{\omega\omega}(\bigoplus_i \tau_i)$  sentences  $\psi_j \in L_{\omega\omega}(\tau_{i_j})$ ,  $j = 1, \dots, k$ , and a boolean condition  $B(p_1, \dots, p_k)$  such that  $\Sigma_{i \in I} M_i \models \varphi$  if and only if  $B(M_{i_1} \models \psi_1, \dots, M_{i_k} \models \psi_k)$  holds. Taking a disjunctive normal form of  $B$ , this is seen to mean that  $(M_i)_{i \in I}$  belongs to a union of basic classes of the product topology on  $\Pi_i St_{\tau_i}$ . Hence,  $S^{-1}(Mod(\varphi))$  is open.

### 3 Topological regularity

As noticed before, a topological space is regular if closed sets and exterior points may be separated by open sets. It is *normal* if disjoint closed sets may be separated by disjoint open sets. We do not include the Hausdorff property in our definitions. Thus, normality does not imply regularity here. However, a regular compact space is normal. Actually, a regular Lindelöf space is already normal (cf. [32], exc. 32.4).

Consider the following equivalence relation in a space  $X$ :

$$x \equiv y \Leftrightarrow cl\{x\} = cl\{y\}$$

where  $cl$  denotes topological adherence. Clearly,  $x \equiv y$  if and only if  $x$  and  $y$  belong to the same closed (open) subsets (of a given basis). Let  $X_{/\equiv}$  be the quotient space and  $\eta : X \rightarrow X_{/\equiv}$  the natural projection. Then  $X_{/\equiv}$  is  $T_0$  by construction but not necessarily Hausdorff. The following claims are easily verified:

a)  $\eta : X \rightarrow X_{/\equiv}$  induces an isomorphism between the respective lattices of Borel subsets of  $X$  and  $X_{/\equiv}$ . In particular, it is open and closed, preserves disjointedness, preserves and reflects compactness and normality.

b) The assignment  $X \mapsto X_{/\equiv}$  is functorial, because  $\equiv$  is preserved by continuous functions and thus any continuous map  $f : X \rightarrow Y$  induces a continuous assignment  $f_{/\equiv} : X_{/\equiv} \rightarrow Y_{/\equiv}$  which commutes with composition.

c)  $X \mapsto X_{/\equiv}$  preserves products; that is,  $(\Pi_i X_i)_{/\equiv}$  is canonically homeomorphic to  $\Pi_i (X_{i/\equiv})$  with the product topology.

d) If  $X$  is regular, the equivalence class of  $x$  is  $cl\{x\}$  (this may fail in the non regular case).

e) If  $X$  is regular,  $X_{/\equiv}$  is Hausdorff: if  $x \not\equiv y$  then  $x \notin cl\{y\}$  by (d); thus there are disjoint open sets  $U, V$  in  $X$  such that  $x \in U$ ,  $cl\{y\} \subseteq V$ , and their images under  $\eta$  provide an open separation of  $\eta x$  and  $\eta y$  in  $X_{/\equiv}$  by (a).

f) If  $K_1$  and  $K_2$  are disjoint compact subsets of a regular topological space  $X$  that can not be separated by open sets, then there exist  $x_i \in K_i$ ,

$i = 1, 2$ , such that  $x_1 \equiv x_2$ . Indeed,  $\eta K_1$  and  $\eta K_2$  are compact in  $X_{/\equiv}$  by continuity and thus closed because  $X_{/\equiv}$  is Hausdorff by (e) above. They can not be disjoint; otherwise, they would be separated by open sets whose inverse images would separate  $K_1$  and  $K_2$ . Pick  $\eta x = \eta y \in \eta K_1 \cap \eta K_2$  with  $x \in K_1, y \in K_2$ .

Clearly, for the elementary topology on  $St_\tau$ , the relation  $\equiv$  coincides with elementary equivalence of structures and  $St_{\tau/\equiv}$  is homeomorphic to the Stone space of complete theories.

As noticed before, the spaces  $St_\tau$  with the elementary topology are uniformizable. It is easy to check that the family of relations  $\equiv_{n,\tau_0}$  ( $\tau_0$ -elementary equivalence of structures up to quantifier rank  $n$ ), with  $n \in \omega$  and finite  $\tau_0 \subseteq \tau$ , forms a uniformity basis.

## 4 An intrinsic characterization of the elementary topology

In this section,  $x, y, \dots$  will denote first-order structures in  $St_\tau$ ,  $x \approx y$  will denote isomorphism.

$x \sim_{n,\tau} y$  means that there is a sequence  $\emptyset \neq I_0 \subseteq \dots \subseteq I_n$  of sets of  $\tau$ -partial isomorphism of finite domain so that, for any  $i < j \leq n$ ,  $f \in I_i$  and  $a \in x$  (respectively,  $b \in y$ ), there is  $g \in I_j$  such that  $g \supseteq f$  and  $a \in \text{Dom}(g)$  (respectively,  $b \in \text{Im}(g)$ ). The later is called the *extension property*.

$x \sim_\tau y$  means the above holds for an infinite chain  $\emptyset \neq I_0 \subseteq \dots \subseteq I_n \subseteq \dots$

Fraïssé's characterization of elementary equivalence [13] says that for finite relational vocabularies:  $x \equiv_{n,\tau} y$  if and only if  $x \sim_{n,\tau} y$ . To have it available for vocabularies containing function symbols it is enough to add to the quantifier rank the complexity of terms in atomic formulas.

It is well known that for countable  $x, y$ :  $x \sim_\tau y$  implies  $x \approx y$ .

Given a vocabulary  $\tau$  let  $\tau^*$  be a disjoint renaming of  $\tau$ . If  $x, y \in St_\tau$  have the same power, let  $y^*$  be an isomorphic copy of  $y$  sharing the universe with  $x$  and renamed to be of type  $\tau^*$ . In this context,  $(x, y^*)$  will denote the  $\tau \cup \tau^*$ -structure that results of expanding  $x$  with the relations of  $y^*$ .

**Lemma 1** *There is a vocabulary  $\tau^+ \supseteq \tau \cup \tau^*$  such that for each finite vocabulary  $\tau_0 \subseteq \tau$  there is a sequence of elementary classes  $\Delta_1 \supseteq \Delta_2 \supseteq \Delta_3 \supseteq \dots$  in  $St_{\tau^+}$  such that if  $\pi = \pi_{\tau^+, \tau \cup \tau^*}$  then (1)  $\pi(\Delta_n) = \{(x, y^*) : |x| = |y| \geq \omega\}$ ,*

$x \equiv_{n, \tau_0} y\}$ , (2)  $\pi(\bigcap_n \Delta_n) = \{(x, y^*) : |x| = |y| \geq \omega, x \sim_{\tau_0} y\}$ . Moreover,  $\bigcap_n \Delta_n$  is the reduct of an elementary class.

**Proof.** Let  $\Delta$  be the class of structures  $(x, y^*, <, a, I)$  where  $<$  is a discrete linear order with minimum but no maximum and  $I$  codes for each  $c \leq a$  a family  $I_c = \{I(c, i, -, -)\}_{i \in x}$  of partial  $\tau_0$ - $\tau_0^*$ -isomorphisms from  $x$  into  $y^*$ , such that for  $c < c' \leq a$ :  $I_c \subseteq I_{c'}$  and the extension property holds for this pair. Describe this by a first-order sentence  $\theta_\Delta$  of type  $\tau^+ \supseteq \tau_0 \cup \tau_0^*$  and set  $\Delta_n = \text{Mod}_L(\theta_\Delta \wedge \exists^{\geq n} x(x \leq a))$ . Then condition (1) in the Lemma is granted by Fraïssé's characterization and the fact that  $x$  being infinite has room to code all partial isomorphisms of finite domain, and condition (2) is granted because  $(x, y^*, <, a, I) \in \bigcap_n \Delta_n$  if and only if  $<$  contains an infinite increasing  $\omega$ -chain below  $a$ , a  $\Sigma_1^1$ -condition.  $\square$

A topology on  $St_\tau$  will be called *invariant* if its open (closed) classes are closed under isomorphic structures. In the following we will always assume this property of the topologies considered. Of course, it is superfluous if we identify isomorphic structures.

**Theorem 1** *Let  $\Gamma_\tau$  be a regular compact topology finer than the elementary topology on each class  $St_\tau$ , such that the countable structures are dense in  $St_\tau$  and reducts and renamings are continuous for these topologies. Then  $\Gamma_\tau$  is the elementary topology for all  $\tau$ .*

**Proof.** We first show that any pair of disjoint closed classes  $C_1, C_2$  of  $\Gamma_\tau$  may be separated by an elementary class. Assume this is not the case. Since the  $C_i$  are compact in the topology  $\Gamma_\tau$  then they are compact for the elementary topology and, by regularity of the latter, there exist  $x_i \in C_i$  such that  $x_1 \equiv x_2$  in  $L_{\omega\omega}(\tau)$  by claim (f) in the previous section. The  $x_i$  must be infinite, otherwise they would be isomorphic contradicting the disjointedness of the  $C_i$ . By normality of  $\Gamma_\tau$ , there are towers  $U_i \subseteq C'_i \subseteq U'_i \subseteq C''_i$ ,  $i = 1, 2$ , separating the  $C_i$  with  $U_i, U'_i$  open and  $C'_i, C''_i$  closed in  $\Gamma_\tau$  and disjoint. Let  $I$  be a first-order sentence of type  $\tau' \supseteq \tau$  such that  $(z, ..) \models I \Leftrightarrow z$  is infinite, and let  $\pi$  be the corresponding reduct operation. For fixed  $n \in \omega$  and finite  $\tau_0 \subseteq \tau$  let  $t$  be a first-order sentence describing the common  $\equiv_{n, \tau_0}$ -equivalence class of  $x_1, x_2$ . As

$$(x_i, ..) \in \text{Mod}_{\tau'}(I) \cap \pi^{-1} \text{Mod}(t) \cap \pi^{-1} U_i, \quad i = 1, 2$$

and this class is open in  $\Gamma_{\tau'}$  by continuity of  $\pi$ , then by the density hypothesis there are countable  $x_i \in U_i$ ,  $i = 1, 2$ , such that  $x_1 \equiv_{n, \tau_0} x_2$ . Thus for some expansion of  $(x_1, x_2^*)$ ,

$$(x_1, x_2^*, \dots) \in \Delta_{n, \tau_0} \cap \pi_1^{-1}(C'_1) \cap (\rho\pi_2)^{-1}(C'_2), \quad (1)$$

where  $\Delta_{n, \tau_0}$  is the class of Lemma 1,  $\pi_1, \pi_2$  are reducts, and  $\rho$  is a renaming:

$$\begin{aligned} \pi_1(x_1, x_2^*, \dots) &= x_1 & \pi_1 : St_{\tau^+} &\rightarrow St_{\tau \cup \tau^*} \rightarrow St_{\tau} \\ \pi_2(x_1, x_2^*, \dots) &= x_2^* & \pi_2 : St_{\tau^+} &\rightarrow St_{\tau \cup \tau^*} \rightarrow St_{\tau^*} \\ \rho(x_2^*) &= x_2 & \rho : St_{\tau^*} &\rightarrow St_{\tau}. \end{aligned}$$

Since the classes (1) are closed by continuity of the above functors then  $\bigcap_n \Delta_{n, \tau_0} \cap \pi_1^{-1}(C'_1) \cap (\rho\pi_2)^{-1}(C'_2)$  is non-empty by compactness of  $\Gamma_{\tau^+}$ . But  $\bigcap_n \Delta_{n, \tau_0} = \pi(V)$  with  $V$  elementary of type  $\tau^{++} \supseteq \tau^+$ . Then

$$V \cap \pi^{-1}\pi_1^{-1}(U'_1) \cap \pi^{-1}(\rho\pi_2)^{-1}(U'_2) \neq \emptyset$$

is open of  $\Gamma_{L^{++}}$  and by the density condition it must contain a countable structure  $(x_1, x_2^*, \dots)$ . Thus  $(x_1, x_2^*, \dots) \in \bigcap_n \Delta_{n, \tau_0}$ , with  $x_i \in U'_i \subseteq C''_i$ . It follows that  $x_1 \sim_{\tau_0} x_2$  and thus  $x_1 \upharpoonright \tau_0 \approx x_2 \upharpoonright \tau_0$ . Let  $\delta_{\tau_0}$  be a first-order sentence of type  $\tau \cup \tau^* \cup \{h\}$  such that:  $(x, y^*, h) \models \delta_{\tau_0} \Leftrightarrow h : x \upharpoonright \tau_0 \approx y \upharpoonright \tau_0$ . By compactness,

$$\left( \bigcap_{\tau_0 \subseteq_{fin} \tau} Mod_{\tau \cup \tau^* \cup \{f\}}(\delta_{\tau_0}) \right) \cap \pi_1^{-1}(C''_1) \cap (\rho\pi_2)^{-1}(C''_2) \neq \emptyset$$

and we have  $h : x_1 \approx x_2$ ,  $x_i \in C''_i$ , contradicting the disjointedness of the  $C''_i$ . Finally, if  $C$  is a closed class of  $\Gamma_{\tau}$  and  $x \notin C$ ,  $cl_{\Gamma_{\tau}}\{x\}$  is disjoint from  $C$  by regularity of  $\Gamma_{\tau}$ . Then  $cl_{\Gamma_{\tau}}\{x\}$  and  $C$  may be separated by open classes of the elementary topology, which implies  $C$  is closed in this topology.  $\square$

If we ask the continuity of further operations, we may trade off the compactness hypothesis in Theorem 1 for normality, or the Lindelöf property, due to the following topological version of Th. 15 in [5].

**Theorem 2** *Let  $\Gamma_{\tau}$  be a regular normal (or Lindelöf) topology on each  $St_{\tau}$  such that reducts, renamings, restrictions, and disjoint sums are continuous. Then  $\Gamma_{\tau}$  is compact.*

**Proof.** By hypothesis, the disjoint sum embedding  $S : \prod_i St_{\tau_i} \rightarrow St_{\oplus_i \tau_i}$  is continuous for the Tychonoff product of the topologies  $\Gamma_{\tau_i}$  in the domain and the topology  $\Gamma_{\oplus_i \tau_i}$  in the co-domain. On the other hand, the map  $D : St_{\oplus_i \tau_i} \rightarrow \prod_i St_{\tau_i}$ :

$$M \longmapsto (\text{renam}_{\tau_i, \tau_i}(M \upharpoonright P_i^M \upharpoonright \mathcal{I}_i))_i$$

decomposing a structure into a family of restricted renamed reducts, is also continuous for these topologies since each projection  $St_{\oplus_i \tau_i} \rightarrow St_{\tau_i}$  is continuous by hypothesis. Clearly,  $D \circ S$  is the identity on  $\prod_i St_{\tau_i}$  (module isomorphism) and by the functorial properties of  $X \longmapsto X_{/\equiv}$ ,  $D_{/\equiv} \circ S_{/\equiv}$  is the identity in  $(\prod_i St_{\tau_i})_{/\equiv}$ . Therefore,  $S_{/\equiv}$  is injective and its image  $C$  in  $St_{\oplus_i \tau_i / \equiv}$  is a retract of this space. Since the latter space is Hausdorff by regularity of  $\Gamma_{\oplus_i \tau_i}$ , and retracts of Hausdorff spaces are closed, then  $C$  is closed. As  $St_{\oplus_i \tau_i}$  is normal by hypothesis, so is  $St_{\oplus_i \tau_i / \equiv}$  and thus  $C$  is normal also. Therefore,  $\prod_i (St_{\tau_i / \equiv}) \approx (\prod_i St_{\tau_i})_{/\equiv} \approx C$  is normal. In particular, all powers  $(St_{\tau / \equiv})^I$  are normal. By Noble's theorem stating that a Hausdorff space  $X$  is compact if and only if the power space  $X^\kappa$  is normal for all  $\kappa$  (see [27], Corollary 2.2), we conclude that  $St_{\tau / \equiv}$  and thus  $St_\tau$  is compact for  $\Gamma_\tau$ . Finally, recall that regular Lindelöf spaces are normal.  $\square$

## 5 Lindström's theorem revisited

A model theoretic logic is a pair  $(L, \models)$  where  $L$  is an assignment  $\tau \longmapsto L_\tau$  from first-order vocabularies to classes of "sentences", and  $\models$  is a relation in  $\cup_\tau S_\tau \times L_\tau$  satisfying Lindström's axioms (cf. [2], II. Definition 1.1.1):

- *Isomorphism*: If  $M \approx N$  then  $M \models \varphi$  iff  $N \models \varphi$ .
- *Reduct*: If  $\sigma \subseteq \tau$  then  $L_\sigma \subseteq L_\tau$  and for any  $\varphi \in L_\sigma$ ,  $M \in St_\tau : M \upharpoonright \sigma \models \varphi$  iff  $M \models \varphi$ .
- *Renaming*: A bijection  $\alpha : \sigma \rightarrow \tau$  respecting kind and arity induces a map  $t : L_\sigma \rightarrow L_\tau$ , so that  $M \models \varphi$  iff  $\hat{\alpha}M \models t\varphi$ , where  $\hat{\alpha} : St_\sigma \rightarrow St_\tau$  renames the structures according to  $\alpha$ .

Our results will hold under relaxed versions of the reduct and renaming axioms where the inclusion  $L_\sigma \subseteq L_\tau$  and the substitution map  $t : L_\sigma \rightarrow L_\tau$  are replaced with maps  $L_\sigma \rightarrow P(L_\tau)$  from symbols to theories.

Define  $L \leq_w L'$  ( $L'$  is a *weak extension* of  $L$ ) if each sentence of  $L_\tau$  is



equivalent to a theory of  $L'_\tau$ . We will say that  $L$  is *equivalent in theories* to  $L'$  if  $L \leq_w L'$  and  $L' \leq_w L$ .

Closure of  $L$  under boolean connectives  $\neg, \vee, \wedge$ , or relativization, is defined as usual. The same for compactness. As noticed in [17], the downward Löwenheim-Skolem theorem splits for negationless logics in several non equivalent versions:

- $\text{LSk}_1$ : If any sentence has a model it has a countable model (the familiar version).
- $\text{LSk}_2$ : Any sentence true in all countable models is true in all models.
- $\text{LSk}_3$ : Two sentence equivalent in all countable models are equivalent (this property implies the other two).

**Examples.**  $\Sigma_1^1$  (second order existential logic) satisfies the first but not the second version, and the opposite is true of  $\Pi_1^1$  (second order universal logic) because  $\{\mathbb{R}\}$  is  $\Pi_1^1$ -definable. The fragment  $L_{\omega\omega}(Q_0^+)$  of  $L_{\omega\omega}(Q_0)$  obtained by closing  $L_{\omega\omega}$  under the quantifier  $Q_0$  (there are infinitely many...) and  $\wedge, \vee, \exists, \forall$ , but not  $\neg$ , satisfies all three versions.

**Definition.** For any model-theoretic logic  $L$  let  $\Gamma_\tau(L)$  the topology on  $St_\tau$  obtained by taking the classes  $Mod(\theta)$ ,  $\theta \in L_\tau$ , as a sub-basis of closed classes.

This topology is invariant, and reducts and renamings become automatically continuous due to the (relaxed) Lindström's axioms. Model theoretic compactness of  $L$  is equivalent to topological compactness of  $\Gamma_\tau(L)$  by Alexander sub-basis lemma (cf [32]). Closure under connectives refers to properties of the sub-basis; for example, closure under disjunctions grants it is a closed basis.  $\text{LSk}_2$  corresponds in the later case to topological density of the countable structures.  $\text{LSk}_1$ , instead, is basis dependent if the logic is not closed under negations. Finally, notice that  $L \leq_w L'$  means that  $\Gamma_\tau(L')$  is finer than  $\Gamma_\tau(L)$ .

Call a logic *regular* if the topology it induces is regular<sup>1</sup>. In a regular compact logic closed under disjunctions and conjunctions,  $\text{LSk}_1$  implies  $\text{LSk}_2$ . Indeed, if  $M \notin Mod(T)$ , then by regularity  $M$  belongs to a closed class  $C$  contained in  $Mod(T)^c$ . By compactness there are finitely many closed basics  $C_i$  such that  $C \subseteq \bigcap_i C_i \subseteq Mod(\varphi)^c$ . By the closure conditions  $\bigcap_i C_i$  contains a countable model if  $\text{LSk}_1$  holds. Thus,  $Mod(\varphi)^c$  has a countable model and  $\text{LSk}_2$  holds.  $\square$

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<sup>1</sup>No to be confused with the notion of regular logic in [2].

The next version of Lindström's theorem follows immediately from Theorem 1 and the previous observation.

**Theorem 3** (Positive Lindström's theorem) *Any regular compact logic  $L$  extending weakly  $L_{\omega\omega}$ , closed under disjunctions and satisfying  $LSk_2$ , is equivalent in theories to  $L_{\omega\omega}$ . If  $L$  is also closed under conjunctions, the results holds under  $LSk_1$ .*

**Examples.** a) Trivial examples of logics satisfying the hypothesis of the first part of this theorem are  $th_{\omega\omega,\tau}^\kappa = \{T : T \text{ is a first-order theory of type } \tau, |T| \leq \kappa\}$ . From these, only  $th_{\omega\omega,\tau}^\omega$  satisfies  $LSk_1$ . This shows that for negationless logics regularity and compactness do not grant finite dependence nor any bound on the dependence number.

b) Regularity can not be dropped from the hypothesis of Theorem 3. For example,  $L_{\omega\omega}(Q_0^+)$  satisfy  $LSk_2$  and  $LSk_1$ , and it is compact by an ultraproduct argument (see next section), but the class  $Mod(\exists x Q_0 y(y < x))$  is not first-order axiomatizable because its complement is not closed under ultrapowers. We conclude that this logic can not be regular.

c) Similarly,  $\Sigma_1^1$  can not be regular because it satisfies the other hypothesis of Theorem 3 (with  $LSk_1$ ) but its sentences are not all reducible to first-order theories. However, its topology is normal due to Robinson's consistency lemma. This shows that we can not replace regularity with normality in any of our previous results.

Topological regularity of  $\Gamma_\tau(L)$  does not have a simple model theoretic description, but in compact logics closed under  $\vee$  and  $\wedge$ , it implies the following weak form of negation: *for any  $\varphi \in L_\tau$  there is  $\{\theta_i\}_{i \in I} \subseteq L_\tau$  such that  $M \not\models \varphi$  if and only if  $M \models \vee_i \theta_i$ .* The second claim of Theorem 3, but not the first, may be shown utilizing this hypothesis instead of regularity.

Lindström's original theorem is an immediate corollary of Theorem 3 (for a wider notion of extension).

**Theorem 4** *A compact weak extension of first-order logic closed under boolean connectives and satisfying  $LSk_1$  is equivalent (in sentences) to  $L_{\omega\omega}$ .*

**Proof.** Closure of  $L$  under boolean connectives means that the closed subbasis of  $\Gamma_\tau(L)$  is actually a clopen basis and thus the space is regular. By Theorem 3, for any  $\theta \in L_\tau$  the complementary classes  $Mod(\theta)$  and  $Mod(\neg\theta)$  are equivalent to first-order theories, thus  $\theta$  is elementary.  $\square$

Versions of Theorems 3 and 4 in terms of  $[\omega_1, \infty]$ -compactness instead of compactness may be obtained from Theorem 2 if we add closure under relativizations and the uniform reduction property for disjoint sums.

## 6 Compactness and ultraproducts

The relation between compactness and ultraproducts survives in negationless logics because it is a purely topological convergence phenomenon. Recall that given an ultrafilter  $U$  over  $I$ , a family  $\{a_i\}_{i \in I}$   $U$ -converges to an element  $x$  in a space  $X$ , in symbols  $\{a_i\}_{i \in I} \rightarrow_U x$ , if and only if  $\{i \in I : a_i \in V\} \in U$  for any open (sub-basic) neighborhood  $V$  of  $x$ . It is well known that a space is compact iff any  $I$ -family of  $X$  has an  $U$ -limit.

Given a logic  $L$ , convergence of structures, say  $\{M_i\}_{i \in I} \rightarrow_U M$ , expressed in terms of open sub-basics of  $\Gamma_\tau(L)$  means:  $M \not\models \varphi \Rightarrow \{i \in I : M_i \not\models \varphi\} \in U$ ; that is

$$\text{for all } \varphi \in L_\tau : \{i \in I : M_i \models \varphi\} \in U \Rightarrow M \models \varphi. \quad (2)$$

Therefore, a logic is compact if and only if for any family of structures  $\{M_i\}_{i \in I}$  of the same type and ultrafilter  $U$  over  $I$  there exists  $M$  such that (2) holds. Thus, any logic  $L$  satisfying the following *one-way ultraproduct property* must be compact:

$$\{i \in P : M_i \models \varphi\} \in U \Rightarrow \Pi_i M_i / U \models \varphi. \quad (3)$$

**Example.** Let  $\mathcal{K}$  be a family of Lindström quantifiers closed under ultraproducts (for example, cardinality quantifiers  $Q_\alpha$ , Magidor-Malitz quantifiers, the Hartig quantifier, etc.), then the logic  $L_{\omega\omega}(\mathcal{K}^+)$  obtained by closing  $L_{\omega\omega}$  under the quantifiers in  $\mathcal{K}$  and  $\wedge, \vee, \exists, \forall$  may be shown to satisfy (3) by a simple induction.

Reciprocally, compactness of  $L$  implies that for any family  $\{M_i\}_{i \in I}$  of structures and any ultrafilter  $U$  over  $I$  there is an extension  $M \geq \Pi_i M_i / U$  such that for any  $\varphi \in L_{\tau \cup \{c, d, \dots\}}$ ,  $f, g, \dots \in \Pi_i M_i$

$$\{i \in P : M_i \models \varphi[f(i), g(i), \dots]\} \in U \Rightarrow M \models \varphi[f/U, g/U, \dots], \quad (4)$$

because the family of expansions  $M_i^+ = (M_i, f(i))_{f \in \Pi_i M_i}$ , where  $f(i)$  interprets  $c_f$ , must have a  $U$ -limit  $(M, a_f)_{f \in \Pi_i M_i}$ , and applying (4) to atomic sentences yields an embedding  $f/U \mapsto a_f : \Pi_i M_i / U \leq M$ .

In this vein, the topological proof given in [7] of Shelah's characterization of  $[\lambda, \kappa]$ -compact boolean closed logics in terms of ultraproducts, [22], may be readily adapted to negationless logics. Regularity of the logic is not required.

Recall that a logic  $L$  is  $[\lambda, \kappa]$ -compact if, for any family of theories  $\{T_\alpha\}_{\alpha < \kappa}$  in  $L_\tau$ , if  $\cup_{j \in S} T_{\alpha_j}$  is satisfiable for all  $S \subseteq \kappa$  of power less than  $\lambda$  then  $\cup_{\alpha < \kappa} T_\alpha$  is satisfiable. An ultrafilter  $U$  is  $(\lambda, \kappa)$ -regular if there is  $F \subseteq U$  of power  $\kappa$  such that  $\cap J = \emptyset$  for any  $J \subseteq F$  of power  $\lambda$ .

**Theorem 5** (Positive abstract compactness theorem) *Let  $L$  be a logic  $L$  closed under disjunctions. If there is a  $(\lambda, \kappa)$ -regular ultrafilter  $U$  over  $\kappa^{<\lambda}$  such that for any family  $\{M_i\}_{i \in I}$  of structures there is an extension  $M \geq \prod_i M_i / U$  such that for any  $\varphi \in L_{\tau \cup \{c, d, \dots\}}$  and  $f, g, \dots \in \prod_i M_i$  (4) holds, then  $L$  is  $[\lambda, \kappa]$ -compact. If  $L$  is closed under relativizations the reciprocal holds.*

## 7 Imperfect information logic as $[0,1]$ -valued logic

One of the most interesting logics lacking classical negation is *imperfect information logic* in its various versions. These derive from independence friendly logic  $IF$ , introduced originally by Hintikka as a "friendly" way of expressing Henkin quantifiers [19]. Its most general syntax extends that of  $L_{\omega\omega}$  allowing for each finite set of variables  $Y$  and formulas  $\varphi, \psi$ , expressions  $\exists x_{/Y} \varphi$ ,  $\forall x_{/Y} \varphi$ ,  $\varphi \vee_{/Y} \psi$ ,  $\varphi \wedge_{/Y} \psi$ , where the decoration  $_{/Y}$  expresses independence with respect to the variables in  $Y$ , as in the usual definition of uniform continuity of a function:

$$\forall x \quad \forall \varepsilon > 0 \quad \exists \delta_{/x} \quad \forall y \quad (|x - y| < \delta \rightarrow |fx - fy| < \varepsilon)$$

|  
"there is  $\delta$  not depending on  $x$ "

The meaning of  $\forall x_{/Y} \varphi$ ,  $\varphi \vee_{/Y} \psi$ ,  $\varphi \wedge_{/Y} \psi$  is more clearly explained by the game semantics governing this logic [23]. To each sentence of  $\varphi$  and structure  $M$  of the same type is associated a two-players game  $G(\varphi, M)$  which is played from the root down along a branch of the syntactical tree of  $\varphi$ . The two players that we call  $E$  and  $A$  make their moves at each node depending on its label as follows:

$\vee_{/Y}$  :  $E$  chooses the left or right descendant (subformula)

$\wedge_{/Y}$  :  $A$  chooses the left or right descendant  
 $\exists x_{/Y}$  :  $E$  assigns a value  $a \in M$  to the variable  $x$   
 $\forall x_{/Y}$  :  $A$  assigns a value  $a \in M$  to the variable  $x$   
 $\neg$  : interchange roles and proceed to next node

Each move is made in ignorance of the value previously assigned to the variables in  $Y$ . When a terminal node is reached,  $E$  wins if the atomic formula labeling the node is true for the valuation of variables constructed by the players along the way; otherwise  $A$  wins (if the roles have been reversed by the last negation reached, it is the opposite).

A sentence  $\varphi$  is said to be *true* if  $E$  has a winning strategy in this game, *false* if  $A$  has a winning strategy. We write, respectively,  $M \models^+ \varphi$  and  $M \models^- \varphi$  (positive and negative satisfaction).

For first order sentences (where  $/\emptyset$  is the only decoration) the game is determined and thus one of the players has a winning strategy according to the classical truth value. This may be seen observing that the existence of a winning strategy for  $E$  (for  $A$ ) corresponds to the satisfaction of a particular (dual) Skolem normal form of  $\varphi$ . For the new sentences of the language this is a game of imperfect information which may be undetermined. That is, it may happen that  $M \models^+ \varphi$  and  $M \models^- \varphi$  and thus  $\varphi$  does not have a truth value in the model  $M$ .

Game semantics of  $IF$  behaves so differently from ordinary semantics that it raises several non-trivial issues. The valid formulas of the form  $\forall x \exists y \forall z \exists w_{/x} \varphi$  ( $\varphi$  first-order) form already a non arithmetical set. Validity is not even  $\Sigma_2$  definable in set theory (Väänänen [29]). A serious study was possible only after streamlining its syntax and Hodges discovery that it has a compositional semantics [18], contradicting Hintikka's previous claims. Väänänen has introduced an essentially equivalent but conceptually different dual version based on the notion of *dependence* between variables, expressed at the atomic level [30]. Some strengthenings have been considered also by Abramsky and Väänänen [1]. We will consider a variant  $IF^*$  which eliminates all restrictions on the use of the quantifier slashes and where  $\vee, \wedge$  are not slashed (cf. [8]). Our observations below are translatable to dependence logic.

Since a winning strategy is a choice of appropriate Skolem functions making true certain first-order condition, the disjoint classes

$$Mod^+(\varphi) = \{M : M \models^+ \varphi\}, \quad Mod^-(\varphi) = Mod^+(\neg\varphi) = \{M : M \models^- \varphi\}$$

are  $\Sigma_1^1$ . Therefore,  $IF^*$  has the compactness property and the Löwenheim-Skolem property with respect to  $\models^+$  and  $\models^-$ . Moreover, in finite models it defines exactly the  $NP$  classes. Burgess [4] has observed that for any pair of disjoint  $\Sigma_1^1$ -classes  $K_1, K_2$  the union of which contains all one element structures there is an  $IF^*$  sentence  $\varphi$  such that  $Mod^+(\varphi) = K_1, Mod^-(\varphi) = K_2$ .

The semantics of  $IF^*$  may be seen as a partial two-valued semantics taking the truth value 1 in  $Mod^+(\varphi)$ , the value 0 in  $Mod^-(\varphi)$ , and no value outside of these classes. It is natural to ask if it is possible to interpolate in a sensible manner a  $[0,1]$ -valued semantic assigning a value strictly between 0 and 1 to the structures where the game is undetermined. An intriguing positive answer in the realm of finite models is given by *equilibrium semantics*, suggested by M. Ajtai (cf. [3]), studied by Sevenster and Sandu [28] for  $IF$ , and by Galliani [15] for Väänänen's dependence logic.

Given a particular run of  $G(\varphi, M)$  in which the players use strategies  $\sigma, \tau$ , the payoff of the run for  $E$  is  $u(\sigma, \tau) = 1$  if she wins, 0 otherwise. For  $A$  the payoff is  $1 - u(\sigma, \tau)$ . Probabilistic distributions on sets of strategies are called *mixed strategies*. Given a pair of mixed strategies  $p, q$  on the respective sets of strategies  $S_1, S_2$  of each player, the expected value of the random variable  $u(\sigma, \tau)$ , assuming the players choose their strategies independently, is:

$$u^*(p, q) = \sum_{\sigma, \tau} p(\sigma)q(\tau)u(\sigma, \tau).$$

Since this is a constant sum game, the theory of games [31] grants that if  $M$  is finite there exists an *equilibrium pair* of mixed strategies  $(p_0, q_0)$  such that  $u^*(p', q_0) \leq u^*(p_0, q_0) \leq u^*(p_0, q')$  for all  $p', q'$  so that the players will not improve their payoff by changing them. Obviously all such pairs give the same value which (less obviously) is identical to

$$u^*(p_0, q_0) = \max_p \min_q u^*(p, q) = \min_q \max_p u^*(p, q). \quad (5)$$

Equilibrium semantics takes this *equilibrium value* as the true value of  $\varphi$  in the model  $M$ . We will call it  $\varphi^M$ . If  $M \models^+ \varphi$  and  $\sigma$  is a winning strategy for  $E$  then any pair  $(\sigma, \tau)$  is an equilibrium pair and  $\varphi^M = 1$ ; similarly,  $\varphi^M = 0$  if  $M \models^- \varphi$ .

For example,  $\varphi^M = 1 - \frac{1}{|M|}$  for the sentence  $\forall x \exists y / x(x \neq y)$  (cf. [3]).

This semantics may be seen to satisfy on finite models:

C1.  $\varphi^M = 1$  if and only if  $M \models^+ \varphi$ ,  $\varphi^M = 0$  if and only if  $M \models^- \varphi$

C2.  $\neg\varphi^M = 1 - \varphi^M$

C3.  $(\varphi \vee \psi)^M = \max\{\varphi^M, \psi^M\}$

C4. For any rationals  $r, \varepsilon \in [0, 1]$ ,  $\varepsilon < 1$ , and any sentence  $\varphi \in IF^*$  there is  $\varphi_{r,\varepsilon} \in IF^*$  such that  $\varphi^M \geq r$  iff  $\varphi_{r,\varepsilon}^M \geq \varepsilon$ .

C2 is straightforward, C1 and C4 are shown in [28], and C3 may be proven by game theoretic considerations. It is an open question whether C4 holds for  $\varepsilon = 1$ ; that is, there is  $\varphi_r$  such that

C4\*. For any rational  $r \in [0, 1]$  there is  $\varphi_r \in IF^*$  such that

$$\varphi^M \geq r \text{ iff } \varphi_r^M = 1.$$

Equivalently, whether the complexity of the query  $\varphi^M \geq r$  with respect to the size of  $M$  is in  $NP$ . Sevenster and Sandu [28] have shown that  $\Sigma_2^1 \cap \Pi_2^1$  is an upper bound for the complexity of these queries. Thus, a collapse of the polynomial hierarchy to  $NP$  would make (??) true. Finding a counterexample seems a tough job because it would separate  $NP$  from  $\Sigma_2^1 \cap \Pi_2^1$ , solving an outstanding problem in algorithmic complexity theory.

Our main question is whether it is possible to have a  $[0, 1]$ -valued semantic of  $IF^*$  in infinite models satisfying C1 to C4 and preserving useful model theoretic properties of  $IF^*$ , as compactness and the downward Löwenheim-Skolem property.

The first candidate is equilibrium semantics itself. To extend equilibrium semantics to infinite models we must consider pairs of probabilistic measures  $p, q$  on  $S_1, S_2$  for which  $u(\sigma, \tau)$  is measurable in the product measure, so that the expected payoff

$$u^*(p, q) = \iint_{(\sigma, \tau) \in S_1 \times S_2} u(\sigma, \tau) dpdq$$

exists. If equilibrium pairs exist, the equilibrium value is given by the generalization of (5):

$$\sup_p \inf_q u^*(p, q) = \inf_q \sup_p u^*(p, q).$$

However, the results may be wild. The existence of equilibrium pairs may depend on the cardinality of the model and the sup-inf identity may fail for humble sentences of  $IF^*$ :

•  $\forall x \exists y / x(x \neq y)$  does not have equilibrium pairs in  $\mathbb{N}$ , but:  $\sup_p \inf_q u^*(p, q) = \inf_q \sup_p u^*(p, q) = 1$ .

- $\forall x \exists y / x(x \neq y)$  has equilibrium pairs with value 1 in  $\mathbb{R}$ ; any pair of atomless probabilistic measures is an equilibrium pair.
- For  $\forall x \exists y / x(x \leq y)$  in  $(\mathbb{N}, \leq)$  :  $\sup_p \inf_q u^*(p, q) < \inf_q \sup_p u^*(p, q)$ .

Notwithstanding this grim panorama, there are genuine  $IF^*$  sentences having mixed equilibrium pairs for all models.

**Example.** Consider the sentence  $Inf : \exists u \forall x \exists y / u \exists z / ux(z = x \wedge y \neq u)$  for which  $Mod^+(Inf) = \{A : A \text{ infinite}\}$  and  $Mod^-(Inf) = \{A : |A| = 1\}$ . It has mixed equilibrium pairs with value 1 in all the infinite models and a well defined equilibrium value  $Inf^M < 1$  in each finite model  $M$ .

Call an  $IF^*$  sentence *total* if it has equilibrium pairs for all models. This family of sentences has pleasant properties and is worth studying, but it has a serious flaw for our purposes. Notice that C1 is precisely the condition we would not like to abandon if we wish to extend faithfully game semantics.

**Fact 1.** *Equilibrium semantics on total sentences satisfy C2, C3, C4 in all models but not necessarily C1.*

**Proof.** C2 is straightforward, C3 may be proved by game theoretic considerations and the proof in [28] of C4 may be seen to hold for infinite structures. Now, there is a sentence  $incf \in IF^*$  such that  $Mod^+(incf) = \{M : M \not\approx (\mathbb{R}, +, \cdot, 0, 1, <)\}$  and  $Mod^-(incf) = \emptyset$  because being an incomplete ordered field is a  $\Sigma_1^1$ -condition. The formula  $incf \vee \forall x \exists y / x(x \neq y)$  has equilibrium value 1 in all models because in structures non isomorphic to  $\mathbb{R}$  there is a pure winning strategy for  $E$  choosing the left formula of the disjunction, and we have seen that the right formula has equilibrium value 1 in  $\mathbb{R}$ . However,  $\mathbb{R} \not\models^+ incf \vee \forall x \exists y / x(x \neq y)$ , violating C1.  $\square$

We could restrict further the formulas of  $IF^*$  or we could search for  $[0,1]$ -valued semantics for all formulas of  $IF^*$  not necessarily related to equilibrium values. If we choose the second course of action we should assume the natural generalization of Lindström's axioms for  $[0,1]$ -valued semantics:

C0  $\varphi^M = \varphi^N$  if  $M \approx N$ ;  $\varphi^M = \varphi^{M|\tau}$  if  $\tau \subseteq \mu$ ;  $\varphi^M = \alpha \varphi^{\hat{\alpha}M}$  if  $\alpha$  is a renaming.

An obvious possibility in this direction is three-valued semantics:

$$M \mapsto \varphi^M = \begin{cases} 1 & \text{if } M \models^+ \varphi \\ 0 & \text{if } M \models^- \varphi \\ \frac{1}{2} & \text{otherwise} \end{cases}$$



Indeed, it satisfies C0-C4, and C4\* in all models. The straightforward proofs are left to the reader. But this is not the end of our search because this semantics does not satisfy compactness nor the Löwenheim-Skolem property. In fact, our search is doomed because no  $[0,1]$ -valued semantics  $M \mapsto \varphi^M$  satisfies the later properties and C0, C1 at the same time in all formulas of  $IF^*$ . To see this we must specify first what we mean by these properties. Define for any  $r \in [0, 1] \cap \mathbb{Q}$ :

$$Mod_{\geq r}(\varphi) = \{M : \varphi^M \geq r\}, \quad Mod_{\leq r}(\varphi) = \{M : \varphi^M \leq r\}.$$

*Compactness:* If  $\{Mod_{\geq(\leq)r_i}(\varphi_i)\}_{i \in I}$  has the finite intersection property then it has non-empty intersection.

*Löwenheim-Skolem property:* If  $Mod_{\geq(\leq)r}(\varphi)$  is non-empty then it contains a countable model.

**Fact 2:** *There is no  $[0,1]$ -valued semantics on all formulas of  $IF^*$  satisfying simultaneously C0, C1 and compactness or the Löwenheim-Skolem property.*

**Proof.** Consider a semantics satisfying C0, C1. Then for the sentence *incf* introduced in the previous example we must have  $\{(M, a) : M \approx \mathbb{R}\} = Mod_{\leq r}(\text{incf})$  for some rational  $r < 1$  due to C1 and C0 (a),(b). But this class is a counterexample to compactness and the Löwenheim-Skolem property.  $\square$

Thus, we are bound to consider proper fragments of  $IF^*$ . In this context, it is worth noticing that the fragment of perfect recall sentences satisfies compactness and the Löwenheim-Skolem property for the three-valued semantics discussed above. The next result shows that there are serious limitations if we wish more than that.

**Theorem 6** *Let  $L$  be an extension of  $L_{\omega\omega}$  closed syntactically under  $\neg, \vee$ , with a  $[0,1]$ -valued semantics extending the  $\{0,1\}$ -valued semantics of  $L_{\omega\omega}$ . Suppose  $L$  satisfies C0, C2-C4, compactness, and the Löwenheim-Skolem property. Then for each  $\varphi \in L$  and rational (real)  $r \in [0, 1]$  there is a first order theory  $T_r$  such that;  $\varphi^M \geq r$  iff  $M \models T_r$ .*

**Proof.** Consider the topology on  $St_L$  having for a sub-basis of closed classes  $Mod_{\geq r}(\varphi)$ .  $\varphi \in L$ ,  $r \in [0, 1] \cap \mathbb{Q}$ . By C2 it includes the classes  $Mod_{\leq r}(\varphi)$  and due to C3, C4, it is closed under unions:  $Mod_{\geq r_1}(\varphi_1) \cup Mod_{\geq r_2}(\varphi_2) = Mod_{\geq \varepsilon}(\varphi_{1r_1}) \cup Mod_{\geq \varepsilon}(\varphi_{2r_2}) = Mod_{\geq \varepsilon}(\varphi_{1r_1} \vee \varphi_{2r_2})$ ; hence, it is a closed basis. The topology is regular because  $M \notin Mod_{\geq r}(\varphi)$  implies  $\varphi^M < s < r$  for some rational  $s$  and thus the open classes  $Mod_{\geq s}(\varphi)^c$  and  $Mod_{\leq s}(\varphi)^c$  separate  $M$

and  $Mod_{\geq r}(\varphi)$ . Countable structures are dense because  $\varphi^M < r$  implies  $\varphi^M \leq s < r$ . Finally, C0 grants the continuity of reducts and renamings. Therefore, this is the elementary topology by Theorem 1.  $\square$

**Corollary.** *If  $L$  is a fragment of  $IF^*$  with a  $[0,1]$ -valued semantics satisfying C0-C4, compactness, and the Löwenheim-Skolem property, then  $Mod^+(\varphi)$  and  $Mod^-(\varphi)$  are recursively first-order axiomatizable for any  $\varphi \in L$ .*

**Proof.**  $Mod^+(\varphi) = Mod_{\geq 1}(\varphi) = Mod(T_1)$  by C1 and the previous theorem. Moreover, the first-order consequences of  $T_1$  coincide with the consequences of a  $\Sigma_1^1$ -sentence.  $\square$

Assuming C4\*, a simpler proof of the corollary may be obtained because  $Mod^+(\varphi)$  being  $\Sigma_1^1$  is closed under ultraproducts, and its complement  $Mod^+(\varphi)^c = \cup_{r < 1} Mod_{\leq r}(\varphi) = \cup_{r < 1} Mod^+((\neg\varphi)_r)$  is closed under ultrapowers; hence, it is first order axiomatizable (cf. [10]). Moreover, under C4\* all the  $T_r$  of Theorem 6 result recursively axiomatizable.

## 8 Final remarks

Further Lindström's characterizations of first order logic also hold for regular logics, the proofs being essentially topological with minimal model theoretic contents. Thus, any regular weak extension of  $L_{\omega\omega}$  satisfying compactness and the (countable) Tarski's chain property, or having relativizations and satisfying the uncountable omitting types theorem, is equivalent in theories to  $L_{\omega\omega}$ . Many other model theoretic results known for boolean logics are essentially topological and may also be lifted to logics without classical negation.

The case of equilibrium semantics for  $IF^*$  logic shows that  $[0,1]$ -valued semantics makes perfect sense for classical structures, and topological regularity appears naturally in this setting. Theorem 6 is actually a Lindström's theorem for  $[0,1]$ -valued logics on classical structures. This viewpoint may be translated to the realm of  $[0,1]$ -valued structures to obtain Lindström's theorems and other results mentioned in this paper for extensions of continuous logic and Lukasiewicz-Pavelka logic. These results will appear elsewhere (cf. [9]).

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