Implicit operations in MV-algebras and the connectives of Łukasiewicz logic *

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Dedicated to Daniele Mundici

Abstract

It is shown that a conservative expansion of infinite valued Łukasiewicz logic by new connectives univocally determined by their axioms does not necessarily have a complete semantics in the real interval [0,1]. However, such extensions are always complete with respect to valuations in a family of MV-chains, Rational Łukasiewicz logic being the largest one that has a complete semantics in [0,1]. In addition, the latter logic does not admit expansions by axiomatic implicit connectives that are not already explicit. Similar results are obtained for n-valued Łukasiewicz logic and for the logic of abelian lattice ordered groups. These and related results are obtained by the study of compatible operations implicitly defined by identities in the varieties of MV-algebras and abelian ℓ-groups; the pertaining algebraic results having independent interest.

1 Introduction

Much research effort has been devoted to enrich propositional Łukasiewicz logic with new connectives in order to enhance its geometric expressiveness and algebraic significance. These connectives are usually introduced as new operations in the real interval $[0,1]$, in consonance with the role of Łukasiewicz logic as one of the basic models of fuzzy logic. We present here a different approach that seems natural from the proof theoretic and algebraic perspectives and may contribute to clarify the possibilities of this quest.

Consider a conservative extension $L(C)$ of an algebraizable deductive calculus $L = (L, \vdash_L)$, by axiom schemes which define univocally a new n-ary connective symbol C. That means that the duplicate system $L(C) \cup L(C')$ deduces

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$C(p_1...p_n) \leftrightarrow C'(p_1...p_n)$, where $\leftrightarrow$ is the equivalence formula associated to the algebraizability of L. Then we say that C is an (axiomatic) implicit connective of L. If there is a formula $\varphi \in L$ such that $\vdash_{L(C)} C(p_1...p_n) \leftrightarrow \varphi(p_1...p_n)$, we say that C is explicit; otherwise, it is a proper implicit connective of L.

It is shown in [9] that any implicit connective of classical propositional calculus is explicit, but that is not the case for Heyting intuitionistic calculus where one has instead the approximation $\vdash_{Heyt(C)} \neg\neg C(p_1...p_n) \leftrightarrow \varphi(p_1...p_n)$. We do not have at the moment a clear picture of the implicit connectives of intuitionistic logic; however, the intermediate calculus G_n given by n-valued Gödel logic possesses a proper implicit connective S such that the extension $\mathrm{G}_n(S)$ does not allow proper implicit connectives. Something similar is shown to hold for n-valued Łukasiewicz logic Ł_n in [8].

We study in this paper the implicit connectives of infinite-valued Łukasiewicz calculus Ł. This logic has infinitely many proper implicit connectives: among others, the division connectives introduced in [3] and utilized in [15] to define Rational Łukasiewicz logic RŁ. The latter logic is complete with respect to its natural interpretation in the real interval $[0, 1]$, according to [15], and it is shown in [3] to satisfy a natural extension of McNaughton's theorem, and to be the minimum extension of Ł having the interpolation property.

Our main results here are the following:

Any implicit connective of RŁ *is explicit* (Theorem 7).

Thus, RŁ is maximal with respect to extensions by implicit connectives. However, it is not the largest extension of Ł by implicit connectives. We exhibit such extensions which are sound but not complete with respect to values in $[0, 1]$, and thus they can not be interpreted faithfully into RŁ (Theorem 3). On the other hand, we show that any extension of Ł by implicit connectives is complete with respect to a family of MV-chains, thus qualifying as a fuzzy logic in the broad sense (Theorem 2, cf. [13]). Among those, RŁ is the largest one having a complete semantics with values in $[0, 1]$:

Any extension of Ł *by implicit connectives having a complete semantics in* $[0, 1]$ *has a faithful syntactic interpretation into* RŁ (Theorem 8).

The latter result implies, for example, that the product connective of combined product logic ŁΠ (cf. [20]) is not an implicit connective of Ł since it is not interpretable into RŁ; that is, it can not be characterized univocally by any axiomatization whatsoever.

We review also the case of n-valued Łukasiewicz calculus Ł_n, showing similar results, and exhibiting examples of implicit connectives whose logic is not complete with respect to a single MV-chain.

Our main tool are the results of [8] which imply that any extension $\text{Ł}(\mathcal{C})$ of Ł by a family $\mathcal{C}$ of implicit connectives is algebraizable by a variety of enriched MV-algebras, where the operations interpreting the connectives in $\mathcal{C}$ are implicitly defined by identities and are compatible with all the MV-algebra congruences. Therefore, studying implicit connectives of Łukasiewicz logic amounts to studying compatible operations implicitly defined by identities in the variety of MV-algebras. Our main algebraic result in this direction may have independent interest (DMV-algebras are the enriched MV-algebras of RŁ):

Any compatible operation defined implicitly by identities in the variety of DMV-algebras is given by a term of the variety (Theorem 5).

For the variety of lattice ordered abelian groups, related to MV-algebras by Mundici's functor [21], we have:

Any compatible operation defined implicitly by identities in divisible lattice ordered abelian groups is given by a $\mathbb{Q}$-vector lattice term. (Theorem 10).

The last result allows us to prove analogues of the previous results for the Logic of equilibrium introduced in [14].

We refer the reader to [7], [6], and [12] as standard references for the concepts of universal algebra, algebraizable logics, and model theory utilized in this paper.

2 Preliminaries

We start with some general preliminaries on implicit operations in varieties of algebras and their relation to implicit connectives.

Let $\mathbb{V}$ be a variety of algebras of type τ and let $\mathcal{E}(\mathcal{C})$ be (the universal closure of) a set of identities of type $\tau \cup \mathcal{C}$ where $\mathcal{C}$ is a family of new function symbols.

Definition 1 $\mathcal{E}(\mathcal{C})$ *defines implicitly* $\mathcal{C}$ in $\mathbb{V}$, if in each algebra $A \in \mathbb{V}$ there is at most one family $\{\nabla^A : A^n \longrightarrow A\}_{\nabla\in\mathcal{C}}$ such that $(A, \nabla^A)_{\nabla\in\mathcal{C}} \models \mathcal{E}(\mathcal{C})$. We say then that $\mathcal{C}$ is an *implicit family of operations of* $\mathbb{V}$, or an *implicit operation* in case it has single member.

The class

$$\mathbb{V}(\mathcal{C}) = \{(A, \nabla^A)_{\nabla\in\mathcal{C}} : A \in \mathbb{V},\ (A, \nabla^A)_{\nabla\in\mathcal{C}} \models \mathcal{E}(\mathcal{C})\}$$

is a new variety of type $\tau \cup \mathcal{C}$. The class $Red_{\mathcal{C}}$ of *reducts* of $\mathbb{V}(\mathcal{C})$, that is, those algebras of $\mathbb{V}$ where each $\nabla \in \mathcal{C}$ exists, does not need to be all of $\mathbb{V}$. In case $Red_{\mathcal{C}}$ generates $\mathbb{V}$ then $\mathbb{V}(\mathcal{C})$ is *conservative* over $\mathbb{V}$, that is, any identity of type τ holding in $\mathbb{V}(\mathcal{C})$ already holds in $\mathbb{V}$.

The following lemmas collect some basic facts about implicit operations.

Lemma 1 *Let $\mathcal{C}$ be an implicit family of operations of $\mathbb{V}$. Then*
1. Each $\nabla \in \mathcal{C}$ has an explicit first order definition $\theta_\nabla(\mathbf{x}, y)$ of type τ. That is, for any $A \in Red_{\mathcal{C}}$ and $\mathbf{x}$, y in A

$$y = \nabla^A(\mathbf{x}) \Leftrightarrow A \models \theta_\nabla(y, \mathbf{x}).$$

2. The class $Red_{\mathcal{C}}$ is first order axiomatizable.
3. If each $\nabla \in \mathcal{C}$ exists in A_i for all $i \in I$ then it exists and is computed componentwise in the product $\Pi_i A_i$. The same is true for reduced products $\Pi_i A_i/_F$.

Proof. 1. This is a simultaneous form of Beth's definability theorem. Without loss of generality, assume that $\mathcal{E}(\mathcal{C})$ contains the defining identities of $\mathbb{V}$ and $\mathcal{E}(\mathcal{C}')$ is a duplicate of $\mathcal{E}(\mathcal{C})$ with disjoint copies of the symbols in $\mathcal{C}$. Pick $\nabla \in \mathcal{C}$ and fix distinct variables $y, \mathbf{x}$, then $\mathcal{E}(\mathcal{C}) \cup \mathcal{E}(\mathcal{C}') \models \nabla(\mathbf{x}) = \nabla'(\mathbf{x})$ by hypothesis, and $\mathcal{E}(\mathcal{C})$ may be assumed to be a single sentence by compactness of first order logic. Thus, the above may be written $\mathcal{E}(\mathcal{C}) \wedge y = \nabla(\mathbf{x}) \models \mathcal{E}(\mathcal{C}') \to y = \nabla'(\mathbf{x})$, and Craig's interpolation lemma yields an interpolant $\theta_\nabla(y, \mathbf{x})$ which does not contain the operation symbols in $\mathcal{C}$ or $\mathcal{C}'$. Standard logical manipulations give then $\mathcal{E}(\mathcal{C}) \models y = \nabla(\mathbf{x}) \leftrightarrow \theta_\nabla(y, \mathbf{x})$, which proves the claim.

2. All operations $\nabla \in \mathcal{C}$ exist in A if and only if A satisfies the set of sentences $\{\forall \mathbf{x} \exists! y \theta_\nabla(y, \mathbf{x})\}_{\nabla \in \mathcal{C}} \cup \mathcal{E}(\nabla/\theta_\nabla)_{\nabla \in \mathcal{C}}$, where $\mathcal{E}(\nabla/\theta_\nabla)_{\nabla \in \mathcal{C}}$ is the result of rewriting the identities in $\mathcal{E}(\mathcal{C})$ so that all the occurrences of $\nabla \in \mathcal{C}$ appear in the form $y = \nabla(\mathbf{x})$ and then replacing these by $\theta_\nabla(y, \mathbf{x})$. For example, $\nabla_1(\nabla_1(v, x), x) = \nabla_2 v$ should be rewritten: $\forall y \forall y'[(y = \nabla_1(v, x) \wedge y' = \nabla_2 v) \to y' = \nabla_1(y, x)]$, and then $\forall y \forall y'[\theta_{\nabla_1}(y, v, x) \wedge \theta_{\nabla_2}(y', v) \to \theta_{\nabla_1}(y', y, x)]$.

3. If $(A, \nabla^A)_{\nabla \in \mathcal{C}} \models \mathcal{E}(\mathcal{C})$ for all $i \in I$ then $\Pi_{i/F}(A_i, \nabla^{A_i})_{\nabla \in \mathcal{C}} \models \mathcal{E}(\mathcal{C})$ for any filter F over I because identities are preserved by reduced products. □

Definition 2 An implicit operation ∇ of $\mathbb{V}$ will be *compatible* if for any $A \in Red_{\mathcal{C}}$ the congruences of A are congruences of $(A, \nabla^A)_{\nabla \in \mathcal{C}}$.

Not every implicit operation of a variety is compatible. For example, the identities

$$nD_n(x) = x, \quad D_n(nx) = x$$

($n \geq 2$) define an implicit operation in the variety of abelian groups since any other operation f satisfying the second equation must satisfy $f(x) = f(nD_n(x))) = D_n(x)$. It may be seen that D_n exists exactly in the n-divisible abelian groups having no elements of order n, where $D_n(x) = \frac{1}{n}x$ is well defined. But this operation is not compatible because we have $k \equiv 0 \pmod{\mathbb{Z}}$ in the group $(\mathbb{Q}, +, -, 0)$ for any integer k, but $D_n(k) \not\equiv D_n(0) \pmod{\mathbb{Z}}$ if n does not divide k.

Lemma 2 *Let $\mathcal{C}$ be an implicit family of compatible operations of $\mathbb{V}$.*
1. If $h : A \to B$ is an onto homomorphism of $\mathbb{V}$ and all $\nabla \in \mathcal{C}$ exist in A then all of them exist in B and $h\nabla^A(a_1, .., a_n) = \nabla^B(h(a_1), .., h(a_n))$.
2. Reducts of subdirectly irreducible algebras of $\mathbb{V}(\mathcal{C})$ are subdirectly irreducible in $\mathbb{V}$.

Proof. 1. If $h : A \to B$ is an onto homomorphism and ∇^A is compatible with $Ker(h)$, then the function $f_\nabla(h(a)) = h(\nabla^A(a))$ is well defined in B. Therefore, $h : (A, \nabla^A)_{\nabla \in \mathcal{C}} \to (B, f_\nabla)_{\nabla \in \mathcal{C}}$ becomes an homomorphism. As $(A, \nabla^A)_{\nabla \in \mathcal{C}} \models \mathcal{E}(\mathcal{C})$, then $(B, f_\nabla)_{\nabla \in \mathcal{C}} \models \mathcal{E}(\mathcal{C})$ and by definition $f_\nabla = \nabla^B$.

2. Since $(A, \nabla^A)_{\nabla \in \mathcal{C}}$ and A have the same congruences, a monolith of the first structure is a monolith of the second. □

Our interest in compatible implicit operations is explained by their relation to implicit connectives of algebraizable logics given by Theorem 1 below. We will consider only logics $\mathcal{L}$ which are strongly algebraizable in the sense of Blok and Pigozzi (cf. [6]) with respect to an equivalence formula $\leftrightarrow$ and a constant formula 1 of the calculus. This means that

$$\varphi \ \vdash_{\mathcal{L}} \varphi \leftrightarrow 1, \qquad \varphi \leftrightarrow 1 \vdash_{\mathcal{L}} \varphi \tag{a}$$

and there is a variety of algebras $\mathbb{V}$, of the same signature as the logic, such that the following algebraic completeness theorem holds:

$$\{\varphi_i \leftrightarrow \psi_i\}_{i\leq n} \vdash_{\mathcal{L}} \varphi \leftrightarrow \psi \text{ if and only if } \{\varphi_i = \psi_i\}_{i\leq n} \models_{\mathbb{V}} \varphi = \psi; \tag{b}$$

equivalently, due to (a),

$$\{\varphi_i\}_{i\leq n} \vdash_{\mathcal{L}} \varphi \text{ if and only if } \{\varphi_i = 1\}_{i\leq n} \models_{\mathbb{V}} \varphi = 1, \tag{c}$$

the usual completeness with respect to valuations in all the algebras of $\mathbb{V}$. Notice that we use, as we will keep on using throughout the paper, the formulas of the calculus as terms of the variety.

Most familiar logics are algebraizable in this sense. By finiteness of the deductions in $\mathcal{L}$ and compactness of first order logic applied to $\mathbb{V}$, (c) holds for infinite theories $\{\varphi_i\}_{i\in I}$. In fact, this *strong algebraic completeness* may be achieved by taking valuations only in the subdirectly irreducible algebras of $\mathbb{V}$:

$$\{\varphi_i\}_{i\in I} \vdash_{\mathcal{L}} \varphi \text{ if and only if } \{\varphi_i = 1\}_{i\in I} \models_{S.I.(\mathbb{V})} \varphi = 1. \tag{sc}$$

Now, let $\mathcal{L}(\mathcal{C})$ be an extension of $\mathcal{L}$ by a system of axiom schemes $\mathcal{A}(\mathcal{C})$ involving a family of new connective symbols $\mathcal{C}$, and let

$$\mathcal{A}^*(\mathcal{C}) = \{\varphi = 1 : \varphi \in \mathcal{A}(\mathcal{C})\};$$

then we may define the variety

$$\mathbb{V}(\mathcal{C}) = \{(A, f_\nabla)_{\nabla\in\mathcal{C}} : A \in \mathbb{V},\ (A, f_\nabla)_{\nabla\in\mathcal{C}} \models \mathcal{A}^*(\mathcal{C})\}.$$

One has by construction that $\vdash_{\mathcal{L}(\mathcal{C})} \varphi$ implies $\models_{\mathbb{V}(\mathcal{C})} \varphi = 1$, but the reciprocal does not necessarily hold. That is, we can not claim that $\mathcal{L}(\mathcal{C})$ is algebraizable by $\mathbb{V}(\mathcal{C})$. However, algebraicity is obtained in the following case.

Definition 3 $\mathcal{L}(\mathcal{C})$ *defines implicitly* $\mathcal{C}$ *over* $\mathcal{L}$ if $\vdash \nabla\mathbf{p} \leftrightarrow \nabla'\mathbf{p}$ for each $\nabla \in \mathcal{C}$, where $\mathcal{A}(\mathcal{C}')$ is a duplicate of $\mathcal{A}(\mathcal{C})$ with a new connective symbol ∇' replacing each $\nabla \in \mathcal{C}$.

Theorems 1 and 4 in [8] yield:

Theorem 1 *If $\mathcal{L}$ is algebraizable by a variety of algebras $\mathbb{V}$, and $\mathcal{L}(\mathcal{C}) = \mathcal{L} \cup \mathcal{A}(\mathcal{C})$ defines implicitly a family of connectives $\mathcal{C}$ over $\mathcal{L}$, then $\mathcal{A}^*(\mathcal{C})$ defines an implicit family of compatible operations of $\mathbb{V}$ (that we denote $\mathcal{C}$ also) and $\mathcal{L}(\mathcal{C})$ is algebraizable by $\mathbb{V}(\mathcal{C})$, by means of the same formulas $\leftrightarrow$ and 1 as $\mathcal{L}$.*

This theorem fails in various ways if the extension does not define implicitly $\mathcal{C}$. It may happen that $\mathcal{L}(\mathcal{C})$ is not algebraizable at all, or that it is algebraizable by algebras not having reducts in $\mathbb{V}$, or that it is algebraizable for algebras with reducts in $\mathbb{V}$ but the interpretation of the connectives in $\mathcal{C}$ is not compatible. See [8] for examples.

3 Implicit Connectives of Łukasiewicz Logic

Infinitely valued Łukasiewicz calculus Ł has the primitive connectives $\to$, $\neg$, and the following axioms plus the Modus Ponens rule:

$$
\begin{aligned}
&p \to (q \to p)\\
&(p \to q) \to ((q \to r) \to (p \to r))\\
&((p \to q) \to q) \to ((q \to p) \to p)\\
&(\neg p \to \neg q) \to (q \to p).
\end{aligned}
$$

Its expressive power is better revealed by the use of the following explicitly defined connectives:

$$
\begin{aligned}
&p \curlyvee q := (p \to q) \to q\\
&p \curlywedge q := \neg(\neg p \curlyvee \neg q)\\
&p \leftrightarrow q := (p \to q) \curlywedge (q \to p)\\
&1 := p \to p\\
&0 := \neg(p \to p)\\
&p \oplus q := \neg p \to q\\
&p \odot q := \neg(p \to \neg q).
\end{aligned}
$$

For each integer $n \geq 2$, the abbreviations:

$$np := \underbrace{p \oplus ... \oplus p}_{n} \quad \text{and} \quad p^n := \underbrace{p \odot ... \odot p}_{n},$$

are unambiguous up to equivalence due to associativity and commutativity of $\oplus$ and $\odot$. The set $\{\oplus, \neg\}$ serves as a complete set of connectives because $\vdash_{\text{Ł}}$ $(p \to q) \leftrightarrow (\neg p \oplus q)$. We will assume familiarity with this calculus. For a full account we refer the reader to [10].

Utilizing $\{\oplus, \neg, 0\}$ as primitive connectives (0 superfluous but convenient), Lukasiewicz logic is algebraizable with respect to the defined connectives $\leftrightarrow$ and 1 by the variety of MV-*algebras,* $\mathcal{MV}$, variety generated as a quasivariety by the Lukasiewicz algebra

$$[0,1]_{MV} = ([0,1], \oplus, \neg, 0), \qquad x \oplus y = \min\{x+y, 1\}, \qquad \neg x = 1 - x.$$

Any MV-algebra has a natural lattice order defined by $x \leq y$ iff $(x \to y) = 1$, where $\curlyvee, \curlywedge$ become the join and meet, and 0, 1 become minimum and maximum, respectively. Chang's representation theorem [11] says that the subdirectly irreducible algebras of this variety are MV-*chains* (linearly ordered MV-algebras) and thus any MV-algebra is a subdirect product of MV-chains. Moreover, $[0,1]_{MV}$ generates $\mathcal{MV}$ as a quasivariety and thus we have Chang's completeness theorem: [1]

$$\{\varphi_i\}_{i \leq n} \vdash_{\text{Ł}} \varphi \text{ if and only if } \{\varphi_i = 1\}_{i \leq n} \models_{[0,1]} \varphi = 1.$$

It is well known that Ł is not strongly complete with respect to $[0,1]_{MV}$; that is, the above does not hold for infinite theories. However, Ł is strongly complete

[1] Chang's theorem is usually stated with an empty set of premises but it is equivalent to the given version because Ł has a form of the Deduction Theorem.

for valuation in all MV-chains, by (sc) in the previous section. In fact, it is enough to take the divisible MV-chains because any MV-chain is embeddable in a divisible one.

Strong completeness with respect to a family of totally ordered algebras has been proposed as a test for being a 'fuzzy logic' in [13]. Our first observation is that any extension of Łukasiewicz logic by implicit connectives qualifies as a fuzzy logic in this sense.

Theorem 2 *Any extension* Ł$(\mathcal{C})$ *of* Ł *by implicit connectives is strongly complete with respect to the class of* $\mathcal{MV}(\mathcal{C})$*-chains.*

Proof. By Theorem 1, Ł$(\mathcal{C})$ is algebraizable by the variety$\mathcal{MV}(\mathcal{C})$ where $\mathcal{C}$ is an implicit family of compatible operations of $\mathcal{MV}$. Thus, by (sc), Ł$(\mathcal{C})$ is strongly complete with respect to valuations in the subdirectly irreducible algebras of $\mathcal{MV}(\mathcal{C})$, which by Lemma 2-2 have subdireclty irreducible reducts in $\mathcal{MV}$ and thus are chains. □

However, we will see later (Theorem 3) that Ł$(\mathcal{C})$ does not need to be complete, even the less strongly complete, with respect to values in the algebra $[0,1]_{MV}$.

3.1 Division Connectives, Rational Łukasiewicz Logic.

For $n \geq 2$, the axiom schemes:

(A1) $n\delta_n p \to p$
(A2) $p \to n\delta_n p$
(A3) $(p \to nq) \to (\delta_n p \to q)$

define an implicit connective δ_n of Ł. To see this, assume the same axioms for a different connective symbol λ:

(A1$_\lambda$) $n\lambda p \to p$, (A2$_\lambda$) $p \to n\lambda p$, (A3$_\lambda$) $(p \to nq) \to (\lambda p \to q)$.

Then A3 gives $(p \to n\lambda p) \to (\delta_n p \to \lambda p)$ and Modus Ponens with A2$_\lambda$ yields $\delta_n p \to \lambda p$. Similarly, A3$_\lambda$ and A2 give $\lambda p \to \delta_n p$. In sum, $\vdash_{\text{Ł}(\delta_n)\cup\text{Ł}(\lambda)} \delta_n p \leftrightarrow \lambda p$.

This axiom system is equivalent to the one given in [15] with a different version for the third axiom. These connectives were introduced semantically in [3] and are explicitly definable from the propositional existential quantifier introduced in [1].

According to Theorem 1, Ł(δ_n) is algebraized by the variety $\mathcal{MV}(\delta_n)$, where δ_n is a compatible operation defined implicitly by the inequalities

$n\delta_n(x) \leq x$
$x \leq n\delta_n(x)$
$(x \to ny) \leq (\delta_n(x) \to y)$.

The reader may verify, after some computation, that these reduce to the single identity:

$$(n-1)\delta_n(x) = x \odot \neg\delta_n(x).$$

This operation exists exactly in the *n-divisible* MV-algebras introduced in [17]. In particular, $\delta_n(x) = \frac{1}{n}x$ in $[0,1]_{MV}$, and it does not exist in any finite non trivial algebra.

The calculus $\text{RŁ} = \text{Ł}(\delta_n)_{n\geq 2}$, obtained by adding the axioms of δ_n to Ł for all $n \geq 2$, is called *Rational Łukasiewicz logic* in [15], and the corresponding variety $\mathcal{MV}(\delta_n)_{n\geq 2}$, consisting of divisible MV-algebras enriched with all the operations δ_n, is called the variety of DMV-algebras, $\mathcal{DMV}$ for short.

It follows immediately from Lemma 2-2 that each DMV-algebra is a subdirect product of DMV-chains. Moreover, Theorem 2, together with the first order completeness of the theory of non trivial divisible MV-chains (see [17]), yields a quick proof of completeness of RŁ with respect to values in $[0,1]$:

Proposition 1 (Th 4.3, [15]) $\text{RŁ} = \text{Ł}(\delta_n)_{n\geq 2}$ *is complete with respect to valuations in* $([0,1]_{MV}, \delta_n)_{n\geq 2}$.

Proof. By Theorem 2, RŁ is algebraicallly complete with respect to all DMV-chains. But any no trivial divisible MV-chain is elementarily equivalent to $[0,1]_{MV}$ by first order completeness. By first order definability of the δ_n, this means that all non trivial DMV-chains are elementarily equivalent to $([0,1]_{MV}, \delta_n)_{n\geq 2}$. Hence, any quasi-identity holds in all DMV chains if and only if it holds in this algebra. □

3.2 Approximate Division Connectives.

We exhibit now a family of implicit connectives whose calculus is sound but not complete for values in $[0,1]_{MV}$. It is clear from the proof of uniqueness of δ_n in the previous example that the pair of axioms

(B1) $p \to n\delta_n^* p$
(B2) $(p \to nq) \to (\delta_n^* p \to q)$

already define an implicit connective of Ł. Regarding its algebraic interpretation we have:

Proposition 2 δ_n^* *exist in a MV-chain* M *if and only if* $\min\{y \in M : ny \geq x\}$ *exists for all* $x \in M$, *in which case* $\delta_n^*(x)$ *is that minimum.*

Proof. The identities defining the variety $\mathcal{MV}(\delta_n^*)$ become:

(E1) $x \leq n\delta_n^* x$, (E2) $\neg x \oplus ny \leq \neg\delta_n^* x \oplus y$.

Assume they hold in a chain M and $x \leq ny$ there. Then $\neg x \oplus ny = 1$ and thus $\neg\delta_n^* x \oplus y = 1$ by E2, which means $\delta_n^* x \leq y$. Together with E1, this shows $\delta_n^* x = \min\{y : ny \geq x\}$. Reciprocally, assume the function $f(x) = \min\{y : ny \geq x\}$ exists in a chain. Then f satisfies (E1) by definition. For the second equation, consider first $f(x) \leq y$, then $\neg f(x) \oplus y = 1$ and thus E2 holds trivially. Consider now $y < f(x)$, then $ny < x$ by definition of f and thus $u = x \odot \neg ny > 0$. Moreover, $x = ny \oplus u \leq ny \oplus nu = n(y \oplus u)$, which implies $y \oplus u \geq f(x)$. Suppose E2 is false, then $\neg x \oplus ny > \neg f(x) \oplus y$ and taking negations

$u = x \odot \neg ny < f(x) \odot \neg y$. Adding y to both sides gives: $y \oplus u < y \oplus (f(x) \odot \neg y) = f(x)$, a contradiction. $\square$

Therefore, δ_n^* exists in $[0,1]_{MV}$ where it coincides with $\delta_n(x) = \frac{1}{n}x$, and it exists also in the finite Lukasiewicz chains

$$L_k = (\{0, \tfrac{1}{k-1}, .., 1\}, \oplus, \neg), \qquad k \geq 2,$$

as well as in all finite MV-algebras by Lemma 1 (3), because these are products of L_k's. However, the reader may check that δ_n^* does not exist in any of the Komori algebras K_m, [16]. Observe that if δ_n exists in a MV-algebra M then δ_n^* also exists in M and coincides there with δ_n, because δ_n^M satisfies the defining identities of δ_n^*. With this observation it is easy to show:

Theorem 3 Ł(δ_n^*) *is sound but not complete for values in* $([0,1]_{MV}, \delta_n^*)$.

Proof. Soundness is clear because δ_n^* exists in $[0,1]_{MV}$. Now, $\nvdash_{L(\delta_n^*)} (n-1)\delta_n^* 1 \leftrightarrow \neg\delta_n^* 1$ because the equation $(n-1)x = \neg x$ does not have solutions in L_n. But $([0,1]_{MV}, \delta_n^*)$ can not refute this because δ_n^* coincides in $[0,1]_{MV}$ with δ_n and $\delta_n 1 = \frac{1}{n}$ satisfies the given equation. $\square$

We do not know if Ł(δ_n^*) is complete with respect to a single chain.

4 Lattice-ordered Abelian Groups and MV-algebras

Abelian lattice ordered groups, *ℓ-groups* for short, are abelian groups with a lattice order compatible with the group operations. They may be presented as a variety $\ell\mathcal{G}$ in the vocabulary $\{+, -, 0, \curlyvee, \curlywedge\}$ where $-$ represents difference and $\curlyvee, \curlywedge$ represent the join and meet of the lattice order, respectively. The homomorphism must preserve not only the group structure and the order but also $\curlyvee$ and $\curlywedge$. We refer the reader to [5] for full details, but emphasize here the following facts:

Fact 1. ℓ-groups are closed under lexicographic products. We will utilize the notation $G \otimes H$ to denote lexicographic product (left priority).

Fact 2. Linearly ordered abelian groups may be expanded naturally to ℓ-groups. All subdirectly irreducible ℓ-groups are linearly ordered.

Fact 3. Any ℓ-group may be embedded in a divisible abelian ℓ-group, the usual divisible hull of the group with a naturally extended order.

We will need also the following model theoretic fact. Recall that a first order theory has *elimination of quantifiers* if for any formula $\theta(\mathbf{x})$ of the language of the theory there is a quantifier free formula $\psi(\mathbf{x})$ which is equivalent to $\theta(\mathbf{x})$ in all models of the theory (see [12]).

Fact 4. The theory of non trivial linearly ordered divisible groups (or ℓ-groups) is complete and has elimination of quantifiers with respect to the language $\{+, -, 0, <\}$ ([22], [10], Cor. 3.1.17 [18]), also with respect to the language $\{+, -, 0, \curlyvee, \curlywedge\}$ because in the context of total order $x < y$ is equivalent to the formula $x \curlywedge y \neq y$.

Consider now the relation between ℓ-groups and MV-algebras.

Definition 4 A *unital* ℓ-group will be a pair (G, u) where G is an ℓ-group and $u \geq 0$. If for any $x \in G$ there is n such that $nu \geq x$, then u is a *strong unit.*

We will need the following refinement of Fact 4.

Lemma 3 *The theory of linearly ordered divisible unital ℓ-groups (G, u) with $u > 0$ is complete.*

Proof. This theory trivially inherits elimination of quantifiers from the theory of linearly ordered divisible ℓ-groups (Fact 4). Therefore, it is model complete (that is, any embedding between its models is elementary, see [12]). To obtain completeness it is enough to notice that $(\mathbb{Q}, +, -, 0, <, 1)$ is a prime model of the theory (it is embeddable in all other models). Indeed, if $u > 0$, the unique group homomorphism $(\mathbb{Z},1) \to (G, u)$ sending 1 to u is injective and preserves the order, and it may be extended canonically to $\mathbb{Q}$ maintaining the same characteristics. □

Notice that the previous result does not hold if we add two distinguished constants $0 < u_1 < u_2$ to ℓ-groups since $(\mathbb{Q}, 1, 2) \not\equiv (\mathbb{Q}, 1, 3)$.

Unital ℓ-groups form a variety $\ell\mathcal{G}_*$ whose morphisms are the ℓ-group homomorphism preserving the constant u. The functor $\Gamma : \ell\mathcal{G}_* \to \mathcal{MV}$ associates to each unital ℓ-group an MV-algebra by generalizing the definition of the Lukasiewicz algebra $[0, 1]_{MV}$:

$\Gamma(G, u) = ([0, u], \oplus, \neg, 0), \qquad x \oplus y := (x + y) \curlywedge u, \qquad \neg x := u - x$
$\Gamma(h) = h \restriction [0, u].$

Mundici [21] has shown that his functor has a left adjoint

$$\Sigma : \mathcal{MV} \to \ell\mathcal{G}_*$$

such that $\Gamma \circ \Sigma = I_{\mathcal{MV}}$ and Σ establishes an equivalence of categories between $\mathcal{MV}$ and the subcategory of $\ell\mathcal{G}_*$ where u is a strong unit. In particular, any MV-algebra is of the form $M = \Gamma(\Sigma M)$. The following may be easily verified by construction or in general categorical grounds:

Lemma 4 Γ *and* Σ *preserve divisibility, linear order, and injectivity of homomorphisms. Hence, for any MV-algebras M, N :*
1. M is divisible iff ΣM is divisible.
2. M is a chain iff ΣM is linearly ordered.
3. $h : M \to N$ is an injective homomorphism iff $\Sigma h : M \to N$ is injective.

Clearly, $\Gamma(G,u)$ is first order definable in (G,u). This definability is best expressed by the following translation (cf. [10]). To any first order formula $\theta(\mathbf{x})$ in the language $\{\oplus,\neg,0\}$ of MV-algebras associate $\theta^*(\mathbf{x},u)$, in the language $\{+,-,0,\curlyvee,\curlywedge\}$ of ℓ-groups, by the following substitution of atomic terms

$$0 \longmapsto 0, \qquad x\oplus y \longmapsto (x+y)\curlywedge u, \qquad \neg x \longmapsto u-x,$$

and restriction of quantifiers to the interval $[0,u]$. Then, for any unital ℓ-group (G,u) and any list of parameters $\mathbf{a}$ in $[0,u]$,

$$\Gamma(G,u)\models\theta[\mathbf{a}] \text{ iff } (G,u)\models\theta^*[\mathbf{a},u].$$

In particular, for any MV-algebra M and choice of parameters $\mathbf{a}$ in M,

$$M\models\theta[\mathbf{a}] \text{ iff } \Sigma M\models\theta^*[\mathbf{a},1_M]. \tag{t}$$

The first order theory of non trivial divisible MV-chains was already mentioned to be complete, [17]. This is an immediate consequence of Lemma 3 and the translation (t), and it implies automatically the completeness of the theory of DMV-chains by definability of the δ_n. In fact, these theories inherit also full elimination of quantifiers from divisible linearly ordered ℓ-groups. Since this is not immediate because elimination of quantifiers is sensible to the vocabulary utilized, and we have not seen it mentioned in the literature, we provide a proof utilizing the following criterion:

Lemma 5 (Corollary 3.1.6, [18]) *T has elimination of quantifiers if and only if for any pair of models B, C of T having a common substructure A, not necessarily a model of T, and for any formula $\theta(\mathbf{x},y)$ and choosing $\mathbf{a}$ of a list of parameters in A, it holds that $B\models\exists y\theta[\mathbf{a},y]$ implies $\mathbf{C}\models\exists y\theta[\mathbf{a},y]$.*

Theorem 4 *The theory of non trivial divisible MV-chains (DMV-chains) has elimination of quantifiers in the language $\{\oplus,\neg,0\}$.*

Proof. Let M,N be non trivial divisible MV-chains and A a common MV-subalgebra. By Lemma 4, we have injections $\Sigma A\leq\Sigma M$ and $\Sigma A\leq\Sigma N$ between totally ordered unital ℓ-groups with ΣM and ΣN non trivial and divisible. Now let $\theta(\mathbf{x},y)$ be any formula of type $\{\oplus,\neg,0\}$ and $\mathbf{a}\in A^r$, $m\in M$ be such that $M\models\theta[\mathbf{a},m]$. Then $\Sigma M\models(0\leq m\leq 1_M)\wedge\theta^*[\mathbf{a},m,1_M]$ by (t). Since $\mathbf{a}$ and $1_M=1_N=1_A$ belong to ΣA, by Fact 4 and the above criterion (Lemma 5), there is $n\in N$ such that $\Sigma N\models(0\leq n\leq 1_M)\wedge\theta^*[\mathbf{a},n,1_M]$. Thus $N\models\varphi[\mathbf{a},n]$ by (t) again. Once more by Lemma 5, we conclude that the theory of divisible MV-chains has elimination of quantifiers. The claim about DMV-chains is immediate from the first order definability of the δ_n. $\Box$

5 Implicit Operations of MV-algebras and Maximality of RŁ

The next result holds for each member of any implicit family of compatible operations. For the sake of simplicity, we consider a single operation only.

Theorem 5 *Any compatible operation implicitly defined by identities in DMV-algebras is given by a term of type $\{\oplus, \neg, \delta_n\}_{n\geq 2}$. Moreover, it exists in all the DMV-algebras or in the trivial algebra only.*

Proof. If such an operation ∇ exists in the trivial algebra only, 0 is the desired term. Assume it exists in a non trivial DMV-algebra M. Then, by compatibility of ∇ (Theorem 1), this operation exists in any non trivial subdirectly irreducible factor of M (Lemma 2-1), which must be a non trivial DMV-chain by Lemma 2-2. By completeness of the theory of these chains and the first order definability of Red_∇ (Lemma 1-2), ∇ exists in all non trivial DMV-chains, in particular in the chain $([0,1]_{MV}, \delta_n)_n$. Let $\theta(y, \mathbf{x})$ be the explicit first order definition of ∇ given by Lemma 1-1, which we may assume to be given in the language of MV-algebras since the δ_n are first order definable, and let $\theta^*(y, \mathbf{x}, u)$ be its translation to the language of unital ℓ-groups where u is the unit constant. By (t), $\theta^*(y, \mathbf{x}, 1)$ defines $\nabla^{[0,1]} : [0,1]^n \to [0,1]$ as a partial function in the unital ℓ-group $(\mathbb{R}, 1)$. Since the join and meet $\curlyvee, \curlywedge$ are interdefinable with the order $<$, we may put $\theta^*(y, \mathbf{x}, 1)$ in the language $\{+, -, 0, <, 1\}$, then in quantifier free form using Fact 4, and finally in full disjunctive normal form $\bigvee_\alpha \theta_\alpha(y, \mathbf{x}, 1)$. Each θ_α is a conjunction of atomic formulas $t = 0$, $t < 0$ or their negations, where the term t has the form $k_o y + ... + k_n x_n + k_{n+1} 1$, $k_i \in \mathbb{Z}$. Negations may be eliminated because in linearly ordered groups: $t \neq 0 \Leftrightarrow (t < 0 \vee -t < 0)$ and $t \not< 0 \Leftrightarrow (t = 0 \vee -t < 0)$. Separating the atomic formulas where y appears with non zero coefficient, and solving for y, $\theta_\alpha(y, \mathbf{x}, 1)$ becomes equivalent in $(\mathbb{R}, 1)$ to:

$$\bigwedge_i y = t_i(\mathbf{x}, 1) \wedge \bigwedge_j y < s_j(\mathbf{x}, 1) \wedge \bigwedge_k u_k(\mathbf{x}, 1) = 0 \wedge \bigwedge_r v_r(\mathbf{x}, 1) < 0,$$

where some of the conjunctions may be empty and the terms t_i, s_j, u_k, v_r have now rational coefficients.

If the first large conjunction $\bigwedge_i$ is empty and there are values $b \in [0,1]$, $\mathbf{a} \in [0,1]^n$, satisfying $\theta_\alpha(b, \mathbf{a}, 1)$, then by density of $<$ in $[0,1]$ there are infinitely many values $y \in [0,1]$ satisfying $\theta_\alpha(y, \mathbf{a}, 1)$. This contradicts the functionality of $\theta^*(y, \mathbf{x}, 1)$. Therefore, the first large conjunction is non-empty (or θ_α is unsatisfiable and thus superfluous in the disjunctive normal form). Fixing one equation in the first conjunction, say $y = t_0(\mathbf{x}, 1)$, and substituting the other occurrences of y by t_0 throughout the formula, θ_α becomes

$$y = t_0 \wedge \bigwedge_i t_0 = t_i \wedge \bigwedge_j t_0 < s_j \wedge \bigwedge_k u_k = 0 \wedge \bigwedge_r v_r < 0$$

which may be rearranged to $\psi_\alpha(y, \mathbf{x}, 1)$:

$$y = t_0 \wedge \bigwedge_k u_k = 0 \wedge \bigwedge_r v_r < 0.$$

Thus $\theta^*(y, \mathbf{x}, 1)$ is equivalent in $(\mathbb{R}, 1)$ to a disjunction $\bigvee_\alpha \theta'_\alpha(y, \mathbf{x}, 1)$ which describes a definition by cases of $\nabla^{[0,1]}$:

$$\nabla^{[0,1]}\mathbf{x} = \begin{cases} t_0(\mathbf{x},1) & \text{if } \bigwedge_k u_{0k}(\mathbf{x},1) = 0 \wedge \bigwedge_r v_{0r}(\mathbf{x},1) < 0 \\ \vdots & \\ t_m(\mathbf{x},1) & \text{if } \bigwedge_k u_{mk}(\mathbf{x},1) = 0 \wedge \bigwedge_r v_{mr}(\mathbf{x},1) < 0 \end{cases} \tag{d}$$

where t_i, u_{ik}, v_{ir} are linear terms with rational coefficients, and the regions R_i defined by the conditions in the right hand side determine a partition of $[0,1]^n$. This could have been obtained also utilizing the fact that the theory of linearly ordered $\mathbb{Q}$-vector spaces is o-minimal (that is, any definable subset of the universe is a finite union of order intervals), see Corollary 7.6, Chap. 1, in [23].

Our aim now is to show that $\nabla^{[0,1]}$ is continuous. By the initial observations, ∇ exists in the DMV-chain $M = (\Gamma(\mathbb{R} \otimes \mathbb{R}, (1,1)), \delta_n)_{n\geq 2}$, and it is defined as a partial function $\nabla^M : [(0,0),(1,1)]^n \to [(0,0),(1,1)]$ in the unital ℓ-group $(\mathbb{R} \otimes \mathbb{R}, (1,1))$ by the formula $\theta^*(y, \mathbf{x}, u)$. Since the latter group is elementarily equivalent to $(\mathbb{R}, 1)$ by Lemma 3 then it satisfies the sentence

$$\forall y \forall \mathbf{x} \in [0,u]^{n+1} (\theta^*(y, \mathbf{x}, u) \leftrightarrow \bigvee_\alpha \theta'_\alpha(y, \mathbf{x}, u)),$$

which says precisely that definition (d) by cases holds for $\nabla^M \mathbf{x}$ in $(\mathbb{R}\otimes\mathbb{R}, (1,1))$ with the unital constant (1,1) in the place of 1 (notice that being these groups torsion free and divisible, the rational coefficients in the θ'_α are first order definable).

Moreover, the first projection $\pi_1 : (\mathbb{R} \otimes \mathbb{R}, (1,1)) \to (\mathbb{R}, 1)$ is an onto homomorphism of unital ℓ-groups, whose restriction to M gives an onto homomorphism $\pi_1 : M \to ([0,1]_{MV}, \delta_n)_n$ of MV-algebras, and a fortiori of DMV-algebras by compatibility of the δ_n.

We are ready to show that $\nabla^{[0,1]}$ is continuous. Suppose that is not the case; then there is a convergent sequence $\mathbf{a}_m \to \mathbf{a}$ in $[0,1]^n$ such that $\nabla^{[0,1]}(\mathbf{a}_n)$ does not converge to $\nabla^{[0,1]}(\mathbf{a})$. We may assume that $\{\mathbf{a}_n\} \subseteq R_i$ for some i because there are finitely many regions. Then $\nabla^{[0,1]}(\mathbf{a}_n) = t_i(\mathbf{a}_n, 1) \to t_i(\mathbf{a}, 1)$ by continuity of t_i and thus

$$\nabla^{[0,1]}(\mathbf{a}) \neq t_i(\mathbf{a}, 1). \tag{e}$$

Similarly, $u_{ik}(\mathbf{a},1) = \lim_n u_{ik}(\mathbf{a}_m,1) = 0$ and $v_{ir}(\mathbf{a},1) = \lim_n v_{ir}(\mathbf{a}_m,1) \leq 0$ by continuity of u_{ik} and v_{ir}. Take a point $\mathbf{b} \in R_i$ and consider the point $\mathbf{a} * \mathbf{b} = ((a_1, b_1), ..., (a_n, b_n)) \in M^n$ where $\mathbf{a} = (a_1, .., \mathbf{a}_n)$, $\mathbf{b} = (b_1, .., b_n)$. Then

$$u_{ik}(\mathbf{a} * \mathbf{b}, (1,1)) = (u_{ik}(\mathbf{a}, 1), u_{ik}(\mathbf{b}, 1)) = (0,0)$$
$$v_{ir}(\mathbf{a} * \mathbf{b}, (1,1)) = (v_{ir}(\mathbf{a}, 1), v_{ir}(\mathbf{b}, 1)) \leq_{lex} (0, v_{ir}(\mathbf{b}, 1)) <_{lex} (0,0)$$

for all k, r. That is, $\mathbf{a} * \mathbf{b}$ belongs to the region R_i in $\mathbb{R} \otimes \mathbb{R}$ and thus by (d)

$$\nabla^M(\mathbf{a} * \mathbf{b}) = t_i(\mathbf{a} * \mathbf{b}, (1,1)) = (t_i(\mathbf{a}, 1), t_i(\mathbf{b}, 1)).$$

On the other hand, since ∇ is a compatible implicit operation of DMV-algebras and $\pi_1 : M \to ([0,1]_{MV}, \delta_n)_n$ is an onto homomorphism then

$$t_i(\mathbf{a}, 1) = \pi_1 \nabla^M(\mathbf{a} * \mathbf{b}) = \nabla^{[0,1]}(\pi_1(a_1, b_1), ..., \pi_1(a_n, b_n))) = \nabla^{[0,1]}(\mathbf{a})$$

by Lemma 2-1, contradicting (e). We conclude, that $\nabla^{[0,1]}$ is continuous.

By the analogue of McNaughton theorem for DMV algebras (Lemma 9 in [3]), $\nabla^{[0,1]}$ must be given by a term φ of type $\{\oplus, \neg, \delta_n\}_{n\geq 2}$. Then the identities in $\mathcal{E}(\nabla)$ are satisfied by φ in all the algebras of the variety generated by $([0,1]_{MV}, \delta_n)_{n\in\omega}$, that is, in all the DMV-algebras. This means, by uniqueness, that ∇ exists and is given by φ in all these algebras. $\square$

We may conclude that RŁ does not admit proper implicit connectives:

Theorem 6 *Any implicit connective of Rational Łukasiewicz logic is explicit. More precisely, if* RŁ($\mathcal{C}$) *is an extension of* RŁ *by implicit connectives then for each* $\nabla \in \mathcal{C}$ *there is* $\varphi \in$ RŁ *such that* $\vdash_{RL(\mathcal{C})} \nabla(\mathbf{p}) \leftrightarrow \varphi(\mathbf{p})$.

Proof. Due to Theorem 1, for any implicit family $\mathcal{C}$ of connectives of Ł, the logic RŁ($\mathcal{C}$) is algebraized by $\mathcal{DMV}(\mathcal{C})$, where $\mathcal{C}$ is an implicit family of compatible operation of DMV-algebras. By Theorem 5, for each $\nabla \in \mathcal{C}$ there is a term φ of DMV algebras such that $\models_{\mathcal{DMV}(\mathcal{C})} \nabla(\mathbf{x}) = \varphi(\mathbf{x})$, and by algebraizability this implies $\vdash_{\mathrm{RL}(\mathcal{C})} \nabla(\mathbf{p}) \leftrightarrow \varphi(\mathbf{p})$, where $\varphi \in$ RŁ. $\square$

An inspection of the proof of Theorem 5 shows that it actually proves:

Theorem 7 *Any member of a family of compatible operations defined implicitly by identities in MV-algebras is given by a term of type* $\{\oplus, \neg, \delta_n\}_{n\geq 2}$ *in all DMV-algebras where the family exists (if any).*

This result will allow us to show that RŁ is the largest extension of Ł by implicit connectives having a sound and complete semantics with values in $[0,1]$, module bi-interpretations leaving Ł fixed.

Definition 5 Call a function T : Ł($\mathcal{C}$) $\to$ Ł($\mathcal{D}$) between extensions of Ł by implicit connectives a *faithful translation* over Ł if there are formulas $\varphi_\nabla \in$Ł($\mathcal{D}$), $\nabla \in \mathcal{C}$, such that for any $\alpha, \alpha_i \in$ Ł($\mathcal{C}$),

1. $T(\alpha) = \alpha(\nabla/\varphi_\nabla)_{\nabla\in\mathcal{C}}$
2. $\{\alpha_i\}_{i\leq n} \vdash_{L(\mathcal{C})} \alpha$ iff $\{T(\alpha_i)\}_{i\leq n} \vdash_{L(\mathcal{D})} T(\alpha)$.

This amounts to say that Ł($\mathcal{C}$) is bi-interpretable with a full fragment of Ł($\mathcal{D}$) by a translation that fixes Ł.

Theorem 8 *An extension* Ł($\mathcal{C}$) *of* Ł *by implicit connectives is sound and complete with respect to valuations in* $([0,1]_{MV}, f_\nabla)_{\nabla\in\mathcal{C}}$ *for some interpretation of the connectives in* $\mathcal{C}$ *if and only if there is a faithful translation* T : Ł($\mathcal{C}$) $\to$ RŁ.

Proof. For simplicity, we consider a single connective. Assume the hypothesis for Ł(∇). By soundness, f_∇ satisfies the identities corresponding to the axioms defining implicitly ∇, and thus $\nabla^{[0,1]} = f_\nabla$ exists in $[0,1]_{MV}$. By Theorem 7, there is a DMV-term φ such that $f_\nabla = \varphi$ in $[0,1]_{MV\delta} = ([0,1]_{MV}, \delta_n)_n$. Hence, we have the following chain of equivalences: $\{\alpha_i\}_{i\leq n} \vdash_{Ł(\nabla)} \alpha$ iff $\{\alpha_i = 1\}_{i\leq n} \models_{([0,1]_{MV}, f_\nabla)} \alpha = 1$ (completeness of Ł(∇)), iff $\{\alpha_i(\nabla/\varphi) = 1\}_{i\leq n} \models_{[0,1]_{MV\delta}} \alpha(\nabla/\varphi) = 1$ (previous observation), iff $\{\alpha_i(\nabla/\varphi)\}_{i\leq n} \vdash_{RŁ} \alpha(\nabla/\varphi)$ (completeness of RŁ). Therefore, $T(\alpha) := \alpha(\nabla/\varphi)$ is the required translation.

Reciprocally, if there is a faithful translation T :Ł(∇) $\rightarrow$ RŁ as described, then $\{\alpha_i\}_{i\leq n} \vdash_{Ł(\nabla)} \alpha$ iff $\{\alpha_i(\nabla/\varphi)\}_{i\leq n} \vdash_{RŁ} \alpha(\nabla/\varphi)$ (hypothesis), iff $\{\alpha_i(\nabla/\varphi) = 1\}_{i\leq n} \models_{[0,1]_{MV\delta}} \alpha(\nabla/\varphi) = 1$ (completeness of RŁ), iff $\{\alpha_i(\nabla) = 1\}_{i\leq n} \models_{([0,1]_{MV}, \varphi^{[0,1]})} \alpha(\nabla) = 1$. Thus, Ł($\nabla$) is complete with respect to $([0,1]_{MV}, \varphi^{[0,1]})$. □

By Theorem 3, the previous result implies that the logic Ł(δ_n^*) of approximate division introduced in Section 3 can not be faithfully embedded in RŁ, even less in Ł(δ_n), for $n \geq 2$. Therefore, RŁ is not the maximum extension of Ł by implicit connectives.

Observe that Ł(δ_n) cannot be embedded in Ł(δ_n^*), even as a weak fragment. Otherwise, the image γ of $\delta_n 1$ by a possible translation would satisfy $\vdash_{Ł(\delta_n^*)} (n-1)\gamma \leftrightarrow \neg\gamma$, which is impossible because the corresponding equation, $(n-1)x = \neg x$ has no solution in L_n. Therefore, Ł(δ_n^*) and Ł(δ_n) are incomparable extensions of Ł with respect to faithful translations.

6 Implicit Connectives of n-valued Łukasiewicz Logic

For $n \geq 2$, Łukasiewicz n-valued calculus Ł$_n$ (cf. [10]) is algebraized by the variety $\mathcal{MV}_n$ of n-valued MV-algebras, generated in turn (as a quasivariety) by the Łukasiewicz chain L_n. By Jónsson's lemma (Th. 6.8, [7]), the subdirectly irreducible algebras of $\mathcal{MV}_n$ are the subalgebras of L_n because this variety is congruence distributive and these algebras are simple. Moreover, they are the only chains of the variety.

For $n \geq 3$, the axiom

$$(n-2)c \leftrightarrow \neg c$$

defines an implicit constant connective of Ł$_n$. In fact, it defines an implicit connective already in Ł because the quasi-identity

$$\forall x \forall y (mx = \neg x \wedge my = \neg y \implies x = y)$$

holds in $[0,1]_{MV}$, and by completeness $mc \leftrightarrow \neg c$, $mc' \leftrightarrow \neg c' \vdash_Ł c \leftrightarrow c'$. According to Theorem 6, c is reducible to $\delta_{n-1}(1)$ in RŁ.

Returning to Ł$_n$, c is realized algebraically in L_n as the element $\frac{1}{n-1}$, but it does not exists in any proper subalgebra of L_n. Thus $(L_n, \frac{1}{n-1})$ is the only subdirectly irreducible algebra of the corresponding variety $\mathcal{MV}_n(c)$ by Lemma 2-2, and therefore Ł$_n(c)$ is sound and strongly complete with respect to values in this algebra by Theorem 1.

Theorem 9 *Any implicit connective of* Ł$_n(c)$ *is explicit.*

Proof. For any implicit extension Ł$_n(c,\nabla)$, the only subdirectly irreducible algebra of $\mathcal{MV}_n(c,\nabla)$ is $(L_n, \frac{1}{n-1}, \nabla^{L_n})$ by Lemma 2-2 and the previous observations, and thus this algebra generates the variety. But $(L_n, \frac{1}{n-1})$ is a primal algebra because it is term equivalent to the basic Post algebra of order n. Then ∇ is a term φ of $\mathcal{MV}_n(c)$ in $(L_n, \frac{1}{n-1})$. The identity $\nabla = \varphi$ is inherited by the variety $\mathcal{MV}_n(c,\nabla)$, and thus $\vdash_{L_n(c,\nabla)} \nabla \leftrightarrow \varphi$ by algebraizability. □

It is possible to show, as in Theorem 8, that $L_n(c)$ is the largest extension of L_n by implicit connectives which is complete with respect to values in the algebra L_n. However, there are such extension that are not complete with respect to L_n, or any single given chain. For example, the axioms

nc^*
$np \to (c^* \to p)$

define an implicit connective of Ł$_n$ which is realized in each subalgebra of L_n as the minimum positive element of that subalgebra. Thus Ł$_n(c^*)$ is complete with respect to the family of chains $(L_k, \frac{1}{k-1})$, $(k-1)|(n-1)$. But it is not complete with respect to any particular one of them. We illustrate the case when $n = 5$.

Proposition 3 Ł$_5(c^*)$ *is a conservative extension of* Ł$_5$ *not complete with respect to any single chain.*

Proof. The only chains of $\mathcal{MV}_5(c^*)$ are $(L_5, \frac{1}{4})$ and $(L_3, \frac{1}{2})$. The logic is not complete with respect to the first chain because $\nvdash_{L_5(c_4^*)} 3c^* \leftrightarrow \neg c^*$, which may be falsified only in $(L_3, \frac{1}{2})$, nor is it complete with respect to the second one because $\nvdash_{L_5(c_4^*)} 2c^*$, which may be falsified only in $(L_5, \frac{1}{4})$. □

7 Implicit Operations of ℓ-groups and Implicit Connectives of Abelian Logic

For each $n \geq 2$, the single identity

$$nD_n(x) = x$$

defines an implicit operation $D_n(x) = \frac{1}{n}x$ in ℓ-groups because these groups are torsion free. This operation may be seen to be compatible because the congruences of ℓ-groups are determined by their convex subgroups and $\frac{1}{n}x$ belongs to the interval determined by 0 and x. The variety $\ell\mathcal{G}(D_n)_{n\geq 2}$ consists of all divisible ℓ-groups endowed with these operations. This is essentially the variety of $\mathbb{Q}$-*vector lattices* (lattice ordered $\mathbb{Q}$-vector spaces satisfying $r(x \vee y) = rx \vee ry$

for any positive $r \in \mathbb{Q}$). Clearly, there is an analogue of Mundici's functor which sends $\ell\mathcal{G}(D_n)_{n\geq 2}$ onto the variety of DMV-algebras.

The proof of Theorem 5 may be readily adapted to show the following result, which we state for a single operation but holds equally for families.

Theorem 10 *Any compatible operation ∇ defined implicitly by identities over the variety $\ell\mathcal{G}(D_n)_{n\geq 2}$ is given by a term of the variety.*

Proof. If ∇ exists in a non trivial $G \in \ell\mathcal{G}(D_n)_{n\geq 2}$, then by compatibility it exists in each one of the non trivial subdireclty irreducible factors of a subdirect decomposition of G. From Lemma 2-2 and Fact 2 in Section 4, these are divisible linearly ordered ℓ-groups. By completeness of the theory of these groups (Fact 4) and first order definability of ∇, this operation exists in all non trivial linearly ordered groups of $\ell\mathcal{G}(D_n)_{n\geq 2}$, in particular in $(\mathbb{R}, D_n)_n$. Arguing as in the proof of Theorem 5, the first order definition $\theta(y, \mathbf{x})$ of ∇ takes the form in $\mathbb{R}$:

$$\nabla^{\mathbb{R}}\mathbf{x} = \begin{cases} t_1(\mathbf{x}) & \text{if } \bigwedge\limits_k u_{1k}(\mathbf{x}) = 0 \wedge \bigwedge\limits_r v_{1r}(\mathbf{x}) < 0 \\ \vdots & \\ t_m(\mathbf{x}) & \text{if } \bigwedge\limits_k u_{mk}(\mathbf{x}) = 0 \wedge \bigwedge\limits_r v_{mr}(\mathbf{x}) < 0 \end{cases} \qquad \text{(q)}$$

where t_i, u_{1k}, and v_{1r} are linear expressions with rational coefficients, and the left right conditions determine a partition of $\mathbb{R}$ into disjoint non empty regions R_i.

To prove that $\nabla^{\mathbb{R}}$ is continuous, notice first that ∇ exists and must obey (q) in the group $(\mathbb{R} \otimes \mathbb{R}, D_n)_n$ because this fact is expressible by first order sentences and $(\mathbb{R} \otimes \mathbb{R}, D_n)_n \equiv (\mathbb{R}, D_n)_n$. Moreover, the first projection $\pi_1 : \mathbb{R} \otimes \mathbb{R} \to \mathbb{R}$ is an epimorphism of ℓ-groups which may be seen to preserve the D_n. Thus $\pi_1 : (\mathbb{R} \otimes \mathbb{R}, D_n)_n \to (\mathbb{R}, D_n)_n$ is an epimorphism and it must preserve ∇ by the compatibility hypothesis; that is,

$$\pi_1 \nabla^{\mathbb{R}\otimes\mathbb{R}}((x_1, y_1), ..., (x_n, y_n)) = \nabla^{\mathbb{R}}(x_1, ..., x_n). \qquad \text{(r)}$$

As in the proof of Theorem 5, were $\nabla^{\mathbb{R}}$ not continuous we could find a region R_i and points $\mathbf{a} \notin R_i$, $\mathbf{b} \in R_i$ in $\mathbb{R}^n$ such that $\nabla^{\mathbb{R}}(\mathbf{a}) \neq t_i(\mathbf{a})$ and $\mathbf{a} * \mathbf{b} = ((a_1, b_1), ..., (a_n, b_n))$ would belong to the region R_i in $(\mathbb{R} \otimes \mathbb{R})^n$. Hence,

$$\nabla^{\mathbb{R}\otimes\mathbb{R}}(\mathbf{a} * \mathbf{b}) = t_i(\mathbf{a} * \mathbf{b}) = (t_i(\mathbf{a}), t_i(\mathbf{b}))$$

and thus $\nabla^{\mathbb{R}}(\mathbf{a}) = \pi_1 \nabla^{\mathbb{R}\otimes\mathbb{R}}(\mathbf{a} * \mathbf{b}) = t_i(\mathbf{a})$, by (r), a contradiction.

We have then a piecewise linear function with rational coefficients. Let m be the common denominator of the coefficients of the $t_i(\mathbf{x}), u_{1k}(\mathbf{x}), v_{1r}(\mathbf{x})$. Then $m\nabla^{\mathbb{R}}(\mathbf{x})$ is a piecewise linear continuous function with integer coefficients in $\mathbb{R}$ and thus it must be given by an ℓ-group term (folklore, see final remark in [4]), say $m\nabla^{\mathbb{R}}(\mathbf{x}) = u(\mathbf{x})$. Hence, $\nabla^{\mathbb{R}}(\mathbf{x}) = \frac{1}{m}u(\mathbf{x}) = D_m u(\mathbf{x})$, a term of $\ell\mathcal{G}(D_n)_{n\geq 2}$. By first order completeness, ∇ is given by $D_m u(\mathbf{x})$ in all subdirectly irreducible algebras of $\ell\mathcal{G}(D_n)_{n\geq 2}$. Therefore, the set of identities $\mathcal{E}(\nabla / D_m u(\mathbf{x}))$ holds in all the algebras of the variety $\ell\mathcal{G}(\nabla, D_n)_{n\geq 2}$ and, by uniqueness, $\nabla(\mathbf{x}) = D_m u(\mathbf{x})$ in $\ell\mathcal{G}(\nabla, D_n)_{n\geq 2}$. □

A *Logic of equilibrium*, $\mathcal{B}al$, is described in [14] which is algebraizable by the variety of ℓ-groups (being thus a version of so called *Abelian logic*, [19]). It has the axiom schemes:

$(p \to q) \to ((r \to q) \to (r \to p))$
$(p \to (q \to r)) \to (q \to (p \to r)$
$((p \to q) \to q) \to p$
$((p \to q)^+ \to (q \to p)^+) \to (p \to q)$
$p^{++} \to p^+$

and inference rules

$$p \to q, p \vdash q, \quad p, q \vdash p \to q, \quad p \vdash p^+, \quad (p \to q)^+ \vdash (p^+ \to q^+)^+.$$

The defined connectives

$$0 := p \to p, \quad -p := q \to 0, \quad p + q := -p \to q, \quad p \curlyvee q := (p \to q)^+ + q$$

form a complete set since $p \to q \dashv\vdash -p + q$ and $p^+ \dashv\vdash p \curlyvee 0$, and they allow the interpretation of $\mathcal{B}al$ in ℓ-groups so that we get algebraic completeness:

$$\{\varphi_i\}_{i\leq n} \vdash_{\mathcal{B}al} \varphi \text{ iff } \{\varphi_i = 0\}_{i\leq n} \models_{\ell\mathcal{G}} \varphi = 0.$$

In fact, algebraic completeness holds with respect to values in $\mathbb{Z}$ (or $\mathbb{Q}$, or $\mathbb{R}$), because these groups generate $\ell\mathcal{G}$ as a quasi-variety, [5]. The 'equivalence' and constant formula mediating algebraicity are just $p \to q$ (equivalently, $-q + p$), and thus this connective must satisfy symmetry and transitivity, and $1 := 0$. Since $p + ... + p = 0 \models_{\ell\mathcal{G}} p = 0$, because all ℓ-groups are torsion free, we have by algebraic completeness: $n\varphi \vdash_{\mathcal{B}al} \varphi$ for any $n \geq 2$. Also, $-np + nq = 0 \models_{\ell\mathcal{G}} n(-p + q) = 0$, which implies $n\varphi \to n\psi \vdash_{\mathcal{B}al} n(\varphi \to \psi)$.

It follows easily from the previous observations that the single axiom

$$p \to nD_n p$$

defines implicitly the connective D_n over $\mathcal{B}al$. Indeed:

$$p \to nD_n p,\ p \to n\lambda p \vdash\ nD_n p \to n\lambda p \vdash\ n(D_n p \to \lambda p) \vdash\ D_n p \to \lambda p.$$

By Theorem 1, the logic $R\mathcal{B}al = \mathcal{B}al(D_n)_{n\geq 2}$, that we could call *Rational logic of equilibrium*, is algebraized by the variety $\ell\mathcal{G}(D_n)_{n\geq 2}$ introduced above. Together with Theorem 10, this implies:

Corollary 1 *Every implicit connective of $R\mathcal{B}al$ is explicit.*

Note that $R\mathcal{B}al$ is complete with respect to values in $\mathbb{R}$ (or $\mathbb{Q}$). One may show, as in Theorem 8, that it is the largest extension of $\mathcal{B}al$ by implicit connectives with this property:

Theorem 11 *An extension of $\mathcal{B}al$ by implicit connectives is sound and complete with respect to valuations in $(\mathbb{R}, f_\nabla)_\nabla$ for some interpretation f_∇ of the new connectives if and only if it has a faithful translation into $R\mathcal{B}al$.*

8 Final Remarks

We have not considered in this paper extensions of Łukasiewicz logic by connectives implicitly defined by axiom schemes and new inference rules. In this case, the extension is still algebraizable by a (perhaps proper) quasivariety of enriched MV-algebras (Th. 1 in [8]), but the algebraic interpretation of the connectives is not necessarily compatible. For example, the following system:

$\beta p \curlyvee \neg\beta p$
$p \to \beta p$
$q \curlyvee \neg q, (p \to q) \vdash (\beta p \to q)$

defines implicitly a connective β over Ł . It may be shown that the extension Ł(β) is algebraizable by the MV-algebras which support the operation

$\beta(x) :=$ smallest boolean y greater or equal than x.

Thus, β exists in all MV-chains, particularly in $[0, 1]$, where it takes the form of the Baaz delta operator, [2]:

$$\beta(x) = \begin{cases} 0 & \text{if } x = 0 \\ 1 & \text{if } x > 0. \end{cases}$$

Clearly, this operation is not compatible in the non-simple MV-chains, and thus, by Theorem 1, β can not be defined implicitly by means of axioms only.

This fact marks a sharp difference with classical propositional calculus, because it may be shown that the latter does not admit new connectives defined implicitly by axiom schemes and inference rules.

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