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Xavier Caicedo

Definability and automorphisms in abstract logics

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Abstract. In any model theoretic logic, Beth's definability property together with Feferman-Vaught's uniform reduction property for pairs imply recursive compactness, and the existence of models with infinitely many automorphisms for sentences having infinite models. The stronger Craig's interpolation property plus the uniform reduction property for pairs yield a recursive version of Ehrenfeucht-Mostowski's theorem. Adding compactness, we obtain the full version of this theorem. Various combinations of definability and uniform reduction relative to other logics yield corresponding results on the existence of non-rigid models.

The celebrated theorem of Ehrenfeucht and Mostowski states that any first order theory having infinite models must have non rigid models. In its strongest form, it claims that any linearly ordered set may be embedded as a set of indiscernibles in a model of the theory, so that all the automorphisms of the order extend to automorphisms of the model (cf. [EM], [Ho]). The known proofs of the theorem utilize compactness and Ramsey's theorem or ultrafilter arguments combining both to get models with the given ordered set as a set of indiscernibles, and they use the fact that first order logic is generated by the existential quantifier to Skolemize over the indiscernibles. These techniques have been adapted successfully to obtain analogs of the Ehrenfeucht-Mostowski theorem for the infinitary logics $L_{\kappa\omega}$ by Chang [Ch] and others, following Morley [Mo], and for logics with additional quantifiers like $L(Q_\alpha)$, $L(Q_\omega^{cof})$ or $L(aa)$ by Ebbinghaus [E1] and Otto [O].

In this paper, we show that the familiar definability properties, together with the Feferman-Vaught uniform reduction property for pairs (URP), imply versions of increasing strength of the Ehrenfeucht-Mostowski theorem for arbitrary model theoretic logics. Thus, Beth's definability property and URP imply the existence of models with infinite automorphism group for any relativized projective class having infinite or arbitrarily large finite models. Recursive compactness follows (Sec. 2). Under Craig's interpolation lemma, this may be improved to a recursive version of Ehrenfeucht-Mostowski's theorem, implying, for example, that any recursive theory with infinite models has models with the ordered rational numbers as indiscernibles (Sec. 3). Adding compactness yields the full Ehrenfeucht-Mostowski theorem except, perhaps, for its functorial character (Sec. 4). Other combinations

X. Caicedo: Departamento de Matemáticas, Universidad de los Andes A.A. 4976, Bogotá, Colombia. e-mail: `xcaicedo@uniandes.edu.co`

of weak forms of definability and uniform reduction relative to other logics yield corresponding results on the existence of non-rigid models, and allow some applications. These results should be seen in the light of [S1, S2], and [Ma], where some connections between definability, uniform reduction, and automorphism properties are considered.

1. Preliminaries, κ-uniform reduction for pairs

For the concept of a model theoretic logic L, we follow [E2]. L will be assumed to contain first order logic, and to be closed under finite conjunctions, renamings, and relativizations. An additional proviso will be that $L(\tau)$ be closed under conjunctions of power $|\tau|$ if there is a sentence $\varphi \in L(\tau)$ for which $|\tau|$ is minimum. Unless stated otherwise, smallness of L will not be assumed; that is, the class $L(\tau)$ of sentences of type τ might be a proper class. For simplicity, we will deal with single sorted logics only, but all the given results hold in the many sorted case.

See [E2] or [Ma] for a description of the definability properties: *Beth definability, weak Beth definability,* and *Craig interpolation*, as well as their relativized versions involving two logics: *Weak-Beth(L,L**), *Beth(L,L**), and *Craig(L,L**). Recall that any logic L has smallest closures $WB(L)$ and $B(L)$ satisfying, respectively, weak-Beth and Beth's properties (which is not the case for the interpolation property). The condition *Weak-Beth(L,L**) means the same as $WB(L) \leq L^*$, but *Beth(L,L* *) is weaker than $B(L) \leq L^*$.

The *disjoint sum* of two vocabularies τ_1 and τ_2 is the vocabulary $\tau_1 \oplus \tau_2 = \{P_1\} \cup \tau_1 \cup \{P_2\} \cup \tau_2'$, where τ_2' is a copy of τ_2, disjoint from τ_1, and P_1, P_2 are new unary relation symbols. The *disjoint pair* (full cardinal sum, strong cardinal sum) of two structures $\mathfrak{A}_i \in Str(\tau_i)$, $i = 1, 2$, is the structure $[\mathfrak{A}_1, \mathfrak{A}_2] = \langle A_1 \cup A_2', \mathfrak{A}_1, \mathfrak{A}_2' \rangle$ of type $\tau_1 \oplus \tau_2$, with $\{P_1\} \cup \tau_1$ interpreted by the universe A_1 and the relations of $\mathfrak{A}_1$, and $\{P_2\} \cup \tau_2'$ interpreted by the universe A_2' and the relations of a copy $\mathfrak{A}_2'$ of $\mathfrak{A}_2$, disjoint from $\mathfrak{A}_1$.

We introduce, in the next definition, a relativized hierarchy of intermediate properties between the *uniform reduction property for pairs* (URP) and the *pair preservation property* (PPP), cf. [Ma]. As usual, $\mathfrak{A} \equiv_\Phi \mathfrak{B}$ means that $\mathfrak{A} \models \varphi \Leftrightarrow \mathfrak{B} \models \varphi$ for all $\varphi \in \Phi$, and we write $\mathfrak{A} \equiv_\varphi \mathfrak{B}$ for $\mathfrak{A} \equiv_{\{\varphi\}} \mathfrak{B}$.

Definition. A logic L *has the κ-uniform reduction property for pairs* (κ-URP) *in* L^* if for any $\varphi \in L(\tau_1 \oplus \tau_2)$ there are sets of formulae $\Phi_i \subseteq L^*(\tau_i)$ of power less than κ such that, for any structures $\mathfrak{A}_i$, $\mathfrak{B}_i$ of type τ_i, $i = 1, 2$:

$$\mathfrak{A}_i \equiv_{\Phi_i} \mathfrak{B}_i,\ i = 1, 2, \text{ imply } [\mathfrak{A}_1, \mathfrak{A}_2] \equiv_\varphi [\mathfrak{B}_1, \mathfrak{B}_2].$$

∞-URP means that the above holds with sets Φ_i of arbitrary power, and PPP means that it holds with $\Phi_i = L^*(\tau_i)$. In this context, we say that the sets Φ_1, Φ_2 *reduce φ uniformly*. Whenever $L = L^*$, the second logic will not be mentioned.

Clearly, $URP = \omega$-$URP \Rightarrow \ldots \kappa$-$URP \Rightarrow \ldots \infty$-$URP \Rightarrow PPP$, the last two properties being equivalent for small L^*. For compact L^*, the hierarchy collapses because $URP = \infty$-URP by Claim 2.2 in [S1].

It is easily shown, using appropriate systems of partial isomorphisms, that $L_{\omega\omega}$, $L_{\omega\omega}(Q_\alpha)$, and $L_{\omega\omega}(Q_\omega^{cof})$ have URP (in themselves), and the infinitary logic $L^\alpha_{\infty\lambda}$, consisting of sentences of $L_{\infty\lambda}$ of quantifier rank less than α, has PPP. However, $L_{\kappa\lambda}$ has PPP in itself if and only if κ is strongly inaccessible or ∞ (cf. Malitz [M]). This is partially repaired in the next example.

Recall that $\beth_\alpha(\kappa)$ is defined inductively as $\beth_0(\kappa) = \kappa$, $\beth_{\alpha+1}(\kappa) = 2^{\beth_\alpha(\kappa)}$, and $\beth_\alpha(\kappa) = \sup_{\beta<\alpha} \beth_\beta(\kappa)$ for limit α. We write $\beth_\alpha$ for $\beth_\alpha(\omega)$.

Example 1. If κ is regular then $L_{\kappa\lambda}$ has $\beth_\kappa$-URP in $L_{\beth_\kappa\lambda}$. Thus, $L_{\infty\lambda}$ and $L_{\infty\infty}$ have ∞-URP. This may be shown by a Feferman-Vaught construction like in Malitz Th. 2.1 [M], but a simpler argument is the following. By regularity of κ, any sentence of $L_{\kappa\lambda}$ has both occurrence number and quantifier rank less than κ. Consider $\varphi \in L^\alpha_{\infty\lambda}(\tau_1 \oplus \tau_2)$, with $\alpha, |\tau_i| < \kappa$. Since $L^\alpha_{\infty\lambda}$ has PPP, the equivalence $\equiv_\varphi$ is reducible to equivalence with respect to subclasses $\Phi_i \subseteq L^\alpha_{\infty\lambda}(\tau_i)$, $i = 1, 2$, but a straightforward induction on α shows that $|L^\alpha_{\infty\lambda}(\tau_i)| \leq \beth_{\alpha+1}(|\tau_i|) \leq \beth_{|\tau_i|+\alpha+1} < \beth_\kappa$.

Any logic has PPP in some extension, for example in its join with the logic $L_{\infty\infty}$, but it may happen that it does not have ∞-URP in any extension (see Example 3 below).

2. Automorphisms from Beth

Given a structure $\mathfrak{A}$ of type τ, $\mathfrak{A}\restriction\mu$ will denote its reduct to a sub-vocabulary $\mu \subseteq \tau$, and $\mathfrak{A}\restriction P^{\mathfrak{A}}$ will denote its restriction to the subuniverse $P^{\mathfrak{A}}$, where $P \in \tau$. A class K of structures of type τ is a *relativized projective class* (RPC) *of* L if it has the form $K = \{\mathfrak{A}\restriction P^{\mathfrak{A}}\restriction\tau : \mathfrak{A} \models \varphi\}$, where $\varphi \in L_{\mu\cup\{P\}}$ and $\tau \subseteq \mu \cup \{P\}$. We say that K is RPC_Δ if a theory is allowed instead of the sentence φ in the definition of K. We will write $K \in RPC(L)$, respectively $K \in RPC_\Delta(L)$, to express these facts.

A *recursive theory of* L will be one of the form $\{\varphi\} \cup T$, where $\varphi \in L$, and there are finitary quantifiers Q^i definable in L such that $T \subseteq L_{\omega\omega}(Q^1, ..., Q^n)$ is recursive in the usual sense. It is easy to see, adapting well known arguments due to Kleene, Craig and Vaught [CV], that the models of such a theory form a RPC class of L. Therefore, all the results stated in this paper for RPC classes will hold for recursive theories.

For any class of structures K, set:

$$||K|| = sup\{|\mathfrak{A}| : \mathfrak{A} \in K\}.$$

Thus, $||K|| = \infty$ means that K has arbitrarily large structures.

Theorem 1. *If $Beth(L, L^*)$ holds and L^* has κ-URP in some extension, then any $K \in RPC(L)$ with $||K|| \geq \kappa$ has non rigid models. If, in particular, L^* has ∞-URP in some extension then no proper class of rigid structures is RPC in L.*

Proof. If $K \in RPC(L)$ is rigid with $||K|| \geq \kappa$, then the class K^p of structures isomorphic to $\langle A \cup S, \mathfrak{A}, \in\restriction(A \times S)\rangle$, where $\mathfrak{A} \in K$ and $S \subseteq \mathcal{P}(A) \cup \mathcal{PP}(A)$ is extensional, inherits from K rigidity and the property of being RPC in L. Moreover, $||K^p|| \geq \sup\{2^{2^\delta} : \delta < \kappa\}$. Assume that $K^p = \{\mathfrak{A}\restriction P^{\mathfrak{A}}\restriction\mu : \mathfrak{A} \models \varphi\}$, where $\mu \subseteq \tau$ and $\varphi \in L(\tau)$ with minimum $|\tau|$. Since the logic has relativizations, and conjunctions of power $|\tau|$, there is a sentence $\theta(F) \in L(\tau^+ \cup \{F\})$, $\tau^+ = (\tau \cup \{c\}) \oplus (\tau \cup \{c\})$, defining the following class of structures

$$\{\langle[\langle\mathfrak{A}, a\rangle, \langle\mathfrak{B}', b\rangle], f\rangle : \mathfrak{A}, \mathfrak{B} \models \varphi, \text{ and } f : \mathfrak{A}\lceil P^{\mathfrak{A}}\lceil\mu \to \mathfrak{B}'\lceil P'^{\mathfrak{B}'}\lceil\mu' \text{ is an isomorphism}\}.$$

Here, $'$ denotes a disjoint copy of each symbol, structure, or relation, and F is interpreted by f, which is an isomorphism in the sense that it sends the interpretation of $R \in \mu$ to the interpretation of the corresponding $R' \in \mu'$. The sentence $\theta(F)$ defines F implicitly, due to the rigidity of K^p. By hypothesis, there is an explicit definition $\rho(c, c') \in L^*(\tau^+)$ of F and a set $\Phi \subseteq L^{**}(\tau \cup \{c\})$ of power $|\Phi| = \delta < \kappa$ uniformly reducing $\rho(c, c')$, in some extension L^{**} of L^*. Since there is $\mathfrak{A} \models \varphi$ such that $|P^{\mathfrak{A}}| \geq 2^{2^\delta}$ and the equivalence $\equiv_\Phi$ has index at most 2^δ, we may find distinct $a, b \in P^{\mathfrak{A}}$ such that $\langle\mathfrak{A}, a\rangle \equiv_\Phi \langle\mathfrak{A}, b\rangle$. Let $\mathfrak{A}'$ be a disjoint copy of $\mathfrak{A}$ with a necessarily unique isomorphism $h : \mathfrak{A} \to \mathfrak{A}'$. Then $\mathfrak{M}_1 = [\langle\mathfrak{A}, a\rangle, \langle\mathfrak{A}', h(a)\rangle] \equiv_{\rho(c,c')} [\langle\mathfrak{A}, a\rangle, \langle\mathfrak{A}', h(b)\rangle] = \mathfrak{M}_2$. But $\langle\mathfrak{M}_i, h\lceil P^{\mathfrak{A}}\rangle \models \theta(F)$; thus, $\langle\mathfrak{M}_i, h\lceil P^{\mathfrak{A}}\rangle \models F(c) = c'$ if and only if $\mathfrak{M}_i \models \rho(c, c')$ for $i = 1, 2$. This is a contradiction, since $\langle\mathfrak{M}_1, h\lceil P^{\mathfrak{A}}\rangle \models F(c) = c'$ and $\langle\mathfrak{M}_2, h\lceil P^{\mathfrak{A}}\rangle \not\models F(c) = c'$. □

As an immediate application of the previous theorem we have the following strengthening of the undefinability of well order in $L_{\infty\omega}$: *no proper class of rigid structures is RPC in* $L_{\infty\omega}$. Indeed, $Beth(L_{\kappa\omega}, L_{(2^{<\kappa})^+\kappa})$ holds by a result of Malitz (Th. 5.1 [M]) and thus $Beth(L_{\infty\omega}, L_{\infty\infty})$; moreover, $L_{\infty\infty}$ has ∞-URP after Example 1. A finer version of this result is the following one, due originally to Chang [Ch] for successor κ.

Example 2. If κ is regular, there is $r < \beth_{(2^{<\kappa})^+}$ such that any $K \in RPC(L_{\kappa\omega})$ with $||K|| \geq r$ has non rigid models. Indeed, combining the above mentioned result of Malitz with the fact that the logic $L_{(2^{<\kappa})^+\kappa}$ has $\beth_{(2^{<\kappa})^+}$-URP in some extension (Example 1), Theorem 1 implies that any rigid $K \in RPC(L_{\kappa\omega})$ has $||K|| < \beth_{(2^{<\kappa})^+}$, and thus $||K||^+ < \beth_{(2^{<\kappa})^+}$ since the last cardinal is limit. But there are essentially $2^{<\kappa}$ distinct RPC classes in $L_{\kappa\omega}$ if we identify isomorphic vocabularies. Thus we may take $r = \sup\{||K||^+ : K \in RPC(L), K \text{ rigid}\}$, which is less than $\beth_{(2^{<\kappa})^+}$ by regularity of $(2^{<\kappa})^+$.

Example 3. $B(L(aa))$, the Beth closure of Stationary Logic, does not have ∞-URP in any extension. Under the continuum hypothesis, the same is true of the infinitely deep logic $M_{\omega_2\omega_1}$. This follows from Theorem 1, since $L(aa)$ has a rigid sentence with arbitrarily large models (Otto [O]), and the second logic has the interpolation property under CH (Hyttinen [Hy]), and obviously defines well-order. We may conclude, further, that these two logics are not contained in $L_{\infty\infty}$ and do not have PPP in any small extension.

Considering $\kappa = \omega$ in Theorem 1, we obtain:

Theorem 2. *If $Beth(L, L^*)$ holds and L^* has URP in some extension, then any RPC class of L having infinite or arbitrarily large finite models has models with infinite automorphism group.*

Proof. Under the given hypothesis, any $K \in RPC(L)$ with $||K|| \geq \omega$ must have infinite models. Otherwise, the rigid class $\{(\mathfrak{A}, <) : \mathfrak{A} \in K$ and $<$ is a linear order of $A\}$ would contradict Theorem 1 for $\kappa = \omega$. Adding sentences to insure that K has only infinite models and applying Theorem 1 again, inductively, we obtain that the following RPC classes of L:

$$K_n = \{\langle \mathfrak{A}, a_i, f_i \rangle_{1 \leq i \leq n} : \mathfrak{A} \in K,\ f_j \in Aut\langle \mathfrak{A}, a_i, f_i \rangle_{1 \leq i \leq j-1},\ f_j(a_j) \neq a_j \text{ for } 1 \leq j < i \leq n\},$$

are non empty for all $n \geq 1$. Since the f_i are distinct by construction, the RPC class $K' = \{G : G$ is a group acting faithfully on some $\mathfrak{A} \in K\}$ has arbitrarily large finite models, and thus infinite models by the initial remark. □

Theorem 2, Claim 2.2 in [S1], and an obvious application of compactness yield:

Corollary 3. *If L is a compact logic satisfying Beth property and ∞-URP (PPP for small L), then any RPC_Δ class of L having infinite models has models with arbitrarily large automorphism group.*

Example 4. Any RPC_Δ class of $B(L(Q^{cf}_{\leq 2^\omega}))$ having infinite models has models with arbitrarily large automorphism group. Because this logic satisfies compactness and URP (Shelah [S1]).

Notice that under the hypothesis of Theorem 2 the class $\{\mathfrak{A} : \mathfrak{A} \approx \langle \omega, < \rangle\}$ can not be RPC in L. Thus, L is *recursively compact,* in the sense that compactness holds for recursive theories of L, by a well know argument of Lindström. In fact, recursive compactness follows already from the weak Beth property by the following theorem, sharpening Claim 3.6 in [S2], where $wo(L)$ denotes the *well ordering number* of L.

Theorem 4. *If L has the weak Beth property and $\beth_\alpha$-URP in some extension, then $wo(L) \leq \max(\omega, \alpha)$. In case α is a recursive ordinal, then $wo(L) = \omega$, and thus L is recursively compact. If L has the weak Beth property and ∞-URP in some extension then well-order is not definable in L.*

Proof. If $wo(L) > \max(\omega, \alpha)$ then L pins down $\alpha + 2$ by a RPC class K of well ordered sets in L. Hence, $K^* = \{\mathfrak{A} \approx \langle S, \in \rangle : S \subseteq V_\beta,\ S$ extensional, $\beta \in K\}$ is RPC in L and rigid, with $||K^*|| \geq |V_{\omega+\alpha+2}| \geq \beth_{\alpha+2}$. Moreover, between any pair of structures in K^* there is an implicitly defined maximal partial isomorphism with transitive domain and co-domain. This allows the use of weak Beth instead of Beth to obtain a contradiction, as in the proof of Theorem 1. Finally, if α is recursive then L can not pin down ω. Otherwise, it would pin down $\max(\omega, \alpha)$, a contradiction. □

Example 5. (Gostanian and Hrbáček [GH]). *If L pins down an infinite regular cardinal κ, then $WB(L) \nleq L_{\kappa\kappa}$. If the class of well-orders is RPC in L then $WB(L) \nleq L_{\infty\infty}$.* This follows immediately from Theorem 4 and Example 1.

Example 6. *$WB(L(Q_0))$ does not have URP in any logic, nor PPP in any $L_{\beth_\alpha\beth_\alpha}$ for recursive α.* Otherwise, Theorem 4 would be contradicted, because the first logic pins down all recursive ordinals and PPP in $L_{\beth_\alpha\beth_\alpha}$ would become $\beth_{\alpha+1}$-URP. This shows how far is the weak Beth closure from preserving PPP.

Example 7. (à la Lindström) *If $L \leq L_{\omega\omega}(Q^i : i \in I)$ has the weak-Beth property, Löwenheim number ω, and PPP, then $L \equiv L_{\omega\omega}$.* Indeed, by the Löwenheim property and finite dependence we may assume $|I| \leq \beth_2$, and thus PPP becomes $\beth_3$-URP. Then, L is recursively compact by Theorem 4, which is enough, under finite dependence, to complete the proof of Lindström's first theorem [L].

3. A recursive Ehrenfeucht-Mostowski theorem

Assuming Craig's interpolation property instead of Beth's property, we may say more about the non rigid models provide by Theorem 1.

Lemma 5. *Let G be the set of partial isomorphisms of a finite linearly ordered set $\langle I, <\rangle$. Assume $Craig(L, L^*)$ holds and L^* has κ-URP in some extension, for κ strong limit. If $K \in RPC(L)$ has, for each $\delta < \kappa$, a model with $|P^{\mathfrak{A}}| > \delta$, then it has a model $\mathfrak{A}$ with $I \subseteq P^{\mathfrak{A}}$ such that any $h \in G$ extends to an automorphism of $\mathfrak{A}$.*

Proof. Let $\langle I, <\rangle = \langle\{0, ..., m\}, <\rangle$, $G = \{h_i\}_{i\leq n}$, and let $\mathbf{d}_i = (d_{i0}, .., d_{ik_i})$, $\mathbf{r}_i = (r_{i0}, .., r_{ik_i})$ be the ordered domain and range of h_i, respectively. If K is the relativized reduct of a sentence φ, consider the following two projective classes of L, where $\mathbf{a}_i = (a_{i0}, ..., a_{ik_i})$, $\mathbf{b}_i = (b_{i0}, ..., b_{ik_i})$, $\mathbf{e}_i = (e_{i0}, ..., e_{ik_i})$:

$$\begin{aligned} K_1 = \{[\langle\mathfrak{A}, <, \mathbf{a}_i, \mathbf{b}_i\rangle_{i\leq n}, \langle\mathfrak{B}_0, \mathbf{e}_0\rangle, ..., \langle\mathfrak{B}_n, \mathbf{e}_n\rangle] : \langle\mathfrak{A}, \mathbf{a}_i\rangle \approx \langle\mathfrak{B}_i, \mathbf{e}_i\rangle, i \leq n,\\ \mathfrak{A} \models \varphi, \text{ and } \langle\{a_{ij}, b_{ij}\}, <, \mathbf{a}_i, \mathbf{b}_i\rangle_{i\leq n} \approx \langle\{0, ..., m\}, <, \mathbf{d}_i, \mathbf{r}_i\rangle_{i\leq n}\}\\ K_2 = \{[\langle\mathfrak{A}, <, \mathbf{a}_i, \mathbf{b}_i\rangle_{i\leq n}, \langle\mathfrak{B}_0, \mathbf{e}_0\rangle, ..., \langle\mathfrak{B}_n, \mathbf{e}_n\rangle] : \langle\mathfrak{A}, \mathbf{b}_i\rangle \approx \langle\mathfrak{B}_i, \mathbf{e}_i\rangle, i \leq n\}. \end{aligned}$$

By hypothesis, there are, for any sentence $\theta \in L^*$ in the common vocabulary of K_1, K_2, sets $\Phi_i(c_0, ..., c_{k_i})$ of power $\delta < \kappa$ in some extension L^{**}, such that: $\langle\mathfrak{B}_i, \mathbf{e}_i\rangle \equiv_{\Phi_i} \langle\mathfrak{B}'_i, \mathbf{e}'_i\rangle$ for $i \leq n$ implies

$$[\mathfrak{C}, \langle\mathfrak{B}_0, \mathbf{e}_0\rangle, ..., \langle\mathfrak{B}_n, \mathbf{e}_n\rangle] \equiv_\theta [\mathfrak{C}, \langle\mathfrak{B}'_0, \mathbf{e}'_0\rangle, ..., \langle\mathfrak{B}'_n, \mathbf{e}'_n\rangle].$$

Also, there is $\mathfrak{A} \models \varphi$ with $|P^{\mathfrak{A}}| > \beth_{\Sigma k_i+m}(\delta) \geq \beth_{k_0}(\beth_{\Sigma_{i\geq 1}k_i+m}(2^\delta))$. Ordering $P^{\mathfrak{A}}$, and applying Erdös-Rado theorem (or Ramsey's theorem if $\kappa = \omega$) to the partition induced by the relation $\mathbf{x} \sim \mathbf{y} \Leftrightarrow \langle\mathfrak{A}, \mathbf{x}\rangle \equiv_{\Phi_0} \langle\mathfrak{A}, \mathbf{y}\rangle$ in ordered sequences of length k_0, we may find in $P^{\mathfrak{A}}$ a subsequence of length greater than $\beth_{\Sigma_{i\geq 1}k_i+m}(2^\delta)$, indiscernible for $k_0 + 1$-tuples with respect to $\Phi_0(c_0, ..., c_{k_0})$. Continuing inductively, we find in $P^{\mathfrak{A}}$ a subsequence of length greater than $\beth_m(2^\delta) \geq m + 1$, indiscernible with respect to $\Phi = \cup_{i\leq n}\Phi_i(c_0, ..., c_{k_i})$. Choose a_{ij}, b_{ij} in that sequence so that $\langle\{a_{ij}, b_{ij}\}, <, \mathbf{a}_i, \mathbf{b}_i\rangle_{i\leq n} \approx \langle\{0, ..., m\}, <, \mathbf{d}_i, \mathbf{r}_i\rangle_{i\leq n}$. Then $\mathbf{a}_i$ and $\mathbf{b}_i$ are both increasing and, by Φ-indiscernibility, $\langle\mathfrak{A}, \mathbf{a}_i\rangle \equiv_{\Phi_i} \langle\mathfrak{A}, \mathbf{b}_i\rangle$ for $i \leq n$. Thus,

$$[\langle\mathfrak{A}, <, \mathbf{a}_i, \mathbf{b}_i\rangle_{i\leq n}, \langle\mathfrak{A}, \mathbf{a}_0\rangle, .., \langle\mathfrak{A}, \mathbf{a}_n\rangle] \equiv_\theta [\langle\mathfrak{A}, <, \mathbf{a}_i, \mathbf{b}_i\rangle_{i\leq n}, \langle\mathfrak{A}, \mathbf{b}_0\rangle, .., \langle\mathfrak{A}, \mathbf{b}_n\rangle],$$

showing that K_1 and K_2 are inseparable in L^*. By the interpolation hypothesis, there is $[\langle\mathfrak{A}^*, <, \mathbf{a}_i^*, \mathbf{b}_i^*\rangle_{i\leq n}, \langle\mathfrak{B}_1, \mathbf{e}_1\rangle, ..., \langle\mathfrak{B}_n, \mathbf{e}_n\rangle] \in K_1 \cap K_2$. Then $\mathfrak{A}^* \models \varphi$, there are isomorphisms $f_i^* : \langle\mathfrak{A}^*, \mathbf{a}_i^*\rangle \approx \langle\mathfrak{B}_i, \mathbf{e}_i\rangle \approx \langle\mathfrak{A}^*, \mathbf{b}_i^*\rangle$, and $\langle\{a_{ij}^*, b_{ij}^*\}, <, \mathbf{a}_i^*, \mathbf{b}_i^*\rangle_{i\leq n} \approx \langle\{0, ..., m\}, <, \mathbf{d}_i, \mathbf{r}_i\rangle_{i\leq n}$. Thus f_i^* is an automorphism of $\mathfrak{A}^*$ extending $h_i^*(a_{ij}^*) = b_{ij}^*$, which we may identify with h_i by the last isomorphism. □

Taking $\kappa = \omega$ in the previous lemma, we get a recursive version of the Ehrenfeucht-Mostowski theorem.

Theorem 6. *Let G be a recursive group of automorphisms of a recursive linearly ordered set $\langle I, <\rangle$. If $Craig(L, L^*)$ holds and L^* has URP in some extension, then any RPC class of L having models with infinite $P^{\mathfrak{A}}$ has a model $\mathfrak{A}$, with $I \subseteq P^{\mathfrak{A}}$, such that any $h \in G$ extends to an automorphisms of $\mathfrak{A}$.*

Proof. Assume $\langle I, <\rangle = \langle\omega, \lhd\rangle$, where $\lhd$ is a recursive linear order of ω, and $G = \{h_n\}_{n\in\omega}$, where $h(i, j) = h_i(j)$ is recursive and h_0 is the identity function. Add to the vocabulary of the sentence φ defining the given RPC class as a relativized reduct the new symbols: c (constant), S (unary function), F (binary function), and $<$ (binary relation). Writing c_n for $S^n(c)$ and $f_n(x)$ for $F(c_n, x)$, we must show that $\{\varphi\} \cup T$ is satisfiable, where T is the first order recursive theory:

$$\{c_i < c_j\}_{i\lhd j} \cup \{P(c_i),\ f_i(c_j) = c_{h_i(j)},\ \text{“}f_i \text{is an automorphism”}\}_{i,j\in\omega}.$$

By recursive compactness of L (Theorem 4), it is enough to show that there is, for each n, a model of $\{\varphi\} \cup \{c_i < c_j\}_{i\lhd j,\ i,j\leq n} \cup \{P(c_i),\ f_i(c_j) = c_{h_i(j)}, \text{“}f_i \text{ is an automorphism”}\}_{i,j\leq n}$. That is, a model of φ with the interpretation of P containing $I_n = \{h_i(j) : i, j \leq n\} \supseteq \{0, ..., n\}$, and such that all the partial isomorphisms $h_i \restriction \{0, 1, .., n\}$, $i = 0, ..., n$, of $\langle I_n, \lhd\rangle$ are extendible to automorphisms of the model. Such model is provided by the previous lemma. □

Corollary 7. *If $Craig(L, L^*)$ holds and L^* has URP in some extension, then any $K \in RPC(L)$ with infinite models has a model having $\langle\mathbb{Q}, <\rangle$ as a set of indiscernibles.*

Proof. The automorphisms of $\langle\mathbb{Q}, <\rangle$ consisting of finite unions of linear functions defined on rational intervals form a recursive group which puts any pair of ordered n-tuples of $\mathbb{Q}$ in the same orbit. □

4. A full Ehrenfeucht-Mostowski theorem

The next theorem could be meaningless beyond $L_{\omega\omega}$, since no other logic is known to satisfy the hypothesis, unless we drop relativizations. But it provides, at least, a new proof of the Ehrenfeucht-Mostowski theorem, and shows how close to $L_{\omega\omega}$ would be a compact logic satisfying interpolation and URP.

Theorem 8. *Let L be a compact logic satisfying interpolation and URP (equivalently, ∞-URP, or just PPP for small L). If $K \in RPC_\Delta(L)$ has infinite models, then any linearly ordered set $\langle I, <\rangle$ may be embedded in some $\mathfrak{A} \in K$ such that any automorphism of $\langle I, <\rangle$ extends to an automorphism of $\mathfrak{A}$.*

Proof. Assume $K = \{\mathfrak{A} \restriction P^{\mathfrak{A}} \restriction \mu : \mathfrak{A} \models \Phi\}$ with $\Phi \subseteq L(\tau)$. Let $\{c_i : i \in I\}$ be a set of new constants and $\{f_h : h \in Aut(I, <)\}$ a set of new unary function symbols. By compactness, it is enough to show the satisfiability of each finite part of the theory:

$$\Phi \cup \{c_i < c_j\}_{i<j} \cup \{\text{“}f_h \text{ is an automorphism”}, f_h(c_i) = c_{h(i)}\}_{i \in I,\ h \in Aut(I,<)}.$$

This follows from the existence of infinite models of K, applying Lemma 5 with $\kappa = \omega$. □

It is well known that, under compactness, interpolation implies Robinson's joint consistency property, and that the converse is true for small logics. Furthermore, for logics having bounded dependence number (smaller than the first measurable cardinal μ_0, if this exists), Robinson's property implies compactness (cf. [Mu]). Therefore, using that $Th(\mathfrak{A})$ is a set in a small logic, we have the following improvement of Th. 4.5.1 in [Ma], inadvertently stated there without the dependence hypothesis.

Corollary 9. *Let L be a small logic with Robinson's property, PPP, and bounded dependence number (below μ_0, if measurable cardinals exist). Then, for any given linearly ordered set $\langle I, <\rangle$ and structure $\mathfrak{A}$, there is an L-extension $\mathfrak{A}^*$ of $\mathfrak{A}$ containing I such that any automorphism of $\langle I, <\rangle$ extends to an automorphism of $\mathfrak{A}^*$.*

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