Equivalence and quantifier rules for logic with imperfect information

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Abstract

In this paper, we present a prenex form theorem for a version of Independence Friendly logic, a logic with imperfect information. Lifting classical results to such logics turns out *not* to be straightforward, because independence conditions make the formulas sensitive to signalling phenomena. In particular, nested quantification over the same variable is shown to cause problems. For instance, renaming of bound variables may change the interpretations of a formula, there are only restricted quantifier extraction theorems, and slashed connectives cannot be so easily removed. Thus we correct some claims from Hintikka [8], Caicedo & Krynicki [3] and Hodges [11]. We refine definitions, in particular the notion of equivalence, and sharpen preconditions, allowing us to restore (restricted versions of) those claims, including the prenex form theorem of Caicedo & Krynicki [3], and, as a side result, we obtain an application to Skolem forms of classical formulas. It is a known fact that a complete calculus for IF-logic is impossible, but with our results we establish several quantifier rules that form a partial calculus of equivalence for a general version of IF-logic reflecting general properties of information flow in games.

1 Introduction

In the last decade of the previous century, Hintikka and Sandu presented their so-called *(Information) Independence Friendly Logic*, henceforth IF-logic (see e.g. [8] and [10]). This logic extends earlier work in Branching Quantification [6] and Game Theoretical Semantics (e.g. the papers collected in [9]). It can most easily be regarded as an extension of classical first order logic interpreted by means of a game semantics. The syntactical extension consists of a slash operator that can impose quantifications and connectives to be shielded from the scope of other quantifications. E.g. in the formula $\forall x \exists y_{/x} \varphi(x, y)$, the slash operator in $\exists y_{/x}$ indicates that there exists a y that is *independent of* x, such that $\varphi(x, y)$. In the game interpretation for that formula the player verifying the formula (we call her ∃loise) has to pick a value for y, or change its previous value, in *ignorance* of the value chosen for x by the falsifying player (we call him ∀belard). The imperfect information in the games makes them possibly undetermined: e.g. if played in a model with at least two elements, neither ∃loise nor ∀belard has a winning strategy in the game for the formula $\forall x \exists y_{/x}[x = y]$. Because truth and falsity are defined in terms of existence of winning strategies for the respective players, this simple example already shows that the law of the excluded middle fails for IF-logic.

In this paper we present a version of IF-logic of maximal generality, in the sense that other versions presented in the literature ([8], [11], [3]) can be represented in our logic. The language is closed under unrestricted combination of formulas by connectives, unrestricted application of quantifiers to a formula, and unrestricted substitution of formulas. We give a game semantics fully symmetric for both players and a compositional semantics for this language, in the style of Caicedo & Krynicki [3], which in turn is rooted in the semantics of Hodges [11].

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With game semantics, where syntactic *independence conditions* are interpreted as *imperfect information*, IF-like logics can be seen as languages representing information flow and information hiding between agents.

Hodges [11] showed with a simple example that unexpected and counterintuitive situations arise in this setting: in contrast with the example given above, ∃loise has a winning strategy for the formula $\forall x \exists z \exists y_{/x}[y = x]$ by choosing z to be equal to x, and then choosing y equal to z. Although she is supposed not to know the value of x when choosing y, she can pass it on by storing it in z. So values assigned to variables that do not occur at all in a formula may influence its evaluation. We speak of *signalling* when in the interpretation of a formula φ values are used of variables that do no occur free in φ, so in the example z *signals* the value of x.

Signalling plays a role in the sensitivity of IF-logic to 'reuse' of variables. Just like in classical first order formulas, nested quantification over the same variable could occur in an IF-formula. In classical first order logic, one can safely rename variables to avoid such nesting, because the result of the renaming is equivalent. In the IF-setting, where variables act as storage places for information, the reuse of a variable by a nested quantification results in overwriting its old value. Such a 'reuse' blocks the possibility to signal with that variable and may turn a formula into being false. And if we rename the innermost of the quantifications with a fresh variable, we create a new place to store the new value, with the consequence that the old value is preserved. In this way, renaming can turn a formula into being true by creating new signalling possibilities.

Another effect of signalling is that several quantifier rules are not as one might expect them to be. For example, interchanging two subsequent existential quantifiers may no longer be neutral, adding 'vacuous' quantifiers may change the truth value, and extraction of quantifiers needs to be done carefully. These effects may arise even in situations where there is no 'reuse' of variables.

The generality of our logic allows to point out similar previously overlooked situations and to propose a general treatment. We consistently take the domains of the valuations into account, in order to express and resolve the issues indicated above. We introduce a refined notion of equivalence that allows us to prove a series of quantifier rules and logical laws. For example, we show under which preconditions it *is* safe to rename variables, and from a series of non-trivial results, we will be able to conclude that for *sentences* nested quantification over one variable *can* be avoided. Also, we obtain a prenex form theorem for the logic, and as a side result, an application to Skolem forms of classical formulas. Our rules establish a partial calculus of equivalences for our version of IF-logic, which apply also to the restricted versions of this logic, and include the classical laws on quantifiers transformation as a special case.

We will be rather precise when we present our proofs (the reader might judge 'pedantically precise') to assure we do not overlook other unexpected features of the logic, because we learnt not to rely on intuition too easily.

2 The logic

2.1 Syntax

We define the language *IF**, which is a variant of the IF-languages as defined in e.g. [8], [3] or [11].

Definition 2.1 (The language *IF).** *Given a first order signature σ, the set of IF*-formulas is defined by induction:*

(at) *The terms and atomic formulas are defined as in first order logic with equality.*
($\neg$) *If $\varphi \in IF^*$ then $\neg\varphi \in IF^*$*
($\exists$) *If $\varphi \in IF^*$ and Y is a finite set of variables, then $\exists x_{/Y}\varphi \in IF^*$*
($\vee$) *If $\varphi_1, \varphi_2 \in IF^*$ and Y is a finite set of variables, then $(\varphi_1 \vee_{/Y} \varphi_2) \in IF^*$.*

We will follow the usual convention of dropping the most external brackets of a formula, and adding inner brackets for better readability. Also, we will use two styles of brackets: [...] to indicate the scope of a quantifier and (...) to indicate priority among connectives.

Of course, the language has conjunction and universal quantifiers, but the proofs about the logic become shorter if we do not have these as primitive constructions. We introduce them as abbreviations.

Definition 2.2 (Abbreviations). *$\forall x_{/Y}\varphi$ is the abbreviation for $\neg\exists x_{/Y}\neg\varphi$, and $\varphi_1 \wedge_{/Y} \varphi_2$ for $\neg(\neg\varphi_1 \vee_{/Y} \neg\varphi_2)$.*
If $Y = \varnothing$, we omit $_{/\varnothing}$ and write e.g. $\exists x\psi$ and $\varphi \wedge \psi$, furthermore we write $_{/\{x,y\}}$ as $_{/xy}$ (e.g. $\exists z_{/xy}$).

We need to redefine some standard notions for our language:

Definition 2.3. *If φ is an IF*-formula, the set of* ***free variables*** *$Fr(\varphi)$ and the set of* ***bound variables*** *$Bd(\varphi)$ are defined inductively as follows:*

(at) *If φ is atomic, then $Fr(\varphi)$ is the set of variables occurring in φ, and $Bd(\varphi) = \varnothing$.*
($\vee$) $Fr(\varphi_1 \vee_{/Y} \varphi_2) = Fr(\varphi_1) \cup Y \cup Fr(\psi_2)$, $\quad Bd(\varphi_1 \vee_{/Y} \varphi_2) = Bd(\varphi_1) \cup Bd(\varphi_2)$
($\exists$) $Fr(\exists x_{/Y}\varphi) = (Fr(\varphi) \setminus \{x\}) \cup Y$, $\quad Bd(\exists x_{/Y}\varphi) = Bd(\varphi) \cup \{x\}$
(pairs) $Fr(\varphi, \psi) = Fr(\varphi) \cup Fr(\psi)$ *and* $Bd(\varphi, \psi) = Bd(\varphi) \cup Bd(\psi)$

Examples: $Bd(\exists x_{/x}[x{=}x]) = \{x\} = Fr(\exists x_{/x}[x{=}x])$ and $Fr(x{=}1 \vee_{/y} x{\neq}1) = \{x, y\}$.

Notation 2.4. *By $\varphi(x)$ we will denote a formula that may contain x as a free variable. It does not need to contain x free, and it may contain other free variables. In the context of $\varphi(x)$ we denote by $\varphi(y)$ the result of replacing all free occurrences of x by y.*

Definition 2.5. *An* ***IF*-sentence*** *is an IF*-formula without free variables.*

In our language there are no restrictions on the use of quantifiers: within one formula, there may be several quantifiers binding the same variable, including nested occurrences. In this respect there is no formal difference with the language definitions of Caicedo & Krynicki [3], Hodges [11] or Hintikka [8] (in the latter, IF-logic is defined as built from classical formulas, which also incorporates the possibility that nested quantifications over the same variable occur). A difference of our language definition with the literature is that in $\exists x_{/Y}$ we do not require that x does not occur in Y. In fact, allowing x to occur in Y is not different from allowing x to occur both free and bound, as in the classical formula $R(x) \vee \exists x P(x)$: in $\exists x_{/x} P(x)$ the x occurring below the slash is free and the one in $P(x)$ is bound. The quantification $\exists x_{/x}$ is will be interpreted as saying that a new value for x should be assigned *independently* of the value it previously had. On the other hand, we will show that *for sentences* nested quantifications over the same variable, and hence quantifications $\exists x_{/Y}$ with $x \in Y$, can be avoided without loss of generality.

Commonly, saying that ψ is a subformula of φ, means that the formula ψ has one or more occurrences in φ. In this paper, we will use the phrase 'a subformula ψ of φ' exclusively to designate a *specific* occurrence of a formula in φ.

In our game semantics we will need to associate with each subformula ψ of a formula φ a set of variables according to the following definition.

Definition 2.6. *The set of* ***variables free in*** ψ ***relatively to*** φ*, notation* $Fr_\varphi(\psi)$*, is defined inductively from the top down as follows:*

(φ) $Fr_\varphi(\varphi) = Fr(\varphi)$
($\neg$) *If* ψ *is preceded by* $\neg$*, then* $Fr_\varphi(\psi) = Fr_\varphi(\neg\psi)$*.*
($\vee$) *If* ψ *is a disjunct, and* ψ' *the other disjunct, then* $Fr_\varphi(\psi) = Fr_\varphi(\psi \vee_{/Y} \psi') = Fr_\varphi(\psi' \vee_{/Y} \psi)$*.*
($\exists$) *If* ψ *is preceded by* $\exists x_{/Y}$*, then* $Fr_\varphi(\psi) = Fr_\varphi(\exists x_{/Y}\psi) \cup \{x\}$

Notice that two different occurrences of one formula as a subformula could have different sets of free variables relatively to the formula in which they occur, let φ be the sentence $\forall x[\exists y_{/x}[y=x] \wedge \exists z \exists y_{/x}[y=x]]$, let ψ_1 be the left occurrence of the formula $\exists y_{/x}[y=x]$, and ψ_2 the right one. Then, going top-down into φ, we see that both ψ_1 and ψ_2 are within the scope of the quantifier binding x, but ψ_2 is also within the scope of the quantifier binding z. So $Fr_\varphi(\psi_1) = \{x\}$, while $Fr_\varphi(\psi_2) = \{x, z\}$.

2.2 The game

Definition 2.7. *Let* φ *be an IF*-formula.*

A ***suitable model*** $\mathcal{A}$ *for* φ *is a model of a signature containing the language of* φ *(so it provides an interpretation of the non logical symbols in* φ*). The domain of* $\mathcal{A}$ *is denoted as* A*. We will use* $\mathcal{B}$ *for the model with domain* $\{0,1\}$*, and the interpretations 0 and 1 for the constants* ***0*** *and* ***1*** *respectively.*

A ***valuation*** *in* $\mathcal{A}$ *is a function* $v\colon X \to A$ *where* X *is a finite set of variables.*

A ***suitable set of valuations*** *for* φ *is a set of valuations* $V \subseteq A^X$ *where* $Fr(\varphi) \subseteq X$*.*

In this section it will be described how a game is used to evaluate an *IF**-formula φ in a given suitable model $\mathcal{A}$ with respect to some suitable set of valuations V. There are two players: ∀belard, who tries to refute the formula, and ∃loise, who tries to verify the formula. Initially ∃loise makes the moves, but after an occurrence of ¬ (an overt occurrence, or a hidden one in e.g. ∀) the players switch turns. In the course of a play of the game the players will encounter subformulas of φ like $\psi \vee_{/Y} \vartheta$ or $\exists x_{/Y}\ \psi$ and valuations v. The subscript indicates that the choice of the next move has to be made independently of the values of variables in Y for v. This requirement does not show in the definition of the moves of the game, but it will put a restriction on the strategies that are allowed, as we will see in Section 2.3.

Definition 2.8. *A* ***semantic game*** G *is a triple* $\langle \mathcal{A}, \varphi, V\rangle$ *where* φ *is an IF*-formula,* $\mathcal{A}$ *a suitable model for* φ*, and* V *a suitable set of valuations for* φ*.*

Definition 2.9. *Let* G *be the semantic game* $\langle \mathcal{A}, \varphi, V\rangle$*. A* ***position*** *of* G *is a triple* $\langle \psi, v, t\rangle$*, where* ψ *is a subformula of* φ*,* v *is a valuation in* $\mathcal{A}$ *defined for* $Fr_\varphi(\psi) \cup dom(V)$*, and* $t \in \{\exists, \forall\}$*. The value of* t *says whose turn is to play in that position; the opposite player is denoted by* t^**.*

A ***play of the game*** *G is a sequence of positions obtained according to the following rules:*

1. *Any triple $\langle\varphi, v, \exists\rangle$ where $v\in V$ is an* ***initial position****.*
2. *If the position is of the form $\langle\neg\psi, v, t\rangle$ then the players change turns and the game is continued from position $\langle\psi, v, t^*\rangle$.*
3. *If the position is of the form $\langle\varphi_1\vee_{/Y}\varphi_2, v, t\rangle$ then t chooses L or R. If L is chosen, the game continues from position $\langle\varphi_1, v, t\rangle$, otherwise from position $\langle\varphi_2, v, t\rangle$.*
4. *If the position is of the form $\langle\exists x_{/Y}\psi, v, t\rangle$ then t chooses a value a and the game is continued from position $\langle\psi, v', t\rangle$ where v' is the valuation such that $v'(x){=}a$ and otherwise is the same as v (if v is defined for x then $v(x)$ is overwritten, otherwise $dom(v)$ is expanded).*
5. *If the position is of the form $\langle\psi, v, t\rangle$, where ψ is an atom, the game ends. If $\mathcal{A}\models\psi[v]$ then t has won the game, otherwise t^* has won.*

Note that player t in a position $\langle\psi, v, t\rangle$ of a play of $\langle\mathcal{A},\varphi, V\rangle$ is determined only by the position of ψ in φ, and does not depend on the valuation v or the actual play. In other words, each subformula is associated to the same player, in any play of the game.

Example 2.10 (Universal Quantifiers). Consider the following semantic game: $\langle\mathcal{A}, \forall x\exists y [x{=}y], \{\lambda\}\rangle$. The initial quantifier is an abbreviation for $\neg\exists x\neg$. So the game starts with the players interchanging turns. Thus ∀belard has to choose a value for x with the aim to make $\neg\exists y[x{=}y]$ true (because that will make the original formula false), so a value that makes $\exists y[x{=}y]$ false. Then the players change turns again, and ∃loise has to choose a value for y with the aim to make $x{=}y$ true. If she is wise, she chooses for y the same value as ∀belard has chosen (this strategy is denoted by $y{:=}x$). Thus she wins.

The definition of $\forall x_{/Y}\psi$ as an abbreviation for $\neg\exists x_{/Y}\neg\psi$ has the effect that in position $\langle\forall x_{/Y}\psi, v, \exists\rangle$, ∀belard has to chose a value which makes ψ false, thus frustrating ∃loise 's aims. Likewise in $\langle\varphi_1\wedge\varphi_2, v, \exists\rangle$ he has to choose a conjunct that falsifies the original formula. More precisely, if we consider $\wedge$ and $\forall$ as primitive symbols, the description of a play of the game should include the following clauses:

6. If the position is of the form $\langle\varphi_1\wedge_{/Y}\varphi_2, v, t\rangle$ then t^* chooses L or R and the game is continued, respectively, from position $\langle\varphi_1, v', t\rangle$ or $\langle\varphi_2, v', t\rangle$.

7. If the position is of the form $\langle\forall x_{/Y}\psi, v, t\rangle$ then t^* chooses a value a and the game is continued from position $\langle\psi, v', t\rangle$.

2.3 Strategies

A semantic game may have many different plays. We are not so much interested whether one of the players accidentally wins (or loses) a particular play, but whether she/he has a strategy to win against all the initial positions and all plays of the opponent; that will be the criterion whether the formula is true or not. To define strategies properly, we define first the notion of a function being independent of a set of variables.

Definition 2.11. *Let v and w be valuations, and Y a set of variables. A valuation v is called* ***a Y-variant of*** *w, relation denoted $v{\sim_Y}w$, if the valuations v and w are defined for the same* *variables and assign the same value to variables outside Y; the values assigned to variables in Y may differ. A valuation v is called a **Y-expansion of** w if $dom(v) = dom(w) \cup Y$, $dom(w) \cap Y = \varnothing$, and v and w assign the same values on $dom(w)$.*

Definition 2.12. *A function f having for domain a set of valuations V is called **Y-independent** (independent of Y) if for all $v, w \in V$: from $v \sim_Y w$ it follows that $f(v) = f(w)$.*

It may happen that $f\colon V \to A$ is not Y-independent but that its restriction $f \upharpoonright W\colon W \to A$ to a subset $W \subseteq V$ becomes Y-independent. Some trivial cases of independence follow immediately from the definition:

Theorem 2.13. *Any function is independent of the empty set of variables. If $V \subseteq A^X$ is a singleton, then for any function f with domain V and any $Y \subseteq X$ it holds that f is Y-independent.*

Definition 2.14. *A **choice function** for the subformula $\varphi_1 \vee_{/Y} \varphi_2$ in a semantic game $\langle \mathcal{A}, \varphi, V \rangle$ is a Y-independent function $c_{\varphi_1 \vee_{/Y} \varphi_2}\colon A^{Fr_\varphi(\varphi_1 \vee_{/Y} \varphi_2)} \to \{L, R\}$. A choice function for a subformula $\exists x_{/Y} \psi$ in a semantic game $\langle \mathcal{A}, \varphi, V \rangle$ is a Y-independent function $c_{\exists x_{/Y} \psi}\colon A^{Fr_\varphi(\exists x_{/Y} \psi)} \to A$.*

Definition 2.15. *A **strategy** S_φ for $\exists$loise in a semantic game $\langle \mathcal{A}, \varphi, V \rangle$ is a collection of choice functions that for each subformula ψ of φ where $\exists$loise has to play, provides a choice function c_ψ. Likewise for $\forall$belard.*

*A **winning strategy** for $\exists$loise in a semantic game $\langle \mathcal{A}, \varphi, V \rangle$ is a strategy that guarantees $\exists$loise to win any play of the game, whatever $\forall$belard plays, if she uses the choice functions to make her moves. That means, at position $\langle \varphi_1 \vee_{/Y} \varphi_2, v, \exists \rangle$ she chooses the value $c_{\varphi_1 \vee_{/Y} \varphi_2}(v)$ (L or R), and at position $\langle \exists x_{/Y} \varphi, v, \exists \rangle$ she chooses $a = c_{\exists x_{/Y} \varphi}(v)$. Likewise for $\forall$belard.*

Definition 2.16 (Truth and falsity). *An IF*-formula φ is said to be **true** in $\mathcal{A}$ for the set of valuations V if there is a winning strategy for $\exists$loise in the semantic game $\langle \mathcal{A}, \varphi, V \rangle$. It is called **false** in $\mathcal{A}$ for V if there is a winning strategy for $\forall$belard in that game, and **undetermined** if none of the players has a winning strategy.*

Definition 2.17 (Truth and falsity for sentences). *An IF*-sentence φ is **true** in $\mathcal{A}$ if $\exists$loise has a winning strategy in the game $\langle \mathcal{A}, \varphi, A^\varnothing \rangle$, **false** in $\mathcal{A}$ if $\forall$belard has a winning strategy in that game, and **undetermined** in $\mathcal{A}$ otherwise.*

2.4 Notations for valuations

In the course of this paper we will use several notations concerning variables, valuations and sets of valuations, and it is convenient to list them together.

Xy	the set of variables $X \cup \{y\}$
λ	(*the empty valuation*) the valuation that is defined for no variable at all, so $A^\varnothing = \{\lambda\}$ for any A
$\{xy\colon ab\}$	(*is an example of the explicit notation we use for a valuation*) the valuation that assigns a to x and b to y
$\{xy\colon aa, bb\}$	(*analogous to the previous example*) the set of valuations that consists of the valuations $\{xy\colon aa\}$ and $\{xy\colon bb\}$
vw	the valuation $v \cup w$; defined only if $dom(v) \cap dom(w) = \varnothing$

$dom(V)$	the set of variables X such that $V \subseteq A^X$
$V \times W$	$\{vw \mid v \in V \text{ and } w \in W\}$; defined if $dom(V) \cap dom(W) = \varnothing$
$v_{x:a}$	if $x \in dom(v)$: the x-variant obtained from v by changing the value assigned to x into a; if $x \notin dom(v)$: the x-expansion of v that assigns a to x
$v_{xy:ab}$	xy-variant or xy-expansion, similar with $v_{x:a}$
$V_{x:a}$	$\{v_{x:a} \mid v \in V\}$
$V_{x:A}$	$\{v_{x:a} \mid v \in V, a \in A\}$ $(= \cup_{a \in A} V_{x:a})$
V_x	typical symbol for any subset of $V_{x:A}$; we call it an x-variant of V if $x \in dom(V)$, or an x-expansion of V otherwise
v_{-x}	the valuation that is not defined for x and that for all other variables is the same as v; note that if $dom(v) = \{x\}$ then $v_{-x} = \lambda$
V_{-x}	$\{v_{-x} \mid v \in V\}$; if $W = A^x$ then $W_{-x} = \{\lambda\}$ and not $\varnothing$
$V_{x:f}$	$\{v_{x:f(v)} \mid v \in V\}$
$V_{[z/x]}$	$(V_{z:f})_{-x}$, where $f(v) = v(x)$, i.e. the set of valuations obtained from V by giving z the role of x; only defined if $x \in dom(V)$ and $z \notin dom(V)$

3 Discussion

Now that all basic notions have been introduced, we can compare our notion with three closely related approaches.

3.1 Hintikka 1996, Hintikka & Sandu 1997

In Hintikka's original game interpretation for IF-sentences, the convention is adopted that ''moves connected with existential quantifiers are always independent of earlier moves with existential quantifiers'' [8, p.63]. So $\exists x \exists y[x=y]$ is interpreted in IF in the same way as $\exists x \exists y_{/x}[x=y]$ would be interpreted in IF^* and the obvious strategy function $y{:=}x$ at the second quantifier is not available for ∃loise. But even then she has a winning strategy: $x{:=}a$, $y{:=}a$ for a fixed element a of the structure.

This convention can be understood from the perspective of Skolem functions, which in [8] are taken as the syntactic counterpart of strategies in the game semantics. In classical first order logic, the arguments of a Skolem function f replacing an existentially quantified variable y, are usually taken to be just the universally quantified variables in whose scope y occurred, and not the existentially quantified ones: $\forall x \exists z \exists y[x=y]$ Skolemizes to $\exists g \exists f \forall x[x=f(x)]$, and not to $\exists g \exists f \forall x[x=f(g(x),x)]$. In this case (as for classical first order logic in general) it is easy to see that both are equivalent. However, Hodges' example ($\forall x \exists z \exists y_{/x}[x=y]$, see p. 92) shows that the situation changes when switching to IF-logic: while its truth is equivalent to $\exists g \exists f \forall x[x=f(g(x))]$ in the game reading of Hodges, it is equivalent to $\exists g \exists c \forall x[x=c]$ (where c is a Skolem constant) in the Skolem-reading of Hintikka [8]. An explanation of the effect of the convention is given with Theorem 13.4.

A prerequisite for Skolemization is the absence of nested quantification over the same variable. This prerequisite is almost always implicit or is avoided by assuming that the formula is in prenex form. It is not explicit whether IF-logic allows for nested quantification.

This aspect is relevant for the claim by Hintikka that IF-logic is a conservative extension of classical predicate logic [8, p. 65]. Either the claim is incorrect because nested quantification is not allowed, or the following example is an counterexample. For a proof that *IF**is a conservative extension, see Thm 4.11.

Example 3.1 (IF-logic is not a conservative extension of full predicate logic).
First, consider the following classical validity:

(1) $\forall x \exists y[x = y \wedge \exists z[y = z]]$.

This is also an IF-validity: despite the convention that ∃loise's choice for z cannot depend on her choice for y, she can use the value of x to signal the value of y by choosing y equal to x, and also z equal to x.

Now consider the following variation of (1), containing an extra quantification over x, which is a classical validity as well.

(2) $\forall x \exists y[x = y \wedge \forall x \exists z[y = z]]$,

When ∃loise now gets to choose a value for z, ∀belard has in the mean time chosen another value for x (at the innermost $\forall x$). This means, x can no longer be used to signal the value of y, which was chosen to be equal to the first value for x. So, except in one-element models, ∃loise has no winning strategy: the formula is not an IF-validity.

3.2 Hodges 1997a

Following Caicedo & Krynicki [3], our semantic games $\langle \mathcal{A}, \varphi, V \rangle$ have as many initial positions as there are valuations in V. A winning strategy for ∃loise does not pick the initial position but must be winning for *all* initial positions. Similarly for ∀belard . One may think that the initial position (called an opening deal by Hodges [11]) is chosen from V by a random dealer or a third party. Which view one takes does not affect the definition of the game.

One could consider $\langle \mathcal{A}, \varphi, V \rangle$ as a collection of games $\langle \mathcal{A}, \varphi, \{v\} \rangle$, $v \in V$, to be played in parallel with a 'uniform' strategy. In the literature on *IF*-logic this position is espoused by Hodges, who understands by 'game' one of the kind $\langle \mathcal{A}, \varphi, \{v\} \rangle$ and calls $\langle \mathcal{A}, \varphi, V \rangle$ a 'contest'. A uniform strategy prescribes the same choice for the games $\langle \mathcal{A}, \varphi, \{v\} \rangle$ and $\langle \mathcal{A}, \varphi, \{w\} \rangle$ if $v \sim_Y w$. Hodges calls V a 'trump' if ∃loise has a winning strategy for the contest $\langle \mathcal{A}, \varphi, V \rangle$, and a 'cotrump' if ∀belard has one. Properly formulated, this conception of a collection of games is equivalent to ours, and leads to Hodges' compositional semantics. But one has to be careful not to identify $\langle \mathcal{A}, \varphi, V \rangle$ with the plain collection $\langle \mathcal{A}, \varphi, \{v\} \rangle$, $v \in V$, witness the example below where ∃loise has a winning strategy for each one of the latter games but not for the former.

Example 3.2. This example is based upon example 3.1 by Hodges in [11]. Let φ be $\exists x_{/y}[x = y]$. Let $\{y\colon 0\}$ and $\{y\colon 1\}$ denote the valuations that assign 0, respectively 1, to y. Consider now the game $G_0 = \langle \mathcal{B}, \varphi, \{y\colon 0\} \rangle$ (recall that $\mathcal{B} = \{0, 1\}$). Any choice function c_φ has the singleton set $\{y\colon 0\}$ as domain. Therefore it is a constant function and thus necessarily y-independent. The strategy $x{:=}0$ (i.e. choose for x the value 0) is then a winning strategy in G_0. Likewise, $x{:=}1$ is the winning strategy in game $\langle \mathcal{B}, \varphi, \{y\colon 1\} \rangle$.

Let $\{y\colon 0,1\}$ denote the set of valuations consisting of the valuations $\{y\colon 0\}$ and $\{y\colon 1\}$. Consider now the game $\langle \mathcal{B}, \varphi, \{y\colon 0,1\} \rangle$. The only y-independent choices for x are, again, constant functions. However, if ∃loise plays the constant function $x{:=}0$, she loses if the initial position is $\langle\varphi, \{y\colon 1\}, \exists\rangle$. Likewise $x{:=}1$ loses in the other initial position. So there is no strategy that such that ∃loise wins in both initial positions of the game with $\{y\colon 0,1\}$, whereas she has winning strategies in both games with $v \in \{y\colon 0,1\}$.

Although the basic concepts are the same as in our approach, there is a technical difference that causes Hodges' semantics to be not equivalent with ours. This will be explained in section 6.

3.3 Väänänen 2002

It is common in the literature of IF-logic to consider only formulas in *negation normal form* (negations are only applied to atomic subformulas), this forces to treat also $\wedge$ and $\forall$ as primitive symbols. In this context, Väänänen [17] interprets IF-logic by means of a perfect information asymmetric game where ∃loise chooses strategy functions instead of sides or individual values of variables, and ∀belard chooses sides in disjunctions and conjunctions. In our notation, a position in Väänänen's game $\langle\mathcal{A}, \varphi, V\rangle$ is a pair $\langle\psi, W\rangle$ where ψ is a subformula of φ and W is a set of valuations, the only initial position being $\langle\varphi, V\rangle$. Both players make a move at position $\langle\psi_1 \vee_{/Y} \psi_2, W\rangle$: first ∃loise chooses a Y-independent function $f\colon W \to \{L, R\}$ and then ∀belard chooses whether the game continues from $\langle\psi_1, f^{-1}(L)\rangle$ or $\langle\psi_2, f^{-1}(R)\rangle$. At position $\langle\exists x_{/Y}\psi, W\rangle$, ∃loise chooses a Y-independent $f\colon W \to A$ and the game continues from $\langle\psi, W_{x:f}\rangle$. At position $\langle\psi_1 \wedge_{/Y} \psi_2, W\rangle$, ∀belard chooses whether the game continues from $\langle\psi_1, W\rangle$ or $\langle\psi_2, W\rangle$. At position $\langle\forall x_{/Y}\psi, W\rangle$ nobody plays and the game continues from position $\langle\psi, W \times A\rangle$. The game ends at $\langle\psi, W\rangle$ when ψ is atomic or negated atomic, winning ∃loise if ψ is classically true for all valuations in W. It may be shown that ∃loise has a winning strategy in this game if and only if φ is true according to Def. 2.16. However, ∀belard having a winning strategy does not mean φ to be false in the sense of Def. 2.16.

4 Inductive definition of satisfaction

In the previous section we have presented an interpretation of *IF**-formulas in terms of winning strategies. If we want to show that a given formula is true or false with respect to a certain model $\mathcal{A}$ and set of valuations V, we just have to come up with a strategy that witnesses this. However, when we prove general properties of the logic it is much more convenient to have an inductive definition of satisfaction. In this section we will define truth inductively with respect to a *set of valuations*.

The counterpart of independence in strategies will be saturatedness of sets of valuations.

Definition 4.1. *A set $W \subseteq V$ of valuations is* Y***-saturated in*** V *if W is closed under $\sim_Y$ (i.e. for all $w \in W$ and $v \in V$: if $w \sim_Y v$ then $v \in W$). A family of sets $\{V_i\}_{i \in I}$ forms a* Y***-saturated cover of*** V *if $V = \cup_{i \in I} V_i$ and each V_i is Y-saturated in V.*

Definition 4.2 (Inductive satisfaction). *Let φ be an IF*-formula , $\mathcal{A}$ a suitable model and V a set of valuations with $Fr(\varphi) \subseteq dom(V)$. We define positive satisfaction with respect to V, denoted as $\mathcal{A} \models^+ \varphi[V]$, respectively negative satisfaction $\mathcal{A} \models^- \varphi[V]$, by induction in the complexity of φ (the 'unsigned' $\models$ denotes classical satisfaction). The clauses of the definition are:*

(at) *If φ is atomic:*
$\mathcal{A} \models^+ \varphi[V] \iff$ *for all $v \in V$ holds that $\mathcal{A} \models \varphi[v]$.*
$\mathcal{A} \models^- \varphi[V] \iff$ *for no $v \in V$ holds that $\mathcal{A} \models \varphi[v]$.*

(¬) $\mathcal{A}\models^+\neg\varphi[V]\Longleftrightarrow\mathcal{A}\models^-\varphi[V]$,
$\mathcal{A}\models^-\neg\varphi[V]\Longleftrightarrow\mathcal{A}\models^+\varphi[V]$

(∨) $\mathcal{A}\models^+(\varphi_1\vee_{/Y}\varphi_2)[V]\Longleftrightarrow$ *there is a* Y*-saturated cover* $\{V_1,V_2\}$ *of* V *such that* $\mathcal{A}\models^+\varphi_1[V_1]$ *and* $\mathcal{A}\models^+\varphi_2[V_2]$.
$\mathcal{A}\models^-(\varphi_1\vee_{/Y}\varphi_2)[V]\Longleftrightarrow\mathcal{A}\models^-\varphi_1[V]$ *and* $\mathcal{A}\models^-\varphi_2[V]$.

(∃) $\mathcal{A}\models^+\exists x_{/Y}\varphi[V]\Longleftrightarrow$ *there is a* Y*-saturated cover* $\{V_i\}_{i\in I}$ *of* V *and for each* $i\in I$ *there is an* $a_i\in A$ *such that* $\mathcal{A}\models^+\varphi[\cup_{i\in I}(V_i)_{x:a_i}]$.
$\mathcal{A}\models^-\exists x_{/Y}\varphi[V]\Longleftrightarrow\mathcal{A}\models^-\varphi[V_{x:A}]$.

The definitions of $\wedge_{/Y}$ and $\forall x_{/Y}$ as abbreviation yield the following clauses:

(∧) $\mathcal{A}\models^+(\varphi_1\wedge_{/Y}\varphi_2)[V]\Longleftrightarrow\mathcal{A}\models^+\varphi_1[V]$ *and* $\mathcal{A}\models^+\varphi_2[V]$.
$\mathcal{A}\models^-(\varphi_1\wedge_{/Y}\varphi_2)[V]\Longleftrightarrow$ *there is a* Y*-saturated cover* $\{V_1,V_2\}$ *of* V *such that* $\mathcal{A}\models^-\varphi_i[V_i]$, $i=1,2$.

(∀) $\mathcal{A}\models^+\forall x_{/Y}\varphi[V]\Longleftrightarrow\mathcal{A}\models^+\varphi[V_{x:A}]$.
$\mathcal{A}\models^-\forall x_{/Y}\varphi[V]\Longleftrightarrow$ *there is a* Y*-saturated cover* $\{V_i\}_{i\in I}$ *of* V *and an* $a_i\in A$ *such that* $\mathcal{A}\models^-\varphi[\cup_{i\in I}(V_i)_{x:a_i}]$.

Note that if $V=\varnothing$, the inductive definition of satisfaction yields, for any *IF**-formula φ, that $\mathcal{A}\models^+\varphi[\varnothing]$ and $\mathcal{A}\models^-\varphi[\varnothing]$. This might look anomalous, but it is actually necessary for the situation with disjunction, where the empty sets of valuations may occur if V is split into V and $\varnothing$, and both satisfy φ. It is different from saying that formulas are always satisfied by the singleton set $A^{\varnothing}=\{\lambda\}$: this is not the case. In fact, satisfaction with respect to $\{\lambda\}$ is only *defined* for formulas with *no* free variables, i.e. sentences, which leads to:

Notation 4.3 (Evaluation of sentences). *If* φ *is an IF*-sentence and* $\mathcal{A}$ *a suitable model, we write* $\mathcal{A}\models^+\varphi$ *instead of* $\mathcal{A}\models^+\varphi[\{\lambda\}]$ *and* $\mathcal{A}\models^-\varphi$ *instead of* $\mathcal{A}\models^-\varphi[\{\lambda\}]$.

Remark 4.4. *We could define the meaning of a formula* φ *in a structure as as a pair consisting of the collection of sets of valuations that satisfy* φ *and the collection of sets of valuations that refute* φ. *Then the meaning of* φ *is a function of the meanings of the subformulas of* φ *and of the way in which they are combined. So meaning assignment is compositional.*

We will show that the definition of satisfaction and the game interpretation are equivalent. For this purpose we need the following result about decreasing sets of valuations which will be quite useful. Essentially the same lemma is given as *Fact 11.1.1* in Hodges [11, p. 57], for his ‘‘trump’’ semantics.

Notation 4.5. *If a definition (lemma, theorem, ...) holds both for the* $\models^+$ *case and the* $\models^-$ *case, we present it as one definition (lemma, theorem,...) using* $\models^\pm$.

Lemma 4.6 (Downward monotonicity). *Let* φ *be an IF*-formula,* $\mathcal{A}$ *a suitable model, and* V *a suitable set of valuations. Let* $W\subseteq V$*, then:*

$$\mathcal{A}\models^\pm\varphi[V]\Rightarrow\mathcal{A}\models^\pm\varphi[W].$$

Proof. We use induction in the complexity of φ. The atomic case and the inductive step for ¬ are clear.

(∨, +) $\mathcal{A}\models^+(\varphi_1\vee_{/Y}\varphi_2)[V]$ implies $\mathcal{A}\models^+\varphi_i[V_i]$ for a Y-saturated cover $\{V_1,V_2\}$ of V. Then $W_i=V_i\cap W$, $i=1,2$, is a Y-saturated cover of W, and by induction hypothesis $\mathcal{A}\models^+\varphi_i[W_i]$, which grants $\mathcal{A}\models^+(\varphi_1\vee_{/Y}\varphi_2)[W]$.

$(\vee,-)$ $\mathcal{A}\models^{-}(\varphi_1\vee_{/Y}\varphi_2)[V]$ implies $\mathcal{A}\models^{-}\varphi_i[V]$ and thus $\mathcal{A}\models^{-}\varphi_i[W]$, $i=1,2$, by induction hypothesis and so $\mathcal{A}\models^{-}(\varphi_1\vee_{/Y}\varphi_2)[V]$.

$(\exists,+)$ $\mathcal{A}\models^{+}\exists x_{/Y}\ \varphi[V]$ implies $\mathcal{A}\models^{+}\varphi[\cup_{i\in I}(V_i)_{x:a_i}]$ with $\{V_i\}_{i:\in I}$ a Y-saturated cover of V. Then $\{V_i\cap W\}_{i\in I}$ is a Y-saturated cover of W and by induction hypothesis $\mathcal{A}\models^{+}\varphi[\cup_{i\in I}(V_i\cap W)_{x:a_i}]$, thus $\mathcal{A}\models^{+}\exists x_{/Y}\ \varphi[W]$.

$(\exists,-)$ $\mathcal{A}\models^{-}\exists x_{/Y}\ \varphi[V]$ implies $\mathcal{A}\models^{-}\varphi[V_{x:A}]$; hence, $\mathcal{A}\models^{-}\varphi[W_{x:A}]$ and $\mathcal{A}\models^{-}\exists x_{/Y}\ \varphi[V]$. ∎

It will be useful to have variants of the clauses from Def. 4.2. In particular, we will apply the following variants in the proof of the equivalence of strategy interpretation with the inductive definition.

Theorem 4.7 (Alternative for $\vee$). *$\mathcal{A}\models^{+}(\varphi_1\vee_{/Y}\varphi_2)[V]$ if and only if there is a Y-saturated partition V_1, V_2 of V such that $\mathcal{A}\models^{+}\varphi_1[V_1]$ and $\mathcal{A}\models^{+}\varphi_2[V_2]$.*

Proof. $(\Rightarrow)$ The definition of satisfaction (Def. 4.2) guarantees that there is a Y-saturated cover V_1, V_2 of V such that $\mathcal{A}\models^{+}\varphi_i[V_i]$. Define $V_2' = V_2\setminus(V_1\cap V_2)$. Since $(V_1\cap V_2)$ is Y-saturated also V_2' is Y-saturated. Moreover, $\mathcal{A}\models^{+}\varphi_2[V_2']$ by downward monotonicity (Lemma 4.6). Then V_1, V_2' is the required partition.$(\Leftarrow)$ A partition is a cover. ∎

Theorem 4.8 (Alternatives for $\exists x_{/Y}$). *The following are equivalent:*

(1.) $\mathcal{A}\models^{+}\exists x_{/Y}\varphi[V]$.

(2.) *There is a Y-saturated partition $\{V_i\}_{i\in I}$ of V and for each $i\in I$ there is an $a_i\in A$ such that $\mathcal{A}\models^{+}\varphi[\cup_{i\in I}(V_i)_{x:a_i}]$.*

(3.) *There is a Y-independent $f\colon V\to A$ such that $\mathcal{A}\models^{+}\varphi[V_{x:f}]$.*

Proof. $(1\Rightarrow 2)$ The definition of satisfaction (Def. 4.2) guarantees the existence of a Y-saturated cover $\{V_i\}_{i\in I}$ of V, and of a corresponding family $(a_i)_{i\in I}$ of elements of A. By the axiom of choice, we may assume I is well ordered by $<$. Then we may transform $\{V_i\}_{i\in I}$ in a disjoint cover of V by the inductive definition: $V_i' = V_i\setminus\cup_{j<i}V_j$. Clearly, $\cup_{i\in I}V_i' = V$ and each V_i' is Y-saturated because $\cup_{j<i}V_j$ is so. Moreover, $\cup_{i\in I}(V_i')_{x:a_i}\subseteq\cup_{i\in I}(V_i)_{x:a_i}$ (the inclusion may be proper because some V_i' could be empty and thus $\{V_i'\}_{x:a_i}$ could be empty). Therefore, $\mathcal{A}\models^{+}\varphi[\cup_{i\in I}(V_i')_{x:a_i}]$ by downward monotonicity (Lemma 4.6).

$(2\Rightarrow 3)$ The function $f\colon V\to A$ defined by: $f(v)=a_i$ if $v\in V_i$, is well defined because $\{V_i\}_{i\in I}$ is a partition of V, and it is Y-independent because the V_i are Y-saturated. Moreover, $V_{x:f}=\cup_{i\in I}(V_i)_{x:a_i}$ and thus $\mathcal{A}\models^{+}\varphi[V_{x:f}]$.

$(3\Rightarrow 1)$ Define $V_a=f^{-1}(a)$ for any $a\in f(V)$, then $\{V_a\}_{a\in A}$ is a Y-saturated cover of V. Moreover, $\cup_{a\in f(V)}(V_a)_{x:a}=V_{x:f}$ and thus we have $\mathcal{A}\models^{+}\varphi[\cup_{a\in f(V)}(V_a)_{x:a}]$, which means by definition $\mathcal{A}\models^{+}\exists x_{/Y}\varphi[V]$. ∎

The proof of the implication $(1\Rightarrow 2)$ in the previous theorem, passing from a cover to a partition, makes an essential use of the axiom of choice. Therefore, the main result from this section depends on the axiom of choice. This axiom could have been avoided if we had used partitions in the inductive clause for $(\exists)$ of Def. 4.2.

Theorem 4.9 (Equivalence of the inductive and strategy definition). *For any IF*-formula φ, suitable model $\mathcal{A}$ and $V\subseteq A^X$ with $Fr(\varphi)\subseteq X$:*

1. $\mathcal{A}\models^{+}\varphi[V]\iff\exists$*loise has a winning strategy in the game $\langle\mathcal{A},\varphi,V\rangle$.*
2. $\mathcal{A}\models^{-}\varphi[V]\iff\forall$*belard has a winning strategy in the game $\langle\mathcal{A},\varphi,V\rangle$.*

Proof. The theorem is proven by simultaneous induction in the complexity of φ:

(at) No moves have to be played, the result follows directly from the definition.

($\neg$) The players interchange turns, so the result follows immediately from the induction hypothesis.

($\vee,+$) Let $\varphi = \varphi_1 \vee_{/Y} \varphi_2$ and assume $\mathcal{A} \models^+ \varphi[V]$. Then there is a Y-saturated cover V_1, V_2 of V such that $\mathcal{A} \models^+ \varphi_i[V_i]$. By induction hypothesis there is a winning strategy S_{φ_i} for ∃loise in $\langle \mathcal{A}, \varphi_i, V_i\rangle$. Define $c_\varphi(v) = (\textit{if} \;\; v \in V_1 \textit{ then } L \textit{ else } R)$. If $v \in V_1$ and $w \in V$ with $v \sim_Y w$, then $w \in V_1$ because V_1 is Y-saturated, hence $c_\varphi(v) = c_\varphi(w)$. So c_φ is Y-independent. Moreover $\{c_\varphi\} \cup S_{\varphi_1} \cup S_{\varphi_2}$ is a winning strategy for ∃loise in $\langle \mathcal{A}, \varphi, V\rangle$.
Conversely, let $\varphi = \varphi_1 \vee_{/Y} \varphi_2$ and assume S_φ is a winning strategy for ∃loise with c_φ as the choice function for $\vee_{/Y}$. Let $V_1 = c_\varphi^{-1}(L)$ and $V_2 = c_\varphi^{-1}(R)$. Since c_φ is independent of Y, the cover V_1, V_2 of V is Y-saturated. Moreover, S_{φ_i} is a winning strategy for $\langle \mathcal{A}, \varphi_i, V_i\rangle$. So, by ind. hyp. $\mathcal{A} \models^+ \varphi_i[V_i]$ for $i = 1, 2$. Hence $\mathcal{A} \models^+ (\varphi_1 \vee_{/Y} \varphi_2)[V]$.

($\vee,-$) $\mathcal{A} \models^- (\varphi_1 \vee_{/Y} \varphi_2)[V] \iff \mathcal{A} \models^- \varphi_1[V]$ and $\mathcal{A} \models^- \varphi_2[V] \iff$ ∀belard has winning strategies for $\langle \mathcal{A}, \varphi_1, V\rangle$ and $\langle \mathcal{A}, \varphi_2, V\rangle \iff$ ∀belard has a winning strategy (the union of the two) in $\langle \mathcal{A}, \varphi_1 \vee_{/Y} \varphi_2, V\rangle$.

($\exists,+$) Let $\mathcal{A} \models^+ \exists x_{/Y}\varphi[V]$. Then by Thm 4.8 there is a Y-independent function $f\colon V \to A$ such that $\mathcal{A} \models^+ \varphi[V_{x:f}]$, and by induction hypothesis there is a winning strategy S_φ for ∃loise in the game $\langle \mathcal{A}, \varphi, V_{x:f}\rangle$. Define $c_{\exists x_{/Y}\varphi} = f$, then $\{c_{\exists x_{/Y}\varphi}\} \cup S_\varphi$ is a winning strategy for the game $\langle \mathcal{A}, \exists x_{/Y}\varphi, V\rangle$.
Conversely, let $S_{\exists x_{/Y}\varphi}$ be a winning strategy for ∃loise in $\langle \mathcal{A}, \exists x_{/Y}\varphi, V\rangle$ with choice function $f = c_{\exists x_{/Y}\varphi}$ for $\exists x_{/Y}\varphi$. Then f is Y-independent and by definition $S_{\exists x_{/Y}\varphi} \setminus \{f\}$ is a winning strategy for the game $\langle \mathcal{A}, \varphi, V_{x:f}\rangle$. By induction hypothesis, $\mathcal{A} \models^+ \varphi[V_{x:f}]$ and thus, by Thm 4.8, $\mathcal{A} \models^+ \exists x_{/Y}\varphi[V]$.

($\exists,-$) $\mathcal{A} \models^- \exists x_{/Y}\psi[V] \iff \mathcal{A} \models^- \psi[V_{x:A}] \iff$ ∀belard has a winning strategy for the game $\langle \mathcal{A}, \psi, V_{x:A}\rangle \iff$ ∀belard has a winning strategy for the game $\langle \mathcal{A}, \exists x_{/Y}\psi, V\rangle$, viz. the same one. ∎

In the rest of this paper we will use $\models^+$, and $\models^-$ both for satisfaction in terms of strategies (mostly in the explanation of the examples) as for the inductive satisfaction (in formal proofs).

The syntax of IF^* is an extension of the syntax of classical predicate logic, and so is its semantics. Thus, positive satisfaction of first order sentences coincides with classical satisfaction. One may expect a difficult proof, because in the Tarskian bottom-up approach only variables occurring in the subformula play a role, whereas in the game theoretic top-down approach all previously encountered variables in principle play a role, even those that do not occur in the subformula. But the proof is surprisingly simple, and does not need the axiom of choice if we use our cover definition (Def. 4.2). We first prove a more general result for classical first order formulas.

Lemma 4.10. *Let φ be a classical first order formula. Then the following two statements hold:*

1. $\mathcal{A} \models^+ \varphi[V] \iff \mathcal{A} \models \varphi[v]$ *for all* $v \in V$ *(classically).*
2. $\mathcal{A} \models^- \varphi[V] \iff \mathcal{A} \not\models \varphi[v]$ *for all* $v \in V$ *(classically)*

Proof. We prove the statements by induction on the structure of φ. The atomic and negative cases are straightforward.

($\vee,+$) Assume $\mathcal{A}\models^{+}(\varphi_1\vee\varphi_2)[V]$. Then there is a cover V_1, V_2 of V such that $\mathcal{A}\models^{+}\varphi_1[V_1]$ and, by ind. hyp., $\mathcal{A}\models\varphi_1[v]$ for all $v\in V_1$. Likewise for φ_2. Hence $\mathcal{A}\models(\varphi_1\vee\varphi_2)[v]$ for all $v\in V_1\cup V_2$.
Conversely, assume $\mathcal{A}\models(\varphi_1\vee\varphi_2)[v]$ for all $v\in V$. Let $V_1=\{v\in V\,|\,\mathcal{A}\models\varphi_1[v]\}$. Then by ind. hyp.: $\mathcal{A}\models^{+}\varphi_1[V_1]$. Likewise for φ_2. Hence $\mathcal{A}\models^{+}(\varphi_2\vee\varphi_2)[V_1\cup V_2]$.

($\vee,-$) $\mathcal{A}\models^{-}(\varphi_1\vee\varphi_2)[V]\Longleftrightarrow\mathcal{A}\models^{-}\varphi_1[V]$ and $\mathcal{A}\models^{-}\varphi_2[V]\Longleftrightarrow$ (ind. hyp) for all $v\in V$: $\mathcal{A}\not\models\varphi_1[v]$ and $\mathcal{A}\not\models\varphi_2[v]\Longleftrightarrow$ for all $v\in V$: $\mathcal{A}\not\models(\varphi_2\vee\varphi_2)[v]$.

($\exists,+$) Suppose $\mathcal{A}\models^{+}\exists x\psi[V]$. Then by definition there is a cover $(V_i)_{i\in I}$ of V and a family $(a_i)_{i\in I}$ such that $\mathcal{A}\models^{+}\psi[\cup_{i\in I}(V_i)_{x:a_i}]$. By induction hypothesis this means $\mathcal{A}\models\psi[v_{x:a_i}]$ for any $v_{x:a_i}\in\cup_{i\in I}(V_i)_{x:a_i}$ and a fortiori $\mathcal{A}\models\exists x\psi[v]$, for all $v\in V$.
Conversely, suppose $\mathcal{A}\models\exists x\psi[v]$ for all $v\in V$. For each $a\in A$ define $V_a=\{v\in V\,|\,\mathcal{A}\models\psi[v_{x:a}]\}$, then $(V_a)_{a\in A}$ forms a cover or V (perhaps with some empty V_a's). Moreover, each $w\in\cup_{a\in A}(V_a)_{x:a}$ is of the form $w=v_{x:a}$ for some $a\in A$ and $v\in V_a$, then $\mathcal{A}\models\psi[w]$ by definition of V_a. By induction hypothesis, $\mathcal{A}\models^{+}\psi[\cup_{a\in A}(V_a)_{x:a}]$, which by definition means $\mathcal{A}\models^{+}\exists x\psi[v]$.

($\exists,-$) $\mathcal{A}\models^{-}\exists x\psi[V]\Longleftrightarrow\mathcal{A}\models^{-}\psi[V_{x:A}]\Longleftrightarrow$ for all $v\in V_{x:A}$: $\mathcal{A}\not\models\psi[v]\Longleftrightarrow$ for all $v\in V$: $\mathcal{A}\not\models\exists x\psi[v]$. ∎

An immediate consequence of Lemma 4.10, making $V=\{\lambda\}$, is the following result (cf. Ex. 3.1, which gave a counterexample against the analogue for Hintikka's IF).

Theorem 4.11 (*IF is a conservative extension of predicate logic).**
For any classical first order sentence φ: $\mathcal{A}\models^{+}\varphi\Longleftrightarrow\mathcal{A}\models\varphi$.

Finally, two technical lemmas. First a result on formulas from which variables under slashes are removed.

Lemma 4.12. *Let φ' be obtained from IF*-formula φ by removing some or all variables under slashes (e.g. replacing $\exists x_{/yz}$ by $\exists x$). Then*

$$\mathcal{A}\models^{\pm}\varphi[V]\Rightarrow\mathcal{A}\models^{\pm}\varphi'[V]$$

for any suitable model $\mathcal{A}$ and set of valuations V.

Proof. If ∃loise (resp. ∀belard) has a winning strategy for the game associated with φ, the same strategy is good for the game associated with φ' because the strategy choice functions trivially satisfy the remaining independence conditions. ∎

And, finally, a result on interchanging variables. Note that the left-to-right direction also holds if x does occur bound in $\varphi(x)$.

Lemma 4.13 (Interchanging free variables). *If x does not occur bound in $\varphi(x)$, and z does not occur in $\varphi(x)$ nor in dom(V) then:*

$$\mathcal{A}\models^{\pm}\varphi(x)[V] \text{ iff } \mathcal{A}\models^{\pm}\varphi(z)[V_{[z/x]}].$$

Proof. Recall that $V_{[z/x]}$ denotes the set of valuations obtained from V by giving z the role of x. Thus, under the hypotheses, both sides are syntactically and semantically identical, except for the change of x to z. ∎

5 Expanding valuations

The more variables occur in the domain of the valuations, the more information is available. The information provided by an extra variable may make the difference for a player to win the game for a given *IF**-formula or not. For example: while $\mathcal{B} \not\models^+ \exists y_{/z}[y=z][\{z\colon 0,1\}]$, we have $\mathcal{B} \models^+ \exists y_{/z}[y=z][\{zx\colon 00,11\}]$, as ∃loise now has x as a signal for the value of z, enabling her to play the winning z-independent strategy $y{:=}x$.

The following theorem shows that extra variables will not introduce new signals if the valuations are all expanded 'in the same way': if one valuation is expanded with a certain combination of values assigned to new variables, then all valuations are expanded with that same combination of values for the new variables. In terms of the example: ∃loise turned into a winner by expanding one valuation by assigning 0 to x and the other by assigning 1 to x. She does *not* turn into a winner if we expand both valuations in $\{z\colon 0,1\}$ by the assignment of 0 to x ($\mathcal{B} \not\models^+ \exists y_{/z}[y=z][\{zx\colon 00,10\}]$), or by the assignment of 1 to x ($\mathcal{B} \not\models^+ \exists y_{/z}[y=z][\{zx\colon 01,11\}]$), or by both at the same time ($\mathcal{B} \not\models^+ \exists y_{/z}[y=z][\{zx\colon 00,01,10,11\}]$).

Expanding all valuations in the same way is formalized as taking the Cartesian product of two sets of valuations with disjoint domains:

Theorem 5.1 (Expansion by Cartesian products).
Let φ be an IF-formula, $\mathcal{A}$ a suitable model and V a suitable set of valuations. Let Z be a finite set of variables such that $Z \cap dom(V) = \varnothing$, and $W \subseteq A^Z$ with $W \neq \varnothing$. Then:*

$$\mathcal{A} \models^\pm \varphi[V] \quad \Longleftrightarrow \quad \mathcal{A} \models^\pm \varphi[V \times W].$$

Proof. We prove the theorem by induction in the complexity of φ.

(at) Follows by definition since classically $\mathcal{A} \models \varphi[v] \Longleftrightarrow \mathcal{A} \models \varphi[vw]$ for any $v \in V$, $w \in W$.
$(\neg)$ Trivial.
$(\vee,+)$ $\mathcal{A} \models^+ (\varphi_1 \vee_{/Y} \varphi_2)[V]$ implies that $\mathcal{A} \models^+ \varphi_i[V_i]$, $i=1,2$, for a Y-saturated cover V_1, V_2 of V. Hence, $\mathcal{A} \models^+ \varphi[V_i \times W]$ by induction hypothesis, and clearly $\{V_1 \times W, V_2 \times W\}$ is a Y-saturated cover of $V \times W$. Thus $\mathcal{A} \models^+ (\varphi_1 \vee_{/Y} \varphi_2)[V \times W]$. For the converse, notice that the last statement implies by Lemma 4.6 that $(\varphi_1 \vee_{/Y} \varphi_2)[V \times \{w\}]$ for a fixed $w \in W$ (recall $W \neq \varnothing$) . Then $\mathcal{A} \models^+ \varphi[V_i \times \{w\}]$ for the Y-saturated cover $(V_i \times \{w\})_{i=1,2}$ of $V \times \{w\}$, and thus (ind. hyp.) $\mathcal{A} \models^+ \varphi_i[V_i]$. Moreover, V_1, V_2 form a Y-saturated cover of V, hence, $\mathcal{A} \models^+ (\varphi_1 \vee_{/Y} \varphi_2)[V]$.
$(\vee,-)$ Immediate.
$(\exists,+)$ $\mathcal{A} \models^+ \exists x_{/Y} \varphi[V]$ implies that for some Y-independent $f\colon V \to A$, we have $\mathcal{A} \models^+ \varphi[V_{x:f}]$, and by induction hypothesis $\mathcal{A} \models^+ \varphi[V_{x:f} \times W_{-x}]$. But $V_{x:f} \times W_{-x} = (V \times W)_{x:g}$ for the Y-independent function $g\colon V \times W \to A$ defined by $g(vw) = f(v)$. Hence, $\mathcal{A} \models^+ \exists x_{/Y} \varphi[V \times W]$. Conversely, if the last statement holds then $\mathcal{A} \models^+ \varphi[(V \times W)_{x:f}]$ for some Y-independent $f\colon V \times W \to A$. Pick $w \in W$, then $\mathcal{A} \models^+ \varphi[(V \times \{w\})_{x:f'}]$ by Lemma 4.6, where f' is the appropriate restriction of f. But $(V \times \{w\})_{x:f'} = V_{x:g} \times \{w\}_{-x}$, where $g(v) = f'(v,w)$ is obviously Y-independent. By induction hypothesis, $\mathcal{A} \models^+ \varphi[V_{x:g}]$, that is $\mathcal{A} \models^+ \exists x_{/Y} \varphi[V]$.
$(\exists,-)$ It is enough to notice that $V_{x:A} \times W_{-x} = (V \times W)_{x:A}$ and use the induction hypothesis. ■

This lemma has a reassuring and important consequence for sentences.

Theorem 5.2 (Evaluation of a sentence does not depend on V). *Let φ be an IF*-sentence, $\mathcal{A}$ a suitable model, and V a suitable non-empty set of valuations. Then:*

$$\mathcal{A} \models^\pm \varphi \Longleftrightarrow \mathcal{A} \models^\pm \varphi[V].$$

Proof. Notice that $\mathcal{A}\models^{\pm}\varphi \Leftrightarrow \mathcal{A}\models^{\pm}\varphi[\{\lambda\}] \Leftrightarrow \mathcal{A}\models^{\pm}\varphi[\{\lambda\}\times V]$ by Lemma 5.1, since V is assumed non empty. Moreover, $\{\lambda\}\times V = V$ for any V. ∎

Another consequence is that if valuations are expanded for new variables, the satisfied formulas remain satisfied:

Theorem 5.3 (Invariance under expansions). *Let φ be an IF*-formula, $\mathcal{A}$ a suitable model, V a suitable set of valuations, and Z a set of variables such that $Z\cap dom(V)=\emptyset$. Let W be obtained from V by expanding each $v\in V$, in one or several ways, with values for the variables in Z. Then:*

$$\mathcal{A}\models^{\pm}\varphi[V] \quad\Longrightarrow\quad \mathcal{A}\models^{\pm}\varphi[W].$$

Proof. Apply Thm 5.1 to $V\times A^{Z}$ and then apply Thm 4.6. ∎

After the example given at the beginning of the section, it should be clear that the converse direction (that is: invariance under restrictions) does not hold. However, if we change the formula in the right and make the added variables in the domain unusable by ‘slashing them away’, we get an equivalence. In order to express this we introduce the following notation:

Definition 5.4 (Slashed formulas). *Let φ be an IF*-formula and x a variable. Then the formula $\varphi_{/x}$ is obtained from φ by replacing (for any Y) each occurrence of a disjunction $\vee_{/Y}$ by $\vee_{/Yx}$, and each occurrence of $\exists z_{/Y}$ by $\exists z_{/Yx}$.*

Lemma 5.5 (Safely expanding the domain I). *Let φ be an IF*-formula, and V a set of valuations for φ. If x is a variable that does not occur in φ nor in $dom(V)$, then for any x-expansion V_x of V:*

$$\mathcal{A}\models^{\pm}\varphi[V] \quad\Longleftrightarrow\quad \mathcal{A}\models^{\pm}\varphi_{/x}[V_x].$$

Proof. ($\Rightarrow$) If the left hand side holds, this is due to a set of strategy functions f_ψ acting on valuations not having x in their domain, due to the conditions put on this variable. Therefore, the functions $g_\psi(v)=f_\psi(v_{-x})$ provide a strategy for the right hand side.
($\Leftarrow$) If the left hand side does not hold, the same happens with the right hand side, because the possible information that the value of x in V_x might give cannot be used due to the slashing of all quantifiers and connectives in $\varphi_{/x}$ that might use this information. ∎

One might ask whether the lemma can be generalized by dropping one of the conditions on x. However, both are needed:

1. If $x\in dom(V)$, the information encoded by x may get lost when we switch to the x-expansions V_x. Let $V=\{yx\colon 00,11\}$ and $V_x=\{yx\colon 00,01,10,11\}$, then $\mathcal{B}\models^{+}\exists y_{/z}[y=z][V]$, but $\mathcal{B}\not\models^{+}\exists y_{/zx}[y=z][V_x]$.
2. If x occurs in φ, for example: $x\in Bd(\varphi)$, then the equivalence may fail because internal dependencies are disturbed. Let $V=\{\lambda\}$ and take $V_x=\{x\colon 0\}$. Then $\mathcal{B}\models^{+}\forall x\exists y[y=x][V]$ but $\mathcal{B}\not\models^{+}\forall x_{/x}\exists y_{/x}[y=x][V_x]$ because $\mathcal{B}\models^{+}\forall x_{/x}\exists y_{/x}[y=x][\{x\colon 0,1\}]$ would mean $\mathcal{B}\models^{+}\exists y_{/x}[y=x][\{x\colon 0,1\}]$ which is impossible.

Below we quote a result by Hodges in the spirit of this section (reformulated in our terminology), which claims the equivalence between positive satisfaction with respect to a set of valuations and satisfaction with respect to a family of restrictions of this set. One direction of the lemma follows from Lemma 5.3, but we give a counterexample to the other direction. The counterexample illustrates the differences between $\mathcal{A}\models^+\varphi[V]$, and $\mathcal{A}\models^+\varphi[V_i]$ *holds for all* V_i *in a cover of* V.

Quote 5.6 (Paraphrase of Hodges [11], Lemma 7.4, and of Proposition 3 in [12]) *Let* ϑ *be a formula with* $Fr(\vartheta)=X$, *and* $y\notin X$. *Let* $\mathcal{A}$ *be a suitable model and* $T\subseteq A^{X\cup\{y\}}$. *Define for each* $b\in A$ *the set* $T_b=\{u\in A^X \mid u_{y:b}\in T\}$. *Then the following two are equivalent:*

1. $\mathcal{A}\models^+\vartheta[T]$
2. *For each* b *either* $T_b=\varnothing$ *or* $\mathcal{A}\models^+\vartheta[T_b]$.

Proof. of $(1\Rightarrow 2)$: Assume $\mathcal{A}\models^+\vartheta[T]$. Then for each $b\in A$: $\mathcal{A}\models^+\vartheta[T_b\times\{y\colon b\}]$ by downward monotonicity (Lemma 4.6); this holds whether $T_b=\varnothing$ or not. Therefore, $\mathcal{A}\models^+\vartheta[T_b]$ by the Cartesian Product theorem (Thm 5.1).

Counterexample showing $(2\not\Rightarrow 1)$: Let $\vartheta=\exists y[(y=x)\wedge\exists u_{/xy}[u=y]]$ and choose $T=\{xy\colon 00,11\}$. Then $T_0=\{x\colon 0\}$ and $T_1=\{x\colon 1\}$. Now $\mathcal{B}\models^+\vartheta[T_0]$ with strategy $\{y{:=}0, u{:=}0\}$, and $\mathcal{B}\models^+\vartheta[T_1]$ with strategy $\{y{:=}1, u{:=}1\}$. Hodges' theorem predicts that $\mathcal{B}\models^+\vartheta[T]$. However,

$$\mathcal{B}\not\models^+\exists y[y=x\wedge\exists u_{/xy}[u=y]][\{xy\colon 00,11\}],$$

because a winning strategy for ∃loise would oblige her to take $y{:=}x$ with the consequence that $\mathcal{B}\models^+\exists u_{/xy}[u=y][\{xy\colon 00,11\}]$. That is not possible. ∎

6 Equivalence

One of the aims of this paper is to examine the validity in *IF**-logic of analogues of classical equivalences regarding quantifiers. In order to express such laws, we need a notion of equivalence of formulas. A natural one is Game equivalence, shortly G-equivalence, introduced by Caicedo & Krynicki [3], p. 24.

Definition 6.1 (G-equivalence). *Two IF*-formulas* φ *and* ψ *are called* ***G-equivalent***, *relation denoted as* $\boldsymbol{\varphi\equiv_G\psi}$, *if for any model* $\mathcal{A}$ *and any set of valuations* V *suitable for* φ *and* ψ*:*

$$\mathcal{A}\models^+\varphi[V]\Longleftrightarrow\mathcal{A}\models^+\psi[V]\quad\text{and}\quad\mathcal{A}\models^-\varphi[V]\Longleftrightarrow\mathcal{A}\models^-\psi[V].$$

Note that G-equivalence amounts to 'having the same meaning' in the sense of remark 4.4.

In the literature (e.g. [8], [17]) one also finds another equivalence notion that only makes reference to $\mathcal{A}\models^+$. This is clearly a weaker notion as shown by the following example.

Example 6.2. Consider $\forall x\exists y_{/x}[y=x]$ and $\exists y\forall x[x=y]$. Both are true in models with only one element, and not true in models with more elements. Hence they are equivalent for positive satisfaction. On the other hand, we have $\mathcal{B}\models^-\exists y\forall x[x=y]$, because ∀belard choosing x distinct from y is a winning strategy, whereas $\mathcal{B}\not\models^-\forall x\exists y_{/x}[y=x]$. Therefore $\forall x\exists y_{/x}[y=x]\not\equiv_G\exists y\forall x[x=y]$.

The above example also shows that equivalence with respect to positive satisfaction is not preserved by negations, since we have $\mathcal{B}\not\models^+\neg\forall x\exists y_{/x}[y=x]$ but $\mathcal{B}\models^+\neg\exists y\forall x[y=x]$. G-equivalence, on the contrary, is clearly preserved under interchange of players, which corresponds to negation in our semantics. In fact, our approach to *IF**-logic is in all respects symmetric with respect to the two players, and thus, results on G-equivalences are more informative.

Hodges [11] also considers a symmetric notion of equivalence (*op.cit.* Section 8), similar to the notion of G-equivalence above. But there is a subtle difference. Hodges conceives formulas as having a fixed set of variables (maybe including non occurring ones) and valuations have that set as domain (as does Väänänen [17]). For equivalence of formulas given as $\varphi(x_1,\dots,x_n)$ and $\psi(y_1,\dots,y_m)$, this amounts to restricting our quantification "for any V *suitable* for φ and ψ" (meaning $Fr(\varphi,\psi)\subseteq dom(V)$) to "for any $V\subseteq A^{\{x_1,\dots,x_n,y_1,\dots,y_m\}}$". That this makes the latter notion of equivalence weaker, is shown by the following example:

Example 6.3. Consider the formulas $\varphi=\exists x\exists y_{/u}[t=x\wedge u=y]$ and $\psi=\exists z\exists y_{/u}[t=z\wedge u=y]$ (the result of renaming the bound variable x in φ into z). It is easy to verify that for any suitable model $\mathcal{A}$ and any $V\subseteq A^{\{u,t\}}$:

$$\mathcal{A}\models^+\varphi(u,t)[V]\Leftrightarrow\{\langle v(t),v(u)\rangle\mid v\in V\}\text{ is a function}\Leftrightarrow\mathcal{A}\models^+\psi(u,t)[V]$$

and

$$\mathcal{A}\models^-\varphi(u,t)[V]\iff V=\emptyset\iff\mathcal{A}\models^-\psi(u,t)[V]$$

so $\varphi(u,t)$ and $\psi(u,t)$ are equivalent in Hodges' sense. But φ and ψ are not G-equivalent: ∃loise can be seen to have a winning strategy for φ proving

$$\mathcal{B}\models^+\exists x\exists y_{/u}[t=x\wedge u=y][\{utz\colon 010,111\}],$$

because, for the choice of y, she can use the value of z to signal the value of u. But she can't use this trick when playing ψ, because in the first move she has to overwrite the original value of z and make it equal to t. So:

$$\mathcal{B}\not\models^+\exists z\exists y_{/u}[t=z\wedge u=y][\{utz\colon 010,111\}],$$

Note that this shows that $\varphi(u,t,z)$ and $\psi(u,t,z)$ are *not* equivalent in Hodges' sense. It follows that the claim in [11, preceding Thm 7.5] that the meaning of a formula with respect to its free variables determines its meaning with respect to any extended list of variables, is problematic if the new variables are allowed to be bound within the formula (cf. Thms 5.1, 5.2 and 5.3).

We give some basic facts concerning negation and a substitution rule that states that subformulas with the same meaningf can be substituted for each other without changing the meaning of the formula in which the substitution is performed.

Notation 6.4. *The expression $\vartheta[\varphi{:}\psi]$ will denote the result of replacing in ϑ zero, one or several occurrences of a subformula φ by ψ.*

Theorem 6.5 (Basic rules). *Let φ be an IF*-formula. Then:*

1. *Double negation cancels: $\neg\neg\varphi\equiv_G\varphi$.*
2. *De Morgan's laws hold for connectives and quantifiers:*
   $\neg(\varphi\vee_{/Y}\psi)\equiv_G\neg\varphi\wedge_{/Y}\neg\psi$ *and* $\neg(\varphi\wedge_{/Y}\psi)\equiv_G\neg\varphi\vee_{/Y}\neg\psi$
   $\neg\exists x_{/Y}\psi\equiv_G\forall x_{/Y}\neg\psi$ *and* $\neg\forall x_{/Y}\psi\equiv_G\exists x_{/Y}\neg\psi$.

3. *Substitution of equivalents: if* $\varphi \equiv_G \psi$ *then* $\vartheta \equiv_G \vartheta[\varphi{:}\psi]$.
4. *Negation normal form: for any* φ *there is* ψ *in the symbols* $\vee, \wedge, \neg, \exists, \forall$ *where the negations only affect atomic formulas, such that* $\varphi \equiv_G \psi$.

Proof. For 1. and 2. apply the definition of satisfaction (Def. 4.2) and for $\wedge$ and $\forall$ Def. 2.2. The substitution property follows by a straightforward induction in the complexity of ϑ. The negation normal form is obtained by repeated use of 2. and 3. ∎

Moreover, we have an unexpected result on removing slashes from connectives.

Notation 6.6. *Let* φ *be a formula in which the variable* x *does not occur. Then* $\varphi|_x$ *denotes the formula obtained from* φ *by adding the independence condition* $_{/x}$ *to all connectives and quantifiers of* φ*, excepting possibly those that are unslashed in* φ*.*

Note the ambiguous character of the new notation: we may choose at will to add or not to add $_{/x}$ to the unslashed quantifiers and connectives of φ. In any case we have:

Theorem 6.7. *Let* φ *be a formula where the variable* x *does not occur, then*

$$\varphi_{/x} \equiv_G \varphi|_x.$$

Proof. Since not containing x is a property inherited by subformulas, we may prove this by induction on the complexity of φ. The atomic case as well as the inductive step for $\neg$, and for occurrences of $\exists$ and $\vee$ slashed by $_{/x}$ in $\varphi|_x$ are obvious by substitution of G-equivalents, and from left to right the equivalence follows from Lemma 4.12. Therefore, we verify the inductive step: $\mathcal{A} \models^{\pm} (\varphi|_x)[V] \Rightarrow \mathcal{A} \models^{\pm} (\varphi_{/x})[V]$, for a suitable model $\mathcal{A}$ and set of valuations V, when φ is $\varphi_1 \vee \varphi_2$ and $\varphi|_x$ is $\varphi_1|_x \vee \varphi_2|_x$, or when φ is $\exists y\varphi$ and $\varphi|_x$ is $\exists y[\varphi|_x]$.

($\vee, +$) Assume $\mathcal{A} \models^{+} (\varphi_1|_x \vee \varphi_2|_x)[V]$. Then $\mathcal{A} \models^{+} \varphi_i|_x[V_i]$, $i = 1, 2$, where $\{V_1, V_2\}$ is a cover of V. By induction hypothesis, $\mathcal{A} \models^{+} \varphi_{i/x}[V_i]$, and then $\mathcal{A} \models^{+} \varphi_i[(V_i)_{-x}]$ by Lemma 5.5. Define now $V_i' = \{v \in V \mid v_{-x} \in (V_i)_{-x}\}$, then V_i' is clearly x-saturated in V and $V_i \subseteq V_i'$, which shows $\{V_1', V_2'\}$ is a cover of V. Moreover, $(V_i')_{-x} = (V_i)_{-x}$ by definition. Therefore, $\mathcal{A} \models^{+} \varphi_i[(V_i')_{-x}]$ and by Lemma 5.5 again: $\mathcal{A} \models^{+} \varphi_{i/x}[V_i']$. We may conclude then that $\mathcal{A} \models^{+} (\varphi_{1/x} \vee_{/x} \varphi_{2/x})[V]$.

($\vee, -$) Now, $\mathcal{A} \models^{-} (\varphi_1|_x \vee \varphi_2|_x)[V]$ means $\mathcal{A} \models^{-} \varphi_i|_x[V]$, $i = 1, 2$, which by induction hypothesis is the same as $\mathcal{A} \models^{-} \varphi_{i/x}[V]$, $i = 1, 2$, in turn equivalent to $\mathcal{A} \models^{-} (\varphi_{1/x} \vee_{/x} \varphi_{2/x})[V]$.

($\exists, +$) Assume $\mathcal{A} \models^{+} \exists y(\varphi|_x)[V]$, then there is $f\colon V \to A$ such that $\mathcal{A} \models^{+} \varphi|_x[V_{y:f}]$, so by induction hypothesis $\mathcal{A} \models^{+} \varphi_{/x}[V_{y:f}]$. By Lemma 5.5 and the fact that $(V_{y:f})_{-x} = (V_{-x})_{y:f}$, since y is necessarily distinct from x, we have $\mathcal{A} \models^{+} \varphi[(V_{-x})_{y:f}]$. Then $\mathcal{A} \models^{+} \exists y\varphi[V_{-x}]$ and thus $\mathcal{A} \models^{+} (\exists y\varphi)_{/x}[V]$ by Lemma 5.5 again.

($\exists, -$) $\mathcal{A} \models^{-} \exists y\varphi|_x[V]$ means $\mathcal{A} \models^{-} \varphi|_x[V_{y:A}]$ and thus by induction hypothesis $\mathcal{A} \models^{-} \varphi_{/x}[V_{y:A}]$ which implies $\mathcal{A} \models^{-} \exists y_{/x}\varphi_{/x}[V]$, that is, $\mathcal{A} \models^{-} (\exists y\varphi)_{/x}[V]$. ∎

The above equivalence fails if we do not add $_{/x}$ to all slashed connectives and quantifiers of φ. For example, let φ be $\exists u_{/z}[u = z \vee_{/z} u \neq z]$, then $\mathcal{B} \not\models^{+} \exists u_{/zx}[u = z \vee_{/zx} u \neq z]\{zx\colon 00, 11\}$ because ∃loise must choose u constant and there is no way of knowing at $\vee_{/zx}$ whether z equals that constant or not, but $\mathcal{B} \models^{+} \exists u_{/zx}[u = z \vee_{/z} u \neq z]\{zx\colon 00, 11\}$, by the strategy: $u := 0$, *if* $x = 0$ *then* L *else* R. Similarly, $\mathcal{B} \models^{+} \exists u/_z[u = z \vee_{/zx} u \neq z]\{zx\colon 00, 11\}$ by the strategy: *first* $u := x$, *then* L.

However, the analogue of most classical laws for connectives and quantifiers do not hold in full generality for G-equivalence. A result that one might expect is that under certain conditions renaming of bound variables is allowed, as claimed in (Caicedo & Krynicki [3], Lemma 3.1(a)):

Quote 6.8. *Let φ be an IF*-formula and z a variable that does not occur in $\exists x_{/Y}\varphi(x)$. Then: $\exists x_{/Y}\varphi(x) \equiv_G \exists z_{/Y}\varphi(z)$.*

Surprisingly, this is, not the case. There are two types of counterexamples.

Example 6.9 (First type: renaming blocks signals from outside).
In Ex. 6.3 we saw that

$$\exists x \exists y_{/u}[t = x \wedge u = y] \not\equiv_G \exists z \exists y_{/u}[t = z \wedge u = y].$$

This non-equivalence is reflected in the games for the following two sentences. Consider (in an arbitrary model):

(3) $\forall z \forall u \forall t[u \neq z \vee \exists x \exists y_{/u}[t = x \wedge u = y]]$.

A winning strategy for $\exists$loise is to choose *if* $u \neq z$ *then* L, *if* $u = z$ *then* R, and next to choose $x := t$ and $y := z$; with the effect that $y = z$. Let now x be renamed into z. According to the quote given above and substitution of G-equivalents, this should be G-equivalent with:

(4) $\forall z \forall u \forall t[u \neq z \vee \exists z \exists y_{/u}[t = z \wedge u = y]]$.

However, in models with at least two elements $\exists$loise has no winning strategy for (4). The strategy that was winning for (3) no longer works, because the value of z is now the latest value chosen for z. So, (3) and (4) are not equivalent.

Example 6.10 (Second type: new signals can be created). Consider again the sentence (4), and change the z's bound by the outermost $\forall z$ into w, thus obtaining:

(5) $\forall w \forall u \forall t[u \neq w \vee \exists z \exists y_{/u}[t = z \wedge u = y]]$

Now (5) is true because the value of u can be signalled to $\exists y_{/u}$: the strategy $y := w$ is always winning, whereas (4) is not true in models with at least two elements. This example shows also that the above renaming law even fails for sentences.

We have seen in the previous examples, and other examples will follow, that several classical laws do not hold for G-equivalence of *open* formulas, either due to the blocking of signals from outside, or due to the creation of new signalling possibilities. These examples show the rather natural notion of G-equivalence of Def. 6.1 to be more tricky than one would expect: it contains a quantification over all suitable sets of valuations V, i.e. implicitly over all possible domains containing the free variables of the formulas. However, we will introduce a family of equivalence relations, which are weaker in the sense that they express equivalence only with respect to domains that avoid certain variables, and show that those laws hold for the weaker equivalences. In this way, many classical laws will be recovered for sentences, because for them the new relations will all be as strong as G-equivalence.

Definition 6.11. *Let Z be a set of variables.*

(Z-closed) *An IF*-formula φ is said to be Z-closed if $Fr(\varphi) \cap Z = \varnothing$.*

(Z-equivalence) $\varphi \equiv_Z \psi$ *if and only if both formulas are Z-closed and for any model $\mathcal{A}$ and set of valuations V suitable for φ and ψ, with $dom(V) \cap Z = \varnothing$, we have: $\mathcal{A} \models^{\pm} \varphi[V] \Longleftrightarrow \mathcal{A} \models^{\pm} \psi[V]$.*

Remarks. The requirement that both formulas must be Z-closed, ensures that there are V suitable for φ and ψ (without this requirement any non-Z-closed φ and ψ would be trivially Z-equivalent. It should be evident that $\equiv_Z$ is an equivalence relation in the class of Z-closed formulas. If $Y \subseteq Z$, the class of Z-closed formulas is contained in the class of Y-closed formulas, and for any pair φ, ψ of Z-closed formulas:

$$\varphi \equiv_Y \psi \Longrightarrow \varphi \equiv_Z \psi.$$

Moreover, sentences are Z-closed for any Z, and for them $\equiv_Z$ coincides with $\equiv_G$ due to Thm 5.2.

In the following, we will write x-closed, xy-closed, etc. for $\{x\}$-closed, $\{x, y\}$-closed, respectively. Likewise, we write $\varphi \equiv_x \psi$, $\varphi \equiv_{xy} \psi$ instead of $\varphi \equiv_{\{x\}} \psi$, $\varphi \equiv_{\{x,y\}} \psi$.

We may state now two correct renaming laws with respect to these restricted equivalences.

Theorem 6.12 (Renaming bound variables, I). *Let z be a variable not occurring in $Qx_{/Y}\varphi(x)$. If x does not occur bound in $\varphi(x)$ nor in Y then:*

$$Qx_{/Y}\varphi(x) \equiv_{xz} Qz_{/Y}\varphi(z).$$

Proof. Both formulas are xz-closed by hypothesis and construction; that is, x and z are not among their free variables. It is enough to consider the existential quantifier. Let V be suitable for $\exists x_{/Y}\varphi(x)$ and $\exists z_{/Y}\varphi(z)$ in a suitable model $\mathcal{A}$, with $x, z \notin dom(V)$. Due to the last condition on x and z for any function $f\colon V \to A$ we have $(V_{x:f})_{[x/z]} = V_{z:f}$ (if we had $x \in dom(V)$ or $z \in dom(V)$ these equations would be incorrect). Now, $\mathcal{A} \models^+ \exists x_{/Y}\varphi(x)[V]$ iff $\mathcal{A} \models^+ \varphi(x)[V_{x:f}]$ for some Y-independent $f\colon V \to A$. This is equivalent to $\mathcal{A} \models^+ \varphi(z)[(V_{x:f})_{[x/z]}]$ by Lemma 4.13, in turn equivalent to $\mathcal{A} \models^+ \varphi(z)[V_{z:f}]$ by the above observation, and thus equivalent to $\mathcal{A} \models^+ \exists z_{/Y}\varphi(z)[V]$. Negative satisfaction is handled similarly, using that $(V_{x:A})_{[x/z]} = V_{z:A}$. The case of the universal quantifier follows from its definition. ∎

The following result gives a stronger renaming theorem which puts minimal restrictions on the domain of the valuations and on the variable x, but it may introduce new free occurrences of the variable x in the resulting formula.

Theorem 6.13 (Renaming bound variables, II). *Let z be a variable not occurring in $Qx_{/Y}\varphi(x)$ and distinct from x. If x does not occur bound in $\varphi(x)$ then*

$$Qx_{/Y}\varphi(x) \equiv_z Qz_{/Y}[\varphi(z)|_x].$$

Notice that x may belong to Y.

Proof. The first formula is z-closed by hypothesis, and the second may acquire a new free variable x, but remains z-closed because x is distinct from z. Let V be suitable for $\exists x_{/Y}\varphi(x)$ and $\exists z_{/Y}\varphi(z)_{/x}$ in a suitable model $\mathcal{A}$, with $z \notin dom(V)$. By the last condition we have $(V_{x:f})_{[x/z]} = (V_{z;f})_{-x}$, for any function $f\colon V \to A$. Then, $\mathcal{A} \models^+ \exists x_{/Y}\varphi(x)[V]$ iff $\mathcal{A} \models^+ \varphi(x)[V_{x:f}]$ for some Y-independent $f\colon V \to A$. This is equivalent to $\mathcal{A} \models^+ \varphi(z)[(V_{x:f})_{[x/z]}]$ by

Lemma 4.13; that is, $\mathcal{A}\models^+\varphi(z)[(V_{z;f})_{-x}]$ by the above observation. This is equivalent in turn to $\mathcal{A}\models^+\varphi(z)_{/x}[V_{z;f}]$ by Lemma 5.5, which means $\mathcal{A}\models^+\exists z_{/Y}\varphi(z)_{/x}[V]$.

Similarly, $\mathcal{A}\models^-\exists x_{/Y}\varphi(x)[V]$ iff $\mathcal{A}\models^-\varphi(x)[V_{x:A}]$ iff $\mathcal{A}\models^-\varphi(z)[(V_{x:A})_{[x/z]}]$ iff $\mathcal{A}\models^-\varphi(z)[(V_{z:A})_{-x}]$ iff $\mathcal{A}\models^-\varphi(z)_{/x}[V_{z:A}]$ iff $\mathcal{A}\models^-\exists z_{/Y}\varphi(z)_{/x}[V]$.

Finally, note that by the hypothesis x does not occur in $\varphi(z)$. Therefore, we may apply Theorem 6.7 and substitution of G-equivalents to change $\varphi(z)_{/x}$ into $\varphi(z)|_x$. ∎

The condition that x is not bound in $\varphi(x)$ is needed in both renaming theorems. If we rename the outermost $\forall x$ in the formula $\forall x[u\neq x\vee\exists x\exists y_{/u}[t=x\wedge u=y]]$ according to Thm 6.12, the resulting formula $\forall z[u\neq z\vee\exists x\exists y_{/u}[t=x\wedge u=y]]$ is not xz-equivalent to the former, since ∃loise does not have a winning strategy for the first with respect to the model $\mathcal{B}$ and the set of valuation $V=\{0,1\}^{\{u,t\}}$, but she has one for the second formula: choose right in the disjunction if $u=z$, then choose $x:=t$ and $y:=z$. Likewise, if we rename $\forall x$ according to Thm 6.13, the resulting formula: $\forall z[u\neq z\vee_{/x}\exists x_{/x}\exists y_{/ux}[t=x\wedge_{/x}u=y]]$, is not z-equivalent to the first, because ∃loise does not have a winning strategy for the first with respect to $\mathcal{B}$ and $V=\{0,1\}^{\{u,t,x\}}$, but the strategy described for the second formula above is also winning for the third.

Examples. In contrast with 6.9, Renaming I (i.e. Thm 6.12) shows that

(6) $\exists x\exists y_{/u}[t=x\wedge u=y]\equiv_{xz}\exists z\exists y_{/u}[t=z\wedge u=y]$,

and Renaming II (i.e. Thm 6.13) yields

$$\exists x\exists y_{/u}[t=x\wedge u=y]\equiv_z\exists z\exists y_{/ux}[t=z\wedge_{/x}u=y]\equiv_z\exists z\exists y_{/ux}[t=z\wedge u=y].$$

Since Renaming II permits x to belong to Y, we have also

$$\exists x_{/x}\exists y_{/u}[t=x\wedge u=y]\equiv_z\exists z_{/x}\exists y_{/ux}[t=z\wedge u=y],$$

showing the way to eliminate self-slashed quantifiers.

The substitution of Z-equivalent subformulas in Z-closed formulas does not always yield Z-equivalent formulas. For example, using the equivalence from (6) in the sentence (3) of Example 6.9, yields the non G-equivalent sentence (4); hence, these two sentences are not xz-equivalent. The only obstacle to safe substitution in this example is the presence of the outermost quantifier $\forall z$ in (3) because, in the inductive definition of satisfaction, it forces evaluating $\exists x\exists y_{/u}[t=x\wedge u=y]$ at a set of valuations containing z in its domain, for which the equivalence is not granted.

In this line of thought, it should be clear that the substitution principle holds for Z-equivalence if the subformula ψ to be substituted in φ is not under the scope of any quantifier binding a variable appearing in Z (in that case, if we do the inductive evaluation of φ with a set of valuations V such that $dom(V)\cap Z=\emptyset$, it is assured that $(dom(V)\cup Fr_\varphi(\psi))\cap Z=\emptyset$).

Theorem 6.14 (substitution of Z-equivalents). *Let ϑ, φ and ψ be Z-closed formulas, where φ is a subformula of ϑ not under the scope of any quantifier Qz with z in Z. Then $\vartheta[\varphi{:}\psi]$ is Z-closed and*

$$\varphi\equiv_Z\psi\quad\Longrightarrow\quad\vartheta\equiv_Z\vartheta[\varphi:\psi].$$

Proof. Notice first that $\vartheta[\varphi{:}\psi]$ inherits its Z-closedness from ϑ and ψ since $Fr(\vartheta[\varphi{:}\psi]) \subseteq Fr(\vartheta) \cup Fr(\psi)$. Assume $\varphi \equiv_Z \psi$. Then we show by induction in the complexity of ϑ that whenever φ is not under the scope of a Z-quantifier in ϑ, and $\mathcal{A}$ and V are suitable for ϑ and $\vartheta[\varphi{:}\psi]$, with $Z \cap dom(V) = \varnothing$:

$$\mathcal{A} \models^{\pm} \vartheta[V] \Longleftrightarrow \mathcal{A} \models^{\pm} \vartheta[\varphi:\psi][V].$$

The atomic case is easy: if there is no substitution there is nothing to prove, if there is actual substitution then ϑ is φ and $\vartheta[\varphi{:}\psi]$ is ψ. The inductive step for negation is immediate, we verify the remaining steps.

Let ϑ be $\vartheta_1 \vee_{/Y} \vartheta_2$, then φ is not under the scope of any Z-quantifier in ϑ_i (in case is subformula of ϑ_i) because $Bd(\vartheta_i) \subseteq Bd(\vartheta)$, and each ϑ_i is Z-closed because $Fr(\vartheta_i) \subseteq Fr(\vartheta)$. Moreover, any $V_i \subseteq V$ inherits from V suitability for ϑ_i, $\vartheta_i[\varphi{:}\psi]$ and the property $Z \cap dom(V_i) = \varnothing$. Hence, by induction hypothesis: $\mathcal{A} \models^{\pm} \vartheta_i[V_i] \Longleftrightarrow \mathcal{A} \models^{\pm} \vartheta_i[\varphi{:}\psi][V_i]$. Choosing appropriately the V_i, $\mathcal{A} \models^{\pm} (\vartheta_1 \vee_{/Y} \vartheta_2)[V] \Longleftrightarrow \mathcal{A} \models^{\pm} (\vartheta_1 \vee_{/Y} \vartheta_2)[\varphi{:}\psi][V]$ follows from the definition of positive and negative satisfaction for $\vee_{/Y}$.

Let ϑ be $\exists x_{/Y} \sigma(x)$. If φ does not occur as subformula of $\sigma(x)$ there is nothing to prove. Otherwise, $\sigma(x)$ can not be under the scope of a Z-quantifier in ϑ by hypothesis and thus $x \notin Z$. Therefore, $\sigma(x)$ inherits Z-closedness from ϑ, and also $Z \cap dom(V_x) = \varnothing$ for any x-variant or expansion V_x of V. Thus, $\mathcal{A} \models^{\pm} \sigma[V_x] \Longleftrightarrow \mathcal{A} \models^{\pm} \sigma[\varphi{:}\psi][V_x]$ by induction hypothesis. Choosing V_x appropriately as $V_{x:f}$ or $V_{x:A}$ we conclude that $\mathcal{A} \models^{\pm} \exists x_{/Y} \sigma[V]$ iff $\mathcal{A} \models^{\pm} \exists x_{/Y} \sigma[\varphi{:}\psi][V]$. ∎

Example 6.15. Renaming z as x, according to Renaming II, gives:

$$\exists z \exists y_{/u}[t = z \wedge u = y] \equiv_x \exists x \exists y_{/uz}[t = x \wedge u = y]$$

which may be substituted in (4) of Example 6.9 to yield

$$\forall z \forall u \forall t[u \neq z \vee \exists z \exists y_{/u}[t = z \wedge u = y]] \equiv_G \forall z \forall u \forall t[u \neq x \vee \exists x \exists y_{/uz}[t = z \wedge u = y]].$$

This example shows the way to eliminate nested quantifications of the same variable. A general theorem proving this will be given in Section 9.

All the equivalences we exhibit in this paper are of the form $\varphi \equiv_Z \psi$ with $Z \subseteq Bd(\varphi, \psi)$. The following lemma shows that it is enough to consider this case.

Lemma 6.16. *For any Z-closed φ and ψ we have:*

$$\varphi \equiv_Z \psi \Longleftrightarrow \varphi \equiv_{Z \cap Bd(\varphi,\psi)} \psi.$$

Proof. The implication from right to left is trivial. For the other direction, suppose that $\varphi \equiv_Z \psi$ and $\varphi \not\equiv_{Z \cap Bd(\varphi,\psi)} \psi$. The last inequivalence implies that there are $\mathcal{A}$ and V suitable for φ and ψ such that $dom(V) \cap Z \cap Bd(\varphi, \psi) = \varnothing$, and say: $\mathcal{A} \models^{+} \varphi[V]$, $\mathcal{A} \not\models^{+} \psi[V]$. Let $dom(V) \cap Z = \{z_1, .., z_n\}$, then $z_i \notin Bd(\varphi, \psi)$ and by hypothesis $z_1, .., z_n$ are not free in φ, ψ, thus they do not occur in φ, ψ. Let $w_1, .., w_n$ be distinct new variables not in $Z \cup dom(V)$ nor in φ, ψ. Then by Lemma 4.13, $\mathcal{A} \models^{\pm} \varphi[V] \Longleftrightarrow \mathcal{A} \models^{\pm} \varphi[V_{[z_1/w_1]...[z_n/w_n]}]$ and $\mathcal{A} \models^{\pm} \psi[V_{[z_1/w_1]...[z_n/w_n]}] \Longleftrightarrow \mathcal{A} \models^{\pm} \psi[V]$. Since $dom(V_{[z_1/w_1]...[z_n/w_n]}) \cap Z = \varnothing$ by construction and $\varphi \equiv_Z \psi$ by hypothesis, then: $\mathcal{A} \models^{\pm} \varphi[V_{[z_1/w_1]...[z_n/w_n]}] \Longleftrightarrow \mathcal{A} \models^{\pm} \psi[V_{[z_1/w_1]...[z_n/w_n]}]$, hence, $\mathcal{A} \models^{\pm} \varphi[V] \Longleftrightarrow \mathcal{A} \models^{\pm} \psi[V]$, a contradiction. ∎

In Dechesne [4] the relation $\varphi \equiv_{Bd(\varphi,\psi)} \psi$ was introduced under the name *safe equivalence,* and denoted $\varphi \equiv_S \psi$. The above lemma implies that for any pair of $Bd(\varphi,\psi)$-closed formulas φ and ψ, and any Z for which they are also Z-closed: $\varphi \equiv_Z \psi$ implies that $\varphi \equiv_S \psi$. Therefore, all the results in this paper hold for safe equivalence.

7 Quantifier extraction

In order to obtain a prenex form theorem we need a theorem that allows to shift quantifiers to the front of a formula. For classical logic this goes by quantifier extraction rules like $Qx[\varphi] \vee \psi \equiv Qx[\varphi \vee \psi]$, with the condition that x does not occur free in ψ. A generalization to IF^* has to take care of the possibility that slashed quantifiers in ψ receive signals from Qx. We consider the generalization as proposed by Caicedo & Krynicki [3], which uses the notation introduced in 6.6 unambiguously as indicating the result of adding $_{/x}$ to the quantifiers but not to the unslashed connectives.

Quote 7.1 ([3], p. 26) *If x does not occur free in φ then* $Qx_{/Y}[\varphi] \vee \psi \equiv_G Qx_{/Y}[\varphi \vee \psi|_x]$.

After our observations concerning renaming variables, one may become suspicious about the just mentioned version of quantifier extraction. Could the extracted quantifier not block signals coming from outside to ψ? Couldn't the extracted quantifier give rise to new signalling possibilities? The next examples show that these phenomena indeed arise.

Example 7.2 (Extracted quantifier may block outside signals).
We have:

(7) $\mathcal{B} \models^+ \forall z \forall x[x \neq z \vee (\forall x[x \neq x] \vee \exists y_{/z}\, y = z)]$.

because ∃loise has a winning strategy: at the first disjunction she chooses L if $x \neq z$, and R otherwise. At the second disjunction she plays R and then $y := x$. Since $x = z$ it follows that $y = z$. However, after quantifier extraction according to the proposal mentioned above we have:

(8) $\mathcal{B} \not\models^+ \forall z \forall x[x \neq z \vee \forall x[x \neq x \vee \exists y_{/zx}\, y = z]]$.

The strategy given for (7), does not work for (8) because the value of the outermost x is not available at $\exists y_{/zx}$. The only strategy allowed for $\exists y_{/zx}$ is a constant strategy, and in this case no such strategy can be winning. For a proof that ∃loise has no winning strategy see [13]. In sum,

$$\forall x[x \neq x] \vee \exists y_{/z}\, y = z \not\equiv_G \forall x[x \neq x \vee \exists y_{/zx}\, y = z].$$

Example 7.3 (Extracting a quantifier may produce inside signals).
We have

(9) $\mathcal{B} \not\models^+ \forall z[\forall x[x \neq z] \vee \exists u_{/z}[u = z \vee_{/z} u \neq z]]$.

because for $\exists u_{/z}$ and $\vee_{/z}$ ∃loise can only follow constant strategies, so she either always ends with the subformula $u = z$ or always with $u \neq z$. But with a constant choice for u either of them can turn out to be false, depending on the play of ∀belard. So she has no winning strategy. After application of the rule in Quote 7.1 ∃loise has a winning strategy:

(10) $\mathcal{B} \models^+ \forall z \forall x[x \neq z \vee \exists u_{/zx}[u = z \vee_{/z} u \neq z]]$.

Her strategy is to choose at the disjunction L if $x{\neq}z$, and R if $x{=}z$. For $\exists u_{/zx}$ she chooses 0, and at $\vee_{/z}$ she chooses L if $x{=}0$ (there also $u{=}z$), and R if $x{\neq}0$ (then $u{\neq}z$). In sum,

$$\forall z \forall x[x{\neq}z] \vee \exists u_{/z}[u{=}z \vee_{/z} u{\neq}z] \not\equiv_G \forall z \forall x[x{\neq}z \vee \exists u_{/zx}[u{=}z \vee_{/z} u{\neq}z]].$$

Note that this example is not about embedding of quantifiers, but about information flow at disjunctions. These need to be studied carefully. The problem from Example 7.2 (blocking outside signals) can be avoided by restricting the equivalence to $\equiv_x$, whereas Example 7.3 suggests that all disjunctions that come under the scope of the extracted quantifier have to be slashed. Indeed, in that way a formula is obtained that is equivalent with (9). This will follow from Theorem 7.5, but we may verify directly that:

(11) $\mathcal{B} \not\models^+ \forall z \forall x[x{\neq}z \vee_{/x} \exists u_{/zx}[u{=}z \vee_{/zx} u{\neq}z]]$.

However adding a slash to just one of the disjuncts might be sufficient as well because (12) and (13) are, just as (9), not true in $\mathcal{B}$:

(12) $\mathcal{B} \not\models^+ \forall z \forall x[x{\neq}z \vee_{/x} \exists u_{/zx}[u{=}z \vee_{/z} u{\neq}z]]$.
(13) $\mathcal{B} \not\models^+ \forall z \forall x[x{\neq}z \vee \exists u_{/zx}[u{=}z \vee_{/zx} u{\neq}z]]$.

In particular (13) is attractive because no new slashes are introduced in the main disjunction. We tried to find counterexamples to Quote 7.1 resembling Ex. 7.3 in which it was necessary to slash the main connective after extracting the quantifier, but we did not succeed. This is due to a surprising result that will be explained in the next section (Theorem 8.3)

The formulation of Quote 7.1 only deals with situations where the original formula does not have slashed connectives because the authors first apply a theorem that removes slashed connectives. However, we give here a more general quantifier extraction theorem which gives us more insights on *IF**-logic and will allow for more general prenex forms. In the proof of we will use the following observation that follows easily from the definition of saturation.

Lemma 7.4. *Let X be a set of variables and $x \notin X$. Then the following two properties hold:*

1. *For any $v, w \in A^X$ and respective x-expansions v_x, w_x:* $v \sim_Y w \iff v_x \sim_{Yx} w_x$.
2. *Let $W \subseteq V \subseteq A^X$, V_x a x-expansion of V, and $W_x = \{v_x \in V_x \mid v \in W\}$. Then: W is Y-saturated in $V \iff W_x$ is Yx-saturated in V_x.*

Theorem 7.5 (Quantifier extraction over connectives, I). *If x does not occur in ψ nor in $Y \cup Z$, then*

$$Qx_{/Y}[\varphi] \vee_{/Z} \psi \equiv_x Qx_{/Y}[\varphi \vee_{/Zx} \psi_{/x}]$$

and similarly for conjunctions $Qx_{/Y}[\varphi] \wedge_{/Z} \psi$.

Proof. Let $\mathcal{A}$ be a suitable model and $V \subseteq A^X$ such that $x \notin X$. We consider the different cases.

$(\exists,+)$ $(\Rightarrow)$ If $\mathcal{A} \models^+ (\exists x_{/Y}[\varphi] \vee_{/Z} \psi)[V]$ then $\mathcal{A} \models^+ \varphi[(V_1)_{x:f}]$ and $\mathcal{A} \models^+ \psi[V_2]$ for some Z-saturated partition V_1, V_2 of V and some Y-independent function $f\colon V_1 \to A$. Choose a Y-independent extension $h\colon V \to A$ of f. Then $(V_1)_{x:h} = (V_1)_{x:f}$ and the $(V_i)_{x:h}$ form a Zx-saturated cover of $V_{x:h}$ by the previous lemma. Moreover, from $\mathcal{A}\models^+\psi[V_2]$ it follows by Lemma 5.5 that $\mathcal{A}\models^+\psi_{/x}[(V_2)_{x:h}]$, thus, $\mathcal{A}\models^+(\varphi\vee_{/Zx}\psi_{/x})[V_{x:h}]$, and therefore $\mathcal{A}\models^+\exists x_{/Y}[\varphi\vee_{/Zx}\psi_{/x}][V]$.

($\Leftarrow$) If $\mathcal{A}\models^+\exists x_{/Y}[\varphi\vee_{/Zx}\psi_{/x}][V]$, there is a Y-independent $f\colon V\to A$ and a Zx-saturated cover $(V_1)_{x:f},(V_2)_{x:f}$ of $V_{x:f}$ such that $\mathcal{A}\models^+\varphi[(V_1)_{x:f}]$ and $\mathcal{A}\models^+\psi_{/x}[(V_2)_{x:f}]$. Hence, $\mathcal{A}\models^+\exists x_{/Y}\ \varphi[V_1]$, and also $\mathcal{A}\models^+\psi[V_2]$ by Lemma 5.5. Moreover, the V_i form a Z-saturated cover of V by the previous lemma, thus $\mathcal{A}\models^+(\exists x_{/Y}\ [\varphi]\vee_{/Z}\psi_x)[V]$.

($\exists,-$) $\mathcal{A}\models^-(\exists x_{/Y}[\varphi]\vee_{/Z}\psi)[V]\Longleftrightarrow(\mathcal{A}\models^-\exists x_{/Y}\varphi[V]$ and $\mathcal{A}\models^-\psi[V])\Longleftrightarrow(\mathcal{A}\models^-\varphi[V_{x:A}]$ and $\mathcal{A}\models^-\psi_{/x}[V_{x:A}]$ (by Lemma 5.5)) $\Longleftrightarrow\mathcal{A}\models^-(\varphi\vee_{/Zx}\psi_{/x})[V_{x:A}]\Longleftrightarrow\mathcal{A}\models^-\exists x_{/Y}[\varphi\vee_{/Zx}\psi_{/x}][V]$.

($\forall,+$) ($\Rightarrow$) If $\mathcal{A}\models^+(\forall x_{/Y}\varphi\vee_{/Z}\psi)[V]$ then $\mathcal{A}\models^+\forall x_{/Y}\varphi[V_1]$ and $\mathcal{A}\models^+\psi[V_2]$ with V_1,V_2 a Z-saturated cover of V. So, $\mathcal{A}\models^+\varphi[(V_1)_{x:A}]$ and $\mathcal{A}\models^+\psi_{/x}[(V_2)_{x:A}]$ by Lemma 5.5. Clearly, the $(V_i)_{x:A}$ form a Zx-saturated cover of $V_{x:A}$, so $\mathcal{A}\models^+(\varphi\vee_{/Zx}\psi)[V_{x:A}]$. Hence $\mathcal{A}\models^+\forall x_{/Y}[\varphi\vee_{/Zx}\psi][V]$.

($\Leftarrow$) $\mathcal{A}\models^+\forall x_{/Y}[\varphi\vee_{/Zx}\psi_{/x}][V]$ implies $\mathcal{A}\models^+\varphi[W_1]$ and $\mathcal{A}\models^+\psi_{/x}[W_2]$ with W_1,W_2 a Zx-saturated cover of $V_{x:A}$. By Zx-saturation, $W_i=(V_i)_{x:A}$, where $V_i=(W_i)_{-x}$. Therefore, on the one hand: $\mathcal{A}\models^+\varphi[(V_1)_{x:A}]$, which implies $\mathcal{A}\models^+\forall x_{/Y}\varphi[V_1]$, and on the other hand: $\mathcal{A}\models^+\psi_{/x}[(V_2)_{x:A}]$, which implies $\mathcal{A}\models^+\psi[V_2]$ (by Lemma 5.5). Moreover, the V_i inherit Z-saturation from the W_i and cover V. Thus, $\mathcal{A}\models^+(\forall x_{/Y}\ \varphi\vee_{/Z}\ \psi)[V]$.

($\forall,-$) $\mathcal{A}\models^-(\forall x_{/Y}[\varphi]\vee_{/Z}\psi)[V]\Longleftrightarrow(\mathcal{A}\models^-\forall x_{/Y}\ \ \varphi[V]$ and $\mathcal{A}\models^-\psi[V])\Longleftrightarrow(\mathcal{A}\models^-\varphi[V_{x:f}]$ with f independent of Y and $\mathcal{A}\models^-\psi_{/x}[V_{x:f}]$ (by Lemma 5.5)) $\Longleftrightarrow\mathcal{A}\models^-(\varphi\vee_{/Zx}\psi_{/x})[V_{x:f}]\Longleftrightarrow\mathcal{A}\models^-\forall x_{/Y}[\varphi\vee_{/Zx}\psi_{/x}][V]$.

The case of conjunctions follows from De Morgan laws. ∎

8 Omitting slashed variables in connectives under quantifiers

In this section we will prove some results on eliminations of slashed variables in connectives which will lead to a refinement of the quantifier extraction rule. As we saw in the discussion of example 7.3, adding the independence of x at the main disjunction may not be necessary when extracting quantifiers. This is explained by the next lemma.

Lemma 8.1 (Elimination of slash under ∃) *If $Z\subseteq Y$ and x is not in Y then*

$$\exists x_{/Y}[\psi_1\vee_{/Zx}\ \psi_2]\equiv_x\exists x_{/Y}[\psi_1\vee_{/Z}\ \psi_2]$$

In particular, it always holds that $\exists x_{/Y}[\psi_1\vee_{/x}\ \psi_2]\equiv_x\exists x_{/Y}[\psi_1\vee\ \psi_2]$.

In game terms, all the information available when the verifying player chooses x, is still available when this player chooses at the disjunction, meaning she can ‘recalculate’ the value of x if she needs it. We prove the lemma formally:

Proof. Let $\mathcal{A}$ be suitable structure and $V\subseteq A^X$ a set of valuations for the given formulas such that $x\notin X$. It is enough to show the implication from right to left by Lemma 4.12.

(+) $\mathcal{A}\models^+\exists x_{/Y}(\psi_1\vee_{/Z}\ \psi_2)[V]$ implies that $\mathcal{A}\models^+\psi_i[(V_i)_{x:f}]$ holds for some Y-independent $f\colon V\to A$ and Zx-saturated cover $(V_1)_{x:f}$, $(V_2)_{x:f}$ of $V_{x:f}$. The result follows if we notice that the $(V_i)_{x:f}$ are Zx-saturated. This holds because for any $v,w\in V$: $v_{x:f(v)}\sim_{Zx}w_{x:f(w)}$ implies $v\sim_Z w$ (since $x\notin dom(V)$), and thus $v\sim_Y w$ by the assumption that $Z\subseteq Y$. Therefore, $f(v)=f(w)$, and thus $v_{x:f(v)}\sim_Z w_{x:f(w)}$.

(−) Immediate, because the independence conditions for $\exists$ and $\vee$ are not relevant in this case. ∎

This lemma fails for G-equivalence: $\mathcal{B} \models^+ \exists x_{/y}[y=0 \vee_{/y} y \neq 0]\{xy\colon 00, 11\}$, with the strategy: at $\exists x_{/y}$ play $x:=x$, at $\vee_{/y}$ play *if* $x=0$, *then* L *else* R. But $\mathcal{B} \not\models^+ \exists x_{/y}[y=0 \vee_{/yx} y \neq 0]\{xy\colon 00, 11\}$, because at the disjunction there is no way of knowing the value of y.

The analogue of the previous lemma holds for a universal quantifier $\forall x_{/Y}$ under quite a different hypothesis: the 'right disjunct' must be of the form $\psi_{/x}$ where ψ does not contain x and no condition is put on Y and Z.

Lemma 8.2 (Elimination of slash under $\forall$) *If x does not occur in ψ nor in Y, then*

$$\forall x_{/Y}[\varphi(x) \vee_{/Zx} \psi_{/x}] \equiv_x \forall x_{/Y}[\varphi(x) \vee_{/Z} \psi_{/x}].$$

Proof. Let $\mathcal{A}$ be suitable structure and $V \subseteq A^X$ a set of valuations for the given formulas such that $x \notin X$.

(+) We have to prove only that $\mathcal{A} \models^+ (\varphi \vee_{/Z} \psi_{/x})[V_{x:A}]$ implies $\mathcal{A} \models^+ (\varphi \vee_{/Zx} \psi_{/x})[V_{x:A}]$. Assume $\mathcal{A} \models^+ \varphi[W_1]$ and $\mathcal{A} \models^+ \psi_{/x}[W_2]$ with W_1, W_2 a Z-saturated partition of $V_{x:A}$. Let $V_2 := (W_2)_{-x}$, then $\mathcal{A} \models^+ \psi[V_2]$ by Lemma 5.5 (since $x \notin X$), and again by this lemma: $\mathcal{A} \models^+ \psi_{/x}[(V_2)_{x:A}]$. Moreover, $(V_2)_{x:A}$ is Zx-saturated in $V_{x:A}$ because: $v_{x:a} \sim_{Zx} w_{x:b}$ for $v \in V_2$, $w \in V$ implies $v \sim_Z w$ (again because $x \notin X$). But $v_{x:c} \in W_2$ for some c and thus $v_{x:c} \sim_Z w_{x:c}$, which implies $w_{x:c} \in W_2$ since $w_{x:c} \in V_{x:A}$ and W_2 is saturated in $V_{x:A}$; hence, $w \in V_2$. Notice also that $W_2 \subseteq (V_2)_{x:A}$, by construction. Define $W_1' := V_{x:A} \setminus (V_2)_{x:A}$. Then $W_1' \subseteq W_1$ which implies $\mathcal{A} \models^+ \varphi[W_1']$, and W_1' is automatically Zx-saturated in $V_{x:A}$ because $(V_2)_{x:A}$ is. Therefore, $\mathcal{A} \models^+ (\varphi \vee_{/Zx} \psi_{/x})[V_{x:A}]$.

(−) Since in the negative evaluation of a disjunction the variables under the slash do not play any role, we have: $\mathcal{A} \models^-_G (\varphi \vee_{/Zx} \psi_{/x})[V_{x:f}]$ iff $\mathcal{A} \models^-_G (\varphi \vee_{/Z} \psi_{/x})[V_{x:f}]$, for any $f\colon V \to A$. Hence, $\mathcal{A} \models^- \forall x_{/Y}[\varphi \vee_{/Zx} \psi_{/x}][V]$ iff $\mathcal{A} \models^- \forall x_{/Y}[\varphi \vee_{/Z} \psi_{/x}][V]$. ∎

Combining the above results for $Z = \varnothing$ with theorems 7.5 and 6.7, and utilizing substitution of G-equivalents and De Morgan laws, we can now state the following refinement to quantifier extraction; note that the extraction rule for classical logic is a special case.

Theorem 8.3 (Quantifier extraction over connectives II). *If x does not occur in ψ nor in Y, then:*

$$Qx_{/Y}\varphi(x) \vee \psi \equiv_x Qx_{/Y}[\varphi(x) \vee \psi|_x],$$

and similarly for unslashed conjunctions.

9 Regular formulas

Given a set of variables Z, the property of being Z-closed is preserved by all logical operators, but is not inherited by subformulas: $\exists x(x=x)$ is x-closed but $x=x$ is not. The following related property is inherited by subformulas (although it is not preserved by logical operators), and it will be needed in our prenex form theorem.

Definition 9.1 (Regular formulas). *A formula φ is **regular** if the following two conditions hold:*

1. *No variable occurs both bound and free in φ.*
2. *No quantifier for a variable occurs within the scope of another quantifier for the same variable.*

Examples: $\forall x[x=x]\vee\neg\exists x[x\neq x]$ is regular but the following are not regular: $\exists z\exists z[y=z]$ and $x\neq x\vee\exists x[x=x]$, as well as $\exists x_{/x}[x\neq y]$.

For classical logic, Hilbert & Ackermann [7], p. 74, argue that the regular fragment should be taken as the standard version of predicate logic. They imply that irregular formulas would result in unnecessary complications, and that such formulas would not expand the expressive power. Indeed, in classical first order logic a formula with two nested quantifications of the same variable is equivalent with a formula in which one of the two variables is renamed to a fresh variable. As we have seen (Examples 6.9 and 6.10), in *IF** such plain renaming does not yield G-equivalent formulas. However, the example 6.15 suggests a way to regularize *IF**-formulas utilizing Renaming II.

Definition 9.2. *An IF*-formula ϑ' will be called a variant of ϑ if it is obtained from ϑ by renaming some bound variables and (perhaps) introducing some new variables under slashes.*

Theorem 9.3 (Regularization). *For any IF*-formula ϑ there is a regular variant ϑ' with the same free variables as ϑ, such that $\vartheta\equiv_Z\vartheta'$, where $Z=Bd(\vartheta')\setminus Bd(\vartheta)$. Moreover, any set of unslashed connectives and quantifiers of ϑ may be chosen to remain unslashed in ϑ'.*

Proof. Let $Z=\{z_1,\dots,z_n\}$ be a set of distinct variables not occurring in ϑ, one for each subformula of ϑ of the form $Qx_{/Y}\varphi(x)$, which is under the scope of a quantifier $Q'x$ or is part of a subformula where x is free (that is, a counterexample to regularity of ϑ).

Start with a subformula σ of the form described above of minimal length. By minimality, x does not appear bound in $\varphi(x)$, thus $\sigma\equiv_{z_1}Qz_{1/Y}[\varphi(z_1)|_x]$ by Renaming II (Theorem 6.13). Since z_1 does not occur in ϑ, this equivalence yields $\vartheta\equiv_{z_1}\vartheta[\sigma:Qz_{1/Y}[\varphi(z_1)|_x]]$ by z_1-substitution. Notice that the second formula has the same free variables as ϑ: if σ was under the scope of a quantifier $Q'x$, because the latter binds the new slashed occurrences of x in $\varphi(z_1)|_x$. If σ was part of a subformula where x appeared free, because the new slashed occurrences of x in $\varphi(z_1)|_x$ do not increase the set of possible free variables of ϑ.

Applying the same procedure to the formula $\vartheta[\sigma:Qz_{1/Y}[\varphi(z_1)|_x]]$ and continuing in this way, we may rename consecutively all the "irregular" quantified variables of ϑ obtaining a chain $\vartheta\equiv_{z_1}\vartheta_1\equiv_{z_2}\dots\equiv_{z_n}\vartheta_n$, where ϑ_n is regular, has the same free variables that ϑ, and $Bd(\vartheta_n)=Bd(\vartheta)\cup Z$. Since no z_i occurs free in any ϑ_j by construction, we have $\vartheta\equiv_Z\vartheta_1\equiv_Z\dots\equiv_Z\vartheta_n$, and thus $\vartheta\equiv_Z\vartheta_n$ by transitivity.

The last claim of the theorem follows because in $\varphi(z_1)|_x$ we may choose which operators to slash with x among those originally non-slashed, due to Thm 6.7. ∎

A regular formula ϑ may still contain multiple (non nested) quantifications of the same variable, those may be eliminated without adding new free variables utilizing Renaming I (Theorem 6.12):

Theorem 9.4 (Strong regularization). *Any IF*-formula ϑ is Z-equivalent to a regular variant ϑ' with the same free variables as ϑ, in which no variables appears quantified more than once,* *and where $Z \subseteq Bd(\vartheta')$. Moreover, any set of unslashed connectives and quantifiers of ϑ maybe chosen to remain unslashed in ϑ'.*

Proof. By the previous theorem we may assume ϑ is regular, then a subformula $Qx_{/Y}\varphi(x)$ of ϑ can not be under the scope of a quantifier $Q'x$, and moreover $x \notin Y$, thus the equivalence $Qx_{/Y}\varphi(x) \equiv_{xz} Qz_{/Y}\varphi(z)$ holds by Renaming I if z does not occur in ϑ, and it may be safely substituted according to the xz-substitution theorem to yield: $\vartheta \equiv_{xz} \vartheta[Qx_{/Y}\varphi(x) : Qz_{/Y}\varphi(z)]$. Notice that no new free variables are introduced. Applying this as many times as needed to eliminate all repeated bound variables, we obtain $\vartheta \equiv_{x_1...x_n z_1...z_m} \vartheta'$ where ϑ' does not have repeated bound variables, $x_1...x_n$ are the bound variables originally repeated in ϑ, and $z_1...z_m$ are the new renaming variables, $m \geq n$. Finally, note that Renaming I does not introduce new variables under slashes. ∎

From this follows:

Theorem 9.5. *Any IF^*-sentence is G-equivalent to a regular variant without multiple quantifications of the same variable, and with the same unslashed connectives and quantifiers as the original formula.*

Z-equivalence has the substitution property in regular contexts without any further condition. If a subformula of a regular formula is replaced by a Z-equivalent one, and the result is regular, then Z-equivalent formulas are obtained.

Theorem 9.6 (Substitution in regular formulas). *Let ϑ, and $\vartheta[\varphi:\psi]$ be regular formulas. If $\varphi \equiv_Z \psi$ then $\vartheta \equiv_Z \vartheta[\varphi:\psi]$.*

Proof. Let $Z' = Z \cap Bd(\varphi, \psi)$. Assume φ occurs in ϑ and is actually substituted by ψ (otherwise there is nothing to prove), then by regularity of ϑ and $\vartheta[\varphi:\psi]$ these formulas are $Bd(\varphi,\psi)$-closed; hence, Z'-closed. Also by regularity of ϑ and $\vartheta[\varphi:\psi]$, the position of φ in ϑ (the same as the position of ψ in $\vartheta[\varphi:\psi]$) is not under the the scope of quantified variables in $Z' \subseteq Bd(\varphi, \psi)$. Therefore, by Lemma 6.16 and the Z'-substitution theorem 6.14

$$\varphi \equiv_Z \psi \Longrightarrow \varphi \equiv_{Z'} \psi \Longrightarrow \vartheta \equiv_{Z'} \vartheta[\varphi:\psi] \Longrightarrow \vartheta \equiv_Z \vartheta[\varphi:\psi].$$ ∎

10 Prenex and Skolem forms

We have now constructed the building blocks necessary to support a prenex form for IF^*-formulas; one that corrects the corresponding theorem from Caicedo & Krynicki [3].

Theorem 10.1 (Prenex form theorem for *IF^**). *For any IF^*-formula φ there exists a formula φ^P and a set of variables $Z \subseteq Bd(\varphi^P)$ such that φ^P is Z-equivalent to φ, and φ^P is in prenex form with the same free variables, the same number of quantifiers, and the same propositional skeleton as φ.*

Proof. By the Strong Regularization Theorem, (Thm 9.4), for any ϑ there is a regular variant ϑ' such that each variable in ϑ' is quantified over at most once, $Fr(\vartheta) = Fr(\vartheta')$, $Fr(\vartheta) \subseteq Fr(\vartheta')$, and $\vartheta \equiv_{Z'} \vartheta'$ for some $Z' \subseteq Bd(\vartheta')$. By regularity, the hypothesis of Theorem 7.5 applies to any subformula of ϑ' of the form $Qx_{/Y}\varphi(x) \vee_{/Z} \psi$ (that is, x is not in ψ, Y or Z). Applying this theorem to the subformula and using substitution of x-equivalents, which can be applied again by regularity of ϑ, a chain $\vartheta' \equiv_{x_1} ... \equiv_{x_n} \vartheta^P$ is obtained where regularity is maintained, all the formulas have the same free and bound variables, $x_i \in Bd(\vartheta') = Bd(\vartheta^P)$, and ϑ^P is in prenex form. Therefore, $\vartheta \equiv_Z \vartheta^P$ where $Z = \{x_1, .., x_n\} \cup Z' \subseteq Bd(\varphi^P)$. ∎

Theorem 10.2 (Special prenex form theorem for *IF).** *The prenex form may be chosen so that any desired set of unslashed connectives and quantifiers remains unslashed in φ^P.*

Proof. According to Theorem 9.4, the strong regularization φ' may be chosen to preserve any desired set of unslashed connectives and quantifiers, then we may use Theorem 8.3 instead of Theorem 7.5 to extract quantifiers, obtaining φ^P which preserves any desired set of unslashed operators. ∎

Corollary 10.3 (Prenex form for sentences).
Any IF-sentence is G-equivalent with an IF*-sentence in prenex form.*

If we compute the prenex form of a non-prenex classical formula in *IF** according to Theorem 10.2, we may choose to maintain all connectives and quantifiers unslashed, in which case we obtain a classical prenex formula. But we may choose also to maintain the connectives unslashed but apply full slashing to the quantifiers when applying the extraction rules, obtaining thus a non-classical prenex form of a classical formula. The following example shows that then the subsequent Skolemization procedure results in a more economical Skolem form (i.e. where the sum of the arities of the Skolem functions is minimal).

Example 10.4 (*IF-Skolem forms for a classical formula).** Consider the first order sentence:

(14) $\forall x \exists y \forall z \ R(x, y, z) \vee \exists u \forall v \exists w \ Q(u, v, w)$

Classically, there are several ways to put this formula in prenex form, depending on the order in which we extract the quantifiers. If we give the leftmost block of quantifiers the widest scope, we get (15), which yields the Skolem form (16).

(15) $\forall x \exists y \forall z \exists u \forall v \exists w [R(x, y, z) \vee Q(u, v, w)]$
(16) $\forall x \forall z \forall v [R(x, f(x), z) \vee Q(g(x, z), v, h(x, z, v))]$

Giving the rightmost block widest scope first, results in a different Skolem form:

(17) $\forall v \forall z \forall x [R(x, f(v, x), z) \vee Q(c, v, g(v))]$

On the other hand, using the *IF**-prenex form and giving the first block widest scope, we get the following prenex and Skolem forms respectively:

(18) $\forall x \exists y \forall z \exists u_{/xyz} \forall v_{/xyz} \exists w_{/xyz} [R(x, y, z) \vee Q(u, v, w)]$
(19) $\forall x \forall z \forall v [R(x, f(x), z) \vee Q(c, v, g(v))]$

Note that (19) is simpler than both (16) and (17) because the *IF**-prenex procedure does not introduce unnecessary dependencies like the classical prenex procedure. The resulting Skolem form is independent of the order of extraction of quantifiers, up to interchange of universal quantifiers.

11 Vacuous quantifiers

In the next section we will show that a prenex normal form is possible in which no slashed connectives occur. In that process additional quantifiers will be introduced. Such introduction must be done carefully, and in this section we investigate the dangers. For instance, adding a vacuous quantifier (what classically is innocent) may evoke the same signalling phenomena we have encountered with renaming:

1. A vacuous quantifier may block a signal. Consider $\forall x \forall z[x \neq z \vee \exists y_{/x}[y = x]]$. A winning strategy is to play at $\vee$ the strategy *if* $x \neq z$ *then L else R* followed by $y := z$. This strategy, however, cannot be applied after the introduction of a vacuous $\forall z$ quantifier: $\forall x \forall z[x \neq z \vee \forall z \exists y_{/x}[y = x]]$, because then the value of the outermost z is overwritten by that of the innermost. Therefore,

$$\exists y_{/x}[y = x] \not\equiv_G \forall z \exists y_{/x}[y = x].$$

2. A vacuous quantifier may introduce a new signal. This is illustrated by Hodges' example (1), which shows $\exists y_{/x}[y = x] \not\equiv_G \exists z \exists y_{/x}[y = x]$ since there is no winning strategy for $\forall x \exists y_{/x}[y = x]$ in a model with two or more elements but there is always one for $\forall x \exists z \exists y_{/x}[y = x]$.

The first phenomenon is dealt with by restricting the equivalence (in the given example) to $\equiv_z$. To neutralize the new signalling possibilities we see two approaches.

One approach is to make the new variable unusable by slashing with that variable all later choices of the formula. In fact, we do not have to slash all of them but only those already slashed:

Theorem 11.1 (Safely adding vacuous quantifiers I). *Let x be a variable that does not occur in φ or Y. Then:*

$$\varphi \equiv_x \exists x_{/Y}[\varphi|_x] \quad \text{and} \quad \varphi \equiv_x \forall x_{/Y}[\varphi|_x]$$

Proof. Assume $\mathcal{A}$ and V are suitable for φ and $\exists x_{/Y}[\varphi_{/x}]$ and $x \notin dom(V)$. This is possible because $x \notin Y$. After Lemma 5.5 and Thm 6.7 we have: $\mathcal{A} \models^{\pm} \varphi[V] \Longleftrightarrow \mathcal{A} \models^{\pm} \varphi|_x[V_x]$. Choosing $V_x = V_{x:f}$ for an appropriate $f\colon V \to A$ we obtain: $\mathcal{A} \models^{+} \varphi[V] \Longleftrightarrow \mathcal{A} \models^{+} \exists x_{/Y}[\varphi|_x][V]$, and choosing $V_x = V_{x:A}$ we get $\mathcal{A} \models^{-} \varphi[V] \Longleftrightarrow \mathcal{A} \models^{-} \exists x_{/Y}[\varphi|_x][V]$. Similarly for the universal quantifier. ∎

The other approach to neutralize the new signaling possibilities of a vacuous quantifier is to prohibit that the new variable encodes usable information by slashing the new quantifier itself. That is, the independence conditions are put on the added quantifier instead of the formula. However, we have been be able to do that only for *regular* formulas.

First we need a lemma on expanding domains that does not introduce slashes in the formula (as is in the first theorem on expanding, viz. Thm 5.5).

Lemma 11.2 (Safely expanding the domain II). *Let φ be a regular IF*-formula, Z_φ the set of free variables that occur in φ under slashes, A and V suitable for φ , and x a variable not occurring in φ or $dom(V)$. If $dom(V) \cap Bd(\varphi) = \varnothing$ then*

$$\mathcal{A} \models^{\pm} \varphi[V] \quad \Longleftrightarrow \quad \mathcal{A} \models^{\pm} \varphi[V_{x:f}].$$

for any Z_φ-independent function $f\colon V \to A$.

Proof. By induction in the complexity of φ. The atomic case, and the inductive step for $(\neg, \pm)$, are immediate, and the implication from left to right follows from an application of Thm 5.3. So it remains to check only the inductive step for $(\vee, \pm)$ and $(\exists, \pm)$ from right to left. Notice that we assume the induction hypothesis for all possible Z_φ-independent functions.

($\vee$) Let φ be $(\psi_1 \vee_{/Y} \psi_2)$. If $\mathcal{A} \models^+ \varphi[V_{x:f}]$ then there is a Y-saturated cover $(V_1)_{x:f}$, $(V_2)_{x:f}$ of $V_{x:f}$ such that $\mathcal{A} \models^+ \psi_i[(V_i)_{x:f}]$. Since $Z_{\varphi_i} \subseteq Z$ then $f \upharpoonright V_i$ is Z_{φ_i}-independent by hypothesis. Moreover, $dom(V_i) \cap Bd(\psi_i) = \varnothing$ trivially. Hence, by the induction hypothesis we know $\mathcal{A} \models^+ \psi_i[V_i]$. It remains to show that V_1, V_2 is a Y-saturated cover. Assume $v \in V_1$, so $v_{x:f(v)} \in (V_1)_{x:f}$, and let $w \sim_Y v$. Since f is independent of $Y \subseteq Z_\varphi$, it follows that $f(v) = f(w)$ and so $v_{x:f(v)} \sim_Y w_{x:f(w)}$. Therefore $w_{x:f(w)} \in (V_1)_{x:f}$, and thus $w \in V_1$ because $x \notin dom(V)$. This shows that V_1 is Y-saturated, the same holds for V_2, and it follows that $\mathcal{A} \models^+ (\psi_1 \vee_{/Y} \psi_2)[V]$.

The case $\mathcal{A} \models^- \varphi[V_{x:f}]$ follows straightforwardly from the induction hypothesis.

($\exists$) Let φ be $\exists z_{/Y} \psi$ and assume $\mathcal{A} \models^+ \varphi[V_{x:f}]$. Then there is a Y-independent $g\colon V_{x:f} \to A$ such that $\mathcal{A} \models^+ \psi[(V_{x:f})_{z:g}]$. Now x and z are distinct (because x does not occur in φ) and z is not in $dom(V)$ because it is bound in φ . So $(V_{x:f})_{z:g}$ may be seen as $(V_{z:g^*})_{x:f^*}$ where $g^*\colon V \to A$ is defined by $g^*(v) = g(v_{x:f(v)})$ and $f^*\colon V_{z:g^*} \to A$ by $f^*(v_{z:a}) = f(v)$ for all $v \in V$. Moreover, f^* is independent of the set $Z_\psi \subseteq Z_\varphi \cup \{z\}$ by construction, and $dom(V_{z:g^*}) \cap Bd(\psi) = \varnothing$ because z can not occur bound in ψ by regularity of φ. Then it holds that $\mathcal{A} \models^+ \psi[(V_{z:g^*})_{x:f^*}]$, and by the induction hypothesis applied to f^*: $\mathcal{A} \models^+ \psi[V_{z:g^*}]$. Now, g^* is independent of Y because f and g are Y-independent. Hence $\mathcal{A} \models^+ \exists z_{/Y} \psi[V]$.

Assume now $\mathcal{A} \models^- \varphi[V_{x:f}]$. Then $\mathcal{A} \models^- \psi[(V_{x:f})_{z:A}]$. But $(V_{x:f})_{z:A} = (V_{z:A})_{x:f^*}$, and by induction hypothesis $\mathcal{A} \models^- \psi[V_{z:A}]$, that is $\mathcal{A} \models^- \varphi[V]$. ∎

Theorem 11.3 (Safely adding vacuous quantifiers II). *Let φ be a regular IF*-formula, Z the set of free variables occurring under slashes in φ, and x a variable not occurring in φ. Then:*

$$\varphi \equiv_{x, Bd(\varphi)} \exists x_{/Z} \varphi \quad \text{and} \quad \varphi \equiv_{x, Bd(\varphi)} \forall x_{/Z} \varphi$$

Proof. The formulas are $x, Bd(\varphi)$-closed by hypothesis. Let $dom(V) \cap (\{x\} \cup Bd(\varphi)) = \varnothing$, then it follows (lemma 5.1) that $\mathcal{A} \models^{\pm} \varphi[V] \iff \mathcal{A} \models^{\pm} \varphi[V_{x:A}]$. Together with lemma 11.2 this proves the result. ∎

12 Elimination of slashed connectives

If φ has slashed connectives, its prenex form will have slashed connectives. One might prefer a theorem in which a formula is equivalent with one that consists of a prefix with possibly slashed quantifiers, followed by a classical matrix: a propositional formula without any slashes. We will show that such a form is possible. The price to be paid is that the structure of the matrix may be much more complex than of the given formula. First we consider approaches from the literature to elimination of slashed connectives.

A natural solution is proposed by Caicedo & Krynicki [3, p. 24]. We give a simplified formulation by neglecting the case of models with only one element (s and t are variables that do not occur in $\varphi \vee_{/Y} \psi$).

(20) $\varphi \vee_{/Y} \psi \equiv_G \exists s_{/Y} \exists t_{/Ys}[(s = t \wedge \varphi) \vee (s \neq t \wedge \psi)]$.

After all our experience with signalling, one will not be surprised that this proposal suffers from both problems we have seen before. The first problem is that new quantifiers may block signals from outside; as in previous cases this can be solved by using s,t-equivalence.

The second problem is that the new quantifiers may give rise to new possibilities for signalling. Consider the following example (the two identical disjuncts are not a printing error):

(21) $\forall y \forall u[\exists x_{/yu}[x{=}u] \vee_{/y} \exists x_{/yu}[x{=}u]]$.

For each $\exists x_{/yu}$ only a constant strategy yielding a fixed value is possible. So ∃loise may guide the game to at most two distinct values for x. But in models with at least three elements ∀belard has more choices available for his $\forall u$. So (21) is not true in such models.

According to (20), sentence (21) would be equivalent with:

(22) $\forall y \forall u \exists s_{/y} \exists t_{/ys}[(s{=}t \wedge \exists x_{/yu} x{=}u) \vee (s{\neq}t \wedge \exists x_{/yu} x{=}u)]$.

However, the existential quantifiers create new possibilities for ∃loise: in her first moves she can assign the value of u to s and t, and satisfy the left disjunct by choosing for x the value of s.

A careful reader may have noticed that the fact that the main disjunction is slashed for y, plays no role of importance in this example. Indeed, with $\vee$ instead of $\vee_{/y}$, sentence (21) would have been a counterexample to the claim as well, but arguably a less convincing one, as there would be no slashed connectives to eliminate. One may check that (21) also is a counterexample if we use Hintikka's implicit slashing convention.

Even though Hintikka does not explicitly formulate an elimination theorem, the slashed connectives in IF-logic *are* eliminated in the translation procedure from IF-logic to Σ^1_1. In [8, p.52] the second order translation (24) of (23) in which the slashed connective is eliminated:

(23) $\forall x \forall z \exists y_{/z}[S_1(x,y,z) \vee_{/x} S_2(x,y,z)]$
(24) $\exists f \exists g \forall x \forall z[(S_1(x,f(x),z) \wedge g(z){=}0) \vee (S_2(x,f(x),z) \wedge g(z){\neq}0)]$.

Apparently it is assumed here that there is a constant **0** in the language, and implicitly, that the model has at least two elements. Based upon these idea's we may formulate as (restricted!) elimination rule:

$$\varphi \vee_{/Y} \psi \;\equiv_S\; \exists s_{/Y}[[s{=}\mathbf{0} \wedge \varphi] \vee [s{\neq}\mathbf{0} \wedge \psi]].$$

However, one special constant is not enough. Consider the corresponding equivalent of (21):

(25) $\forall y \forall u \exists s_{/y}[[s{=}\mathbf{0} \wedge \exists x_{/uy} x{=}u] \vee [s{\neq}\mathbf{0} \wedge \exists x_{/uy} x{=}u]]$

∃loise can still choose the value of s equal to the value of u. At the disjunction she chooses left if $s{=}0$, and right otherwise. In both cases, she wins by choosing for x the value of s. So, also the rule underlying Hintikka's translation procedure fails due to signalling.

Because of the already mentioned assumption that models contain at least two elements, the problem can be avoided by assuming two distinct special constants:

Theorem 12.1 (Elimination using two constants). *Let φ and ψ be two IF*-formulas, and $s \notin Fr(\varphi \vee_{/Y} \psi)$. Then for all suitable model $\mathcal{A}$ with distinct interpretations for the constants* **0** *and* **1***:*

$$\varphi \vee_{/Y} \psi \;\equiv_s \exists s_{/Y}[[(s{=}\mathbf{0} \wedge \varphi) \vee (s{=}\mathbf{1} \wedge \psi)].$$

This is an improvement (with analogous proof) of Dechesne [4]. But this theorem does *not* provide a solution to our aim of obtaining a prenex form theorem without slashed connectives because it puts requirements on the language and its interpretation, and on the size of models. Below we give a solution that works without such requirements, at the cost of a more complex translation and a long proof that we have been not able to simplify. Its proof uses the following lemma about one element domains.

Lemma 12.2. *Let ϑ^c denote the classical formula resulting of replacing in ϑ all slashed symbols by their unslashed forms. Let X be a set of variables such that $Fr(\vartheta)\subseteq X$. Then for any one element structure $\mathcal{A}$ and the unique valuation $v\colon X\to A$ we have: $\mathcal{A}\models^+\vartheta[\{v\}]\Longleftrightarrow\mathcal{A}\models\vartheta^c[v]$ and $\mathcal{A}\models^-\vartheta[\{v\}]\Longleftrightarrow\mathcal{A}\not\models\vartheta^c[v]$.*

Proof. Clearly, $\mathcal{A}\models^+\vartheta[\{v\}]$ implies $\mathcal{A}\models^+\vartheta^c[\{v\}]$, which in turn implies $\mathcal{A}\models\vartheta^c[v]$ by Thm 4.10. Since any set of valuations arising in the inductive verification of the last statement is a singleton, the functions there arising are independent of any set of variables that may appear in ϑ (by Thm 2.13), and thus verify $\mathcal{A}\models^+\vartheta[\{v\}]$. The second equivalence follows from the first. ∎

First we consider regular disjunctions:

Theorem 12.3 (Elimination of slashed connectives). *Let s,t, and u be distinct variables that don't occur in the regular formula $\varphi\vee_{/Y}\psi$, and let Z be the set of all free variables occurring under the slashes in φ or ψ. Then:*

$$(\varphi\vee_{/Y}\psi)\equiv_{s,t,u,Bd(\varphi,\psi)}(\forall s\forall t[s=t]\wedge(\varphi\vee\psi))\vee$$
$$(\exists s\exists t[s\neq t]\wedge\forall s_{/Z}\forall t_{/Z}[s=t\vee\exists u_{/Y}[(u=s\wedge\varphi)\vee(u=t\wedge\psi)]]).$$

Proof. Denote the left formula in the equivalence by ϑ, and the right one by ϑ'. Let $\mathcal{A}$ and $V\subseteq A^X$ be suitable for these formulas, with X disjoint of $\{s,t,u\}\cap Bd(\varphi,\psi)$. We must prove:

$$\mathcal{A}\models^{\pm}\vartheta[V]\text{ iff }\mathcal{A}\models^{\pm}\vartheta'[V].$$

If $V=\varnothing$ the equivalence is trivial. If $|A|=1$, and $V\neq\varnothing$, then V is a singleton and the equivalence follows from Lemma 12.2 since ϑ^c, ϑ'^c are easily seen to be first order equivalent. Therefore, we will assume $V\neq\varnothing$ and $|A|\geq 2$ for the rest of the proof.

Assume $\mathcal{A}\models^+\vartheta[V]$, then $\mathcal{A}\models^+\varphi[V_1]$, $\mathcal{A}\models^+\psi[V_2]$ for a Y-saturated cover V_1,V_2 of V. Since $s,t\notin X$ we may define a cover W_1,W_2 of $V_{st:A\times A}$ by $W_1=V\times\{st\colon aa\mid a\in A\}$, and $W_2=V\times\{st\colon ab\mid a,b\in A,a\neq b\}$. Define now $f\colon W_2\to A$ by:

$$f(v_{st:ab})=\begin{cases}a & \text{if } v\in V_1\\ b & \text{if } v\in V_2\end{cases}$$

Thus, since $u\notin X$, $(W_2)_{u:f}=(V_1\times\{stu\colon aba\mid a\neq b\})\cup(V_2\times\{stu\colon abb\mid a\neq b\})$, while by Lemma 5.1: $\mathcal{A}\models^+\varphi[V_1\times\{stu\colon aba\mid a\neq b\}]$ and $\mathcal{A}\models^+\psi[V_2\times\{stu\colon abb\mid a\neq b\}]$. Therefore, $\mathcal{A}\models^+((u=s\wedge\varphi)\vee(u=t\wedge\psi))[(W_2)_{u:f}]$. But f is independent of Y by the fact that the V_i are Y-saturated. Thus

$$\mathcal{A}\models^+\exists u_{/Y}[(u=s\wedge\varphi)\vee(u=t\wedge\psi)][W_2],$$

and because $\mathcal{A}\models^+(s=t)[W_1]$ and $W_1\cup W_2=V_{st:\,A\times A}$ it follows that:

$$\mathcal{A}\models^+\forall s_{/Z}\forall t_{/Z}[s=t\vee\exists u_{/Y}[(u=s\wedge\varphi)\vee(u=t\wedge\psi)]][V].$$

Finally, since $\mathcal{A}\models^+\exists s\exists t[s\neq t][V]$ because $|A|\geq 2$, and $\mathcal{A}\models^+(\forall s\forall t[s=t]\wedge(\varphi\vee\psi))[\varnothing]$, we obtain $\mathcal{A}\models^+\vartheta'[V]$.

Conversely, assume $\mathcal{A}\models^+\vartheta'[V]$. Since no $W\neq\varnothing$ satisfies the left disjunct of ϑ' because $|A|\geq 2$, we have, consecutively:

* $\mathcal{A}\models^+ s=t\vee\exists u_{/Y}[(u=s\wedge\varphi)\vee(u=t\wedge\psi)][V_{st:A\times A}]$
* $\mathcal{A}\models^+ \exists u_{/Y}[(u=s\wedge\varphi)\vee(u=t\wedge\psi)][V_{st:\{ab\in A\times A|a\neq b\}}]$
* $\mathcal{A}\models^+ \exists u_{/Y}[(u=s\wedge\varphi)\vee(u=t\wedge\psi)][V_{st:ab}]$, for any fixed $a,b\in A$ with $a\neq b$ (Lemma 4.6)
* $\mathcal{A}\models^+ [(u=s\wedge\varphi)\vee(u=t\wedge\psi)][(V_{st:ab})_{u:f}]$, for some $f\colon V_{st:ab}\to A$, that is Y-independent
* $\mathcal{A}\models^+ (u=s\wedge\varphi)[((V_1)_{st:ab})_{u:f}]$ and $\mathcal{A}\models^+ (u=t\wedge\psi)[((V_2)_{st:ab})_{u:f}]$, for some cover V_1, V_2 of V.

Then necessarily $f(v_{st:ab})=a$ if $v\in V_1$, and $f(v_{st:ab})=b$ if $v\in V_2$. Hence,

$$\mathcal{A}\models^+\varphi[(V_1\times\{stu\colon aba\}] \textit{ and } \mathcal{A}\models^+\psi[(V_2\times\{stu\colon abb\}].$$

and the V_i are automatically Y-saturated. By Lemma 5.1, $\mathcal{A}\models^+\varphi[V_1]$, $\mathcal{A}\models^+\psi[V_2]$, which implies $\mathcal{A}\models^+(\varphi\vee_{/Y}\psi)[V]$.

For negative satisfaction, it is enough to prove that (26) and (27) are equivalent:

(26) $\mathcal{A}\models^+(\neg\varphi\wedge\neg\psi)[V]$
(27) $\mathcal{A}\models^+\exists s_{/Z}\exists t_{/Z}[s\neq t\wedge\forall u[(u\neq s\vee\neg\varphi)\wedge(u\neq t\vee\neg\psi)]][V]$

Now, (26) implies, by Lemma 5.1, that $\mathcal{A}\models^+(\neg\varphi\wedge\neg\psi)[(V_{u:A})_{st:ab}]$ for any fixed $a,b\in A$. Then it follows that:

$$\mathcal{A}\models^+((u\neq s\vee\neg\varphi)\wedge(u\neq t\vee\neg\psi))[(V_{st:ab})_{u:A}]$$

(take the empty set of valuations for for the left disjuncts $u\neq s$, and $u\neq t$). Thus (27) follows by choosing $a\neq b$ and interpreting s and t by constant functions of value a and b, respectively. Conversely, from (27) it follows consecutively

* $\mathcal{A}\models^+(s\neq t\wedge\forall u[(u\neq s\vee\neg\varphi)\wedge(u\neq t\vee\neg\psi)])[(V_{s:f})_{t:g}]$, for some $f\colon V\to A$ and $g\colon V_{s:f}\to A$ that are Z-independent.
* $\mathcal{A}\models^+(u\neq s\vee\neg\varphi)[((V_{s:f})_{t:g})_{u:A}]$ and $\mathcal{A}\models^+(u\neq t\vee\neg\psi)[((V_{s:f})_{t:g})_{u:A}]$
* $\mathcal{A}\models^+(u\neq s\vee\neg\varphi)[((V_{s:f})_{t:g})_{u:f^*}]$ and $\mathcal{A}\models^+(u\neq t\vee\neg\psi)[((V_{s:f})_{t:g})_{u:g^*}]$, where $f^*(w)=w(s)$, $g^*(w)=w(t)$, respectively.
* $\mathcal{A}\models^+\neg\varphi[((V_{s:f})_{t:g})_{u:f^*}]$ and $\mathcal{A}\models^+\neg\psi[((V_{s:f})_{t:g})_{u:g^*}]$, because by construction, any valuation w in $((V_{s:f})_{t:g})_{u:f^*}$ assign the same value $f(w\restriction X)$ to u and s, and similarly those in $((V_{s:f})_{t:g})_{u:g^*}$ identify u and t.
* Because $s,t\notin Z$, the functions f^*, g^* are Z-independent, as were f and g. Applying Lemma 11.2 three times we have $\mathcal{A}\models^+\neg\varphi[V]$ and $\mathcal{A}\models^+\neg\psi[V]$ because φ and ψ are regular, hence (26) holds. ■

Theorem 12.4. *Any formula φ is $Bd(\varphi,\psi)$-equivalent to a regular formula ψ without slashed connectives which may be taken in prenex form.*

Proof. Apply in order: the regularization theorem, the previous theorem combined with the substitution theorem for regular formulas to eliminate all slashed disjunctions, and then the special prenex theorem, maintaining the connectives unslashed. ■

13 Other properties, interchange of quantifiers

If one would aim at other or more specific normal forms, with some standard order of the quantifiers, or disjunctive or conjunctive normal forms, further equivalences and rules are needed. We end this paper with some *caveats* when transferring classical rules to the IF-setting, followed by some extra quantifier rules that *do* hold.

Idempotency **:** $\varphi \not\equiv_G \varphi \vee \varphi$.

We know that $\mathcal{B} \not\models^+ \forall x \exists y_{/x}[x \neq y]$, but $\mathcal{B} \models^+ \forall x\ [\exists y_{/x}[x \neq y] \vee \exists y_{/x}[x \neq y]]$, as ∃loise can play the following winning strategy: at the disjunction, play *if* $x{=}0$ *then L, otherwise* R; at the left $\exists y_{/x}$ always $y{:=}0$; at the right $\exists y_{/x}$ always $y{:=}1$. Essentially the same 'empowering' effect of duplication of subformulas disturbs the following classical rule relevant for making conjunctive normal forms.

Distributivity **:** $(\varphi \wedge \psi) \vee \eta \not\equiv_G (\varphi \vee \eta) \wedge (\psi \vee \eta)$

We consider here an example interpreted on the natural numbers. The language is extended with the predicates $Even(x)$ and $Odd(x)$ (with the obvious interpretations). It is clear that ∃loise has no winning strategy for $\forall x[(Even(x) \vee Odd(x)) \wedge \exists y_{/x}[y \neq x]]$. But for $\forall x[(Even(x) \wedge \exists y_{/x}[y \neq x]) \vee (Odd(x) \wedge \exists y_{/x}[y \neq x])]$ ∃loise has a winning strategy: for the leftmost occurrence of $\exists y_{/x}$ she always plays $y{:=}1$, for the rightmost $y{:=}0$, and for $\vee$ she chooses according to the value of x. Related examples are given by [15], who investigates for which formulas with one free variable $\varphi \models^+ \psi$ holds (it turns out that it does so only for few combinations).

Associativity **:** $(\varphi \vee \psi) \vee_{/x} \eta \not\equiv_G \varphi \vee (\psi \vee_{/x} \eta)$

The winning strategy for $x \neq y \vee (x{=}0 \vee_{/x} x \neq 0)$ is to choose R for $\vee$ if $x{=}y$, and to use y as a signal for the value of x. However there is no winning strategy for ∃loise in $(x \neq y \vee x{=}0) \vee_{/x} x \neq 0$.

Exchanging quantifiers of the different type **:** Van Benthem remarked already in 2002, in the first version of [2], that the apparently correct equivalence $\forall y \exists x_{/y}[x{=}y] \equiv_G \exists x \forall y[x{=}y]$ holds only if restricted to positive satisfaction, because ∀belard has a refuting strategy for the second formula but not for the first in structures with two or more elements.

Exchanging quantifiers of the same type $\forall z \exists x \exists y_{/z}[y{=}z] \not\equiv_G \forall z \exists y_{/z} \exists x[y{=}z]$

The classical rule that allows the exchange of two consecutive existential quantifiers does not hold. We have $\mathcal{B} \models^+ \forall z \exists x \exists y_{/z}[y{=}z]$ because ∃loise has the winning strategy $x{:=}z$, then $y{:=}x$. But $\mathcal{B} \not\models^+ \forall z \exists y_{/z} \exists x[y{:=}z]$ because any strategy for ∃loise must choose $y{:=}a$ constant, the value chosen for x not being of any help. Taking negations we have a similar failure for interchange of consecutive ∀.

This example illustrates again the point that signalling gives unexpected results, even in case there is no embedding of quantifiers binding the same variable. For positive results, see Theorems 13.1 and 13.3 below.

In van Benthem [1] it is pointed out that one may find quantifier exchange rules by a change of perspective. Two consecutive, but independent choices G and H can be viewed as being played in parallel, in game algebraic notation $G \times H$. Game algebra teaches us that $G \times H = H \times G$. As illustration he gives the equivalence

(28) $\forall x \exists y_{/x} \varphi(x, y) \equiv_G \exists y \forall x_{/y} \varphi(x, y)$.

This may be generalized straightforwardly to

$$\forall x_{/Z} \exists y_{/Wx} \varphi \equiv_G \exists y_{/W} \forall x_{/Zy} \varphi$$

when x and y are distinct variables, $y \notin Z$ and $x \notin W$

However, the application of insights from game algebra to IF^* is not as easy as this example suggests. The analogue of van Benthem's rule (28) where both quantifiers are the same, is *not* correct. A counterexample is as follows. We have, in an arbitrary model $\mathcal{A}$:

$$\mathcal{A} \models^+ \forall u \forall z \exists y \exists x_{/z} \exists y_{/zx} [x = z \wedge y = u]$$

because we can use the first y as signal for the value of z in the winning strategy $\{y := z, x := y, y := u\}$. But when we interchange the last two quantifiers

$$\mathcal{A} \not\models^+ \forall u \forall z \exists y \exists y_{/z} \exists x_{/yz} [x = z \wedge y = u]$$

because here the signal is not available at $\exists x_{/z}$, in which stage the value of y equals u. This shows:

$$\exists x_{/z} \exists y_{/zx} [x = z \wedge y = u] \not\equiv_G \exists y_{/z} \exists x_{/yz} [x = z \wedge y = z].$$

However, these two formulas *are* xy-equivalent as a special case of the next theorem which shows the rule holds under restricted equivalence.

Theorem 13.1. *Given sets of variables Z, W and distinct variables x and y not in $Z \cup W$, then*

$$Qx_{/Z}\, Qy_{/Wx} \varphi \equiv_{xy} Qy_{/W}\, Qx_{/Zy} \varphi.$$

Proof. Since $x, y \notin Z \cup W$ both formulas are xy-closed. It is enough to consider the case that $Q = \exists$. If $\mathcal{A} \models^+ \exists x_{/z} \exists y_{/Wx} \varphi[V]$ with $x, y \notin dom(V)$ then $\mathcal{A} \models^+ \varphi[(V_{x:f})_{y:g}]$ where $f\colon V \to A$ and $g\colon V_{x:f} \to A$ are Z-independent and Wx-independent respectively. Define now $g^*\colon V \to A$ and $f^*\colon V_{x:g^*} \to A$ by $g^*(v) = g(v_{x:f(v)})$ and $f^*(v_{y:g^*(v)}) = f(v)$, respectively (f^* is well defined because $y \notin dom(V)$). Then $(V_{x:f})_{y:g} = (V_{y:g^*})_{x:f^*}$. Moreover, g^* is W-independent: $v \sim_W w \Rightarrow v_{x:f} \sim_{xW} w_{x:f} \Rightarrow g^*(v) = g(v_{x:f}) = g(w_{x:f}) = g^*(v)$, and f^* is Zy-independent by construction. Therefore, $\mathcal{A} \models^+ \exists y_{/W} \exists x_{/Zy} \varphi[V]$. The converse direction proceeds symmetrically. On the other hand, the equivalence $\mathcal{A} \models^- \exists x_{/Z} \exists y_{/Wx} \varphi[V] \Longleftrightarrow \mathcal{A} \models^- \exists y_{/W} \exists x_{/Zy} \varphi[V]$ is straightforward. ∎

Another quantifier rule, akin to Lemma 8.1 is given below. Note that it does not hold for G-equivalence since $\exists x_{/y} \exists z_{/y} [y = z] \not\equiv_G \exists x_{/y} \exists z_{/yx} [y = z]$.

Lemma 13.2. *If $Z \subseteq Y$ and $x \notin Y$, then $\exists x_{/Y} \exists y_{/Z} \varphi \equiv_x \exists x_{/Y} \exists y_{/Zx} \varphi$, and similarly for $\forall$.*

Proof. From left to right for positive and negative satisfaction, and from right to left for negative satisfaction the implications are trivial. Now, if $\mathcal{A} \models^+ \exists y_{/Z} \psi[V_{x:f}]$ then $\mathcal{A} \models^+ \psi[(V_{x:f})_{y:g}]$ where $g\colon V_{x:f} \to A$ is Z-independent. It is enough to verify that g is Zx-independent. Indeed: $v_{x:f} \sim_{Zx} w_{x:f} \Rightarrow v \sim_Z w$ (since $x \notin dom(V)$) $\Rightarrow v \sim_Y w$ (because $Y \supseteq Z$) $\Rightarrow f(v) = f(w) \Rightarrow v_{x:f} \sim_Z w_{x:f} \Rightarrow g(v_{x:f}) = g(w_{x:f})$. Hence, $\mathcal{A} \models^+ \exists y_{/Zx} \psi[V_{x:f}]$. ∎

A consequence of the previous results is a felicitous generalization of the classical exchange rule for *identical* quantifiers:

Theorem 13.3. *If* $x, y \notin Z$, *then* $Qx_{/Z}\, Qy_{/Z}\varphi \equiv_{xy} Qy_{/Z}\, Qx_{/Z}\varphi$.

Proof. Consider x and y distinct, otherwise there is nothing to prove. Then $Qx_{/Z}\, Qy_{/Z}\varphi \equiv_x Qx_{/Z}\, Qy_{/Zx}\varphi \equiv_{xy} Qy_{/Z}\, Qx_{/Zy}\varphi \equiv_x Qy_{/Z}\, Qx_{/Z}\varphi$, where the first and last equivalences follow from Lemma 13.2, and the middle one from Theorem 13.1. ∎

Lemma 13.2 and Lemma 8.1 may be generalized in the following way:

Theorem 13.4 (Removing a variable below a slash). *Consider a regular formula* $\exists x_{/Y}\varphi$. *Assume that* $\exists y_{/Zx}$ *occurs positively in a subformula of* φ, *and let* φ' *be the result of replacing that occurrence by* $\exists y_{/Z}$. *If* $Y \supseteq Z$ *and* U *is the set of bound variables of* $\exists x_{/Y}\varphi$ *having the subformula in its scope, then:*

$$\exists x_{/Y}\varphi \equiv_U \exists x_{/Y}\varphi'.$$

A similar result holds for positive occurrences of $\vee_{/Z}$.

The idea of the proof (for positive satisfaction) is that if the subformula $\exists y_{/Z}\psi$ occurs positively then ∃loise has to make a choice there. If she has a winning strategy at that point she may turn it in a winning strategy for $\exists y_{/Zx}$ by executing the strategy for the outermost $\exists x_{/Y}$ again and then use the value of x as an input for the strategy at $\exists y_{/Z}$ (or $\vee_{/Z}$). Because $Z \subseteq Y$, this combination is a Zx-independent strategy. The reciprocal direction is obvious. As to negative satisfaction, a winning strategy of ∀belard for refutation at $\exists y_{/Z}$ should work for $\exists y_{/Zx}$, and viceversa, since it does not take in account the independence restriction. For a positive occurrence of $\vee_{/Z}$ the argument is identical.

Now some form of regularity of $\exists x_{/Y}\varphi$ and some restriction in the domain of the valuations considered are needed in the previous theorem because

$$\exists x_{/y}\exists x\exists z_{/y}[y=z] \not\equiv_x \exists x_{/y}\exists x\exists z_{/yx}[y=z]$$

since ∃loise may choose the value of second $\exists x$ to signal y to $\exists z$ in the first formula, but this is impossible in the second formula. Moreover,

$$\exists x_{/y}\exists u(u=0\wedge\exists z_{/y}[y=z]) \not\equiv_x \exists x_{/y}\exists u(u=0\wedge\exists z_{/yx}[y=z])$$

because under the set of valuations $\{yu\colon 00, 11\}$ ∃loise has a winning strategy for the first formula: $x{:=}u$, $u{:=}0$, $z{:=}x$, but she does not have one for the second. As the informal argument above does not precise these facts, we prefer to provide an inductive proof.

Proof. (of Thm 13.4) Without loss of generality we may assume that φ is in negation normal form. We show by induction in the complexity of φ, starting at $\varphi := \exists y_{/Z}\psi$ that for any $\mathcal{A}$ and V suitable for $\exists x_{/Y}\varphi$, (1.) and (2.) below hold:

(1.) $A \models^+ \varphi[V_{x:f}] \Longleftrightarrow A \models^+ \varphi'[V_{x:f}]$, *if* $dom(V)\cap Bd(\varphi)=\varnothing$ *and* $f\colon V\to A$ *is* Y*-independent.* The basis step is given by Lemma 13.2.

($\vee$) $\mathcal{A}\models^+(\varphi_1\vee_{/W}\varphi_2)[V_{x:f}]$ iff $\mathcal{A}\models^+\varphi_i[(V_i)_{x:f}]$, $i=1,2$, with $\{V_1,V_2\}$ a W-independent cover of V. Then we may apply the induction hypothesis to each $\varphi_i[(V_i)_{x:f}]$, and the result follows.

($\wedge$) Simpler than the previous case.

($\exists$) $\mathcal{A}\models^+\exists u_{/S}\varphi[V_{x:f}]$ implies $\mathcal{A}\models^+\varphi[(V_{x:f})_{u:g}]$ for an S-independent $g\colon V_{x:f}\to A$. Since $x,u\notin dom(V)$ by hypothesis and x is distinct from u by regularity of $\exists x_{/Y}\varphi$, $(V_{x:f})_{u:g}=(V_{u:g^*})_{x:f^*}$ as in the proof of Thm 13.1, where $f^*\colon V_{u:g^*}\to A$ is still Y-independent, then $\mathcal{A}\models^+\varphi'[(V_{u:g^*})_{x:f^*}]$ by induction hypothesis and thus $\mathcal{A}\models^+\varphi'[(V_{x:f})_{u:g}]$ and $\mathcal{A}\models^+\exists u_{/S}\varphi[V_{x:f}]$.

($\forall$) $\mathcal{A}\models^+\forall u_{/W}\varphi[V_{x:f}]$ iff $\mathcal{A}\models^+\varphi[(V_{x:f})_{u:A}]$. But $(V_{x:f})_{u:A}=(V_{u:A})_{x:f^*}$ as in the case of $\exists$. The rest is obvious.

(2.) $\mathcal{A}\models^-\varphi[W]\Longleftrightarrow\mathcal{A}\models^-\varphi'[W]$, *for W arbitrary*. The basis step is obvious because $\mathcal{A}\models^-\exists y_{/Z}\psi[W]\Longleftrightarrow\mathcal{A}\models^-\exists y_{/Zx}\psi[W]$ by definition, and the induction for $\vee,\wedge,\exists,\forall$ is, for the same reason, straightforward.

Finally, from (1.) we get $\mathcal{A}\models^+\exists x_{/Y}\varphi[V]\Longleftrightarrow\mathcal{A}\models^+\exists x_{/Y}\varphi'[V]$. From (2.) taking $W=V_{x:A}$ we get $\mathcal{A}\models^-\exists x_{/Y}\varphi[V]\Longleftrightarrow\mathcal{A}\models^-\exists x_{/Y}\varphi'[V]$. The proof for a positive occurrence of a subformula $\psi_1\vee_{/Z}\psi_2$ is identical utilizing Lemma 8.1 as basis step in (1.). ∎

We have, for example:

$$\exists x_{/y}\exists u(u=x\wedge\exists z_{/y}[y=z])\equiv_{xu}\exists x_{/y}\exists u(u=x\wedge\exists z_{/yx}[y=z]).$$

It follows from the proof that we only need to ask regularity of $\exists x_{/Y}\varphi$ with respect to the bound variables that have the subformula in their scope.

Obviously, we have a similar result for positive occurrences of $\forall y_{/Z}$ or $\wedge_{/Z}$ in $\forall x_{/Y}\varphi$. The connection of this result with the game theoretical *Thompson transformation* of Inflation-Deflation [16] is interesting; see [5].

This generalization explains the issue of the 'implicit slashing' (Hintikka's convention, see p. 97). The above theorem teaches us for which formulas the evaluation could alter by imposing the convention: e.g. for Hodges example $\forall z\exists x\exists y_{/z}[y=z]$, which does not satisfy the condition that $Z\subseteq Y$ (in this case: $Z=\{z\}$ while $Y=\varnothing$).

14 Conclusion

We defined a general logic with imperfect information, with both a game semantics and an equivalent compositional semantics. We had to rethink the basic notions of equivalence, yielding new, slightly restricted, equivalences. We obtained new quantifiers exchange rules and quantifier extraction rules, and corrected several published results. We proved a prenex form theorem for the logic, and as a side result we found a method to obtain a simple Skolem form for classical logic. Our research has shown that signalling is an important method in imperfect information games that has impact on any properties of the logic.

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