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Continuous Operations on Spaces of Structures

Abstract. Given a model theoretic logic L and a vocabulary τ, the space $E_\tau(L)$ of τ-structures topologized by the L-elementary classes as an open base is a natural uniform space, and the uniform continuity of operations arising between these spaces (powers, quotients, all sort of algebraic functors) faithfully reflects the properties of L, or the relations of L with other logics. Most fundamental properties of logics as closure under substitutions or relativizations, uniform reduction for pairs, Craig's interpolation lemma, Δ-interpolation, and Robinson's lemma (for small occurrence number) are all equivalente to the uniform continuity of certain families of projective operations. We give in this paper a survey of possible application of this topological approach.

Contents

M. Krynicki et al. (eds.), Quantifiers: Logics, Models and Computation, Volume I, 263-296.

Introduction

It is well known that the class of first order structures of a fixed similarity type forms a topological space if endowed with the elementary classes as a basis of open sets. In this way first order logical properties of models correspond to topological properties. An obvious example is compactness of first order logic which corresponds to topological compactness of these spaces. If we identify elementarily equivalent structures, the resulting quotient spaces are the Stone spaces of the Boolean algebras of first order formulae. These "first order topologies" or strengthenings of them have been utilized by many authors specially in the study of types of elements; see [Rasiowa-Sikorski 1950], [Ehrenfeucht-Mostowski 1961], [Fraïssé 1972], [Morley 1965], [Morley 1974], [Hanf-Myers 1983], [Baldwin-Plotkin 1974].

We intend to illustrate in this paper the relevance of the topological viewpoint for the more general study of model theoretic logics, [Barwise-Feferman 1985]. It unifies, simplifies, and gives deeper insight in many aspects of abstract model theory. Topological methods have not been utilized in this context except by [Mundici 1986]. The topology of elementary classes in spaces of structures associated to a logic is uniform, and it is precisely the uniform continuity of operations arising between them (cartesian products, quotients, all sort of algebraic constructions, etc.) that reflects the model theoretical properties of the logics. For example, the relativization property as well as other axioms for logics are uniform continuity phenomena. The uniform reduction property for a operations in the sense of [Gaifman 1974] and [Feferman 1974], or the property of being a construction for an interpretation in the sense of [Szczerba 1977] and [Krynicki 1988] are no more than uniform continuity. Following [Feferman 1974] and [Makowsky 1985] interpolation, Δ-interpretation, Beth's definability, and the Robinson's consistency properties are equivalent to the uniform continuity of certain families of operations.

Several results in the literature such as Mundici's characterization of compactness by separation properties, or the uniqueness of a compact logic with a given elementary equivalence relation due to [Lipparini 1985], may be shown to be topological phenomena. The existence of models with automorphisms and the homogeneity property (see [Shelah 1985]) may be shown also to have topological content.

Only familiarity with the very fundamental concepts of abstract model theory and topology (including uniform spaces) is assumed. Our main reference for model theory will be [Bell-Slomson 1971] and [Barwise-Feferman 1985], and for topology [Willard 1968].

We must thank the group of logic of the Department of Mathematics of the University of Helsinki, Finland, for inviting us in May 1989 to give a series of lectures where we presented a first draft of these ideas. We were able to develop them further thanks to another invitation received from the logic group of the Universidade Estadual de Campinas, Brasil, to lecture in June and July of 1989.

1. Model Spaces

Given a (similarity) type τ of single sorted first order structure, let E_τ be the class of all structures of type τ. First order structure will be denoted by plain letters $A, B, C, \ldots$ and their universes by $|A|, |B|, |C| \ldots$. The last notation will be also used to denote their cardinality; the actual meaning will be clear from the context. To complete the abuse of the slash, we shall write $A|\mu$ to denote the *projection* (or *reduct*) of the structure A to a subtype μ of the type of A; and $A|P^A$ will denote the *restriction* (or *relativization*) of A to the subuniverse P^A, where P is a monadic predicate symbol. Notice that $A|P^A$ exists only when P^A is closed under the functions of A.

Let L be a *small single sorted logic closed under substitutions* in the sense of Definition 1.1.1 in [Ebbinghaus 1985], that is, a triple $(Dom(L), L, \models)$ where $Dom(L)$ is a class of single sorted similarity types, L is a function which assigns a set of *sentences* $L(\tau)$ to each $\tau \in Dom(L)$, and $\models$ is a relation in $\bigcup_\tau E_\tau \times L(\tau)$, satisfying properties 1 to 4 below:

1. *Isomorphism property*. If $A \approx B$ then for all $\varphi \in L(\tau)$, $A \models \varphi \Leftrightarrow B \models \varphi$.

2. *Renaming property*. For any arity-preserving bijection $f : \mu \to \tau$ sending relation, function, and constant symbols to the like, and $\varphi \in L(\tau)$ there is $\alpha(\varphi) \in L(\tau)$ such that $A \models \varphi \Leftrightarrow (|A|, f(R)^A)_{R\in\tau} \models \alpha(\varphi)$, for all $A \in E_\tau$.

3. *Reduct property*. If $\mu \subset \tau$, then $L(\mu) \subset L(\tau)$ and $A \models \varphi \leftrightarrow A|\mu \models \varphi$, for all $A \in E_\tau$ and $\varphi \in L(\mu)$.

Given distinct constant symbols $c_1, \ldots, c_n \notin \tau$, a sentences $\varphi \in L(\tau \cup \{c_1, \ldots, c_n\})$ will be called an *n-ary formula of* $L(\tau)$, and its *truth set* in a structure A will be $\phi^A = \{(a_1, \ldots, a_n) : (A, a_1, \ldots, a_n) \models \varphi\}$.

4. *Substitution property*. For n_i-ary formulae φ_i of $L(\tau), i = 1, \ldots, k$ and $\theta \in L(\tau \cup \{R_1, \ldots, R_k\})$ where R_i is n_i-ary, there is $\alpha(\theta) = \theta(R_1/\varphi_1, \ldots, R_k/\varphi_k) \in L(\tau)$ such that: $A \models \alpha(\theta) \Leftrightarrow (A, \varphi_1^A, \ldots, \varphi_k^A) \models \theta$.

We will assume also that L contains first order logic $L_{\omega\omega}$ and is closed under *conjunctions* and *negations*. $Dom(L)$ does not need to contain all similarity types; we only assume it to be closed under sub-types, renamings, expansions by a constant or a monadic predicate, and finite unions. For example a logic may be purely monadic and defined only in countable types. If a logic in addition the following property we will call it a *regular* logic:

5. *Relativization property* For any monadic $P \in L(\tau)$ and $\varphi \in L(\tau - \{P\})$ there is $\varphi^P \in L(\tau)$ such that $A \models \varphi^P \Leftrightarrow A|P^A \models \varphi$ whenever $A|P^A$ is defined.

Logics are preordered by their expressive power: $L \leq M$ if and only if for any $\varphi \in L(\tau)$ there is an $\alpha(\varphi) \in M(\tau)$ such that $A \models \alpha(\varphi) \Leftrightarrow A \models \varphi$ for all $A \in E_\tau$. Also $L \equiv M$ if $L \leq M$ and $M \leq L$; and $L < M$ if $L \leq M$ but $L \not\equiv M$.

1.1 Definition. Given a type τ, let $E_\tau(L)$ be the class E_τ endowed with the topology having as a base the L-elementary classes $Mod(\varphi), \varphi \in L(\tau)$.

The L-elementary classes form a base of clopen sets due to closure under conjunctions and negations. The closed classes in this space are precisely the classes $Mod(T)$ of models of some theory $T \subset L(\tau)$. The space $E_\tau(L)$ becomes a *uniform space* (see [Willard 1968], Definition 35.2) if we take as a *base* for the uniformity the classes

$$U_\Phi = \{(A, B) \, : \, A \equiv_\Phi B\}$$

where Φ runs through the finite subsets of $L(\tau)$, and $A \equiv_\Phi B$ means: $A \models \varphi \Leftrightarrow B \models \varphi$ for any $\varphi \in \Phi$. Evidently, the classes U_Φ satisfy the conditions of a base for a uniformity since:

1) $\Delta \subset U_\Phi$
2) $U_\Phi^{-1} = U_\Phi$
3) $U_\Phi \circ U_\Phi = U_\Phi$
4) $U_\Phi \cap U_\Gamma = U_{\Phi \cup \Gamma}$.

This base generates the elementary topology because the projection of any basic of the uniformity, say $U_\Phi(A) = \{B | (A, B) \in U_\Phi\}$, coincides with the L-elementary class $Mod(Th_\Phi(A))$ where $Th_\Phi(A) = \bigwedge\{\varphi \in \Phi : A \models \varphi\} \cup \{\neg\varphi \in \Phi : A \not\models \varphi\}$; and conversely any nonempty basic open is a projection: $Mod(\varphi) = U_{\{\varphi\}}(A)$ for $A \models \varphi$. We will call the uniformity generated by this base the *canonical uniformity of* $E_\tau(L)$.

If $L(\tau)$ is countable then $E_\tau(L)$ is actually a pseudometric space. For example, if the logic is countably generated $L = L_{\omega\omega}(Q^n | n \in \omega)$ and τ is countable, a explicit pseudometric for the uniformity of $E_\tau(L)$ is given by

$$d(A,B) = inf\{1/n+1 \ : \ A \equiv_{L^n(\tau_n)(Q^1,\ldots,Q^n)} B\}$$

where $\tau_1, \tau_2, \ldots,$ is an enumeration of the finite subtypes of τ, and $L^n(\tau_n)$ denotes the sentences of $L(\tau_n)$ of quantifier rank at most n.

Clearly, $L \leq M$ if and only if for all τ the canonical uniformity of $E_\tau(L)$ is weaker than the canonical uniformity of $E_\tau(M)$, equivalently, if the identity operation $I : E_\tau(M) \to E_\tau(L)$ is uniformly continuous. $L \equiv M$ if and only if both logics have the same canonical uniformities.

REMARK. Since E_τ is always a proper class, model spaces are very large indeed, but this should cause no difficulty. As we are assuming the logic L to be small the set of sentence $L(\tau)$ parametrizes a canonical base for the uniform topology and so the topology itself is (parametrized by) a set. We could work also in the quotient spaces $E_\tau(L)/\equiv_L$ obtained by identifying L-elementarily equivalent structures, which become ordinary topological spaces and share many topological properties with the $E_\tau(L)$. However, we prefer to work in the model spaces because some significant situations are not reflected in the quotients. For example, the images of disjoint projective subclasses of $E_\tau(L)$ are not necessarily disjoint in $E_\tau(L)/\equiv$, which makes it unnatural to even state the interpolation properties in these space.

Recall that a uniform space is *totally bounded* if for each element U of the uniformity (base) the space may be covered with a finite number of projections of U. Our first observation is that model spaces are always totally bounded.

1.2. LEMMA. *$E_\tau(L)$ is always totally bounded with respect to the canonical uniformity.*

PROOF. Given a finite set of sentences Φ, the equivalence relation $\equiv_\Phi$ has finite index. Picking a representative A_i of each equivalence class, the corresponding projections $U_\Phi(A_i) = \{A \ : \ A \equiv_\Phi A_i\}$ will form a finite covering (and partition) of the space. □

2. Compactness

A *net* in a uniform space $(X, \mathcal{U})$ is a family $(x_\alpha)_{\alpha \in I}$, where $(I, \leq)$ is a directed set (this is, for all $\alpha, \beta \in I$ there is $\gamma \in I$ such that $\alpha \leq \gamma, \beta \leq \gamma$). It is a *Cauchy net* if for any $U \in \mathcal{U}$ there is $\alpha \in I$ such that for all $\beta, \gamma \geq \alpha$, $(x_\beta, x_\gamma) \in U$. It *converges* to x if for any $U \in \mathcal{U}$ there is $\alpha \in I$ such that for all $\beta \geq \alpha, (x_\beta, x) \in U$; then x is called a *limit* of the net. Of course, limits do not need to be unique. A uniform space is *complete* if all its Cauchy nets converge.

It is well known that a uniform space is compact if and only if it is complete and totally bounded (see Theorem 39.9 in [Willard 1968]). Hence, to show compactness of a logic L it is enough to prove completeness of the spaces $E_\tau(L)$. In fact, behind the well known proof of compactness of first order logic by means of ultraproducts, [Bell-Slomson 1971], there is a proof of the Cauchy-completeness of the spaces $E_\tau(L_{\omega\omega})$.

2.1 Corollary (Compactness of first order logic). *If $K \subseteq E_\tau(L_{\omega\omega})$ is closed under ultraproducts then it is compact. In particular $E_\tau(L_{\omega\omega})$ is compact.*

Proof. K is also uniform and totally bounded for the subspace uniformity; we have to show it is complete. Let $(A_\alpha)_{\alpha\in\Sigma}$ be a Cauchy net in K, where Σ is a directed set, then for each basic U_Φ of the canonical uniformity of X there is $\alpha_\Phi \in \Sigma$ such that $(A_\beta, A_\gamma) \in U_\Phi$, this is $A_\beta \equiv_\Phi A_\gamma$, for all $\beta, \gamma \geq \alpha_\Phi$. Let $[\alpha) = \{\beta \in \Sigma | \beta \geq \alpha\}$, then the set

$$\mathcal{F} = \{[\alpha_\Phi) \ : \ \Phi \text{ a finite subset of } L(\tau)\}$$

is a filter basis over Σ because Σ is directed. Let $\mathcal{F}^*$ be an ultrafilter extending $\mathcal{F}$, then the net converges to the corresponding ultraproduct of the A_α:

$$(1) \qquad (A_\alpha)_{\alpha\in\Sigma} \to A^* = \Pi_{\alpha\in\Sigma} A_\alpha / \mathcal{F}^*,$$

because given U_Φ we have $A_\beta \models T^0 = Th_\Phi(A\alpha_\Phi)$ for any $\beta \in [\alpha_\Phi) \in \mathcal{F}^*$; by Łoś Theorem on ultraproducts, this implies $A^* \models T^0$ and so $A^*, A_\beta \in U_\Phi$ for all $\beta \geq \alpha_\Phi$, showing (1). By hypothesis the ultraproduct belongs to K showing the net converges in K. □

2.2 Corollary ([Bell-Slomson 1971], Theorem 3.4.). *$K \subseteq E_\tau$ is first order axiomatizable if and only if it is closed under ultraproducts and elementary equivalence.*

Proof. One direction is trivial by Łoś Theorem. Now, let K be closed under ultraproducts and elementary equivalence, it is enough to show that it is topologically closed. If $B \in Cl\ K$ then there is a Cauchy net $(A_\alpha)_\alpha$ in K which converges to B. We have seen that the net converges also to a ultraproduct $\Pi A_\alpha/\mathcal{F}$; hence $B \equiv \Pi A_\alpha/\mathcal{F}^*$. By hypothesis, we have $B \in K$. □

For countable $L(\tau)$ compactness is equivalent to convergence of Cauchy sequences of structures. Fraïssé ([Fraïssé 1972]) shows the countable compactness of $L_{\omega\omega}$ by actually constructing the limit of a Cauchy sequence by means of partial isomorphisms.

Given a logic L, let $E^k_\tau(L)$ be the subspace of $E_\tau(L)$ of structures of power less or equal than k. It is easily seen that if L is δ-compact (this is compactness holds for theories of power $\leq \delta$) and satisfies the downward Löwenheim-Skolem Theorem down to k, for theories of power $\leq \delta$, then $E^k_\tau(L)$ is compact whenever $|L(\tau)| \leq \delta$. For example, $E^k_\tau(L\omega\omega)$ is compact if $|\tau| \leq k$.

Let Q_α denote the cardinality quantifier "there are at least ω_α many ...". The compactness of logics generated by these quantifiers still is an open problem. However, [Flum 1985], Theorem 1.3.4, shows that for countable β the monadic part of the logic

$$L_\beta = L\omega\omega(Q_\alpha \ : \ 1 \leq \alpha \leq \beta, \alpha \text{ successor})$$

is countably compact. As L_β obviously satisfies the Löwenheim Theorem down to ω_β for countable theories and $|L_\beta(\tau)| = \omega$ for countable τ and β, then $E^{\omega_\beta}_\tau(L_\beta)$ is a compact space in this case. Moreover, each monadic structure of finite type of power at most ω_β is characterized up to isomorphism by the cardinalities of the complete intersections of its predicates. Since any limit cardinal is a limit of successor (or finite) cardinals, these cardinalities may be expressed by a set of sentences of L_β. This means that if we identify isomorphic structures, the space $E^{\omega_\beta}_\tau(L_\beta)$ is Hausdorff for finite monadic τ. In sum, $E^{\omega_\beta}_\tau(L_\beta)$ is a compact Haussdorf space for finite monadic τ.

2.3 Corollary (Maximality and interpolation for monadic logics with cardinality quantifiers, [Flum 1985a], Theorem 1.3.2). *For countable β, the monadic part of the logic L_β is maximal among all monadic logics with finite occurrence number which are countably compact and satisfy the Löwenheim-Skolem Theorem down to ω_β for countable theories. Moreover, L_β satisfies the interpolation theorem.*

Proof. Let M be an extension of monadic L_β satisfying the conditions of the corollary and let $\varphi \in M(\tau)$, τ finite. By taking the closure of $\{\varphi\} \cup L_{\omega\omega}(\tau)$ under finite conjunctions and negations we may assume without loss of generality that $\varphi \in M(\tau)$ with $|M(\tau)| \leq \omega$. Hence, $E^{\omega_\beta}_\tau(M)$ is compact by the above remarks. This implies that the closed classes $Mod(\varphi)$ and $Mod(\neg\varphi)$ restricted to $E^{\omega_\beta}_\tau$ are compact in $E^{\omega_\beta}_\tau(M)$ and a fortiori in the weaker topology of $E^{\omega_\beta}_\tau(L_\beta)$. By a standard topological argument disjoint compact classes in a compact Hausdorff space may be separated by finite unions of basic open classes. Applying this to $Mod(\varphi)$ and $Mod(\neg\varphi)$ restricted to $E_\tau^{\omega_\beta}(L_\beta)$, φ is seen to be equivalent to a sentence ψ of L_β in models of power $\leq \omega_\beta$, and by the Löwenheim-Skolem Theorem in all models. Similarly, applying this to the restriction to $E_\tau^{\omega_\beta}(L_\beta)$ of disjoint PC classes of L_β which must be compact we conclude that monadic L_β satisfies the interpolation theorem. □

Flum has generalized his results to uncountable β, [Flum 1985b]. The interpolation result holds in $L\omega\omega(Q_\alpha : \alpha \in I)$, without any condition in the α (see [Caicedo 1985]).

3. Separation

Model spaces are not necessarily Hausdorff but they are always *completely regular*, that is points and closed sets may be separated by real valued continuous functions. In fact, complete regularity is a characterization of uniformizability by a theorem of A. Weil (see Theorem 38.2, in [Willard 1968]). The next separation property is *normality*: disjoint closed sets may be separated by disjoint open sets. In model spaces normality means that given families of sentences $\{\varphi_i\}, \{\psi_j\}$ such that $\bigwedge_i \varphi_i \models \bigvee_j \psi_j$, there are families $\{\sigma_s\}$, $\{\rho_r\}$ such that

$$\bigwedge_i \varphi_i \models \bigvee_s \sigma_s \models \bigwedge_r \rho_r \models \bigvee_j \psi_j. \tag{1}$$

More spaces are not necessarily normal, a proof of the following will be given elsewhere.

3.1 THEOREM. *The model spaces of $L\omega\omega(Q_0)$ are not normal for uncountable types. The model spaces of $L\omega\omega(Q_\alpha)$, $\alpha \geq 1$, are not normal for types of power $\geq \omega_\alpha$.*

However, model spaces are always "countably normal"; hence, (1) holds for countable families of sentences.

3.2 LEMMA. *If $L(\tau)$ is countable then $E_\tau(L)$ is normal. Moreover, the separation of disjoint closed classes may be achieved by a single clopen class.*

PROOF. If $L(\tau)$ is countable then $E_\tau(L)$ is second countable and so Lindeloff, but any Lindeloff regular space is normal (Theorem 16.8 in [Willard 1968]). As the space has a basis of clopen classes the separation may be achieved by a clopen (Theorem 16.16, in [Gillman-Jerison 1960]). □

Model spaces of compact logics are normal because compact regular spaces are always normal, and by compactness the separation may be achieved by a single elementary class. Mundici has shown that this strong form of normality is equivalent to compactness for regular logics with finite occurrence number.

3.3 THEOREM ([Mundici 1986], Th. 2.3). *Let $L = L_{\omega\omega}(Q_i : \in I)$ be closed under relativizations, then L is compact if and only if any pair of disjoint closed classes may be separated by an elementary class.*

Some hypothesis of bounded occurrence is necessary in Mundici's Theorem since, for example, $E_\tau(L_{\kappa\kappa})$ is normal in the sense of Theorem 3.3 when κ is a compact cardinal. A purely topological proof of Mundici's result may be given showing that it holds for any logic L closed under relativizations where $Dom(L)$ satisfies certain closure condition. It holds for example for monadic logics, or for countably generated regular logics defined in countable types only.

3.4 COROLLARY. *Let $L = L_{\omega\omega}(Q^n : n \in \omega)$ be closed under relativizations, then L is countably compact if and only if for all countable τ the clopen classes of $E_\tau(L)$ are L-elementary.*

PROOF. By Lemma 3.2, Theorem 3.3, and the above remarks, restricting the logic to countable types. □

QUESTION. Under which conditions is plain normality of model spaces equivalent to compactness?

4. Continuous Operations on Model Spaces

Let τ and μ be types, a *(partial) operation on structures from E_τ to E_μ* is a isomorphism preserving (possibly partial) function $F : E_\tau \to E_\mu$. This is, $A \approx B$ implies $F(A) \approx F(B)$ for all A, B in the domain of F.

Many constructions and functors of mathematical practice and model theory are operations on structures: for example, the field of quotients of an integral domain, the algebraic closure of a field, the free group generated by a set, etc. See [Gaifman 1974], [Feferman 1972], and [Feferman 1974] for more examples.

4.1 DEFINITION. Let L and M be logics, a partial operation $F : E_\tau \to E_\mu$ is said to have the *uniform reduction property, with respect to L and M*, if for any sentence φ in $M(\mu)$ there is a sentence $\alpha(\varphi)$ in $L(\tau)$ such that:

$$(1) \qquad A \models \alpha(\varphi) \text{ if and only if } F(A) \models \varphi.$$

The function α is called a *translation for F*. It is easy to see that when it exists, α is unique and preserves the Boolean operations, modulo logical equivalence, this is $\alpha(\neg\varphi) \equiv \neg\alpha(\varphi), \alpha(\varphi \wedge \psi) \equiv \varphi \wedge \psi$.

This notion is due to [Gaifman 1974] and [Feferman 1974], see [Makowsky 1985]. It corresponds also to the notion of first order *interpretation* studied by [Szczerba 1977], [Gajda-Krynicki-Szczerba 1987], or to the notion of *L-construction* studied by [Krynicki 1988] for an arbitrary logic L. A pair (F, α) satisfying (1) provides a generalized notion of *uniform interpretation*, this is, a uniform reduction of the theory of $F(A)$ in M to the theory of A in L. If α may be chosen recursive, for example, the decidability of $Th_L(A)$ implies the decidability of $Th_M(F(A))$.

From the topological point of view, uniform reduction is no more than uniform continuity. This important fact has been noticed in the case of first order logic ([Gajda-Krynicki-Szczerba 1987], Th. 5.4). Recall that a function $f : (X, \mathcal{U}) \to (Y, \mathcal{V})$ between uniform spaces is *uniformly continuous* if for any $V \in \mathcal{V}$ there exists $U \in \mathcal{U}$ such that $(a, b) \in U$ implies $(f(a), f(b)) \in V$.

4.2 THEOREM. *F has the uniform reduction property with respect to L and M if and only if $F : E_\tau(L) \to E_\mu(M)$ is uniformly continuous.*

PROOF. If F has the uniform reduction property and $D \subset M(\mu)$ is finite, let $\alpha D = \{\alpha(\varphi) : \varphi \in D\} \subseteq L(\tau)$. Then $A \equiv_{\alpha D} B$ implies $F(A) \equiv_D F(B)$. Conversely, suppose F is uniformly continuous, and let $\varphi \in M(\mu)$. Then there is a finite set of sentences $D \subseteq L(\tau)$ such that $(A, B) \in U_D$ implies $(F(A), F(B)) \in U_{\{\varphi\}}$. Since the space is totally bounded we may choose finitely many structures $A_1, \ldots, A_n$ such that the projections $U_D(A_i) = \{A | A \equiv_D A_i\}$ form a partition of the space $E_\tau(L)$. Let $U_D(A_i) = Mod(t_i)$, then

$$\alpha(\varphi) = \bigvee_i \{t_i \ : \ F(A_i) \models \varphi\}$$

is the required translation. If $A \in E_\tau$, then $A \models t_i$ for a unique i; hence, $A \equiv_D A_i$ and so $F(A) \equiv_{\{\varphi\}} F(A_i)$ for that i. Therefore, $A \models \alpha(\varphi)$ if and only if $F(A_i) \models \varphi$ for that i, if and only if $F(A) \models \varphi$. □

Apart from the axiom of isomorphism, the axioms of logics are just assertions about uniform reduction of certain operations.

4.3 COROLLARY. *The following operations are uniformly continuous for any logic L:*

1. *Projections. For each pair $\mu \subset \tau$, the operation $F(A) = A|\mu$ from E_τ to E_μ.*
2. *L-definitional expansions. For any formulae $\varphi_1, \ldots, \varphi_n$ of $L(\tau)$ with φ_i of arity k_i, and relation symbols R_i of arity k_i not in τ, the operation $F(A) = (A, \varphi_1^A, \ldots, \varphi_n^A)$ from E_τ to $E_{\tau\cup\{R1,\ldots,Rn\}}$.*

Moreover, L is closed under relativizations if and only if the following operations are uniformly continuous in L:

3. *Restrictions. For each monadic predicate $P \in \tau$, the operation $F(A) = A|P^A$ from E_τ to $E_{\tau-\{P\}}$. This operation is not total if τ contains function or constant symbols.*

PROOF. Continuity of 1 follows from the reduct property, and that of 2 and 3 is equivalent to the substitution and relativization properties, respectively. □

The following operations are uniformly continuous for $L_{\omega\omega}$. This may be shown by using partial isomorphisms or by defining explicitly recursive translations as in [Feferman-Vaught 1959], or [Szczerba 1977]. This follows also from more general theorems that we discuss later.

4. *Cartesian operations.* For any $R_1, \ldots, R_m \in \tau$ with R_i of arity nk_i and relation symbols S_i of arity k_i not in τ, the operation from E_τ to $E_{\tau\cup\{S1,\ldots,Sn\}}$:
$$F(A) = (|A|^n, R_1^A/n, \ldots, R^A/n),$$
where R_I^A/n denotes the k_i-ary relation defined naturally in $|A|^n$ by: $R_i^A/n((a_{1,1}, \ldots, a_{1,n}), \ldots, (a_{ki,1}, \ldots, a_{ki,n})) \Leftrightarrow R_i(a_{1,1}, \ldots, a_{1,n}, \ldots, a_{ki,1}, \ldots, a_{ki,n})$.
5. *Quotients.* For each type τ and binary $R \notin \tau$, the operation from $E_{\tau\cup\{R\}}$ to E_τ:
$$F(A, \mathbf{R}) = \begin{cases} A/\mathbf{R} & \text{if } \mathbf{R} \text{ is a congruence relation in } A \\ A & \text{otherwise.} \end{cases}$$

6. *Powers.* For each τ and $n \in \omega$ the nth-*power* operation $F : E_\tau \to E_\tau$:

$$F(A) = A^n = \text{ cartesian product } n \text{ times.}$$

4.4 EXAMPLE (FIRST ORDER INTERPRETATIONS, [Szczerba 1977]).Call *classical construction* to the composite of any number of operations of the form (1) to (5), where the domain has finite type and formulae in (3) are first order. These are uniformly continuous with recursive translations for $L_{\omega\omega}$. Power are classical constructions since they may be obtained as the composite of a first order definitional expansion, a cartesian operation, and a projection. [Szczerba 1977] has noticed that these operations may be put in the canonical form:

$$F(A) = (\psi_A, \varphi_1^A/n, \ldots, \varphi_k^A/n)/\theta^A$$

where ψ is a n-ary formula, the φ_i are nk_i-ary formulae such that $\varphi_i \models \psi$, and θ is $2n$-ary defining a congruence relation in the resulting structure. The pairs (F, α) are presicely the *first-order interpretations* usually studied in the literature.

4.5 EXAMPLE. Classical constructions do not need to be uniformly continuous in other natural logics.

a) Quotients are uniformly continuous in logics with Magidor-Malitz quantifiers (see [Krynicki 1988]), but powers do not, since they do not preserve elementary equivalence as shown in [Badger 1977].

b) It is shown in [Caicedo 1990] that quotients are not uniformly continuous in any proper regular extension of $L_{\omega\omega}$ generated by any combination of monadic or linear order quantifiers, for example in $L_{\omega\omega}(Q_\alpha, Q_\alpha^{cof})$. See §10.

4.6 EXAMPLE. After [Feferman-Vaught 1959], *reduced powers*: $F(A) = A^I/U$, where U is a filter in I, are uniformly continuous for $L_{\omega\omega}$.

Of course, uniform continuity of an operation on structures implies continuity but not conversely, unless the domain logic is compact, when both notions are equivalent (pass to $E_\tau(L)/\equiv_L$ and use Theorem 36.20, [Willard 1968]).

4.7 EXAMPLE. Let $L = L_{\omega\omega}(Q_0)$, then the following operation $F : E_\emptyset(L) \to E_\emptyset(L)$ is continuous but not uniformly continuous:

$$F(A) = \begin{cases} \omega & \text{if } |A| \text{ is even or infinite} \\ A & \text{if } |A| \text{ is odd.} \end{cases}$$

Suppose $F(A) \in Mod(\varphi)$. If $|A|$ is infinite then the neighborhood $Mod(Q_0x(x = x))$ of A is sent by F to $\omega = F(A) \in Mod(\varphi)$. If A is finite, let θ be the sentence declaring the power of A, then F sends $Mod(\theta)$ to the isomorphic copies of $F(A)$ which also are in $Mod(\varphi)$. This shows continuity. However, F is not uniformly continuous because for any n we may find finite structures A, B with $|A|$ even and $|B|$ odd such that $A \equiv B$ in $L^n_{\omega\omega}$ (use partial isomorphisms). As Q_0 acts trivially in finite structures, then $A \equiv B$ in $L^n_{\omega\omega}(Q_0)$. On the other hand $F(A) \nvDash Q_0x(x = x)$, $F(B) \vDash Q_0x(x = x)$.

5. Product of Model Spaces

Given types τ and σ, let $\tau \oplus \sigma = \{P\} \cup \tau \cup \{\underline{P}\} \cup \underline{\sigma}$ be the new type where $\underline{\sigma}$ is a copy of σ, disjoint from τ, and $P, \underline{P}$ are two new monadic predicates. We will call if the *disjoint sum* of the types. Given two structures $A \in E_\tau$, $B \in E_\sigma$, the *disjoint sum of A and B* will be the structure of type $\tau \oplus \sigma$:

$$[A, B] = (|A| \cup |\underline{B}|, A, \underline{B})$$

where $\underline{B}$ is a isomorphic copy of B, disjoint from A, P is interpreted by $|A|$, $\underline{P}$ is interpreted by $|\underline{B}|$, the type τ is interpreted by the relations of A, and $\underline{\sigma}$ is interpreted by the relations of $\underline{B}$.

Disjoint sums are the single sorted version of many sorted pairs used in [Feferman 1972], and [Makowsky 1985]. They are denoted $A + B$ and called *full cardinal sums* in [Dickmann 1985], they are denoted $A \oplus B$ in [Szczerba 1977]. Evidently, if disjoint copies are chosen canonically there is a bijection between true pairs of structures $\langle A, B \rangle$ and disjoint sums $[A, B]$. Hence, module isomorphism, we may identify the cartesian product of the class E_τ and E_σ with the following subclass of $E_{\tau\oplus\sigma}$:

$$E_\tau \times E_\sigma = \{C \in E_{\tau\oplus\sigma} \ : \ C = [A, B], A \in E_\tau, B \in E_\sigma\}.$$

The product $E_\tau \times E_\sigma$ inherits two uniformities from L, which in general do not need to be comparable:

— The *canonical uniformity* that it inherits as subspace ([Willard 1968], 37.1) of $E_{\tau\oplus\sigma}(L)$; we denote the corresponding uniform space by

$$(E_\tau \times E_\sigma)(L).$$

This is a closed subspace of $E_{\tau\oplus\sigma}(L)$ because being a disjoint sum is first order axiomatizable, in fact elementary for finite σ an τ.

— The *product uniformity* ([Willard 1968], 37.6) inherited from $E_\tau(L)$ and $E_\sigma(L)$; we will denote the corresponding uniform product space by

$$E_\tau(L) \times E_\sigma(L).$$

The last space has for canonical basis the classes $Mod(\varphi) \times Mod(\psi) = Mod(\varphi^P \wedge \underline{\psi}^{\underline{P}}) \cap (E_\tau \times E_\sigma)$, where $\varphi \in L(\tau)$, $\psi \in L(\sigma)$, and $\underline{\psi} \in L(\underline{\sigma})$ is a renaming of ψ.

The relative strength of both uniformities depends on fundamental model theoretical properties of L.

5.1 THEOREM. *The canonical uniformity induced by L in $E_\tau \times E_\sigma$ is finer than the product uniformity for all σ, τ if and only if L is closed under relativizations.*

PROOF. "$\Leftarrow$" Closure under relativizations implies that the projections $\pi_1 : (E_\tau \times E_\sigma)(L) \to E_\tau(L)$, $\pi_2 : (E_\tau \times E_\sigma)(L) \to E_\sigma(L)$, $\pi_1(C) = C|P^C|\tau$, $\pi_2(C) = C|\underline{P}^C|\underline{\tau}$ are uniformly continuous. As the product uniformity in $E_\tau \times E_\sigma$ is the smallest making the projections uniformly continuous the result follows.

"$\Rightarrow$" Assume the canonical uniformity is finer than the product uniformity. Then the operation $(A, P^A) \mapsto A|P^A$ factors in the form:

$$(A, P^A) \mapsto^F (|A|, A|P^A, A|(|A| - P^A)) \approx [A|P^A, A|(|A| - P^A)] \mapsto^{\pi_1} A|P^A$$

where $F : E_{\tau\cup\{P\}}(L) \to (E_\tau \times E_\tau)(L)$ is uniformly continuous because it is a definitional expansion, and $\pi_1 : (E_\tau \times E_\tau)(L) \to E_\tau(L)$ is uniformly continuous because it is continuous for the product uniformity in $E_\tau \times E_\tau$ which is weaker by hypothesis than the uniformity of $(E_\tau \times E_\tau)(L)$. □

Recall ([Makowsky 1985]) that a logic L has the *uniform reduction property for pairs* (in short, URP) if and only if for any $\varphi \in L(\tau \oplus \sigma)$ there is a Boolean function $\mathbf{b}(p_1, \ldots, p_n, q_1, \ldots, q_m)$ and sentences $\varphi_i \in L(\tau)$, $\psi_j \in L(\sigma)$ such that for $[A, B] \in E_\tau \times E_\sigma$:

(1) $[A, B] \models \varphi$ if and only if
$\mathbf{b}(A \models \varphi_1, \ldots, A \models \varphi_n, B \models \psi_1, \ldots, B \models \psi_m) = 1.$

By taking disjunctive normal form, it may be seen that (1) is equivalent to the existence of formulae $\varphi_i \in L(\tau)$, $\psi_i \in L(\sigma)$, $i = 1, \ldots, k$, such that

(2) $[A,B] \models \varphi$ if and only if $[A,B] \models \bigvee_i(\varphi_i^P \wedge \underline{\psi}_i^P)$.

5.2 THEOREM. *The product uniformity induced by L in $E_\tau \times E_\sigma$ is finer than the canonical uniformity if and only if L has uniform reduction for pairs.*

PROOF. (2) is equivalent to uniform continuity of the identity operation $I : E_\tau(L) \times E_\sigma(L) \to (E_\tau \times E_\sigma)(L)$. □

5.3 EXAMPLE. a) $L\omega\omega$ and $L\omega\omega(Q_\alpha)$ are closed under relativizations and have URP (use partial isomorphisms), so the canonical and the product topology coincide in products of model spaces for these logics.

b) For $k \geq \omega_1$, including $\kappa = \infty, L_{\kappa\omega}$ has relativizations but it does not have URP (see [Dickmann 1985]). Hence, the canonical topology is strictly finer than the product topology for these logics.

6. Projective Operations

Recall that a class of structures $K \subseteq E_\tau$ is *projective in L* if it is the image of an L-elementary class by a projection. It is *relativized projective in L* if it is the image of an L-elementary class by a *relativized projection* $A \mapsto A|\tau|V^A$. We denote these two properties by L-PC and L-RPC, respectively. More generally, K is L-PC$_\Delta$ (respectively L-RPC$_\Delta$) if it is the projection (respectively the relativized projection) of the class of models of a set of sentences in L.

6.1 DEFINITION. The *graph of* partial operation $F : E_\sigma \to E_\tau$ will be the class:

$$G(F) = \{[A,B] \in E_\tau \times E_\sigma : B \approx F(A)\}.$$

The operation F will be said to be L-PC if $G(F)$ is L-PC as subclass of $E_{\tau\oplus\sigma}$. We define similarly L-RPC, L-PC$_\Delta$, and L-RPC$_\Delta$ operations.

6.2 LEMMA. *Operations of the form*

$$F(A) = (A^n, \varphi_1^A/n, \ldots)|\psi^A|\tau/\theta^A$$

with φ_i, ψ, θ in L, are L-PC$_\Delta$ for infinite types and L-PC for finite types. In particular, classical constructions are L-PC for any logic L.

PROOF. We check projections, restrictions, quotients, and the expanded cartesian products $F(A) = (A^n, \varphi^A/n, \ldots)$ which generate these constructions. Let $Pair(\tau, \underline{\sigma})$ be the sentence saying that a structure belongs to $E_\tau \times E_\sigma$. $\underline{R}$ will denote the copy of $R \in \sigma$ in $\underline{\sigma}$.

a) *Projections.* If $\sigma \subset \tau$, then $C \in G(F) \Leftrightarrow C \approx [A, A|\sigma] \Leftrightarrow \exists f(C, f) \models Pair(\tau, \underline{\sigma}) \wedge f : C|P^C|\sigma \approx C|Q^C|\underline{\sigma}$, where the last expression is an abbreviation for:

"f is a bijection from P^C onto $\underline{P}^C$" $\wedge$
$\bigwedge_{R\in\sigma} \forall x_1, \ldots, x_m[P(x_1) \wedge \ldots \wedge P(x_m \to (\underline{R}(f(x_1), \ldots, f(x_m) \leftrightarrow (R(x_1, \ldots, x_m))]$.

b) *Restrictions.* $C \approx [A, A|U^A] \Leftrightarrow \exists f(C, f) \models Pair(\tau, \underline{\tau} - \{\underline{U}\}) \wedge f : C|U^C|\tau \approx C|\underline{P}^C|\underline{\tau} - \{\underline{U}\}$.

c) *Quotients.* Let S be a binary symbol not in τ, then:

$C \approx [(A, S), A/S^A] \Leftrightarrow \exists V, f(C, V, f) \models Pair(\tau + \{S\}, \underline{\tau}) \wedge$
$[($"S is not a congruence relation in $C|P^C|\tau$" $\wedge f : C|P^C|\tau \approx C|\underline{P}^C|\underline{\tau})$
$\vee$ ("S is a congruence relation in $C|P^C|\tau$" $\wedge$
"V is a choice set for S" $\wedge f : C|V|\tau \approx C|\underline{P}^C|\underline{\tau})]$.

d) *Cartesian expansions.* We check the case $F(A) = (A^2, \varphi^A/2)$. Let E be a k-ary symbol not in τ, then:

$C \approx [A, (A^2, \varphi^A/2)] \Leftrightarrow \exists f(C, f) \models Pair(\tau, \underline{\tau} + \{E\})$
$\wedge$ "f is a bijection from $P^C \times P^C$ to $\underline{P}^C$"
$\wedge \bigwedge_{R\in\tau} \forall x_1 \ldots x_m y_1 \ldots y_m[P(x_1) \wedge P(y_1) \wedge \ldots \wedge P(x_m) \wedge P(y_m) \to$
$(\underline{R}(f(x_1, y_1), \ldots, f(x_m, y_m)) \leftrightarrow (R(x_1, \ldots, x_m) \wedge R(y_1, \ldots, y_m))]$
$\wedge(E(f(x_1, y_1), \ldots, f(x_m, y_m)) \leftrightarrow \varphi(x_1, y_1, \ldots, x_m, y_m))$. □

An uniformly continuous operation $F : E_\tau(L) \to E_\sigma(M)$ will be called *recursively uniformly continuous* if $L(\tau)$ and $M(\sigma)$ have recursively enumerable presentations and the translation $\alpha : M(\sigma) \to L(\tau)$ may be taken to be recursive in this presentations.

6.3 THEOREM. *If L is recursively axiomatizable and has recursively computable relativizations, then any uniformly continuous L-RPC operation is recursively uniformly continuous.*

PROOF. If $G(F) = \{[A, B] : \exists C(C, [A, B]) \models \theta\}$ and F is uniformly continuous then uniform reduction becomes: for any $\varphi \in L(\sigma)$ there exist $\psi \in L(\tau)$ such that $\models \theta \to (\psi^P \leftrightarrow \varphi^{\underline{P}})$. As relativizations are recursively computable, we may generate recursively all the valid formulae of the above form and take for $\alpha(\varphi)$ the first ψ appearing in such a formula. □

7. A Compact Graph Theorem

A continuous operation must preserve logical equivalence of structures. The converse is not true even if the domain logic is compact and the operation is PC as shown by the next example.

7.1 EXAMPLE. Let q_0 be the quantifier defined by $A \models q_0 x\varphi(x)$ if and only if $|A| \geq \omega$. Then the identity operation $I : E\tau(L\omega\omega) \rightarrow E\tau(L\omega\omega(q_0))$ is not continuous, otherwise we would have uniform continuity by compactness of the domain, and so $L\omega\omega(q_0) \leq L\omega\omega$ in type τ which is absurd. On the other hand $A \equiv_{L\omega\omega} B$ implies $A \equiv_{L\omega\omega(q_0)} B$ trivially.

However, if both logics are compact and closed under relativizations, preservation of equivalence by a PC operation implies uniform continuity, as immediate consequence of a simple topological result. Define for any topological space X, and element $a \in X$:

i) $a^0 = \bigcap\{O \ : \ O \text{ is open}, \ a \in O\}$,
ii) $a \equiv_X b$ if and only if $a^0 = b^0$, if and only if $b \in a^0$.

If X is a regular space $a^0 = \bigcap\{Cl(V) : V \text{ open}, \ a \in V\} = Cl(\{a\})$ and so a^0 is closed. For model spaces $A^0 = Mod(Th_L(A))$, and this topological equivalence is just elementary equivalence.

7.2 LEMMA. *Let $f : X \rightarrow Y$ be a partial function between uniform spaces. If the graph of f is compact in the product topology of $X \times Y$ and f preserves $\equiv$, then f is uniformly continuous.*

PROOF. If the graph is compact then the domain is compact, so it is enough to prove continuity. Assume f is not continuous and fix a point $b \in X$ and an open set $V \subseteq Y$ such that $f(b) \in V$ but for all open $W \subseteq X$ with $b \in W$, we have $f(W) \not\subseteq V$. Then any finite subfamily of the following family of closed subsets of $X \times Y$

$$\mathcal{F} = \{Cl(W) \times (Y - V) \ : \ W \text{ open}, \ b \in W\}$$

intersects $G(f)$. By compactness of $G(f)$ there is a point $(a, f(a))$ of $G(f)$ in the intersection $\bigcap \mathcal{F}$, which by regularity of X and the remark above is $b^0 \times V^C$. This implies $a \equiv_X b$ and $f(a) \notin V$; as $f(b) \in V$ then $f(a) \not\equiv_Y f(x)$, a contradiction. □

7.3 THEOREM. *Let L, M, N be compact logics with $L, M \leq N$, and N closed under relativizations. If $F : E\tau(L) \to E\sigma(M)$ is an N-RPC_Δ partial operation then the following are equivalent:*
i) *F is uniformly continuous,*
ii) *F preserves elementary equivalence.*

PROOF. Assume (ii); if $G(F)$ is N-RPC_Δ then it is compact in $(E\tau \times E\sigma)(N)$ because it is the image of a closed (hence compact) class in some $E\mu(N)$ by a relativized projection, which must be continuous by closure under relativizations of N. A fortiori $G(F)$ is compact in the topology of $E_\tau(N) \times E_\sigma(N)$ which is weaker by Theorem 5.1, and in the even weaker topology of $E_\tau(L) \times E_\mu(M)$. Now apply Lemma 7.2. □

In fact, continuity has to be checked in infinite structures only.

7.4 THEOREM. *Under the same hypothesis of* Theorem 7.3, *the following are equivalent:*
i) *F is uniformly continuous.*
ii) *F restricted to infinite structures is uniformly continuous.*

PROOF. (ii) implies F preserves elementary equivalence of infinite structures. Now if A is finite, $A \equiv_L M$ implies $A \approx B$ and so $F(A) \equiv_M F(B)$. □

REMARK. For logics of the form $L_{\omega\omega}(Q : Q \in C)$, Theorems 7.3 and 7.4 hold for $L - \mathrm{RPC}$ operations under the weaker hypothesis of *local recursive compactness*. This means that $L_{\omega\omega}(\varphi)$ is recursively compact in the usual sense for any $\varphi \in L(\tau)$. For example, this holds for any finitely generated logic which is either recursively axiomatizable or satisfies Beth's definability theorem. The proof in this case is more involved but still topological.

Hanf and Myers (in [Hanf-Myers 1983]) have observed that Keisler-Shelah Theorem on ultrapowers implies the preservation of elementary equivalence by first order RPC_Δ operations; hence:

7.5 COROLLARY ([Hanf-Myers 1983]). *Any first order* RPC_Δ *operation is uniformly continuous for $L_{\omega\omega}$. If it is* RPC, *it is recursively uniformly continuous.*

PROOF. Theorem 7.3 plus the above observation, and Theorem 6.2. □

7.6 COROLLARY ([Szczerba 1977]). *Classical constructions are recursively uniformly continuous.*

PROOF. These operations are $L_{\omega\omega}$-PC. □

A operation F will be said to *preserve ultraproducts* if its domain is closed under ultraproducts and $F(\Pi A_\alpha/U) \approx F(\Pi A_\alpha)/U$.

7.7 COROLLARY. *Ultraproducts preserving operations are uniformly continuous for first order logic.*

PROOF. If $[A_\alpha, F(A_\alpha)] \in G(F)$ then $\Pi[A_\alpha, F(A_\alpha)]/\mathcal{F} \approx [\Pi A_\alpha/\mathcal{F}, F(A_\alpha)/\mathcal{F}] \in G(F)$ by hypothesis, showing $G(F)$ is compact in first order topology by Corollary 2.1. On the other hand, if $A \equiv B$ then by Keisler-Shelah Theorem on ultrapowers there is I and an ultrafilter U over I such that $A^I/U \approx B^I/U$. Hence, $F(A) \equiv F(A)^I/U \approx F(A^I/U) \approx F(B^I/U) \approx F(B)^I/U \equiv F(B)$, so $F(A) \equiv F(B)$. □

As another application of Lemma 7.2, we have a topological proof of a result of Shelah. The topological content of this theorem has been noticed in [Mundici 1986], Theorem 4.1. Recall that a logic L is said to have the *pair preservation property* or the *Feferman-Vaught property for pairs* , in short PPP, cf. [Makowsky 1985], if and only if

$$A \equiv_L B, A' \equiv_L B' \Rightarrow [A, A'] \equiv_L [B, B'].$$

Of course, URP (uniform reduction for pairs) implies PPP, but the converse does not hold. For example:
a) $L_{\omega\omega}, L_{\omega\omega}(Q_\alpha)$ have URP and so PPP.
b) $L_{\infty\omega}$ has the PPP (use partial isomorphisms), but not URP.
c) $L_{\omega_\alpha\omega}$ does not have PPP unless ω_α is strongly inaccessible (see [Dickmann 1985]).
d) $L_{\omega\omega}(aa)$ does not have PPP.

7.8 THEOREM (Th. 4.2.13 in [Makowsky 1985]). *Let L be a compact logic closed under relativizations then the following are equivalent:*
i) *L has uniform reduction for pairs;*
ii) *L has the* PPP.

PROOF. (ii) means that the identity operation $I : E_\tau(L) \times E_\sigma(L) \to (E_\tau \times E_\sigma)(L)$ preserves the topological relation $\equiv$ (clearly $(a, b) \equiv (c, d)$ in a product space $X \times Y$ if and only if $a \equiv c$ in X and $b \equiv d$ in Y). Since $G(I)$ is trivially L-PC$_\Delta$, it must be compact in $(E_\tau \times E_\sigma)(L)$ as the continuous image of a closed (hence compact) class in a compact space, and so in $E_\tau(L) \times E_\sigma(L)$ which has a weaker topology by the relativization property. Therefore, by Lemma 7.2, I is uniformly continuous which gives URP. □

8. On a Theorem of Lipparini

Notice that if $M \leq L$ then $\equiv_L \leq \equiv_M$ (this is, $A \equiv_L B \Rightarrow A \equiv_M B$). However, the converse is not true even if L is compact by Example 7.1 in the previous section. The logic $L\omega\omega(q_0)$ in this example is not closed under relativizations. Surprisingly this is the only obstacle for the validity of the equivalence $M \leq L \Leftrightarrow \equiv_L \leq \equiv_M$ when L is compact as shown by Lipparini (in [Lipparini 1985]). The relativizations property becomes again the dividing line between the validity and non-validity of a non-trivial result. We give here a topological proof of this result.

8.1 THEOREM. *Let X be a compact regular space then $C \subseteq X$ is closed in X if and only if C is compact and closed under $\equiv_X$; this is, $b \equiv_X c$ and $c \in C$ imply $b \in C$.*

PROOF. One direction is trivial. Assume C is compact and closed under $\equiv_X$, and $b \in Cl(C)$, then for any open neighborhood V of b we have $Cl(V) \cap C \neq \emptyset$. Hence, $b^0 \cap C = [\bigcap_{x \in V} Cl(V)] \cap C \neq \emptyset$ by compactness of C. Take $c \in b^0 \cap C$, then $b \equiv_X c$ and so $b \in C$. □

In the next lemma we assume the logic to be defined in non-monadic types.

8.2 LEMMA. *Let L, M be regular logics with L compact. If $\equiv_L \leq \equiv_M$, then any L-RPC$_\Delta$ subclass of E_τ is compact in $E_\tau(M)$.*

PROOF. Let C be RPC$_\Delta$ in L and assume it is not compact in M. Then there is a set of sentences $X = \{\varphi_a : a < k\}$ in $M(\tau)$ which does not have a model in C but any subset $S_b = \{\varphi_a : a < b\}, b < k$, does. By putting together a model A_b for each S_b construct a structure $\mathbf{A} = (A, \ldots, R, <, c_a)_{a<k}$ such that $A_b \approx \mathbf{A}|\{y : yRc_b\} \in C$ and it satisfies the sentences of M:

$$(1) \qquad \forall x(c_a < x \rightarrow \varphi_a^{\{y:yRx\}}),\ a < k.$$

Let $T \subseteq L(\sigma)$ with $\sigma \supset \tau$ be a theory defining C as a RPC_Δ class of L via relativization to a monadic predicate V. By compactness of L, the theory

$$Th_L(\mathbf{A})^V \cup \{c_a < c \,:\, a < k\} \cup \{V(c)\} \cup T^{\{y:yRc\}}$$

where c is a new constant has a model $\mathbf{B}$. Evidently $\mathbf{B} \models Th_L(\mathbf{A})^V$ and so $\mathbf{B}_1 = \mathbf{B}|V^{\mathbf{B}} \equiv_L \mathbf{A}$. Hence, by hypothesis $\mathbf{B}_1 \equiv_M \mathbf{A}$ and so $\mathbf{B}_1$ satisfies all sentences in (1). In particular $\mathbf{B}_1 \models \varphi_a^{\{y:yRb\}}$ with $b = c^{\mathbf{B}}$ and so $\mathbf{B}_1|\{y \,:\, yRb\} \models \varphi_a$ for each $a < k$. This is a contradiction since $\mathbf{B}|\{y \,:\, yRb\} \models T$ and so $\mathbf{B}|\{y \,:\, yRb\}|V^{\mathbf{B}}|\tau = \mathbf{B}_1|\{y \,:\, yRb\}|\tau \in C$. □

8.3 THEOREM ([Lipparini 1985]). *Let L and M be logics closed under relativizations with L compact, then $\equiv_L \leq \equiv_M$ if and only if $M \leq L$.*

PROOF. One direction (right to left) is trivial, for the other assume first the special case $\equiv_L = \equiv_M$. Then the identity function $I : E_\tau(M) \to E_\tau(L)$ is continuous because if C is closed in $E_\tau(L)$ it is compact in $E_\tau(M)$ by Lemma 8.2, and it is closed under $\equiv_L$; hence, closed under $\equiv_M$ by hypothesis. Therefore, C is closed in $E_\tau(M)$ by Lemma 8.1. Lemma 8.2 also implies that $E_\tau(M)$ is compact; hence, I is uniformly continuous and so $L \leq M$. Now, the identity function $J : E_\tau(L) \to E_\tau(M)$ is PC^Δ in any logic, and it preserves elementary equivalence by hypothesis. As $L \leq M$ with M compact, then J is uniformly continuous by Theorem 7.3 and so $M \leq L$. Assume now the general case $\equiv_L \leq \equiv_M$. Under the hypothesis it is clear that if $B(L, M)$ denotes the logic obtained by closure of $L \cup M$ under finite Boolean operations then $\equiv_L$ implies $\equiv_{B(L,M)}$ and so $\equiv_L = \equiv_{B(L,M)}$. It is easy to see that $B(L, M)$ is closed under relativizations since these commute with Boolean operations. Therefore, by the first part of the proof, $B(M, L) \equiv L$ and so $M \leq L$. □

9. A Löwenheim-Skolem Theorem for First Order PC Operations

The following result seems intuitively obvious, but the only proof we do know makes use of heavy artillery (it was suggested to us by Lauri Hella and Kerkko Luosto of Helsinki). Call a operation RPC_ω if it is the relativized reduct of the models of a countable theory.

9.1 THEOREM. *If F is a partial first order RPC_ω operation then for any $\mathcal{A}$ in the domain of F:*

a) $|\mathcal{A}| \geq \omega$ *implies* $|F(\mathcal{A})| \leq |\mathcal{A}|$.
b) $|\mathcal{A}| < \omega$ *implies* $|F(\mathcal{A})| < \omega$.

PROOF. Assume $G(F) = \{[\mathcal{A},\mathcal{B}] : \exists V, R(V,[\mathcal{A},\mathcal{B}],R) \models \theta\}$, where θ is a countable set of sentences. If there is $\mathcal{A}$ in the domain of F with $|F(\mathcal{A})| > |\mathcal{A}| \geq \omega$, then calling A and B the universes of $\mathcal{A}$ and $\mathcal{B}$, respectively, there is $\mathbf{C} = (V,[\mathcal{A},\mathcal{B}],\ldots) = (V, A \cup B, A, B, \ldots) \models \theta$ with $|A \cup B| > |\mathcal{A}| \geq \omega$. By the downward Löwenheim-Skolem-Tarski Theorem we may assume without loss of generality that $|V| = |A \cup B|$. Add an extra predicate and a sentence to θ insuring this last equality. By Theorem 3.2.14 in [Chang-Keisler 1973], there are structures $\mathbf{C}' = (V',[\mathcal{A}',\mathcal{B}'],\ldots) = (V', A' \cup B', A', \ldots)$, $\mathbf{C}'' = (V'',[\mathcal{A}'',\mathcal{B}]'',\ldots) = (V'', A'' \cup B'', A'', \ldots)$ with $\mathbf{C}' < \mathbf{C}, \mathbf{C}' < \mathbf{C}''$, $|A' \cup B'| = \omega$, $|A'' \cup B''| = \omega_1$ and $A' = A''$. Therefore, $\mathcal{A}' = \mathcal{A}''$, since the relations of $\mathcal{A}'$, are those of $\mathcal{A}''$ restricted to A'. On the other hand, $\mathbf{C}', \mathbf{C}'' \models \theta$ and so $[\mathcal{A}',\mathcal{B}']$, $[\mathcal{A}'',\mathcal{B}''] \in G(F)$. As $|\mathcal{A}''| = |\mathcal{A}'| = \omega$, we must have $|\mathcal{B}''| > |\mathcal{B}'|$, and so $\mathcal{A}'$ will have two non-isomorphic images. This shows (a).

Now, if $|\mathcal{A}| < \omega$ and $|F(\mathcal{A})| \geq \omega$, we could keep $\mathcal{A}$ fixed with a convenient set of sentences describing it, and at the same time inflate $[\mathcal{A}, F(\mathcal{A})]$ by compactness to a model $[\mathcal{A},\mathcal{B}]$ in $G(F)$ with $\mathcal{B}$ of infinite power greater than $|F(\mathcal{A})|$, obtaining two images for $\mathcal{A}$ of distinct cardinality, which is absurd and shows (b). □

9.2 COROLLARY. *If F is first order* RPC *and τ, σ are finite then module isomorphism, F is partial recursive in finite structures.*

PROOF. Let $B(1), B(2), \ldots$ be a recursive enumeration of all finite structures in E_σ with domain a initial segment of ω, and $\varphi_1, \varphi_2, \ldots$ a recursive listing of their first order descriptions. As F is RPC there is by Corollary 6.3 a recursive α such that for any finite structure A in the domain of $F, A \models \alpha(\varphi)$ if and only if $F(A) \models \varphi$. Then $F(A) \approx B(n)$ if and only if $F(A) \models \varphi_n$, if and only if $A \models \alpha(\varphi_n)$. Hence, as $F(A)$ must be finite:

$$F(A) \approx B(\mu n[A \models \alpha(\varphi_n)])$$

which is partial recursive since satisfaction is a recursive relation for finite structures. □

The following counterexample shows that Theorem 9.1 does not necessarily hold for countably compact logics. The operation:

$$F(A) = \begin{cases} A & \text{if } A \text{ is not a field} \\ \text{Algebraically closed extension} & \\ \text{of } A \text{ of power } |A| + \omega_1 & \text{if } A \text{ is a field,} \end{cases}$$

is PC_ω, in fact PC, in $L_{\omega\omega}(Q_1)$ since with the help of additional predicates and a recursive set of sentences it is possible to state in this logic that $F(A)$ is an algebraically closed field of the same characteristic as A and $|F(A)| = |A| + |B|$ where B has an ω_1-like ordering. However $F(\mathbf{Z}_p)$ is infinite and $|F(\mathbf{Q})| > |\mathbf{Q}|$.

10. Continuity and Interpolation

Given two logics $L \leq M$ the notations $Int(L, M)$ and Δ-$Int(L, M)$, will denote, respectively that L satisfy the *interpolation* and Δ-*interpolation*, properties with respect to M; this is, the interpolant of sentences in L may be found in M (cf. [Ebbinghaus 1985]). Since we are working with single sorted logics, plain interpolation means separation of PC classes, not of RPC classes. Whenever we wish to refer to the stronger *relativized interpolation*, or separation of disjoint RPC classes, we will utilize a subscript R, so $Int_R(L, M)$, Δ-$Int_R(L, M)$. Assuming uniform reduction for pairs both notions coincide.

10.1 Lemma. *If L has relativizations and M satisfies* URP *then $Int(L, M)$ implies $Int_R(L, M)$. Similarly for Δ-interpolation.*

Proof. Let $K_i = \{A|P|\tau : A \models \theta_i\}, i = 1, 2$, be disjoint L-RPC classes. Then the classes $K_i^* = \{[[A_1, A_2], D] : A_1 \models \theta_1, A_2 \models \theta_2, D \approx A_i|P|\tau\}$ are disjoint L-PC. Let $\sigma \in M$ separate them, say $K_1^* \subset Mod(\sigma)$. By URP there is finite $\Phi \subset M$ such that $D_1 \equiv_\Phi D_2$ implies $[[A, B], D_1] \equiv_\sigma [[A, B], D_2]$ for any A, B. Let $D \in K_1$ then $Th_\Phi(D)$ separated K_1 and K_2. □

The following theorem is due in [Gaifman 1974], [Feferman 1974], and [Makowsky 1985]. They state it in the context of many sorted logics, but interpolation for many sorted logics (cf. [Ebbinghaus 1985], 7.1.1) may be reduced to relativized interpolation.

10.2 Theorem ([Makowsky 1985], Th. 4.2.14). *Let L be closed under relativizations, then $Int_R(L, M)$ holds if and only if any partial L-RPC operation $F : E_\tau(M) \to E_\sigma(L)$ is uniformly continuous.*

Relativized Δ-interpolation has a similar characterization.

10.3 THEOREM. *Let L be closed under relativizations, then Δ-$Int_R(L,M)$ if and only if any total L-RPC operation $F : E_\tau(M) \to E_\sigma(L)$ is uniformly continuous.*

The topological content of the plain non-relativized interpolation properties is more delicate. Interpolation will imply continuity only for those PC operations where $F(A)$ may be "constructed" inside A. If A and B are structures of types σ and μ, respectively, with $|B| \subset |A|$, let (A,B) be the structure of type $\sigma \cup \{P\} \cup \underline{\mu}$ where P is a new monadic symbol interpreted by the universe of B, and $\underline{\mu}$ is a renaming of μ, disjoint, of σ interpreted by the relations of B.

10.4 DEFINITION. An operation $F : E_\tau \to E_\mu$ will be called an L-PC (respectively, L-PC$_\Delta$) *expansion* if $|F(A)| \leq |A|$ for all A in its domain, and the class

$$G^*(F) = \{(A,B) : B \approx F(A)\}$$

is L-PC (respectively, L-PC$_\Delta$).

Projections, restrictions, L-definitional expansions, and quotients are L-PC expansions for any logic L. But powers are L-PC expansions in infinite structures only. By Theorem 9.1, the downward Löwenheim-Skolem Theorem for $L_{\omega\omega}$ and next lemma all first order RPC operations are L-PC expansions in infinite structures for any logic L.

10.5 LEMMA. *Any L-PC operation F satisfying $|F(A)| \leq |A|$ is an L-PC expansion in infinite structures. Similarly for L-PC$_\Delta$ operations.*

PROOF. Let $G(F) = \{[A,B] : \exists R([A,B],R) \models \theta(\mu)\}$ then for infinite A, $(A,B) \in G^*(F) \Leftrightarrow \exists C,R,f(A,B,C,R,f) \models$ "$f : (|A|,C) \approx [A,B]$"$\wedge\theta(\underline{\mu})$, where the last sentence is a renaming of θ to the type of $(|A|,C,R)$. □

10.6 THEOREM. *Let L be closed under relativizations, then*
a) *$Int(L,M) \Leftrightarrow$ any partial L-PC expansion $F : E_\tau(M) \to E_\sigma(L)$ is uniformly continuous.*
b) *Δ-$Int(L,M) \Leftrightarrow$ any total L-PC expansion $F : E_\tau(M) \to E_\sigma(L)$ is uniformly continuous.*

PROOF. "$\Leftarrow$" If $B \approx F(A) \Leftrightarrow \exists R(A,B,R) \models \theta$ with $\theta \in L$, then for each $\varphi \in L(\sigma)$ the classes

$$K_1 = \{A : \exists B, R(A,B,R) \models \theta \wedge \varphi^P\}, K_2 = \{A : \exists B, R(A,B,R) \models \theta \wedge \neg\varphi^P\}$$

where P denotes the universe of B, are obviously disjoint L-PC subclasses of $E\tau(M)$. Let $\alpha(\varphi)$ be an interpolant for them in $M(\tau)$, say $K_1 \subset Mod(\alpha(\varphi))$, then for $A \in dom(F)$ we have $A \models \alpha(\varphi) \Leftrightarrow A \in K_1 \Leftrightarrow \exists B, R(A,B,R) \models \theta \wedge \varphi^P \Leftrightarrow F(A) \approx B \models \varphi$. In case the operation is total, K_1 and K_2 are complementary and Δ-interpolation is enough.

"$\Rightarrow$" Given disjoint L-PC classes in E_τ, say $K_i = \{A : \exists R(A,R) \models \theta_i\}$, $i = 1,2$, the operation defined in $K_1 \cup K_2$ by $F(A) = (A, |A|)$ if $A \in K_1$, $F(A) = (A, \emptyset)$ if $A \in K_2$, is easily seen to be an L-PC expansion, total if the classes are complementary. Then $\alpha(\forall x V(x)) \in M(\tau)$ separates K_1 and K_2 since we have for $A \in K_1 \cup K_2$ that $A \in K_1 \Leftrightarrow F(A) = (A, |A|) \Leftrightarrow F(A) \models \forall x V(x) \Leftrightarrow A \models \alpha(\forall x V(x))$. □

10.7 EXAMPLE ([Badger 1977]). Let Q^n_κ denote the n-ary Magidor-Malitz quantifier in the κ-interpretation, $Q^{<\omega}_\kappa = \{Q^n_\kappa : n \in \omega\}$, and $MM = \bigcup_\kappa Q^{<\omega}_\kappa$. Then

$$\Delta\text{-}Int(L_{\omega\omega}(Q^2_\kappa), L_{\infty\omega}(MM))$$

fails for uncountable κ. Badger shows how to construct a pair of structures A, B of uncountable power κ such that $A \equiv_{L_{\infty\omega}(Q^{<\omega}_\kappa)} B$ but $A^2 \not\equiv_{L_{\omega\omega}(Q^{<\omega}_\kappa)} B^2$; it may be seen that in fact $A \equiv_{L_{\infty\omega}(MM)} B$. Hence, the operation: $C \mapsto C^2$ for $|C| \geq \kappa$, $C \mapsto C$ for $|C| < \kappa$, is a total $L_{\omega\omega}(Q^2_\kappa)$-PC expansion but it is not uniformly continuous from $E_\tau(L_{\infty\omega}(MM))$ to $E_\tau(L_{\omega\omega}(Q^2_\kappa))$. Theorem 10.6 (b) implies the claim.

10.8 EXAMPLE. Theorem 10.3 does not hold if there are not relativizations. The logic $L\omega\omega(q)$ where q is the quantifier:

$$qx\, \varphi(x) \Leftrightarrow \text{ the universe has finite or regular cardinal}$$

satisfies interpolation (and compactness, see [Caicedo 1985]) but quotients, which are PC expansions, are not uniformly continuous for this logic by Corollary 11.2 in next section.

However, Theorem 10.6 holds in logics without relativizations for *power preserving* L-PC expansion, those satisfying $|F(A)| = |A|$. For example, interpolation implies the uniform continuity of cartesian operations in infinite structures. Another curious corollary is that Δ-interpolation implies closure under relativizations to predicates of the same power as the universe.

11. A General Non-Interpolation Theorem

Now we give a topological versions of some results in [Caicedo 1990]. A Lindström quantifier Q of type $(n_1, \ldots, n_m)$ will be called κ-*thin* if $(A, R_1, \ldots, R_m) \in Q$ implies that for all $S \subseteq A$ with $|S| = |A| \geq \kappa$ and $1 \leq i \leq m$ there are distinct $a_1, \ldots, a_{ni} \in S$ such that $(a_1, \ldots, a_{ni}) \notin R_i$. Let $\mathbf{T}_\kappa$ be the class of κ-thin quantifiers. Call *thin* those quantifiers Q for which there is α such that for all $\kappa \geq \alpha, Q$ is definable in structures of power κ by a sentence of finite quantifier rank of $L_{\infty\omega}(\mathbf{T}_\kappa)$. The defining sentence may depend on α. Let $\mathbf{T}$ denote the class of all thin quantifiers.

All monadic type quantifiers, all linear order quantifiers (well ordering quantifier, cofinality quantifiers, equicofinality quantifiers, etc. for example), as well as many other natural quantifiers are thin. On the other hand, Magidor-Malitz quantifiers Q^n_κ may be shown not to be thin for $n \geq 2$.

Consider now for each cardinal κ the operation $F_\kappa : E_\tau \to E_{\tau\cup\{R\}}$ given by

$$F_\kappa(A) = (A \times \kappa_\tau, E), \tag{1}$$

where κ_τ denotes the structure with universe κ and all the relations in τ interpreted as the largest possible, and E is the congruence relation $(a, \beta)E(a', \beta')$ if and only if $a = a'$.

11.1 Lemma. *Let C be a set of thin quantifiers then there is α such that for all $\kappa \geq \alpha, F_\kappa : E^\kappa_\tau(L_{\omega\omega}) \to E_{\tau+2}(L_{\omega\omega}(C))$ is uniformly continuous.*

Proof. The operation $[A, B] \mapsto (A \times B, E)$ where $(a, b)E(a', b')$ if and only if $a = a'$ is first-order PC_Δ and so uniformly continuous in $L\omega\omega$. Since $L_{\omega\omega}$ has uniform reduction for pairs, then for fixed B the operation $A \mapsto [A, B]$ is also uniformly continuous. This shows uniform continuity of (1) in first order topology. Now pick κ such that all quantifiers in C are κ-thin (it exists because C is a set). If $|A| \leq \kappa$, one may show that any infinite subset of $F_\kappa(A)^n$ invariant under automorphisms admits $k = |F(A)|$ indiscernibles. With this one may prove that any sentence in $L\omega\omega(C)$ is equivalent in $F(A)$ to a first order sentence which depends only on the cardinal κ. The claim follows. □

11.2 Corollary. *If $L\omega\omega < L \leq L\omega\omega(\mathbf{T})$ then for some τ, the quotient operation $q : E_{\tau+2}(L(\mathbf{T})) \to E_\tau(L)$ is not uniformly continuous (in infinite structures).*

PROOF. Let $\varphi \in L - L_{\omega\omega}$ and $L^* = L_{\omega\omega}(\varphi)$, then the identity operation $I^* : E_\tau(L_{\omega\omega}) \to E_\tau(L^*)$ is not uniformly continuous, as L^* is a set it follows that for some β the operation $I^* : E_\tau^\kappa(L_{\omega\omega}) \to E_\tau(L^*)$ is not uniformly continuous for all $\kappa \geq \beta$. For each $C \subset \mathbf{T}$ choose $\kappa \geq \beta, \alpha$, where α is given by the previous Lemma, then the diagram below commutes

$$\begin{array}{ccccc} & & E_{\tau+2}(L\omega\omega(C)) & & \\ & \overset{F_\kappa}{\nearrow} & & \overset{q}{\searrow} & \\ E_\tau^\kappa(L\omega\omega) & & \xrightarrow{\quad I^* \quad} & & E_\tau(L^*) \end{array}$$

and F_κ is uniformly continuous; hence q cannot be uniformly continuous. Therefore, $q : E_{\tau+2}(L\omega\omega(\mathbf{T})) \to E_\tau(L^*)$ is not uniformly continuous and the claim follows. □

11.3 COROLLARY (Th. 2, [Caicedo 1990]). *No proper extension of $L\omega\omega$ generated by thin quantifiers and closed under relativizations satisfies Δ-interpolation.*

PROOF. Since q is total PC in any logic, the previous Corollary and Theorem 10.6 on the previous section show that Δ-$Int(L, L\omega\omega(\mathbf{T}))$ fails if $L\omega\omega < L$. □

The operation F_κ defined above is uniformly continuous for finite types in any logic closed under interpolation and uniform reduction for pairs, since it factors as the composite of two operations

$$A \mapsto [A, \kappa_\tau] \mapsto (A \times \kappa_\tau, E),$$

where the first is uniformly continuous by URP and the second is (extendible to) a power preserving PC expansion in finite structures. If M is such a logic, let $M(\mathbf{T})$ be the result of closing M under all thin quantifiers, then arguing as Lemma 11.1 and its corollaries we get:

11.4 COROLLARY. *Let M be closed under interpolation and uniform reduction for pairs. If $M < L \leq M(\mathbf{T})$ then quotients are not uniformly continuous in L. If in addition L is closed under relativizations then $Int(L, M(\mathbf{T}))$ fails.*

12. Robinson's Lemma and Continuity

Recall that *Robinson's consistency lemma* for L says that if $A_1|\tau \equiv_L A_2|\tau$ there are structures A_1^*, A_2^* such that $A_i^* \equiv_L A_i$, $i = 1, 2$, and $A_1^*|\tau = A_2^*|\tau$. An useful equivalent formulation is the following : if K_1, K_2 are L-PC$_\Delta$ classes and there are $A_i \in K_i$, $i = 1, 2$, such that $A_1 \equiv_L A_2$, then $K_1 \cap K_2 \neq \emptyset$.

The following result shows that Robinson's Lemma is also a continuity phenomenum. In this section we assume all logics have relativizations.

12.1 Theorem. *The following are equivalent for any logic L.*
i) *Robinson's consistency lemma.*
ii) *Any L-PC$_\Delta$ expansion preserves elementary equivalence.*

Proof. (i) $\Rightarrow$ (ii). Let $F : E_\tau \to E_\sigma$ be a L-PC$_\Delta$ expansion defined by $F(A) = C \Leftrightarrow \exists R(A, B, R) \models \theta, \theta \in L(\tau \cup \sigma \cup \mu)$, and assume that there are structures A_1, A_2 with $A_1 \equiv_L A_2$, $F(A_1) \models \varphi$, and $F(A_2) \not\models \varphi$ for some sentence φ. Then there are relations R_1, R_2 such that $(A_1, F(A_1), R_1) \models \theta \wedge \varphi^Q$ and $(A_2, F(A_2), R_2) \models \underline{\theta} \wedge \neg\underline{\varphi}^Q$, where Q interprets the universe of $F(A_1)$ and $\underline{\theta} \wedge \neg\underline{\varphi}^Q$ is a renaming of $\theta \wedge \neg\varphi^Q$ in the type $\tau \cup \underline{\sigma} \cup \underline{\mu}$, $\underline{\sigma} \cup \underline{\mu}$ a disjoint copy of $\sigma + \mu$. Robinson's Lemma implies the existence of $(A, B_1, R_1^*, B_2, R_2^*) \models \theta \wedge \varphi^Q, \underline{\theta} \wedge \neg\underline{\varphi}^Q$ such that $(A, B_i, R_i^*) \equiv_L (A_i, F(A_i), R_i)$, $i = 1, 2$; hence, $B_1 = B_2 = F(A)$ and $B_1 \models \varphi, B_2 \not\models \varphi$, a contradiction.

(ii) $\Rightarrow$ (i) Assume $A_1|\tau \equiv A_2|\tau$ is a situation where Robinson's Lemma fails, then the L-PC$_\Delta$ classes $K_i = \{A|\tau : A \models Th_L(A_i)\}$, $i = 1, 2$, must be disjoint. Hence,

$$F(A) = \begin{cases} (A, |A|) & \text{if } A \in K_1 \\ (A, \emptyset) & \text{if } A \in K_2 \end{cases}$$

is a well defined L-PC$_\Delta$ expansion, and so $(A, |A|) = F(A_1|\tau) \equiv_L F(A_2|\tau) = (A, \emptyset)$, which is absurd. □

12.2 Corollary. *The following are equivalent for any logic L having occurrence number smaller than the first measurable cardinal.*
i) *Robinson's consistency lemma.*
ii) *Any $L -$ PC$_\Delta$ expansion is uniformly continuous.*

Proof. Under the hypothesis Robinson's Lemma implies compactness ([Makowsky 1985], Cor. 3.3.5). Apply Theorem 7.3. □

By a result of Shelah ([Shelah 1985], Claim 3.3, [Makowsky 1985], Th. 4.5.13.1), any logic L satisfying Robinson's Lemma and the pair preservation property (PPP) must have the *homogeneity property*, this is, for very structure A and elements a, b of the same L-type in A there is an L-elementary extension B of A with an automorphism sending a to b. Robinson's Lemma is in fact equivalent under PPP to a stronger form of the homogeneity property. The equivalence between Robinson's Lemma and this strong homogeneity has been noticed by H. Mildenberger (see [Ebbinghaus 1995]). See also in this connection [Shelah 1985], Lemma 3.6.

12.3 Definition. L has the *strong homogeneity property* if given a structure A and two sequences (perhaps infinite) of relations R, S in A such that $(A, R) \equiv_L (A, S)$, there is a elementary extension (A^*, R^*, S^*) of (A, R, S) with $(A^*, R^*) \approx (A^*, S^*)$.

The homogeneity property results when R and S above reduce to singletons. Strong homogeneity follows in turn from an even stronger property:

12.4 Definition. L is said to have the *general homogeneity property* if given a pair of partial L-RPC_Δ operations $F_1, F_2 : E_\tau \to E_\sigma$ and a structure A in their domain such that $F_1(A) \equiv_L F_2(A)$ there is $B >_L A$ such that $F_1(B) \approx F_2(B)$.

The strong homogeneity property is obtained taking for F_1 the projection of $\tau = \mu + c_1 + c_2$ into $\sigma = \mu + c_1$, and for F_2 the projection of τ into $\mu + c_2$ followed by the renaming c_2/c_1, where μ is the type of A and c_1 and c_2 are disjoint copies of the type of the sequences R, S.

12.5 Theorem. *The following are equivalent for any logic L.*
i) *L has the Robinson's property +* PPP.
ii) *L has the strong homogeneity property +* PPP.
iii) *L has the general homogeneity property.*

Proof. (i) $\Rightarrow$ (iii). First assume F_1 and F_2 are L-PC_Δ operations and $F_1(A) \equiv_L F_2(A)$. Let F_i^* be the restriction of F_i to models of $Th_L((A, e)_{e \in A})$ which is still L-PC_Δ. By PPP, $[A, F_1(A)] \equiv_L [A, F_2(A)]$, the first structure being in $G(F_1^*)$ and the second in $G(F_2^*)$. As these two graphs are PC_Δ by hypothesis, Robinson's Lemma implies there is $[B, D]$ in their intersection. Then, $B >_L A$ and $F_1(B) \approx D \approx F_2(B)$, showing homogeneity. Now, if F_1, F_2 are L-RPC_Δ with $G(F_i) = \{[A, D] : (R_i, [A, D]) \models \theta_i\}$, $i = 1, 2$, apply homogeneity to the operations F_i° defined in the class

$$\{[(R_1, A_1), (R_2, A_2)] : \exists D_1 D_2 (R_i, [A_i, D_i]) \models \theta_i, i = 1, 2, \text{ and } A_1 \approx A_2\}$$

and sending $[(R_1, A_1), (R_2, A_2)]$ to $F_i(A_i)$.

(iii) $\Rightarrow$ (ii) It is enough to show that (iii) implies PPP. If $A \equiv_L B$ and $[C, A] \models \theta$, $[C, B] \models \neg\theta$ then $[C, A, B] \models \theta \wedge \neg\underline{\theta}^Q$ for an appropriate renaming and relativization of θ. By general homogeneity applied to the projections π_2, π_3 there is $[C^*, A^*, B^*] >_L [C, A, B]$ with $A^* \approx B^*$. Hence, $[C^*, A^*] \models \theta$, $[C^*, B^*] \models \neg\theta$ which is absurd.

(ii) $\Rightarrow$ (i) Assume $A_1|\tau \equiv_L A_2|\tau$. If X is a set of infinite power greater than $|A_1|, |A_2|$, and $D = [(X), A_1, A_2]$ then $[D, A_i|\tau] = [(X), A_1, A_2, A_i|\tau] \approx [(X'), A_1, A_2, A_i|\tau]$ with the disjoint union $X' \cup |A_1| \cup |A_2| \cup |A_i|$ being equal to X; then, $[D, A_i|\tau] \approx (X, D, A_i|\tau)$, $i = 1, 2$. Since $[D, A_1|\tau] \equiv_L [D, A_2|\tau]$ by the PPP then $(X, D, A_1|\tau) \equiv_L (X, D, A_2|\tau)$. By the strong homogeneity property there must exist $(X^*, D^*, A_1^*|\tau, A_2^*|\tau) \equiv_L (X, D, A_1|\tau, A_2|\tau)$ such that $(X^*, D^*, A_1^*|\tau) \approx (X^*, D^*, A_2^*|\tau)$. Hence, $A_i^* \equiv A_i$ and $A_1^*|\tau \approx A_2^*|\tau$ yielding Robinson's property. □

12.6 Corollary. ROB + PPP *implies that any L-RPC_Δ operation preserves elementary equivalence.*

Proof. Assume $A \equiv_L B$ and $F(A) \models \varphi, F(B) \models \neg\varphi$. If F is L-RPC_Δ so are the operations π_1, π_3 in $K = \{C \in G(F) \times G(F) : \pi_2(C) \models \varphi, \pi_4(C) \models \neg\varphi\}$. Let $C = [A, F(A), B, F(B)]$ then $\pi_1(C) \equiv_L \pi_3(C)$ and by general homogeneity there is $C^* = [A^*, F(A^*), B^*, F(B^*)] >_L C$ in K such that $A^* \approx B^*$, which is absurd since this implies $F(A^*) \approx F(B^*)$ and $F(A^*) \models \varphi$, $F(B^*) \models \neg\varphi$. □

13. Characterization of Initial Structures

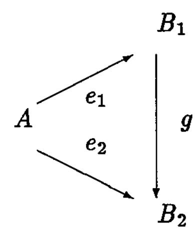
The following is a interesting consequence of the Robinson property. It does not depend on the PPP (a simpler proof could be given using homogeneity but this depends on PPP). A structure A will be called *L-initial* if it has at most one L-elementary embedding in any other structure. For example, $(\omega, <)$ is L-initial for any logic L.

13.1 Theorem. *If L has occurrence number smaller than the first measurable cardinal and satisfies Robinson's Lemma, then a structure A is L-initial if and only if all its elements are L-definable in A.*

PROOF. If all elements of a structure are L-definables, it is obviously L-initial. Now, let A be a L-initial structure and consider the partial operation from $E_{\tau\cup\{c\}}$ to $E_{\tau\cup\{c\}\cup\{c_a:a\in A\}}$:

$$F((B,b)) = (B,b,c_a)_{a\in A} \text{ if and only if } (B,c_a)_{a\in A} \models T$$

where $T = Th_L((A,a)_{a\in A})$. This is a PC_Δ definition because T must be equivalent to a set; to see that it actually defines an operation take an isomorphism $g:(B_1,b_1)\to(B_2,b_2)$ and $F((B_i,b_i) = (B_i,b_i,c_a^i)_{a\in A}$, $i=1,2$, then we have elementary embeddings $e_i : A \to B_i$, $e_i(a) = c_a^i$. By initiality of A, the diagram



commutes. This means that $g:(B_1,b_1,c_a^1)_1 \to (B_2,b_2,c_a^2)_a$ is as also an isomorphism. By Theorem 12.1 this operation must be uniformly continuous. If some $e\in A$ where not definable, consider the sentence $c=c_e$. There must exist a finite set of sentences Φ such that $(B,b)\equiv_\Phi(B',b')$ implies: $F(B,b)\models c=c_e$ if and only if $F(B',b')\models c=c_e$. But $(A,e)\equiv_\Phi(A,d)$ for some $d\neq e$ by non-definability of e. As $(A,e,a)_{a\in A}\models c=c_e$ then $(A,d,a)_{a\in A}\models c=c_e$, this is $d=e$, which is absurd. □

Call an element of a structure L_∞-*definable in* A if it is the only element of A realizing its L-type. If we do not put conditions on the occurrence number of the logic we still get:

13.2 THEOREM. *If L satisfies Robinson's Lemma, then a structure A is L-initial if and only if all its elements are L_∞-definable in A.*

13.3 COROLLARY. *If L is a small logic satisfying Robinson's Lemma then any theory of L having arbitrarily large models has arbitrarily large models A, B with two distinct elementary embeddings from A to B.*

PROOF. As the logic is small the structures with all its elements L_∞-definable are bounded in power. Any model of the theory of a larger power is non-initial and so it provides two distinct embeddings in another model. □

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