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Implicit Connectives of Algebraizable Logics

Abstract. An extensions by new axioms and rules of an algebraizable logic in the sense of Blok and Pigozzi is not necessarily algebraizable if it involves new connective symbols, or it may be algebraizable in an essentially different way than the original logic. However, extension whose axioms and rules define implicitly the new connectives are algebraizable, via the same equivalence formulas and defining equations of the original logic, by enriched algebras of its equivalente quasivariety semantics. For certain strongly algebraizable logics, all connectives defined implicitly by axiomatic extensions of the logic are explicitly definable.

Keywords: Algebraizable logics, quasivarieties, connectives, implicit and explicit definitions.

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The well known functional completeness of the truth table interpretation of the usual connectives of classical propositional calculus has a deductive counterpart: if an axiomatic extension of the calculus defines implicitly a new connective symbol, this must be deductively equivalent to a combination of classical connectives. Although this may seem obvious, its proof is not immediate. Other calculi, like Heyting calculus or intermediate logics, allow implicitly defined connectives which are not explicitly definable in this manner, providing thus natural and interesting enrichments by hidden concepts of the logic, cf. [3]. In this paper, we consider implicit connectives in the general context of algebraizable logics, in the sense of Blok and Pigozzi [1].

An extension by new axioms and rules of an algebraizable logic $\mathcal{L}$ is not necessarily algebraizable if it involves new connective symbols, or it may happen that it is algebraizable in a essentially different way than $\mathcal{L}$ (Sec. 2). However, we prove that extensions defining the new connectives implicitly (with respect to the equivalence provided by the algebraization) are algebraizable, via the same equivalence formulas and defining equations of $\mathcal{L}$, by expanded algebras of its equivalent quasivariety semantics (Sec. 3). The algebraic interpretation of a connective implicitly defined by a purely axiomatic extension (adding new axioms but no new rules) is shown to be

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always compatible with the relative congruences of the quasivariety semantics of $\mathcal{L}$. It follows that such a connective is explicitly definable by a formula of $\mathcal{L}$ if and only if the class of algebras where its interpretation exists forms a quasivariety (Sec. 4, 5). Under certain conditions on the variety semantics of a strongly algebraizable logic, all the connectives defined implicitly by axiomatic extensions are shown to be equivalent to combination of the original connectives (Sec. 6).

1. Algebraizable logics

Given a family of finitary function symbols $\tau = \{\omega_\alpha : \alpha < \kappa\}$, a *sentential logic* or *deductive system* $\mathcal{L}$ of type τ will be a structural finitary consequence relation $\vdash_{\mathcal{L}}$ in the absolutely free algebra Fm_τ of type τ generated by the (propositional) variables $p_1, p_2, ...$ The function symbols ω_α are called *primitive connectives* of $\mathcal{L}$ and the terms of Fm_τ are called *formulas* of $\mathcal{L}$. Usually, $\vdash_{\mathcal{L}}$ is specified by a set of Hilbert style axiom schemes and inference rules.

Definition (cf. Def. 2.8, Cor. 2.9 [1]). A logic $\mathcal{L}$ is (*finitely) algebraizable* by a quasivariety K of type τ (the *equivalent quasivariety semantics* of $\mathcal{L}$) via a finite set of binary formulas $\{p \Leftrightarrow_i q\}_i$ (the *equivalence formulas* of $\mathcal{L}$) and a finite set of identities $\{\delta_j(p) \approx \epsilon_j(p)\}_j$ (the *defining equations* of $\mathcal{L}$) if the following conditions are met:

i) $\phi_1, ..., \phi_n \vdash_{\mathcal{L}} \phi$ iff $\{\delta(\phi_i) \approx \epsilon(\phi_i)\}_{i=1}^n \models_K \delta(\phi) \approx \epsilon(\phi)$

ii) $p \approx q \mathrel{=\!\!\mid\!\models_K} \delta(p \Leftrightarrow q) \approx \epsilon(p \Leftrightarrow q)$.

We are using the abbreviations $p \Leftrightarrow q$ for $\{p \Leftrightarrow_i q\}_i$, $\delta(p \Leftrightarrow q) \approx \epsilon(p \Leftrightarrow q)$ for $\{\delta_j(p \Leftrightarrow_i q) \approx \epsilon_j(p \Leftrightarrow_i q)\}_{i,j}$, etc.

The quasivariety $K = K_{\mathcal{L}}$ is uniquely determined by $\mathcal{L}$, when it exists, and the equivalence formulas and defining equations are also unique in the sense that for any other system $p \Leftrightarrow' q$, $\delta'(p) \approx \epsilon'(p)$ satisfying (*i*-*ii*) we have: $p \Leftrightarrow q \dashv\vdash_{\mathcal{L}} p \Leftrightarrow' q$ and $\delta(p) \approx \epsilon(q) \mathrel{=\!\!\mid\!\models_K} \delta'(p) \approx \epsilon'(q)$.

Among the various characterizations of algebraizability, the following one is quite useful.

Syntactic characterization (Th. 4.7 [1]). *$\mathcal{L}$ is algebraizable via $p \Leftrightarrow q = \{p \Leftrightarrow_i q\}_i$, $\delta(p) \approx \epsilon(p) = \{\delta_j(p) \approx \epsilon_j(p)\}_j$ if and only if:*

1. $\vdash_{\mathcal{L}} p \Leftrightarrow p$
2. $p \Leftrightarrow q \vdash_{\mathcal{L}} q \Leftrightarrow p$
3. $p \Leftrightarrow q, q \Leftrightarrow r \vdash_{\mathcal{L}} p \Leftrightarrow r$

4. $p \Leftrightarrow q \vdash_{\mathcal{L}} \omega(p_1, .., p, .., p_n) \Leftrightarrow \omega(p_1, .., q, .., p_n)$, *for any* $\omega \in \tau$
5. $p \dashv\vdash_{\mathcal{L}} \delta(p) \Leftrightarrow \epsilon(p)$.

Notice that with the help of condition 3, condition 4 spreads to all formulas by induction on complexity:

4′. $p \Leftrightarrow q \vdash_{\mathcal{L}} \varphi \Leftrightarrow \varphi(p/q)$,

and a derived *Detachment Rule* follows from 4′ and 5 (cf. Lemma 2.14 [1]):

DR. $p \Leftrightarrow q,\, p \vdash_{\mathcal{L}} q$.

Classical and intuitionistic propositional calculus, intermediate logics, deductive Łukasiewicz logics, normal modal logic with classic or intuitionistic base, as well as many fragments of these logics are algebraizable by suitable varieties of algebras, via what we will call the *standard system*: $p \Leftrightarrow q = \{p \to q, q \to p\}$ and $\delta(p) \approx \epsilon(p) = \{p \approx p \to p\}$. BCK-logic is algebraizable via the standard systems but requires a proper quasivariety semantics, cf. [11]. Relevance logic in the sense of Anderson and Benlap is algebraizable by a variety with standard $p \Leftrightarrow q$ but $\delta(p) \approx \epsilon(p) = \{p \wedge (p \to p) \approx p \to p\}$.

2. Algebraizable and non-algebraizable extensions

An extension $\mathcal{L}'$ of an algebraizable logic $\mathcal{L}$, by new axioms and rules on the primitive connectives of $\mathcal{L}$, is automatically algebraizable by a sub-quasivariety of $K_{\mathcal{L}}$ and the same equivalence formulas and defining equations of $\mathcal{L}$ (Cor. 4.9 [1]). In fact, it may be shown that the lattice of such extensions of $\mathcal{L}$ is anti-isomorphic to the lattice of sub-quasivarieties of $K_{\mathcal{L}}$ by the natural correspondence:

$$\mathcal{L}' \longmapsto K_{\mathcal{L}'} = Mod\{\textstyle\bigwedge_i \phi_i \approx \psi_i \Rightarrow \phi \approx \psi :\ \{\phi_i \Leftrightarrow \psi_i\}_i \vdash_{\mathcal{L}'} \phi \Leftrightarrow \psi\},$$

$$K' \longmapsto \mathcal{L}_{K'} = \langle Fm\tau,\ \{\{\phi_i\}_i \vdash \phi : \textstyle\bigwedge_i \delta(\phi_i) \approx \epsilon(\phi_i) \models_{K'} \delta(\phi) \approx \epsilon(\phi)\ \}\rangle.$$

However, extensions involving new connective symbols are not necessarily algebraizable. For example, adding the axioms of modal logic $S5$ without the necessitation rule to classical propositional calculus yields a non algebraizable logic (Cor. 5.6 [1]). A more significant example follows.

Example 1 (*non-algebraizable logic of the dual pseudocomplement*). Consider the extension $IPC(D)$ of intuitionistic propositional calculus IPC by the axioms:

D_1. $p \vee Dp$
D_2. $(q \vee D(q \vee p)) \leftrightarrow (q \vee Dp)$
D_3. $\neg D(p \to p)$.

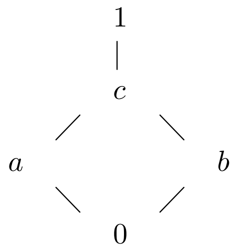
Interpreting D as an operation satisfying the corresponding identities in Heyting algebras:

E_1. $x \vee D(x) = 1$
E_2. $y \vee D(y \vee x) = y \vee D(x)$
E_3. $D(1) = 0$,

it becomes the *dual pseudocomplement* operation:

$$D_H(x) = \min\{y \in H : x \vee y = 1\}.$$

Indeed, these identities are clearly satisfied by D_H and, whenever $x \vee y = 1$, they imply: $y \vee D(x) =_{E_2} y \vee D(y \vee x) = y \vee D(1) =_{E_3} y \vee 0 = y$; that is, $D(x) \leq y$. Not every Heyting algebra supports such an operation, but finite Heyting algebras (at least) do. To see that $IPC(D)$ is not algebraizable. consider the Heyting algebra H:



where $D_H(1) = 0$ and $D_H(x) = 1$ for $x \neq 1$. Then $\langle H, D_H \rangle$ is simple because any congruence Θ distinct from the diagonal must contain $(x, 1)$ with $x \neq 1$, thus $(1, 0) = (D_H(x), D_H(1)) \in \Theta$ and $\Theta = H \times H$. On the other hand, the non trivial Heyting algebra filters of H: $\{a, c, 1\}$, $\{b, c, 1\}$, $\{c, 1\}$, $\{1\}$, are $IPC(D)$-filters because the axioms D_1,D_2,D_3 evaluate to 1 when interpreted by D_H. This means that the Leibniz operator $\Omega_{(H, D_H)}$ from $IPC(D)$-filters into congruences of (H, D_H) can not be injective. Thus $IPC(D)$ is not algebraizable by any quasivariety whatsoever by Th. 5.1(ii) in [1], not even weakly algebraizable in the sense of Czelakowski and Jansana [6].

$IPC(D)$ inherits from IPC, by structurality, all the conditions in the syntactic characterization of algebraizability for standard $\Leftrightarrow$, δ, ϵ, except condition 4 for the connective D. In fact, we may conclude that $p \leftrightarrow q \nvdash_{IPC(D)} Dp \leftrightarrow Dq$, since we know that the logic is not algebraizable.

An extension of an algebraizable logic $\mathcal{L}$ by new connectives may be algebraizable via a system of equivalence formulas and defining equations essentially different from those of $\mathcal{L}$, and the algebras in its quasivariety semantics do not need to be expansions of the algebras in the quasivariety semantics of $\mathcal{L}$, as illustrated next.

EXAMPLE 2. Let $\mathcal{L}' = CPC(\Box)$ be the following extension of classical propositional calculus, with Modus Ponens as the only inference rule:

A1. $\Box\varphi$ for any theorem φ of IPC
A2. $\Box(p \to q) \to (\Box p \to \Box q)$
A3. $\Box(p \to q) \to \Box(\Box p \to \Box q)$
A4. $p \to \Box\neg\neg p$
A5. $\Box\neg\neg p \to p$.

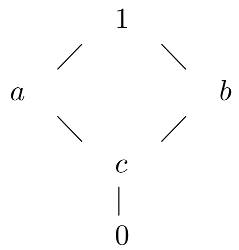
Glivenko's theorem plus A1, A5 grant that $\mathcal{L}'$ extends, in fact, CPC. This logic is algebraizable via

$$p \Leftrightarrow' q := \{\Box(p \to q), \Box(q \to p)\}, \quad \delta'(p) \approx \epsilon'(p) := \{\neg\neg p \approx \top\},$$

where $\top$ stands for $(p \to p)$. Clause 1 in the syntactic characterization of algebraizability follows from A1, clauses 2, 3 and clause 4 for classical connectives follow from A1, A2, and clause 4 for $\Box$ follows from A3. Let us verify clause 5:

$\vdash_{\mathcal{L}'} \Box(\neg\neg p \to \top)$ and $\vdash_{\mathcal{L}'} \Box(\neg\neg p \leftrightarrow (\top \to \neg\neg p))$ by A1; hence,
$\vdash_{\mathcal{L}'} \Box\neg\neg p \leftrightarrow \Box(\top \to \neg\neg p)$ by A2, and thus
$p \vdash_{\mathcal{L}'} \{\Box(\neg\neg p \to \top), \Box(\top \to \neg\neg p)\} \vdash_{\mathcal{L}'} p$ by A4, A5.

However, $p \Leftrightarrow' q$ is not equivalent in $\mathcal{L}'$ to standard $p \Leftrightarrow q$. To see this, notice that the enriched Heyting algebra $\langle B, \Box \rangle$:



$$\Box x = \begin{cases} 1 & \text{if } x = 1 \\ 0 & \text{if } x \neq 1 \end{cases}$$

belongs to the equivalent quasivariety semantics of $\mathcal{L}'$, according to Th. 2.17 in [1], because it satisfies the identities and quasi-identities:

$\neg\neg \mathrm{A}_i \approx \top$ $(i = 1, ..., 5)$
$\neg\neg\Box(p \to p) \approx \top$
$\neg\neg(p \to q) \approx \top,\ \neg\neg p \approx \top \Rightarrow \neg\neg q \approx \top$
$\neg\neg\Box(p \to q) \approx \top,\ \neg\neg\Box(q \to p) \approx \top \Rightarrow p \approx q,$

corresponding, respectively, to the axioms of the logic, the identity $\delta'(p \Leftrightarrow' p) \approx \epsilon'(p \Leftrightarrow' p)$, the Modus Ponens rule, and the quasi-identity $\delta'(p \Leftrightarrow' q) \approx \epsilon'(p \Leftrightarrow' q) \Rightarrow p \approx q$. Indeed, for any values of p, q in $\langle B, \Box \rangle$ we have $\mathrm{A}_i = 1$ for $i = 1, .., 4$, and $\mathrm{A}_5 \neq 0$ ($\Box\neg\neg x \to x = 0 \to x = 1$ if $\neg\neg x \neq 1$, and

$\Box\neg\neg x \to x = 1 \to x = x$ if $\neg\neg x = 1$). Thus, $\neg\neg \mathrm{A}_i = 1$ for $i = 1, .., 5$. The second displayed identity holds trivially and the first quasi-identity holds in any Heyting algebra. The last quasi-identity holds because $\neg\neg\Box(x \to y) = 1$ is possible only if $\Box(x \to y) = 1$, which implies $x \to y = 1$, and hence $x \leq y$. On the other hand, $\langle B, \Box\rangle$ does not satisfy the quasi-identity

$$\neg\neg(p \to q) \approx 1\ ,\ \ \neg\neg(q \to p) \approx 1 \ \Rightarrow\ \neg\neg\Box(p \to q) \approx 1$$

because $\neg\neg(1 \to c) = \neg\neg(c \to 1) = 1$ while $\neg\neg\Box(1 \to c) = 0$. By algebraizability this means

$$\{p \to q,\ q \to p\} \nvdash_{\mathcal{L}'} \{\Box(p \to q), \Box(q \to p)\}.$$

Similarly, $\delta'(p) \approx \epsilon'(p)$ is not equivalent to standard $\delta(p) \approx \epsilon(p)$ because $\neg\neg c = 1$ but $c \neq 1$ in $\langle B, \Box\rangle$.

3. Extensions by implicit connectives

Let $\mathcal{L}$ be an algebraizable logic with a system of equivalence formulas $p \Leftrightarrow q$. We will say that an extension $\mathcal{L}(\mathcal{C})$ by axioms and rules involving a family of new connective symbols $\mathcal{C}$ *defines* $\mathcal{C}$ *implicitly* if:

$$\vdash_{\mathcal{L}(\mathcal{C})\cup\mathcal{L}(\mathcal{C}')} \nabla(p_1, ..., p_{n_\nabla}) \Leftrightarrow \nabla'(p_1, ..., p_{n_\nabla}),\ \text{for each } \nabla \in \mathcal{C},$$

where $\mathcal{C}'$ is a family of disjoint copies of the symbols in $\mathcal{C}$, and $\mathcal{L}(\mathcal{C})\cup\mathcal{L}(\mathcal{C}')$ is the structural extension of the logic $\mathcal{L}(\mathcal{C})$ by the ∇'-duplicates of the axioms and rules of $\mathcal{L}(\mathcal{C})$. We will write $\mathcal{L}(\nabla)$ for $\mathcal{L}(\{\nabla\})$. This generalizes the notion of an implicit connective of intuitionistic propositional calculus introduced in [3].

Theorem 1. *An extension $\mathcal{L}(\mathcal{C})$ of an algebraizable logic $\mathcal{L}$ defining implicitly a family of connectives $\mathcal{C}$ is algebraizable via the same equivalence formulas and defining equations of $\mathcal{L}$.*

Proof. By the syntactic characterization of algebraizability, it is enough to show that $p \Leftrightarrow q \vdash_{\mathcal{L}(\mathcal{C})} \nabla(\mathbf{p}) \Leftrightarrow \nabla(\mathbf{p}(p/q))$ for each $\nabla \in \mathcal{C}$, since the other conditions hold automatically in the extension by structurality. Fix distinct propositional variables p, q and define, by simultaneous induction, two transformations φ^* and φ^+ from $Fm_{\tau\cup\mathcal{C}\cup\mathcal{C}'}$ into formulas of $Fm_{\tau\cup\mathcal{C}}$:

$p^* := q$, and $v^* := v$ if v is variable distinct from p
$\omega(\varphi_1, ..., \varphi_k)^* := \omega(\varphi_1^*, ..., \varphi_k^*)$, for $\omega \in \tau$
$\nabla(\varphi_1, ..., \varphi_n)^* := \nabla(\varphi_1^+, ..., \varphi_n^+)$, for $\nabla \in \mathcal{C}$
$\nabla'(\varphi_1, ..., \varphi_n)^* := \nabla(\varphi_1^*, ..., \varphi_n^*)$, for $\nabla' \in \mathcal{C}'$.

$v^+ = v$ for any propositional variable

$\omega(\varphi_1, ..., \varphi_k)^+ := \omega(\varphi_1^+, ..., \varphi_k^+)$, for $\omega \in \tau \cup \mathcal{C}$.
$\nabla'(\varphi_1, ..., \varphi_n)^+ := \nabla(\varphi_1^*, ..., \varphi_n^*)$, for $\nabla' \in \mathcal{C}'$.

Claim 1: $p \Leftrightarrow q \vdash_{\mathcal{L}} \varphi^* \Leftrightarrow \varphi^+$, for any $\varphi \in Fm_{\tau\cup\mathcal{C}\cup\mathcal{C}'}$. For a variable φ this becomes: $p \Leftrightarrow q \vdash_{\mathcal{L}} q \Leftrightarrow p$ if φ is p, and $p \Leftrightarrow q \vdash_{\mathcal{L}} \varphi \Leftrightarrow \varphi$, otherwise. The inductive step for a connective ω of $\mathcal{L}$ follows by condition (4) in the syntactic characterization of algebraizability, and for the connectives in $\mathcal{C}\cup\mathcal{C}'$ by condition (1) since $\nabla'(\varphi_1, ..., \varphi_n)^* = \nabla(\varphi_1^*, ..., \varphi_n^*) = \nabla'(\varphi_1, ..., \varphi_n)^+$ and $\nabla(\varphi_1, ..., \varphi_n)^* = \nabla(\varphi_1^+, ..., \varphi_n^+) = \nabla(\varphi_1, ..., \varphi_n)^+$.

Claim 2: $\vdash_{\mathcal{L}(\mathcal{C})\cup\mathcal{L}(\mathcal{C}')} \varphi$ implies $p \Leftrightarrow q \vdash_{\mathcal{L}(\mathcal{C})} \varphi^*, \varphi^+$. We verify this by induction on the length of the proof of $\vdash_{\mathcal{L}(\mathcal{C})\cup\mathcal{L}(\mathcal{C}')} \varphi$. For convenience, we consider axioms as inference rules with an empty set of premises. Assume that the last step in the proof of $\vdash_{\mathcal{L}(\mathcal{C})\cup\mathcal{L}(\mathcal{C}')} \varphi$ is the use of an inference rule in $\mathcal{L}(\mathcal{C}')$:

$$\begin{array}{rcll} \rho_j & = & \theta_j(\nabla'_1, .., \nabla'_m, p_1/\psi_1, ..., p_k/\psi_k) & j < m \\ \hline \varphi & = & \theta(\nabla'_1, , .., \nabla'_m, p_1/\psi_1, ..., p_k/\psi_k) & \psi_i \in Fm_{\tau\cup\mathcal{C}\cup\mathcal{C}'}, \end{array}$$

where $\vdash_{\mathcal{L}(\mathcal{C})\cup\mathcal{L}(\mathcal{C}')} \rho_j$. Then $p \Leftrightarrow q \vdash_{\mathcal{L}(\mathcal{C})} \rho_j^*$ by induction hypothesis, and since $\theta_j, \theta \in Fm_{\tau\cup\mathcal{C}'}$ do not contain symbols of $\mathcal{C}$, applying $(\ )^*$ yields

$$\begin{array}{rcll} \rho_j^* & = & \theta_j(\nabla_1, .., \nabla_m, p_1/\psi_1^*, ..., p_k/\psi_k^*) & j < m \\ \hline \varphi^* & = & \theta(\nabla_1, .., \nabla_m, p_1/\psi_1^*, ..., p_k/\psi_k^*), & \end{array}$$

a rule of $\mathcal{L}(\mathcal{C})$. Thus, $p \Leftrightarrow q \vdash_{\mathcal{L}(\mathcal{C})} \varphi^*$, and $p \Leftrightarrow q \vdash_{\mathcal{L}(\mathcal{C})} \varphi^+$ by Claim 1 and the Detachment Rule. Now, if the rule used belongs to $\mathcal{L}(\mathcal{C})$:

$$\begin{array}{rcll} \rho_j & = & \theta_j(\nabla_1, .., \nabla_m, p_1/\psi_1, ..., p_k/\psi_k) & j < m \\ \hline \varphi & = & \theta(\nabla_1, .., \nabla_m, p_1/\psi_1, ..., p_k/\psi_k) & \psi_i \in Fm_{\tau\cup\mathcal{C}\cup\mathcal{C}'}, \end{array}$$

where θ_j, θ do not contain$\nabla' \in \mathcal{C}'$, applying $(\ )^+$ yields a rule in $\mathcal{L}(\mathcal{C})$:

$$\begin{array}{rcll} \rho_j^+ & = & \theta_j(\nabla_1, .., \nabla_m, p_1/\psi_1^+, ..., p_k/\psi_k^+) & j < m \\ \hline \varphi^+ & = & \theta(\nabla_1, .., \nabla_m, p_1/\psi_1^+, ..., p_k/\psi_k^+). & \end{array}$$

Then $p \Leftrightarrow q \vdash_{\mathcal{L}(\mathcal{C})} \rho_j^+$ by the induction hypothesis, and thus $p \Leftrightarrow q \vdash_{\mathcal{L}(\mathcal{C})} \varphi^+, \varphi^*$ as before. Finally, according to Claim 2, the hypothesis $\vdash_{\mathcal{L}(\mathcal{C})\cup\mathcal{L}(\mathcal{C}')} \nabla(\mathbf{p}) \Leftrightarrow \nabla'(\mathbf{p})$ yields a proof of: $p \Leftrightarrow q \vdash_{\mathcal{L}(\mathcal{C})} [\nabla(\mathbf{p}) \Leftrightarrow \nabla'(\mathbf{p})]^* = \nabla(\mathbf{p})^* \Leftrightarrow \nabla'(\mathbf{p})^* = \nabla(\mathbf{p}) \Leftrightarrow \nabla(\mathbf{p}(p/q))$. ∎

Corollary 2. *Under the hypothesis of Theorem 1, $K_{\mathcal{L}(\mathcal{C})}$ consists of the algebras $(A, \nabla_A)_{\nabla\in\mathcal{C}}$, $A \in K_{\mathcal{L}}$, satisfying the identities and quasi-identities: $\bigwedge_i \delta(\rho_i) \approx \epsilon(\rho_i) \Rightarrow \delta(\varphi) \approx \epsilon(\varphi)$ corresponding to the axioms and rules*

$\{\rho_i\}_i \vdash \varphi$ *in* $\mathcal{L}(\mathcal{C}) \setminus \mathcal{L}$. *Moreover, for each* $A \in K_{\mathcal{L}}$ *there is at most one family of functions* $\{\nabla_A : \nabla \in \mathcal{C}\}$ *such that* $(A, \nabla_A)_{\nabla \in \mathcal{C}} \in K_{\mathcal{L}(\mathcal{C})}$.

PROOF. The first statement follows from Th. 2.17 in [1]. Moreover, Theorem 1 also yields: $p \Leftrightarrow q \vdash_{\mathcal{L}(\mathcal{C}')} \nabla'(\mathbf{p}) \Leftrightarrow \nabla'(\mathbf{p}(p/q))$. Thus, $\mathcal{L}(\mathcal{C}) \cup \mathcal{L}(\mathcal{C}')$ is algebraizable via $\Leftrightarrow$, (δ, ϵ), and therefore $\vdash_{\mathcal{L}(\mathcal{C}) \cup \mathcal{L}(\mathcal{C}')} \nabla(\mathbf{p}) \Leftrightarrow \nabla'(\mathbf{p})$ implies $\models_{K_{\mathcal{L}(\mathcal{C}) \cup \mathcal{L}(\mathcal{C}')}} \nabla(\mathbf{p}) \approx \nabla'(\mathbf{p})$ for each $\nabla \in \mathcal{C}$. □

REMARK. The algebraic interpretation of the connectives $\nabla \in \mathcal{C}$ does not necessarily exist in all algebras A of $K_{\mathcal{L}}$. However, the subclass $\mathrm{Red}_{\mathcal{L}}(K_{\mathcal{L}(\mathcal{C})})$ of algebras of $K_{\mathcal{L}}$ where ∇_A exists for all $\nabla \in \mathcal{C}$ is easily seen to be closed under products. Moreover, it is first order axiomatizable because each $\nabla \in \mathcal{C}$ is first order definable in the vocabulary of $K_{\mathcal{L}}$, due to Beth's definability theorem, plus compactness in case $\mathcal{C}$ is infinite.

Recall that $\mathcal{L}(\mathcal{C})$ is a *conservative* extension of $\mathcal{L}$ if $\Gamma \vdash_{\mathcal{L}(\mathcal{C})} \varphi$ implies $\Gamma \vdash_{\mathcal{L}} \varphi$ whenever $\Gamma \cup \{\varphi\} \subseteq Fm_{\mathcal{L}}$.

COROLLARY 3. *An extension* $\mathcal{L}(\mathcal{C})$ *of an algebraizable logic* $\mathcal{L}$ *defining* $\mathcal{C}$ *implicitly is conservative over* $\mathcal{L}$ *if and only if* $Red_{\mathcal{L}}(K_{\mathcal{L}(\mathcal{C})})$ *generates* $K_{\mathcal{L}}$ *as a quasivariety. Equivalently, if and only if each* $A \in K_{\mathcal{L}}$ *is embeddable in an algebra of* $Red_{\mathcal{L}}(K_{\mathcal{L}(\mathcal{C})})$.

PROOF. $\mathrm{Red}_{\mathcal{L}}(K_{\mathcal{L}(\mathcal{C})})$ generates $K_{\mathcal{L}}$ as a quasivariety if and only if any quasi-identity holding in the first class holds in the second. By algebraizability this is equivalent to conservativity. The second claim follows from closure under products of $\mathrm{Red}_{\mathcal{L}}(K_{\mathcal{L}(\mathcal{C})})$. ■

EXAMPLE 3 (*implicit definability of classical connectives*). In any algebraizable logic with standard equivalence $p \Leftrightarrow q = \{p \to q, q \to p\}$ and containing the Modus Ponens rule, the usual axioms for conjunction:

$p \wedge q \to p$
$p \wedge q \to q$
$(p \to q) \to ((p \to r) \to (p \to q \wedge r))$,

define implicitly this connective. Something similar happens with disjunction. Negation is defined implicitly by the axioms:

$(q \to (p \to \neg p)) \to (q \to \neg p)$
$\neg p \to (p \to q)$.

In this way, most familiar logics may be obtained from their implicational fragments by adding implicitly defined connectives.

EXAMPLE 4 (*algebraizable logic of the dual pseudocomplement*). According to Theorem 1, the logic $IPC(D)$ of Example 1 can not define D implicitly because it is not algebraizable. However, its non-axiomatic strengthening $IPC(D)^+$:

D_1. $p \vee Dp$
D_2. $(q \vee D(q \vee p)) \leftrightarrow (q \vee Dp)$
R. $p \vdash \neg Dp$,

(the inference rule R makes D_3 superfluous) defines D implicitly. Indeed,

$\vdash_{\mathrm{D}'_1+IPC+\mathrm{R}} \neg D(D'p \vee p)$; thus,
$\vdash_{\mathrm{D}'_1+IPC+\mathrm{R}} (D'p \vee D(D'p \vee p)) \rightarrow D'p$, but
$\vdash_{IPC} Dp \rightarrow (D'p \vee Dp) \vdash_{\mathrm{D}_2+IPC} Dp \rightarrow (D'p \vee D(D'p \vee p))$; hence,
$\vdash_{\mathrm{D}'_1,\mathrm{D}_2+IPC+\mathrm{R}} Dp \rightarrow D'p$.

Therefore, $IPC(D)^+$ is algebraizable via the standard systems. The reader may wish to verify directly that $p \leftrightarrow q \vdash_{IPC(D)^+} Dp \leftrightarrow Dq$ (warning: the deduction theorem is not available). $K_{IPC(D)^+}$ consists of the enriched Heyting algebras (H, D_H) satisfying the identities E_1,E_2,E_3 of Example 1 and the quasi-identity corresponding to the rule R:

$$x \approx 1 \Rightarrow D(x) \approx 0.$$

Since the identities E_1,E_2,E_3 force D_H to be the dual pseudocomplement, this quasi-identity is already a first order consequence of them, and thus $K_{IPC(D)^+}$ is the variety considered in Example 1. As D exists in all ultraproducts of finite Heyting algebras and any Heyting algebra is embeddable in such an ultraproduct, $IPC(D)^+$ is conservative over IPC, not only for theorems but for deductive inferences also.

Example 5 (*Kuznetsov connective*). The extension $IPC(S)$ of IPC by the axioms

S_1. $p \rightarrow Sp$
S_2. $Sp \rightarrow (q \vee (q \rightarrow p))$
S_3. $(Sp \rightarrow p) \rightarrow p$,

was introduced by Kuznetsov [10], who proved that these axioms are conservative over any intermediate logic, and studied in [3] where it is shown that it has the disjunction property. It is easily seen that $IPC(S)$ defines S implicitly; thus, it is algebraizable. S_H exists in any well founded Heyting algebra H and must be the successor in linearly ordered ones. It does not exist for example in the real interval $[0, 1]$. Conservativity may be obtained as in Example 4.

Example 6 (*n-valued linear Heyting logic with successor*). The intermediate logic IPC_n extending IPC by the axioms $(p \rightarrow q) \vee (q \rightarrow p)$ and $(p_1 \rightarrow$

$p_2) \vee ... \vee (p_n \to p_{n+1})$, is algebraizable by the variety of Heyting algebras generated by the ordered chain H_n of length at least n. The corresponding extension $IPC_n(S)$ by S_1, S_2, S_3 is algebraized by the variety generated by $\langle H_n, S_{H_n}\rangle$ (cf. [3]).

The further extension $IPC_n(S)^+$ of $IPC_n(S)$ by the inference rule

$$S^{n-1}p \vdash S(p \wedge \neg p) \to p$$

has a proper quasivariety semantics since the algebra $\langle H_n, S_{H_n}\rangle$ satisfies the corresponding quasi-identity

$$S^{n-1}(x) \approx 1 \Rightarrow (S(0) \to x) \approx 1,$$

but its homomorphic image $\langle H_{n-1}, S_{H_{n-1}}\rangle$ does not. $K_{IPC_n(S)^+}$ is generated by $\langle H_n, S_{H_n}\rangle$ as a quasivariety, and any maximal chain in a non trivial algebra of this quasivariety must have length at least n.

Example 7 (*division connectives in Łukasiewicz logic*). Łukasiewicz infinite valued logic ŁL is a sublogic of classical propositional calculus given by the axioms

$p \to (q \to p)$
$(p \to q) \to ((q \to r) \to (p \to r))$
$((p \to q) \to q) \to ((q \to p) \to p)$
$(\neg p \to \neg q) \to (q \to p)$

and the Modus Ponens rule. ŁL is algebraizable via the standard systems by the variety of *Wajsberg algebras* generated by $L_\infty = \langle [0,1], \to_L, \neg_L\rangle$, where $x \to_L y = \min(1, 1 - x + y)$ and $\neg_L x = 1 - x$. These algebras become lattice by the relation: $x \leq y$ iff $x \to_L y = 1$ (cf. [5]).

Define $2p := \neg p \to p$, and inductively $(k+1)p := \neg p \to kp$. Then, for each integer $k \geq 2$ the axioms:

Dk$_1$. $p \to k(\delta_k p)$
Dk$_2$. $(p \to kq) \to (\delta_k p \to q)$,

define implicitly a connective δ_k over ŁL. Thus the logic ŁL$(\{\delta_k\}_{k\geq 2})$ is algebraizable. The interpretation of δ_k in Wajsberg algebras is:

$$\delta_k(x) = \min\{y : ky \geq x\}.$$

Therefore, it exists in L_∞ where $\delta_k(x) = \frac{1}{k}x$, and in the finite algebras $L_n = \langle\{0, \frac{1}{n-1}, .., \frac{n-2}{n-1}, 1\}, \to_L, \neg_L\rangle$ where $\delta_k(\frac{j}{n-1}) = \frac{[(j+k-1)/k]}{n-1}$. Being first order definable, δ_k exists in all ultrapowers of these algebras. But any Wajsberg algebra is embeddable in a product of ultrapowers of L_∞ as a consequence of Chang's representation theorem for MV-algebras [4]. Thus Ł$L(\{\delta_k\}_{k\geq 2})$ is conservative over ŁL. The connective δ_k can not be given by a formula of ŁL because the subalgebra $\{0,1\}$ of L_∞ is not closed under δ_k.

Adding to Ł$L(\{\delta_k\}_{k\geq 2})$ the following axiom for each $k \geq 2$:

Dk$_3$. $k\delta_k p \to p$,

we obtain the implicational version of *Rational Łukasiewicz Calculus* (RŁL), introduced by B. Gerla [7]. This logic is algebraized by the Wajsberg version DW of the variety DMV of *divisible MV algebras* in which $\delta_k(x)$ is the unique y such that $ky = x$. Clearly, DW is a proper subvariety of $K_{\text{ŁL}(\{\delta_k\}_{k\geq 2})}$ because it does not contain the finite algebras L_n. However, $L_\infty \in DW$, granting as before that RŁL is conservative over ŁL.

Example 8 (*n-valued Łukasiewicz logic with successor).* Łukasiewicz n-valued logic ŁL_n is the extension of ŁL by the axioms

$np \to (n-1)p$
$(n-1)(jp \to \neg(\neg p \to \neg(j-1)p))$, for $j \leq n-1$, $j \nmid n-1$.

It is algebraized by the variety generated by L_n. The further extension, Ł$L_n(\delta_{n-1})$, by the axioms of δ_{n-1} is algebraized by the variety generated by $\langle L_n, \delta_{n-1}\rangle$. This last algebra is *primal* (every function of the algebra is given by a term) because it has a definable successor $S(x) = \neg x \to \frac{1}{n-1} = \neg x \to \delta_{n-1}(p \to p)$, and the set of operations $\{\to_L, \neg_L, S\}$ is known to be functionally complete for L_n.

4. Compatibility

A function $f : A^n \to A$, where A in an algebra, is *compatible* if any congruence relation of A is a congruence of $\langle A, f\rangle$. It is K-*compatible* for a class K of algebras if this happens for any congruence relation R such that $A/R \in K$.

An extension of a logic $\mathcal{L}$ will be called *axiomatic* if it may be defined by adding a set of axiom schemes to $\mathcal{L}$ but no new inference rules.

In [3] it is shown that the algebraic interpretation of any connective defined implicitly by an axiomatic extension of IPC is always compatible. Although this property fails for non-axiomatic extensions (Example 9 below), it generalizes to axiomatic extensions of any algebraizable logic.

Theorem 4. *Let $\mathcal{L}$ be an algebraizable logic and $\mathcal{L}(\mathcal{C})$ an axiomatic extension defining implicitly a family of connectives $\mathcal{C}$. Then the functions ∇_A, $\nabla \in \mathcal{C}$, are $K_\mathcal{L}$-compatible wherever they exist. Thus, $Red_\mathcal{L}(K_{\mathcal{L}(\mathcal{C})})$ is closed under homomorphic images in $K_\mathcal{L}$.*

Proof. Let $\mathcal{L}(\mathcal{C})$ be obtained by adding a set of axiom schemas $\mathcal{A}(\mathcal{C})$ to $\mathcal{L}$. If $(A, \nabla_A)_\nabla \in K_{\mathcal{L}(\mathcal{C})}$, then any $K_\mathcal{L}$-congruence Θ of A comes from a $\mathcal{L}$-filter F of A by the Leibniz operator $\Theta = \Omega_A(F) = \{(a,b) \in A \times A : a \Leftrightarrow b \in F\}$ (Th. 5.1(i), Lemma 5.2, [1]). If $\alpha \in \mathcal{A}(\mathcal{C})$ then $\models_{K_{\mathcal{L}(\mathcal{C})}} \delta(\alpha) \approx \epsilon(\alpha)$, due to the algebraizability of $\mathcal{L}(\mathcal{C})$ with the same defining equations and equivalence formulas of $\mathcal{L}$ (Theorem 1). Thus $\delta^A(v(\alpha)) = v(\delta(\alpha)) = v(\epsilon(\alpha)) = \epsilon^A(v(\alpha))$ for any valuation v into A, and trivially $(\delta^A(v(\alpha)), \epsilon^A(v(\alpha))) \in \Omega_A(F)$. Hence, $\delta^A(v(\alpha)) \Leftrightarrow \epsilon^A(v(\alpha)) \in F$. But this implies that $v(\alpha) \in F$ since $\delta(p) \Leftrightarrow \epsilon(p) \vdash_\mathcal{L} p$. Thus, F is an $\mathcal{L}(\mathcal{C})$-filter and $\Theta = \Omega_A(F) = \Omega_{(A,\nabla_A)_\nabla}(F)$ is a $K_{\mathcal{L}(\mathcal{C})}$-congruence by Th. 5.1(i) [1]. This shows compatibility. Now, if $h : A \to B$ is an epimorphism in $K_\mathcal{L}$ and ∇_A exists in A for all $\nabla \in \mathcal{C}$, the maps $\nabla_B(h(a_1), ..., h(a_m)) = h(\nabla_A(a_1, ..., a_m)$ are well defined and satisfy the identities satisfied by the ∇_A's. Hence, $B \in \mathrm{Red}_\mathcal{L}(K_{\mathcal{L}(\mathcal{C})})$. ■

Example 9. The logic $IPC(D)^+$ of the dual pseudocomplement (Example 4) defines D implicitly but D_H is not compatible for the algebra H described in Example 1, because $\langle H, D_H \rangle \in K_{IPC(D)^+}$ is simple while H is not. Therefore, the previous theorem fails for non-axiomatic extensions. In fact, although its quasivariety semantics is a variety, the theorem grants that $IPC(D)^+$ is not equivalent to any axiomatic extension of IPC.

5. Explicit definability

A connective defined implicitly by an extension $\mathcal{L}(\nabla)$ of an algebraizable logic $\mathcal{L}$ with a system of equivalence formulas $p \Leftrightarrow q$ is *explicitly definable* by a formula θ of $\mathcal{L}$ if:

$$\vdash_{\mathcal{L}(\nabla)} \nabla(\mathbf{p}) \Leftrightarrow \theta(\mathbf{p}). \tag{i}$$

By algebraizability this means:

$$\models_{K_{\mathcal{L}(\nabla)}} \nabla(\mathbf{p}) \approx \theta(\mathbf{p}). \tag{ii}$$

Obviously, $K_\mathcal{L}$-compatibility of each ∇_A is a necessary condition for this to happen (a term is compatible with any congruence relation). Another necessary condition is that the class of algebras $A \in K_\mathcal{L}$ where ∇_A exists forms a quasivariety, since (ii) implies that these are precisely the algebras satisfying $E(\nabla/\theta)$ for the system $E(\nabla)$ of identities and quasi-identities of $K_{\mathcal{L}(\nabla)}$. A third condition is that $\mathcal{L}(\nabla)$ be an *essentially axiomatic* extension, in the sense that $\mathcal{L}(\nabla)$ is equivalent to a set of axioms over an extension $\mathcal{L}'$ of $\mathcal{L}$ not involving ∇. To see this, let $\mathcal{L}'$ be the logic $\mathcal{L}(\nabla)$ restricted to the formulas of $\mathcal{L}$. Then (i) implies the following chain of equivalences: $\Gamma \vdash_{\mathcal{L}(\nabla)} \varphi$ iff $\Gamma(\nabla/\theta) \vdash_{\mathcal{L}(\nabla)} \varphi(\nabla/\theta)$ iff $\Gamma(\nabla/\theta) \vdash_{\mathcal{L}'} \varphi(\nabla/\theta)$ iff $\Gamma \vdash_{\mathcal{L}' \cup \{\nabla(\mathbf{p}) \Leftrightarrow \theta(\mathbf{p})\}} \varphi$.

All the previous remarks can be generalized to families of connectives. That these three conditions are already sufficient for explicit definability follows by algebraizability from the next general result on quasivarieties.

LEMMA 5. *Let K be a quasivariety of type τ, $E(\mathcal{C})$ a set of identities defining implicitly a family of operations $\mathcal{C}$ in K, and $M_{\mathcal{C}} = \{A \in K : \exists(\nabla_A)_{\nabla\in\mathcal{C}}\ \langle A, \nabla_A\rangle_{\nabla\in\mathcal{C}} \models E(\mathcal{C})\}$. Then each $\nabla \in \mathcal{C}$ is explicitly definable by a term of type τ if and only if each ∇_A is K-compatible, and $M_{\mathcal{C}}$ is closed under subalgebras (equivalently, $M_{\mathcal{C}}$ is a quasivariety).*

PROOF. Notice that $M_{\mathcal{C}}$ is first order axiomatizable by Beth's definability theorem and it is always closed under products. Thus, it is a quasivariety if and only if it is closed under subalgebras. The conditions in the theorem are necessary for explicit definability by the discussion above. Assume now that $M_{\mathcal{C}}$ is a quasivariety and each $\nabla \in \mathcal{C}$ is K-compatible in $M_{\mathcal{C}}$. Let F be the free algebra of $M_{\mathcal{C}}$ on generators g_α, $\alpha < \kappa = |\tau \cup \mathcal{C}| + \omega$. Then writing $\mathbf{g}$ for $g_1, ..., g_n$, and $\mathbf{g}'$ for $g_{\alpha_1} ..., g_{\alpha_k}$, $\nabla_F(\mathbf{g}) = q^F_\nabla(\mathbf{g}, \mathbf{g}')$ where $q_\nabla(x_1, ..., x_n, x_{n+1} ..., x_{n+k})$ is a τ-term. Given $A \in M_{\mathcal{C}}$ of power at most κ and $\mathbf{a} = (a_1, ..., a_n) \in A^n$, let $h : F \to A$ be an onto homomorphism such that $h(g_i) = a_i$ for $i = 1, ..., n$, and $h(g_{\alpha_i}) = a_n$ for $i = n+1, ..., n+k$. Since ∇_F is compatible with the kernel of h, there is a unique function $f^+_\nabla : A^n \to A$ such that h is an homomorphism from $\langle F, \nabla_F\rangle_{\nabla\in\mathcal{C}}$ onto $\langle A, f^+_\nabla\rangle_{\nabla\in\mathcal{C}}$. Thus, $\langle A, f^+_\nabla\rangle_{\nabla\in\mathcal{C}} \models E(\mathcal{C})$ and, by uniqueness, each $f^+_\nabla = \nabla_A$. Hence, $\nabla_A(\mathbf{a}) = f^+_\nabla(h(\mathbf{g})) = h(\nabla_F(\mathbf{g})) = h(q^F_\nabla(\mathbf{g}, \mathbf{g}')) = q^A_\nabla(h(\mathbf{g}), h(\mathbf{g}')) = q^A_\nabla(\mathbf{a}, a_n, ..., a_n)$. In sum, $\nabla_A = t^A_\nabla$ for any $A \in M$ of power at most κ, where $t_\nabla(x_1, ..., x_n) = q_\nabla(x_1, ..., x_n, ..., x_n)$. Thus, $E(\{\nabla/t_\nabla : \nabla \in \mathcal{C}\})$ is satisfied in these algebras and, by the Tarski-Löwenheim-Skolem theorem, in all algebras of $M_{\mathcal{C}}$. By uniqueness again, $\nabla_A = t^A_\nabla$ in all $M_{\mathcal{C}}$. ■

THEOREM 6. *Let $\mathcal{L}(\mathcal{C})$ be an essentially axiomatic extension of an algebraizable logic $\mathcal{L}$ defining implicitly a family of connectives $\mathcal{C}$. All $\nabla \in \mathcal{C}$ are explicitly definable by formulas of $\mathcal{L}$ if and only if the class of algebras in $K_{\mathcal{L}}$ where all of them exist is closed under subalgebras. Equivalently, if and only if it is a subquasivariety of $K_{\mathcal{L}}$.*

PROOF. Let $\mathcal{L}(\mathcal{C})$ be equivalent to a set of axioms $\mathcal{A}(\mathcal{C})$ over an extension $\mathcal{L}'$ of $\mathcal{L}$ not containing symbols of $\mathcal{C}$. Apply the previous lemma to $K = K_{\mathcal{L}'}$ and the set of identities $E(\mathcal{C})$ corresponding to $\mathcal{A}(\mathcal{C})$. The algebraic interpretations of the new connectives in $M_{\mathcal{C}} = \mathrm{Red}_{\mathcal{L}}(K_{\mathcal{L}(\mathcal{C})}) = \mathrm{Red}_{\mathcal{L}'}(K_{\mathcal{L}'+\mathcal{A}(\mathcal{C})})$ are automatically $K_{\mathcal{L}'}$-compatible by Theorem 4. Then, by Theorem 1 and Lemma 5, they are explicitly definable by formulas of $\mathcal{L}$ (= formulas of $\mathcal{L}'$)

if and only if $M_{\mathcal{C}}$ is closed under subalgebras or, equivalently, it is a quasi-variety. ■

Recalling Corollary 3, we have:

COROLLARY 7. *Let $\mathcal{L}$ be an algebraizable logic. A family of connectives $\mathcal{C}$ defined implicitly by an axiomatic conservative extension of $\mathcal{L}$ is explicitly definable if and only if each $\nabla \in \mathcal{C}$ exists in all algebras of $K_{\mathcal{L}}$.*

For example, $IPC(S)$ is a conservative extension of IPC but S does not exist in all Heyting algebras (Example 5). Hence, Sp is not equivalent in $IPC(S)$ to a formula of IPC. On the other hand, ŁL(δ_k) extends conservatively ŁL (Example 7) but δ_k is not definable in ŁL for $k \geq 2$. We may conclude that δ_k does not exist in all Wajsberg algebras.

6. A definability theorem

For some logics, any connective defined implicitly by an axiomatic extension is explicitly definable, as shown in [3] for classical propositional calculus and n-valued linear intuitionistic logic with successor (Example 6). These are particular cases of the next theorem.

We will utilize the fact that any finite algebra of an arithmetical variety is affine complete (Cor. 3.4.1, [9]). Recall that an algebra A is called *affine complete* if any compatible function in A is given by a polynomial.

THEOREM 8. *Let $\mathcal{L}$ be an algebraizable logic such that $K_{\mathcal{L}}$ is an arithmetical variety generated either* (i) *by a finite algebra without proper subalgebras and with linearly ordered congruence lattice, or* (ii) *by finitely many finite simple algebras without proper subalgebras. Then all connectives defined implicitly by axiomatic extensions of $\mathcal{L}$ are explicitly definable.*

PROOF. Let ∇ be implicitly defined over $\mathcal{L}$ by a set of axioms $\mathcal{A}(\nabla)$, and let $E(\nabla)$ be the corresponding set of identities. Since $K_{\mathcal{L}}$ is congruence distributive and the generators do not have subalgebras, Jónsson's lemma (Cor. IV.6.10, [2]) implies that the subdirectly irreducible (s.i.) algebras of $K_{\mathcal{L}}$ are homomorphic images of the generators and thus do not have proper subalgebras either. In case (i), they form a chain of epimorphisms: $S_0 \twoheadrightarrow S_1 \twoheadrightarrow ....$ starting with the generator and, in case (ii), they coincide with the generators. The finite set $\{S_i\}$ of s.i. algebras entering in the generation of the algebras in $M = \mathrm{Red}_{\mathcal{L}}(K_{\mathcal{L}(\nabla)})$ is included in M by Theorem 4, and the quasi-variety generated by M has the form $ISP(\{S_i\})$ by finite generation. Module isomorphism, any algebra $A \in ISP(\{S_i\})$ has a representation $A \leq \Pi_j S_{i_j}$, which is necessarily subdirect because the S_{i_j} do not have proper subalgebras. We claim that A contains a finite subalgebra $B \in M$. To see this in case (i), let S_k have the smallest subindex in the representation of A, then there is an induced embedding $S_k \approx B \leq \Pi_j S_{i_j}$ where $B = A \cap B \leq A$ because B does not have proper subalgebras. In case (ii), pick any finite subalgebra B of A', which exists since $K_{\mathcal{L}}$ is locally finite (Th. II.10.6 in [2]), and make the necessarily subdirect embedding $B \leq \Pi_j S_{i_j}$ irredundant. Then B becomes isomorphic to a direct product of S_{i_j}'s by Lemma 1.2.15 in [9], and hence $B \in M$. In any case, B is affine complete by arithmeticality of $K_{\mathcal{L}}$ and $\nabla(\mathbf{x})$ is compatible (Theorem 4). Thus, $\nabla_B(\mathbf{x}) = t(\mathbf{x}, \mathbf{b})$, where $t(\mathbf{x}, \mathbf{y})$ is a term and $\mathbf{b} \in B^k$. Therefore, $B \models E[\nabla(-)/t(-, \mathbf{b})]$. Let $u(\mathbf{z}, \mathbf{b}) \approx s(\mathbf{z}, \mathbf{b})$ be one of the identities in $E[\nabla(-)/t(-, \mathbf{b})]$. Since the embedding $B \leq A \leq \Pi_j S_{i_j}$ is subdirect for B, given $\mathbf{a} \in A^m$ interpreting $\mathbf{z}$, we may choose $\mathbf{b}_j \in B^m$ such that $\pi_j(\mathbf{b}_j) = \pi_j(\mathbf{a})$, where $\pi_j : \Pi_j S_{i_j} \to S_{i_j}$ is the projection. Since $B \models u(\mathbf{b}_j, \mathbf{b}) \approx s(\mathbf{b}_j, \mathbf{b})$, then $S_{i_j} \models u(\pi_j(\mathbf{b}_j), \pi_j(\mathbf{b})) \approx s(\pi_j(\mathbf{b}_j), \pi_j(\mathbf{b}))$. That is, $S_{i_j} \models u(\pi_j(\mathbf{a}), \pi_j(\mathbf{b})) \approx s(\pi_j(\mathbf{a}), \pi_j(\mathbf{b}))$, and hence $S_{i_j} \models \pi_j(u(\mathbf{a}, \mathbf{b})) \approx \pi_j(s(\mathbf{a}, \mathbf{b}))$. Since this holds for all j, then $A \models u(\mathbf{a}, \mathbf{b}) \approx s(\mathbf{a}, \mathbf{b})$, showing that $A \models E[\nabla(-)/t(-, \mathbf{b})]$ and thus $A \in M$. We have shown $M = ISP(\{S_i\})$ and thus we may apply Theorem 6. ■

EXAMPLE 10. The first condition of the theorem applies to $IPC_n(S)$ (Example 6) because $K_{IPC_n(S)}$ inherits arithmeticality from the variety of Heyting algebras, and is generated by $\langle H_n, S_{H_n} \rangle$, which has linearly ordered congruence lattice and no proper subalgebras.

EXAMPLE 11. Both conditions of the theorem apply to classical propositional calculus, since its variety semantics is generated by the simple boolean algebra without proper subalgebras $\{0, 1\}$. More generally, they apply to n-valued Łukasiewicz logic with successor $\text{ŁL}_n(\delta_{n-1})$ (Example 8) because Wajsberg algebras are arithmetical and $\langle L_n, \delta_{n-1} \rangle$, being primal, is necessarily simple and without proper subalgebras. In particular, all δ_k, $k \geq 2$, are definable in $\text{ŁL}_n(\delta_{n-1})$.

EXAMPLE 12. The second condition of the theorem applies to the algebraizable extension of $\text{ŁL}(\{\delta_k\}_{k\geq 2})$ corresponding to the subvariety of $K_{\text{ŁL}(\{\delta_k\}_{k\geq 2})}$ generated by the primal algebras $\langle L_{n_1}, \delta_k \rangle_{k\geq 2}$, ..., $\langle L_{n_r}, \delta_k \rangle_{k\geq 2}$.

Theorem 8 fails for non-axiomatic extensions:

EXAMPLE 13. For $n \geq 3$, $IPC_n(S)+ \{D_1, D_2, R\}$ defines the dual pseudocomplement D implicitly (cf. Example 4) but not explicitly. Indeed, D can not be given by a formula of $IPC_n(S)$ because D_{H_n} it is not compatible with the congruence relation of $\langle H_n, S_{H_n}\rangle$ that identifies the top element 1 with its immediate predecessor.

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