The Abstract Compactness Theorem revisited [1]

by

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Abstract. The Abstract Compactness Theorem of Makowsky and Shelah for model theoretic logics is shown to be an immediate consequence of a general characterization of topological spaces having $[\kappa, \lambda]$-compact products, when applied to spaces of structures endowed with the natural topology induced by the definable classes of a logic L. In this context, the notion of an ultrafilter $\mathcal{U}$ being related to L corresponds to $\mathcal{U}$-compactness of theses spaces. The given characterization of topological productive $[\kappa, \lambda]$-compactness may have independent interest since it generalizes known results by J. Ginsburg and G. Saks, G. Saks, S. García-Ferreira, and others, for initial κ- compactness.

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INTRODUCTION

It is well known that Łoś theorem on ultraproducts implies the compactness theorem of first order logic (Frayne, Morel, and Scott 1962 [FMS]). Similarly, utilizing appropriate versions of Łoś theorem one may prove $[\kappa, \kappa]$-compactness of the infinitary logic $L_{\kappa\kappa}$ for a measurable cardinal κ, or $[\omega, \omega]$-compactness of $L(Q_{c^+})$ where Q_{c^+} is the quantifier "there are at least $(2^\omega)^+$ ...". It is natural to ask then if any form of compactness in model theoretic logics is associated to some Łoś-like theorem. Makowsky and Shelah 1983 [MSh] have shown the remarkable result that this is always the case for $[\kappa, \lambda]$-compactness of a logic. That is the content of their

Abstract Compactness Theorem. *A model theoretic logic L is $[\kappa, \lambda]$-compact if and only if there is a (λ, κ)-regular ultrafilter $\mathcal{U}$ over some set I which satisfies the following property:*

$(*)$ *For any family of structures $\{A_i : i \in I\}$ of type σ there is an extension A^* of the ultraproduct $\Pi_i A_i/_{\mathcal{U}}$ such that for any formula $\phi(x, ...) \in L(\sigma)$ and sequence of functions $f, ... \in \Pi_i A_i$:*

$$A^* \models \phi[f/_{\mathcal{U}}, ...] \quad \text{iff} \quad \{i \in I : A \models \phi[f(i), ...]\} \in \mathcal{U}.$$

($\mathcal{U}$ may be taken always over $I = \kappa^{<\lambda}$, or uniform over $I = \kappa$ if κ is a regular cardinal and $\kappa = \lambda$).

The theorem implies for example that $L(Q_{\alpha+1})$ is $[\kappa, \omega]$-compact if and only if $\omega_\alpha^\kappa = \omega_\alpha$ (hence, $L(Q_1)$ is not $[\omega, \omega]$-compact). Deeper consequences about the compactness spectrum of logics are discussed in Makowsky 1985 [Ma].

The original version in [MSh] is given in terms of extensions of ultrapowers and assumes expansions of vocabularies by binary relation symbols. We have stated the theorem in terms of ultraproducts because this version implies the original one and holds for a wider family of logics, including monadic logics. For a version in terms of ultralimits see Lipparini 1987 [Li].

An ultrafilter $\mathcal{U}$ satisfying the Loś-like condition $(*)$ of the Theorem is said to be *related to L*.

We show in this paper that the "Abstract Compactness Theorem" and other results on $[\kappa, \lambda]$-compactness of logics are purely topological phenomena. They follow from a characterization of *productive* $[\kappa, \lambda]$-compactness of topological spaces, that is preservation of $[\kappa,\lambda]$-compactness by cartesian products, which generalizes analogous results for productive $[\kappa,\omega]$-compactness by Ginsburg and Saks 1975 [GS], Saks 1978 [Sa], and García-Ferreira 1990 [GF]. For this purpose we consider the spaces of first order structures endowed with the topology induced by the "elementary" classes of a logic L, the key observations being that an ultrafilter $\mathcal{U}$ is related to L if and only these spaces are $\mathcal{U}$-compact in the sense of Saks, and any product of them is $[\kappa,\lambda]$-compact when the logic is $[\kappa,\lambda]$-compact.

In the topological side, our characterization implies that several properties previously known for logic compactness hold true for productive compactness of topological spaces. For example, if κ is smaller than the first measurable cardinal then productive $[\kappa,\kappa]$-compactness of a space implies (productive) countable compactness of the space.

I. $[\kappa,\lambda]$-COMPACTNESS AND $\mathcal{U}$-COMPACTNESS OF TOPOLOGICAL SPACES

The following natural generalization of the notion of compactness of a topological space was first considered by Alexandroff and Urysohn in 1929 [AU] and thoroughly studied later by many people. See the survey papers by Vaughan 1984 [V2] and Stephenson 1984 [St], also Nyikos 1992 [N].

Definition 1.1. A topological space X is $[\kappa,\lambda]$-*compact*, for $\omega \le \lambda \le \kappa \le \infty$, if and only if any set of at most κ closed subsets of X (of arbitrary power if $\kappa = \infty$) such that every sub-family of power less than λ has non-empty intersection (λ-*intersection property)*, has itself non-empty intersection. $[\kappa,\omega]$-compactness and $[\infty,\kappa]$-compactness are usually called *initial* κ-*compactness* and *final* κ-*compactness,* respectively.

The reader should be aware that the notation utilized in model theory for $[\kappa,\lambda]$-compactness, which we will use in this paper, reverses the notation utilized in the topological literature.

$[\infty,\omega]$-compactness is (full) compactness, $[\omega,\omega]$-compactness is *countable compactness*, and $[\infty,\omega_1]$-compactness is the *Lindelöf property.* Although $[\kappa,\kappa]$- compactness does not transfer up or down, for example, ω with the discrete topology is trivially $[\omega_1,\omega_1]$-compact but not $[\omega,\omega]$-compact and $(\omega_1,<)$ with the order topology is $[\omega,\omega]$-compact but not $[\omega_1,\omega_1]$-compact, there are some straightforward transfer relations:

LEMMA 1.2. i) *X is $[\kappa, \lambda]$-compact if and only if it is $[\mu, \mu]$-compact for any $\lambda \leq \mu \leq \kappa$.*

ii) *If X is $[cof(\kappa), cof(\kappa)]$-compact then it is $[\kappa, \kappa]$-compact.*

iii) *If $f : X \to Y$ is continuous and X is $[\kappa, \lambda]$-compact, then f(X) is $[\kappa, \lambda]$-compact.*

Proof. i) One implication is trivial. For the other, notice that a counterexample to $[\mu, \lambda]$-compactness with μ minimum is a counterexample to $[\mu, \mu]$-compactness; (ii) and (iii) follow from the definitions. $\square$

The product of $[\kappa, \lambda]$-compact spaces does not need to be $[\kappa, \lambda]$-compact, even for squares. For example, the real line with the topology generated by the intervals [a,b) is a Lindelöf space but its square is not (see Willard 1968 [W]), and the product of two countably compact spaces is not necessarily countably compact (see Vaughan 1974 [V1], Hart and Mill 1991 [HM]).

On the positive side, Stephenson and Vaughan 1974 [SV] have shown that $[\kappa, \omega]$-compactness is preserved by products if κ is a strong limit singular cardinal, and starting with Scaraborough and Stone 1966 [SS] a deep study of spaces with countably compact or initial κ-compact products, and related properties, has been carried out by Vaughan 1974 [V1], Ginsburg and Saks 1975 [GS], Saks 1978 [Sa], García-Ferreira 1990 [GF], among others. The main tools in this study have been the notion of ultrafilter convergence and compactness, introduced by Berstein 1970 [Be] for ultrafilters over ω, and extended later by Saks to ultrafilters over uncountable powers.

Definition 1.3. Let $\mathcal{U}$ be an ultrafilter over a set I, then an I-family $\{a_i : i \in I\}$ in a topological space X is said to *$\mathcal{U}$-converge* to a point $x \in X$ if and only if $\{i \in I : a_i \in V\} \in \mathcal{U}$ for any open neighborhood V of x. We say also that x is an *$\mathcal{U}$-limit* of $\{a_i : i \in I\}$, and write $\{a_i : i \in I\} \to_{\mathcal{U}} x$. A space X will be called *$\mathcal{U}$-compact* if and only if any I-family of X has an $\mathcal{U}$-limit in X.

$\mathcal{U}$-limits are not necessarily unique since we do not assume the Hausdorff condition. Evidently, $\{a_i : i \in I\} \to_{\mathcal{U}} x$ if and only if x is an adherence point in X of the ultrafilter $a(\mathcal{U}) = \{S \subseteq X : \{i \in I : a_i \in S\} \in \mathcal{U}\}$ in the ordinary sense of topology. Hence, X is fully compact if and only it is $\mathcal{U}$-compact for any ultrafilter. Contrasting with $[\kappa, \lambda]$-compactness, $\mathcal{U}$-convergence and compactness are preserved by products.

LEMMA 1.4. i) *$\{(a_{i,\alpha})_\alpha : i \in I\} \to_{\mathcal{U}} (a_\alpha)_\alpha$ in $\Pi_\alpha X_\alpha$, if and only if $\{a_{i,\alpha} : i \in I\} \to a_\alpha$ in X_α for each α.*

ii) $\Pi_\alpha X_\alpha$ *is* $\mathcal{U}$*-compact if and only if each* X_α *is* $\mathcal{U}$*-compact.*

In fact, it follows from Saks work for a related compactness property $C[\kappa,\lambda]$ concerning the existence of *complete accumulation points* (property in general stronger than $[\kappa,\lambda]$-compactness but equivalent to it for $\lambda=\omega$ or $\kappa=\lambda$ regular), that productive $[\kappa,\omega]$ -compactness of a space is equivalent to $\mathcal{U}$-compactness with respect to particular families of uniform ultrafilters depending on the space. García-Ferreira has shown that $\mathcal{U}$-compactness with respect to a single decomposable ultrafilter on κ is enough. We sumarize this in the next proposition. The last item also follows immediately from Saks work but we have not seen it stated anywhere.

PROPOSITION 1.5 i) (Th. 6.2, Saks 1978 [Sa]; Th. 5.13, Stephenson 1984 [St]) *All powers of* X *are* $[\kappa,\omega]$*-compact if and only if there is a sequence of ultrafilters* $\{\mathcal{U}_\mu : \omega \le \mu \le \kappa\}$, $\mathcal{U}_\mu$ *uniform over* μ*, such that* X *is* $\mathcal{U}_\mu$*-compact for each* μ.

ii) (Prop. 2.15, García-Ferreira 1990 [GF]). *All powers of* X *are* $[\kappa,\omega]$*-compact if and only if there is a decomposable ultrafilter* $\mathcal{U}$ *over* κ *such that* X *is* $\mathcal{U}$*-compact.*

iii) *For regular* κ*,* X *has* $[\kappa,\kappa]$*-compact powers if and only if* X *is* $\mathcal{U}$*-compact for some uniform ultrafilter* $\mathcal{U}$ *over* κ*.*

By Donder 1988 [D], it is consistent that a uniform ultrafilter over κ is always decomposable. Therefore, by (ii) and (iii) above, it is consistent that productive $[\kappa,\kappa]$-compactness for a regular cardinal κ is equivalent to productive $[\kappa,\omega]$-compactness.

In order to obtain the Abstract Compactness Theorem (in the next section), we extend the above characterizations to $[\kappa,\lambda]$-compactness for arbitrary κ, λ, utilizing (λ,κ)-regular ultrafilters.

Definition 1.6 (cf. Keisler 1964 [Ke]). An ultrafilter $\mathcal{U}$ over a set I is (λ,κ)-regular if and only if there is a family $\mathcal{F}\subseteq\mathcal{U}$ of power κ such that $\cap\mathcal{J}=\emptyset$ for any sub-family $\mathcal{J}$ of power λ. A (κ,ω)*-regular* ultrafilter is usually called κ*-regular* in the literature.

It is easy to verify that a uniform ultrafilter over κ is $(\kappa,\kappa)-$regular, and a decomposable ultrafilter over κ is $(\mu,\mu)-$regular for any regular cardinal $\mu\le\kappa$.

Call a class T of spaces *productively* $[\kappa,\lambda]$*-compact* (in short, p-$[\kappa,\lambda]$- compact) if the product of any family of spaces in T is $[\kappa,\lambda]$-compact. In particular, a space X will be productively $[\kappa,\lambda]$-compact if X^β is $[\kappa,\lambda]$-compact for any cardinal β. The next theorem is our main result.

THEOREM 1.7. *The following are equivalent for any class* T *of topological spaces:*

i) *T is productively $[\kappa,\lambda]$-compact.*

ii) *There exists some (λ,κ)-regular ultrafilter $\mathcal{U}$ such that all the spaces in T are $\mathcal{U}$-compact ($\mathcal{U}$ may be taken always over $\kappa^{<\lambda}$, or uniform over κ if this cardinal is regular and equal to λ.)*

The proof is defered to the last section of the paper but we note now the following immediate consequences:

COROLLARY 1.8. *Let T be a class of topological spaces.*

i) *If κ is smaller than the first measurable cardinal (or arbitrary if no such cardinal exists) and T is p-$[\kappa,\kappa]$-compact, then it is p-$[\omega,\omega]$-compact.*

ii) *If T is p-$[\kappa^+,\kappa^+]$-compact, then it is p-$[\kappa,\kappa]$-compact.*

iii) *Let $\mathrm{cof}(\kappa)\geq\lambda$ or $\kappa=\infty$. Then T is p-$[\kappa,\lambda]$-compact if and only if it is p-$[\mu,\mu]$-compact for any regular cardinal μ, $\lambda\leq\mu\leq\kappa$.*

Proof. i) Let $\mathcal{U}$ be the (κ,κ)-regular ultrafilter over $\kappa^{<\lambda}$ given by the Theorem such that X is $\mathcal{U}$-compact. By (κ,κ)-regularity, $\mathcal{U}$ is non principal. If $\mathcal{U}$ is not (ω,ω)-regular, then it is ω-complete. But the smallest set carrying a ω-complete non principal ultrafilter is measurable, and $\kappa^{<\lambda}$ is below the first measurable by hypothesis.

ii) A uniform ultrafilter on κ^+ is (κ,κ)-regular by results of Kanamori 1976 [K] and Kunen-Pikry 1971 [KP].

iii) If the compactness condition holds for regular μ, $\lambda\leq\mu\leq\kappa$, then p-$[\delta,\delta]$-compactness follows from (ii) for $\lambda\leq\delta<\kappa$ because $\lambda\leq\delta^+\leq\kappa$ and for $\delta=\kappa$ by Lemma 1.2(ii). Now apply Lemma 1.2(i). □

II. SPACES OF STRUCTURES

For the definition of model theoretic logic see Lindström 1969 [L] or Ebbinghaus 1985 [E]. The *domain* of a logic L, $\mathrm{Dom}(L)$, is the class of first order *vocabularies* σ for which the class of sentences $L(\sigma)$ is defined. $\mathrm{Dom}(L)$ will be assumed to allow expansions of vocabularies by arbitrarily many monadic relation symbols and constants, and to be closed under disjoint unions. $L(\sigma)$ will be always a set; that is, we consider *small logics* only. Appart of Lindström's axioms ((i)-(v) in [E]), we only assume closure under negations, conjunctions, and relativizations. $A|\tau$ and $A|P^A$ will denote, respectively, the reduct of a structure A to a sub-vocabulary τ, and the substructure of A induced by the subset P^A where P is a monadic relation symbol.

For each $\sigma \in \mathrm{Dom}(L)$, a logic L induces a topology in the class E_σ of structures of type σ, having for open basis the classes $\mathrm{Mod}(\phi), \phi \in L(\sigma)$. The topologically closed classes are then the L-axiomatizable classes $\mathrm{Mod}(T)$ for some theory $T \subseteq L(\sigma)$. The resulting large topological space of structures will be denoted $E_\sigma(L)$. Although it is proper class, its basis is parametrized by the set $L(\sigma)$ of sentences and so the topology is also parametrized by a set. Therefore, we may quantify over open classes, classes of open classes, etc., via the parameters, and apply without misgivings most of the ordinary concepts and results of topology to these spaces. The spaces $E_\sigma(L)$ are uniformizable by the canonical uniformity having for basis the classes:

$$U_F = \{(A, B) \in E_\sigma^2 : A \models \phi \text{ if and only if } B \models \phi, \text{ for any } \phi \in F\}.$$

where F runs through the finite theories $F \subseteq L(\sigma)$, cf. Caicedo 1993 [C1], 1995 [C2].

Definition 2.1. A logic L is said to be $[\kappa, \lambda]$-*compact* if whenever $\{T_\alpha\}_{\alpha<\kappa}$ is a family of theories in $L(\sigma)$ such that $\cup_{\beta<\delta} T_{\alpha_\beta}$ is satisfiable for any $\delta < \lambda$, then $\cup_{\alpha<\kappa} T_\alpha$ is satisfiable.

The above property is just topological $[\kappa, \lambda]$- compactness of the spaces $E_\sigma(L)$. The equivalence between this topological notion and the original definition of $[\kappa, \lambda]$-compactness in Makowsky and Shelah 1983 [Ma-Sh] was first noticed by Mannila 1983 [M]. It holds for any logic satisfying the closure conditions we have imposed on $\mathrm{Dom}(L)$.

Given a family of vocabularies $\{\sigma_i : i \in I\}$ in $\mathrm{Dom}(L)$, let $\oplus_i \sigma_i$ be the disjoint union of the vocabularies $\sigma_i \cup \{P_i\}$ where each P_i is a monadic symbol not in σ_i. $\mathrm{Dom}(L)$ is closed under this operation. The function

$$F : E_{\oplus_i \sigma_i} \to \Pi_i E_{\sigma_i} \ , \ F(A) = ((A|P_i^A)|\sigma_i)_{i\in I}$$

is onto because $(A_i)_{i\in I} = F(\oplus_i A_i)$, where $\oplus_i A_i = (\cup_i |A_i|, A_i, ...)_{i\in I}$ is the structure of type $\oplus_i \sigma_i$ having for universe the union of the universes of the A_i, with each σ_i interpreted by the relations of A_i and P_i interpreted by the universe $|A_i|$. Our key observation is the following:

LEMMA 2.2. *For any logic L, $F : E_{\oplus_i \sigma_i}(L) \to \Pi_i E_{\sigma_i}(L)$ is (uniformly) continuous. Hence, L is $[\kappa, \lambda]$-compact if and only if the family of topological spaces $\{E_\sigma(L) : \sigma \in$ Dom(L)$\}$ is productively $[\kappa, \lambda]$-compact.*

Proof. Any open subbasic of the product topology in $\Pi_i E_{\sigma_i}(L)$ has the form $U_{j,\phi} = \{(A_i)_i : A_j \models \phi\}$ with $j \in I$ and $\phi \in L(\sigma_j)$. Hence, $F^{-1}(U_{j,\phi}) = \{A : (A|P_j^A)|\sigma_j \models \phi\} = \{A : A \models \phi^{P_j}\}$ which is a basic open class due to the reduct axiom and the existence of relativizations in L. By Lemma 1.2 (iii), the $[\kappa, \lambda]-$compactness of the space $E_{\oplus_i \sigma_i}(L)$ is inherited by the product space $\Pi_i E_{\sigma_i}(L)$. □

Now, expressing $\mathcal{U}$-convergence in the spaces $E_\sigma(L)$ in terms of the basic open classes $Mod(\phi)$ and using that the logic has negations, we have for any ultrafilter $\mathcal{U}$ over I and structures A, A_i $(i \in I)$ in $E_\sigma(L)$

$$\{A_i : i \in I\} \to_{\mathcal{U}} A \ \ \textit{if and only if::} A \models \phi \Leftrightarrow \{i \in I : A_i \models \phi\} \in U \textit{for any } \phi \in L(\sigma) \tag{1}$$

(without negations we would have only left to right implication). Therefore, equivalence (1) in the Abstract Compactness Theorem applied to sentences yields immediately $\{A_i : i \in I\} \to_{\mathcal{U}} A^*$, and so the spaces $E_\sigma(L)$ must be $\mathcal{U}$-compact if $\mathcal{U}$ is related to L. In fact,

LEMMA 2.3. *A ultrafilter $\mathcal{U}$ is related to a logic L if and only if the spaces $E_\sigma(L)$ are $\mathcal{U}$-compact for any $\sigma \in$Dom(L).*

Proof. Assume that the spaces $E_\sigma(L)$ are $\mathcal{U}$-compact. Given a family of structure $\{A_i : i \in I\}$ of type σ, consider the vocabulary $\sigma_{\Pi_i A_i} = \sigma \cup \{c_f : f \in \Pi_i A_i\}$, where each c_f is a constant symbol, and define for each fixed $j \in I$ the following expansion of type $\sigma_{\Pi_i A_i}$ of A_j :

$$A_j^* = (A_j, f(j), ...)_{f \in \Pi_i A_i}$$

where c_f is interpreted by $f(j)$. Since $E_{\sigma_{\Pi_i A_i}}(L)$ is $\mathcal{U}$-compact by hypothesis, the family $\{A_j^*\}_j$ has an $\mathcal{U}$-limit $(A^*, a_f, ...)$. By (2) this means:

$$\begin{aligned} A^* \models \phi[a_f, ...] &\Leftrightarrow (A^*, a_f, ...) \models \phi(c_f, ...) \\ &\Leftrightarrow \{j \in I : A_j^* \models \phi(c_f, ...)\} \in \mathcal{U} \ \Leftrightarrow \{j \in I : A_j \models \phi[f(j), ...]\} \in \mathcal{U} \end{aligned} \tag{2}$$

for all $\phi(x,...) \in L(\sigma)$. Applied to atomic formulae equivalence (3) yields an embedding $f/_{\mathcal{U}} \mapsto a_f$ from $\Pi_i A_i/_{\mathcal{U}}$ into A^*. Therefore, A^* may be taken to be a true extension of $\Pi_i A_i/_{\mathcal{U}}$ with $a_f = f/_{\mathcal{U}}$, and (3) becomes then condition (1), showing that $\mathcal{U}$ is related to L. □

Theorem 1.7 applied to the family of spaces $T = \{E_\sigma(L) : \sigma \in Dom(L)\}$, together with lemmas 2.2, 2.3, gives the topological proof of the **Abstract Compactness Theorem**.

After Lemma 2.2, any property of productive $[\kappa,\lambda]$-compactness of topological spaces may be translated directly to $[\kappa,\lambda]$-compactness of logics, without passing by the Abstract Compactness Theorem. For example, the following results from Makowsky-Shelah 1983 [Ma-Sh] are direct applications of the respective parts of Corollary 1.8. It follows from this topological proof that they hold for monadic logics, since Lemma 2.2 only needs expansions of vocabularies by monadic predicate symbols and relativizations of sentences to monadic predicates. Their original proofs rely instead in the possibility of expanding vocabularies by non monadic relation symbols.

Example 2.4.

i) (Lemma 2.6 [Ma-Sh]) *If κ is below the first measurable cardinal then $[\kappa,\kappa]$-compactness of a logic implies $[\omega,\omega]$-compactness.*

ii) (Th. 3.10 [Ma-Sh]). *$[\kappa^+,\kappa^+]$-compactness of a logic implies $[\kappa,\kappa]$-compactness.*

iii) (Th. 3.11 [Ma-Sh]). *For regular κ, $[\mu,\mu]$-compactness of a logic for all regular μ, $\lambda \le \mu \le \kappa$, implies $[\kappa,\lambda]$-compactness.*

Remark 2.5. The condition on expansion of vocabularies by constants is not needed in Lemma 2.2, and it may be replaced by closure under the existential quantifier in Lemma 2.3 if the formulae of the logic contain only finitely many free variables because then the constants may be simulated by monadic predicates.

Remark 2.6. To see that the Abstract Compactness Theorem may be stated in terms of ultrapowers as in [Ma-Sh] when L allows expansions of vocabularies by binary relation symbols and relativization to a variable of a binary relation (for example if L is closed under substitutions), it is enough to note that the existence of an extension satisfying (1) for any $\mathcal{U}$-ultrapower implies the same for any $\mathcal{U}$-ultraproduct. Given a family $\{A_i : i \in I\} \subseteq E_\sigma$, code it in the single structure $A = (\sqcup_i A_i, \sqcup_i Q^{A_i}, ..., I, R)_{Q\in\sigma}$ containing I as a predicate, and the relation $R = \cup_{i\in I}\{i\}\times A_i$ so that $A|\{x : R(i,x)\} =$

A_i. If the extension $B^* \supseteq A^I/_\mathcal{U}$ satisfies (1), define $P^* = \{v \in B^* : B^* \models R[g/_\mathcal{U}, v]\}$ where $g \in A^I$ is the identity function. Then $\Pi_i A_i/_\mathcal{U} \leq B^*|P^*$, and for any formula $\phi(x, ...) \in L(\sigma)$ and function $f \in \Pi_i A_i$:

$$B^*|P^* \models \phi[f/_\mathcal{U}, ...] \Leftrightarrow B^* \models \phi[f/_\mathcal{U}, ...]^{\{v:R(g/_\mathcal{U},v)\}}$$

$$\Leftrightarrow \{i \in I : A_i \models \phi[f(i), ...]^{\{v:R(i,v)\}}\} \in \mathcal{U}$$

$$\Leftrightarrow \{i \in I : A_i|\{v : R(i, v)\} \models \phi[f(i), ...)]\} = \{i \in I : A_i \models \phi[f(i), ...]\} \in \mathcal{U}.$$

Therefore $B^*|P^*$ is the desired extension of $\Pi_i A_i/_\mathcal{U}$ satisfying (1).

III. CHARACTERIZATION OF PRODUCTIVE $[\kappa, \lambda]$ -COMPACTNESS

In this section we prove Theorem 1.7 in a wide version (Theorem 3.4 below) including characterizations of productive $[\kappa, \lambda]$-compactness in terms of small products, which generalize similar known results for initial κ-compactness.

LEMMA 3.1. *If X is $\mathcal{U}$-compact for a (λ, κ)-regular ultrafilter $\mathcal{U}$, then X is $[\kappa, \lambda]$-compact.*

Proof. Let $\{I_\alpha\}_{\alpha<\kappa}$ be a family of elements of $\mathcal{U}$ such that the intersection of any λ of the I_α's is empty. We may assume $I = I_0$. Given a family $\{F_\alpha\}_{\alpha<\kappa}$ of closed sets in X with the λ-intersection property, define $F_t = \cap_{t\in I_\alpha} F_\alpha$ for each $t \in I$. This set is non-empty, because t belongs to less than λ many sets I_α by hypothesis. Choose $a_t \in F_t$, then $J_\alpha = \{t \in I : a_t \in F_\alpha\} \in \mathcal{U}$ because $J_\alpha \supseteq I_\alpha$ by construction of F_t. By hypothesis, $\{a_t\}_{t\in I}$ $\mathcal{U}$-converges to some x of X; hence, given an open neighborhood V of x, $J = \{t \in I : a_t \in V\} \in \mathcal{U}$. Therefore, $\{t : a_t \in V \cap F_\alpha\} = J \cap J_\alpha \in \mathcal{U}$ for any α. Consequently, this set is non-empty, showing that x belongs to the adherence of any F_α. $\square$

Definition 3.2. Let $P(\kappa, \lambda) = \{S \subseteq \kappa : |S| < \lambda\}$; hence, $|P(\kappa, \lambda)| = \kappa^{<\lambda}$.

LEMMA 3.3. i) *If X is $[\kappa, \lambda]$-compact, then every I-family in X, with $I = P(\kappa, \lambda)$, $\mathcal{U}$-converges for some (λ, κ)-regular ultrafilter $\mathcal{U}$ over I (which depends on the family).*

ii) *If κ is a regular cardinal and X is $[\kappa, \kappa]$-compact then every κ-family in X $\mathcal{U}$-converges for some uniform ultrafilter $\mathcal{U}$ over κ.*

Proof. i) Given $\{a_t : t \in I\}$ in X, let $A_t = \{a_s : t \subseteq s\}$. The family of closed sets $\{\mathrm{cl}(A_t) : t \in P(\kappa,\omega)\}$ has the λ-intersection property, because

$$\bigcap_{i<\delta} \mathrm{cl}(A_{t_i}) \supseteq \mathrm{cl}\left(\bigcap_{i<\delta} A_{t_i}\right) = \mathrm{cl}\left(A_{\cup_{i<\delta} t_i}\right) \neq \emptyset ,$$

whenever $\delta < \lambda$ since $\bigcup_{i<\delta} t_i \in I$. By $[\kappa,\lambda]$-compactness there is an element $x \in \bigcap_{t \in P(\kappa,\omega)} \mathrm{cl}(A_t)$. Hence, $V \cap A_t \neq \emptyset$ for any neighborhood V of x and any $t \in P(\kappa,\omega)$. This implies that $a^{-1}(V) \cap [t] = \{s \in I : a_s \in V \text{ and } s \supseteq t\}$ is non empty and in fact the family $F = \{a^{-1}(V) \cap [t] : V \text{ open neighborhood of } x,\ t \in P(\kappa,\omega)\}$ has the finite intersection property. Extend F to an ultrafilter $\mathcal{U}$ over I. By construction, $\{a_s : s \in I\}$ $\mathcal{U}$-converges to x. Moreover, for any ordinal $\alpha \in \kappa : I_\alpha = \{s \in I : \alpha \in s\} = a^{-1}(X) \cap [\{\alpha\}) \in \mathcal{U}$. But the intersection of λ-many distinct I_α's is empty because no $s \in I$ may contain λ many ordinals. This shows that $\mathcal{U}$ is (λ,κ)-regular.

ii) For any κ-sequence $(a_\alpha)_{\alpha<\kappa}$ in X, let $A_\alpha = \{a_\beta : \beta \geq \alpha\}$. By regularity of κ and $[\kappa,\kappa]$-compactness, there is $x \in \cap_{\alpha<\kappa} \mathrm{cl}(A_\alpha)$. Hence, $V \cap A_\alpha \neq \emptyset$ for any $\alpha \in \kappa$ and open neighborhood V of x. By transfinite induction and regularity of κ we get an increasing sequence of ordinals $\{\beta_\gamma : \gamma < \kappa\}$ such that $a_{\beta_\gamma} \in V \cap A_{\beta_\gamma}$ for all $\gamma < \kappa$. This means that all sets in the family $F = \{a^{-1}(V) \cap [\alpha) : x \in V, \alpha < \kappa\}$ have power κ; hence, F may be extended to an uniform ultrafilter $\mathcal{U}$ over κ, such that $(a_\alpha)_{\alpha\in\kappa}$ $\mathcal{U}$-converges to a. $\square$

THEOREM 3.4. *The following are equivalent for any class T of topological spaces:*

i) *T is productively $[\kappa,\lambda]$-compact.*

ii) *There exists some (λ,κ)-regular ultrafilter $\mathcal{U}$ such that all the spaces in T are $\mathcal{U}$-compact ($\mathcal{U}$ may be taken over $P(\kappa,\lambda)$, or uniform over κ when $\kappa = \lambda$ is regular).*

iii) *Any product of $2^{2^{|P(\kappa,\lambda)|}}$ many copies of spaces in T is $[\kappa,\lambda]$-compact.*

iv) *(If $cof(\kappa) \geq \lambda$) Any product of 2^{2^κ} many copies of spaces in T is $[\kappa,\lambda]$-compact.*

Proof. (ii) $\Rightarrow$ (i). If each X_r is $\mathcal{U}$-compact then $\Pi_r X_r$ is $\mathcal{U}$-compact by Lemma 1.4(ii), and by (λ,κ)-regularity of $\mathcal{U}$ it follows from Lemma 3.1 that $\Pi_r X_r$ is $[\kappa,\lambda]$-compact. This works also if $\mathcal{U}$ is uniform over $\kappa = \lambda$ because then it must be (κ,κ)-regular

(i) $\Rightarrow$(iii) $\Rightarrow$ (ii). Assume that any product of $2^{2^{|I|}}$ many spaces in T is $[\kappa,\lambda]$-compact, but there is no (λ,κ)-regular ultrafilter $\mathcal{U}$ over $I = P(\kappa,\lambda)$ such that all the elements of T are $\mathcal{U}$-compact. Let Σ be the family of all (λ,κ)-regular ultrafilters over I and choose for each $\mathcal{U} \in \Sigma$ an I-family $\{a_{\mathcal{U},i} : i \in I\}$ in some space $X_\mathcal{U} \in T$ which does not $\mathcal{U}$-converge. For each i, let $\sigma_i = (a_{\mathcal{U},i})_\mathcal{U} \in \Pi_{\mathcal{U}\in\Sigma} X_\mathcal{U} = X^*$. As Σ has power at most $2^{2^{|I|}}$, this space is $[\kappa,\lambda]$-compact by hypothesis; then by Lemma 3.3, there is an ultrafilter $\mathcal{W} \in \Sigma$ such that $\{\sigma_i : i \in I\}$ $\mathcal{W}$-converges to some $\sigma = (a_\mathcal{U})_\mathcal{U} \in X^*$.

By the continuity of the $\mathcal{W}$-projection (Lemma 1.4 (i)), $\{a_{\mathcal{W},i} : i \in I\}$ $\mathcal{W}$-converges to $a_{\mathcal{W}}$ in $X_{\mathcal{W}}$, a contradiction.

(iv) $\Rightarrow$ (i). First assume that $\kappa = \lambda$ is regular, then as in the previous proof, working with uniform ultrafilters over $I = \kappa$, utilizing Lemma 3.3 (ii), and recalling that a uniform ultrafilter on κ is (κ, κ)-regular, we get (ii) and a fortiori (i). Now, if $cof(\kappa) \geq \lambda$ and μ is regular, $\lambda \leq \mu \leq \kappa$, then the hipotesis implies that the product of $2^{2^{\mu}}$ spaces in T is $[\mu, \mu]$-compact, and by the previous observation T is productively $[\mu, \mu]$-compact. Hence X is p-$[\kappa, \lambda]$-compact by Corollary 1.8(iii) which depends only on the equivalence (i) $\Leftrightarrow$ (ii), already proven.□

Notice that the equivalence (i)$\Leftrightarrow$(iv) of the the previous theorem for the cases $\lambda = \omega$ or $\kappa = \lambda$ regular follows already from Th. 2.3 in Saks 1978 [Sa]. Making $T = \{X\}$, we obtain generalizations of Th. 5.14 in Stephenson 1984 [St], and Prop. 2.15 in García-Ferreira 1990 [GF]:

COROLLARY 3.5. *The following are equivalent for any topological space X:*

i) X^{β} *is* $[\kappa, \lambda]$*-compact for all* β.

ii) X *is* $\mathcal{U}$*-compact for some ultrafilter* $\mathcal{U}$ *(over* $I = P(\kappa, \lambda)$*, or uniform over* κ *if* $\kappa = \lambda$ is regular).

iii) $X^{2^{2^{|P(\kappa,\lambda)|}}}$ *is* $[\kappa, \lambda]$*-compact (* $X^{2^{2^{\kappa}}}$ *is* $[\kappa, \lambda]$*-compact, if* $\mathrm{cof}(\kappa) \geq \lambda$).

iv) $X^{|X|^{|P(\kappa,\lambda)|}}$ *is* $[\kappa, \lambda]$*-compact (* $X^{|X|^{\kappa}}$ *is* $[\kappa, \lambda]$*-compact, if* $\mathrm{cof}(\kappa) \geq \lambda$).

Proof. Only (iv) $\Rightarrow$ (i) needs proof. In the proof of Theorem 3.4, (iii) $\Rightarrow$ (ii), (iv) $\Rightarrow$ (i), one family $\{a_i : i \in I\}$ of X may serve as counterexample for the non convergence of various ultrafilters $\mathcal{U}$ over I; hence, we need to take the factor X in the power X^* only once for each possible family. That is, $X^* = X^{|X|^{|I|}}$. □

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