Every minimal dual discriminator variety is minimal as a quasivariety

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Abstract. Let (†) denote the following property of a variety $\mathcal{V}$: *Every subquasivariety of* $\mathcal{V}$ *is a variety.* In this paper, we prove that every idempotent dual discriminator variety has property (†). Property (†) need not hold for nonidempotent dual discriminator varieties, but (†) does hold for *minimal* nonidempotent dual discriminator varieties. Combining the results for the idempotent and nonidempotent cases, we obtain that every minimal dual discriminator variety is minimal as a quasivariety

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1. Introduction

In this paper, we consider a question that originates in algebraic logic. A propositional deductive system is called *structurally complete* if every admissible rule of the logic is derivable. Informally, this means that the rules of inference cannot be properly enlarged without properly increasing the set of theorems. As explained in [1], structural completeness for algebraizable logics corresponds to a property of the corresponding quasivariety: a quasivariety is *structurally complete* if it is generated as a quasivariety by its free members. The hereditary version of this property is of interest in algebraic logic: a quasivariety $\mathcal{Q}$ is *hereditarily structurally complete*, or *deductive*, if every subquasivariety of $\mathcal{Q}$ is generated by its free members. This is equivalent to the property that every subquasivariety of $\mathcal{Q}$ is a *relative subvariety*, i.e. is of the form $\mathcal{Q} \cap \mathcal{V}$ for some variety $\mathcal{V}$. When $\mathcal{Q}$ is itself a variety, this property means that every subquasivariety of $\mathcal{Q}$ is a subvariety.

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We take as our starting point the algebraic versions of these concepts. We consider the question: when is it the case that every subquasivariety of some variety is a variety? Our main result is about minimal varieties, and in this case the question becomes: when is a minimal variety minimal as a quasivariety?

In this paper we prove the following.

Theorem 1.1. *If $\mathcal{V}$ is a minimal dual discriminator variety, then $\mathcal{V}$ is minimal as a quasivariety.*

The argument for Theorem 1.1 is divided into two parts. In the first part we prove that any subquasivariety of an idempotent dual discriminator variety is a variety. Then we prove that any nonidempotent, minimal, dual discriminator variety is minimal as a quasivariety.

The dual discriminator operation on a set A is the ternary operation

$$q(a,b,c) = \begin{cases} a & \text{if } a = b; \\ c & \text{else.} \end{cases}$$

This operation is a ternary majority operation on A.

A variety $\mathcal{V}$ is a dual discriminator variety if it is generated as a variety by a subclass $\mathcal{K} \subseteq \mathcal{V}$ for which there exists a ternary term $q(x,y,z)$ such that q interprets as the dual discriminator operation on each member of $\mathcal{K}$. This forces the class of nontrivial members of the universal class $\mathbb{SP}_U(\mathcal{K})$ to coincide with the class of simple members of $\mathcal{V}$, and also to coincide with the class of subdirectly irreducible members of $\mathcal{V}$.

The standard examples of dual discriminator varieties are the variety of distributive lattices and the variety of Boolean algebras. In each of these cases, a suitable class $\mathcal{K}$ consists of the algebras in the variety of size ≤ 2, and the term $q(x,y,z) = (x \vee y) \wedge (x \vee z) \wedge (y \vee z)$ interprets as the dual discriminator on $\mathcal{K}$. Other examples can be created by starting with an arbitrary class $\mathcal{K}$ of similar algebras, adjoining a ternary operation $q(x,y,z)$ that is the dual discriminator operation on $\mathcal{K}$, and generating a variety with the resulting class $\mathcal{K}^q$.

The main theorem of this paper was proved for the more restrictive class of discriminator varieties in Proposition 5.4 of [4]. It is known that a variety is a discriminator variety if and only if it is both congruence permutable and a dual discriminator variety [7, Lemma 2.2(iii)].

2. The idempotent case

Our goal in this section is to prove that any subquasivariety of an idempotent dual discriminator variety is a variety. We begin by enumerating some of the properties of idempotent dual discriminator varieties.

The property that is most important for us, which is recorded in the next lemma, is that if $\mathcal{V}$ is an idempotent dual discriminator variety, $\mathbf{A} \in \mathcal{V}$, and

$a, b \in A$, then the principal congruence $\mathrm{Cg}^{\mathbf{A}}(a,b)$ has a uniquely determined complement $\mathrm{Cg}^{\mathbf{A}}(a,b)^*$, which is the kernel of an endomorphism of $\mathbf{A}$.

Lemma 2.1. *Let $\mathcal{V}$ be an idempotent dual discriminator variety, and suppose $q(x,y,z)$ is a ternary term whose interpretation in all simple members of $\mathcal{V}$ is the dual discriminator operation. Let $\mathbf{A} \in \mathcal{V}$ and let $a, b \in A$. The following hold:*

(1) *The function $h_{a,b}(x) := q^{\mathbf{A}}(a,b,x)$ is an endomorphism of $\mathbf{A}$.*
(2) *The congruence $\mathrm{Cg}^{\mathbf{A}}(a,b)^* := \ker(h_{a,b})$ is the unique complement of $\mathrm{Cg}^{\mathbf{A}}(a,b)$ in $\mathrm{Con}(\mathbf{A})$.*

Proof. The fact that $h_{a,b}$ is an endomorphism is [7, Theorem 3.2(2)]. Item (2), except for the uniqueness statement, is proved in [7, Theorem 3.8]. But dual discriminator varieties are congruence distributive, since the dual discriminator operation is a majority operation for the variety, and in congruence distributive varieties complements of congruences are unique when they exist. □

The congruence $\mathrm{Cg}^{\mathbf{A}}(a,b)^*$, complementary to $\mathrm{Cg}^{\mathbf{A}}(a,b)$, is called the *co-principal congruence* associated to (a,b).

We write $\mathbb{H}$, $\mathbb{S}$, and $\mathbb{P}$ to denote the class operators for the formation of the class of homomorphic images, subalgebras, and products, respectively. We let $\mathbb{Q}(\mathbf{A})$ denote the quasivariety generated by $\mathbf{A}$. If $\mathcal{Q}$ is a quasivariety and $\mathbf{A} \in \mathcal{Q}$ we write $\mathrm{Con}_{\mathcal{Q}}\mathbf{A}$ to denote the set of *relative congruences* of $\mathbf{A}$, that is the set of all $\theta \in \mathrm{Con}(\mathbf{A})$ such that $\mathbf{A}/\theta \in \mathcal{Q}$.

Lemma 2.2. *Let $\mathcal{V}$ be an idempotent dual discriminator variety, let $\mathcal{Q} \subseteq \mathcal{V}$ be a subquasivariety, and let $\mathbf{A} \in \mathcal{Q}$.*

(1) *The join of two co-principal congruences of $\mathbf{A}$ is co-principal: specifically, for all $a, b, c, d \in A$ it is the case that*
$$\mathrm{Cg}^{\mathbf{A}}(a,b)^* \vee \mathrm{Cg}^{\mathbf{A}}(c,d)^* = \mathrm{Cg}^{\mathbf{A}}(q^{\mathbf{A}}(a,b,c), q^{\mathbf{A}}(a,b,d))^*.$$
(2) *For all $a, b \in A$, $\mathbf{A}/\mathrm{Cg}^{\mathbf{A}}(a,b)^* \in \mathbb{S}(\mathbf{A}) \subseteq \mathcal{Q}$. Hence $\mathrm{Cg}^{\mathbf{A}}(a,b)^* \in \mathrm{Con}_{\mathcal{Q}}\mathbf{A}$.*
(3) *If γ is a maximal congruence of $\mathbf{A}$, then $\gamma \in \mathrm{Con}_{\mathcal{Q}}\mathbf{A}$.*

Proof. Item (1) is proved in [7, Corollary 3.9]. Item (2) follows from the fact that $\mathrm{Cg}^{\mathbf{A}}(a,b)^*$ is the kernel of an endomorphism of $\mathbf{A}$.

Now we consider Item (3). Fix a maximal congruence γ of $\mathbf{A}$, and let
$$\mathcal{I} := \{\theta^* : \theta^* \subseteq \gamma \text{ and } \theta \text{ is a principal congruence}\}$$
be the set of co-principal congruences contained in γ. We aim to show that γ is the directed union of the elements of $\mathcal{I}$. The fact that $\mathcal{I}$ is directed by join follows from Item (1). Since $\gamma \supseteq \bigcup \mathcal{I}$, it suffices to prove that the union of $\mathcal{I}$ majorizes every principal congruence below γ.

Choose a pair $(a,b) \in \gamma$. We have $\mathrm{Cg}^{\mathbf{A}}(a,b)^* \nsubseteq \gamma$, because otherwise
$$A^2 = \mathrm{Cg}^{\mathbf{A}}(a,b) \vee \mathrm{Cg}^{\mathbf{A}}(a,b)^* \subseteq \gamma,$$

which is false. Hence there is a pair $(c,d) \in \mathrm{Cg}^{\mathbf{A}}(a,b)^* \setminus \gamma$. Since $\mathrm{Cg}^{\mathbf{A}}(c,d) \cap \mathrm{Cg}^{\mathbf{A}}(c,d)^* = \Delta^A \subseteq \gamma$, and γ is meet prime in $\mathrm{Con}(\mathbf{A})$, it follows that $\mathrm{Cg}^{\mathbf{A}}(c,d)^* \subseteq \gamma$. We have

$$\begin{aligned}(c,d) \in \mathrm{Cg}^{\mathbf{A}}(a,b)^* &\Leftrightarrow \mathrm{Cg}^{\mathbf{A}}(a,b) \wedge \mathrm{Cg}^{\mathbf{A}}(c,d) = \Delta^A \\ &\Leftrightarrow (a,b) \in \mathrm{Cg}^{\mathbf{A}}(c,d)^* \ (\subseteq \gamma).\end{aligned}$$

So, for every $(a,b) \in \gamma$ there is a co-principal congruence $\mathrm{Cg}^{\mathbf{A}}(c,d)^*$ such that $(a,b) \in \mathrm{Cg}^{\mathbf{A}}(c,d)^* \subseteq \gamma$, which is what was required to establish that $\gamma = \bigcup \mathcal{I}$.

The lattice $\mathrm{Con}_{\mathcal{Q}}\mathbf{A}$ of relative congruences of $\mathbf{A}$ with respect to the quasivariety $\mathcal{Q}$ is the lattice of closed sets of an algebraic closure operator (cf. [5, Lemma 2.2]). Therefore $\mathrm{Con}_{\mathcal{Q}}\mathbf{A}$ is closed under the formation of unions of up-directed subsets. By Item (2) of this lemma we have that $\mathcal{I} \subseteq \mathrm{Con}_{\mathcal{Q}}\mathbf{A}$, and by Item (1) of this lemma $\mathcal{I}$ is directed by join, so $\gamma = \bigcup \mathcal{I} \in \mathrm{Con}_{\mathcal{Q}}\mathbf{A}$. □

We are now ready to prove the main result in this section.

Theorem 2.3. *If $\mathcal{V}$ is an idempotent dual discriminator variety, then any subquasivariety of $\mathcal{V}$ is a subvariety.*

Proof. Let $\mathcal{Q}$ be a subquasivariety of $\mathcal{V}$. We shall argue that $\mathcal{Q}$ is closed under the formation of quotients. Choose $\mathbf{A} \in \mathcal{Q}$. We proved in Lemma 2.2 (3) that the maximal congruences of $\mathbf{A}$ belong to $\mathrm{Con}_{\mathcal{Q}}(\mathbf{A})$. Since every subdirectly irreducible algebra in $\mathcal{V}$ is simple, every congruence on $\mathbf{A}$ is an intersection of maximal congruences. But $\mathrm{Con}_{\mathcal{Q}}(\mathbf{A})$ is closed under intersection, so we must have $\mathrm{Con}(\mathbf{A}) \subseteq \mathrm{Con}_{\mathcal{Q}}(\mathbf{A})$. This means exactly that if $\theta \in \mathrm{Con}(\mathbf{A})$, then $\mathbf{A}/\theta \in \mathcal{Q}$, or equivalently that $\mathbb{H}(\mathcal{Q}) \subseteq \mathcal{Q}$. □

Theorem 2.3 establishes Theorem 1.1 when $\mathcal{V}$ is idempotent.

3. The non-idempotent case

C. Bergman and R. McKenzie proved the following result about locally finite varieties that are minimal as quasivarieties.

Proposition 3.1 ([2, Corollary 2])**.** *Let $\mathcal{V}$ be a locally finite variety. Then $\mathcal{V}$ is a minimal quasivariety if and only if*

(1) *$\mathcal{V}$ has a unique subdirectly irreducible algebra $\mathbf{A}$, and*
(2) *$\mathbf{A}$ is embeddable in every nontrivial member of $\mathcal{V}$.*

The algebra $\mathbf{A}$, if it exists, is finite and strictly simple.

Using a fine analysis of finite strictly simple algebras, it is shown in [8] that Item (1) implies Item (2) for any locally finite variety.

We want to extend the "if" direction of Proposition 3.1 to a version that applies to varieties that need not be locally finite. Let $\mathcal{V}_S$ and $\mathcal{V}_{SI}$ denote the classes of simple members and subdirectly irreducible members of $\mathcal{V}$ respectively.

Proposition 3.2. *If $\mathcal{V}$ is a variety and*

(1) *$\mathcal{V}_{SI} \cup \{$trivial algebras$\}$ is a universal class that is minimal for properly containing the class $\{$trivial algebras$\}$, and*
(2) *every nontrivial member of $\mathcal{V}$ contains a subalgebra in $\mathcal{V}_{SI}$,*

then $\mathcal{V}$ is minimal as a quasivariety.

Proof. Assume that φ is a quasi-identity that holds in some nontrival algebra $\mathbf{A} \in \mathcal{V}$. By Item (2), there exists $\mathbf{S} \in \mathcal{V}_{SI}$ embeddable in $\mathbf{A}$. Necessarily $\mathbf{S}$ satisfies φ. By Item (1) we derive that $\mathcal{V}_{SI}$ satisfies φ, and therefore $\mathcal{V} = \mathbb{SP}(\mathcal{V}_{SI})$ also satisfies φ. □

Let's apply this result to minimal filtral varieties.

Recall that a variety $\mathcal{V}$ is *filtral* if it is semisimple (meaning $\mathcal{V}_{SI} = \mathcal{V}_S$) and, for every $\mathbf{A} \in \mathcal{V}$ and every representation of $\mathbf{A}$ as a subdirect product of simple algebras, every congruence on $\mathbf{A}$ is determined by a filter on the index set of the product (cf. [3, 9]).

It is known that a variety $\mathcal{V}$ is filtral if and only if it is congruence distributive and has the property that principal congruences are complemented on members of $\mathcal{V}$ (cf. [6, Theorem 4.14]). It is also known that if $\mathcal{V}$ is filtral, then $\mathcal{V}_S \cup \{$trivial algebras$\}$ is a universal class. This allows us to derive the following result from Proposition 3.2.

Theorem 3.3. *Let $\mathcal{V}$ be a minimal filtral variety. If every nontrivial member of $\mathcal{V}$ has a simple subalgebra, then $\mathcal{V}$ is minimal as a quasivariety.*

The proof of Theorem 3.3 is recorded after the proof of the next lemma.

Lemma 3.4. *Suppose $\mathcal{V}$ is a minimal filtral variety. If φ is a universal or existential sentence in the language of $\mathcal{V}$, then either $\mathcal{V}_S \models \varphi$ or $\mathcal{V}_S \models \neg\varphi$.*

Proof. Suppose to the contrary that there is a universal sentence φ such that $\mathcal{K} := \{\mathbf{A} \in \mathcal{V}_S : \mathbf{A} \models \varphi\}$ satisfies $\emptyset \neq \mathcal{K} \subsetneq \mathcal{V}_S$. If $\mathcal{W}$ is the variety generated by $\mathcal{K}$, by Jónsson's lemma, the class of subdirectly irreducibles members in $\mathcal{W}$ agrees with $\mathcal{K}$. Thus $\mathcal{W} \subsetneq \mathcal{V}$, which contradicts the minimality of $\mathcal{V}$. □

Proof of Theorem 3.3. Lemma 3.4 and $\mathcal{V}_{SI} = \mathcal{V}_S$ establish that Item (1) of Proposition 3.2 holds for any minimal filtral variety. If every nontrivial member of $\mathcal{V}$ has a simple subalgebra, then Item (2) of Proposition 3.2 will also hold. □

Next we provide a condition which identifies circumstances when Item (2) of Proposition 3.2 holds for filtral varieties.

Proposition 3.5. *Let $\mathcal{V}$ be a minimal filtral variety. Suppose there are a term $t(\bar{x})$ and $\mathbf{S} \in \mathcal{V}_S$ such that*

(1) *$t^{\mathbf{S}}$ is a constant function and*
(2) *the range of $t^{\mathbf{S}}$ is not a singleton subuniverse of $\mathbf{S}$.*

Then $\mathcal{V}$ is minimal as a quasivariety.

Proof. Our aim is to prove that if some term $t(\bar{x})$ has the properties listed, then there must be a simple member $\mathbf{P} \in \mathcal{V}_S$ that is embeddable in every nontrival member of $\mathcal{V}$. The conclusion of Proposition 3.5 then follows from Theorem 3.3.

Suppose that $t(\bar{x})$ and $\mathbf{S} \in \mathcal{V}_S$ satisfy the hypotheses. Then, by Lemma 3.4, it follows that $\mathcal{V}_S \vDash t(\bar{x}) = t(\bar{y})$, i.e, the interpretation of t is a constant function in each simple algebra. Furthermore, if

$$\Delta(y) := \{\alpha(y) : \mathbf{S} \vDash \alpha(t(\bar{x})) \text{ and } \alpha(y) \text{ is } \pm \text{atomic}\},$$

then $\mathcal{V}_S \vDash \Delta(t(\bar{x}))$. Since every member of $\mathcal{V}$ is (isomorphic with) a subdirect product of simple algebras, it is easy to see that

$$\mathcal{V} \setminus \{\text{trivial algebras}\} \vDash \Delta(t(\bar{x})). \tag{‡}$$

Let $s \in \mathbf{S}$ be such that $t^{\mathbf{S}}(\bar{x}) = s$, and let $\mathbf{P}$ be the subalgebra of $\mathbf{S}$ generated by s. Note that (2) says that $\mathbf{P}$ is nontrivial, and, since $\mathcal{V}_S$ is closed under substructures, it follows that $\mathbf{P} \in \mathcal{V}_S$.

We are now ready to see that $\mathbf{P}$ is embeddable in any nontrivial $\mathbf{A} \in \mathcal{V}$. Choose $\bar{a} \in A^n$ and let $a := t^{\mathbf{A}}(\bar{a})$. Now, (‡) implies that the subalgebra of $\mathbf{A}$ generated by a is isomorphic with $\mathbf{P}$. □

Note that Proposition 3.5 holds, in particular, when $\mathcal{V}$ has constant terms 0 and 1 such that $0 \neq 1$ for all nontrivial algebras in $\mathcal{V}$. This motivates the following:

Problem 3.6. Is every minimal filtral variety minimal as a quasivariety?

Our next result shows that the minimality question for filtral varieties can be settled by inspecting the two-variable quasi-identities.

Lemma 3.7. *Let $\mathcal{V}$ be a minimal filtral variety which is not minimal as a quasivariety. Then, there exists a quasi-identity ρ in two variables, say $\forall xy(\alpha(x,y) \to \beta(x,y))$, such that $\mathcal{V} \nvDash \rho$ and $\mathbf{A} \vDash \rho$ for some nontrivial $\mathbf{A} \in \mathcal{V}$. Furthermore, if $\mathcal{V}$ is not idempotent, then there is such a quasi-identity in just one variable.*

In this lemma (and later), we write quasi-identities compactly as

$$\forall xy(\alpha(x,y) \to \beta(x,y)).$$

In such an expression, α is a conjunction of atomic formulas and β is atomic.

Proof. Let $\mathcal{V}$ be a minimal filtral variety which is not a minimal as a quasivariety. Then, there is a quasi-identity $\varphi := \forall \bar{x}(\alpha_0(\bar{x}) \to \beta_0(\bar{x}))$ such that $\mathcal{V} \nvDash \varphi$ and $\mathbf{A} \vDash \varphi$ for some nontrivial $\mathbf{A} \in \mathcal{V}$. From the fact that $\mathcal{V} \nvDash \varphi$ it is clear that $\mathcal{V}_S \nvDash \varphi$, which by Lemma 3.4 yields $\mathcal{V}_S \vDash \neg\varphi$. Now, let $\mathbf{S}$ be a simple algebra generated by some $\{s, s'\} \subseteq S$ (there is always such an algebra because $\mathcal{V}_S$ is closed under subalgebras). Since $\mathbf{S} \vDash \neg\varphi$, there exist terms $t_1(x,y), \ldots, t_n(x,y)$ such that

$$\mathbf{S} \vDash \alpha_0(t_1(s,s'), \ldots, t_n(s,s')) \wedge \neg\beta_0(t_1(s,s'), \ldots, t_n(s,s')).$$

Define

$$\alpha(x,y) := \alpha_0(t_1(x,y),\dots,t_n(x,y)),$$
$$\beta(x,y) := \beta_0(t_1(x,y),\dots,t_n(x,y)).$$

The quasi-identity $\forall xy(\alpha(x,y) \to \beta(x,y))$ will hold in $\mathbf{A}$ and fail in $\mathbf{S}$.

In the case that $\mathcal{V}$ is not idempotent, in the argument above we can take $\mathbf{S}$ generated by a single element. □

Any dual discriminator variety is filtral since (i) it is congruence distributive (the dual discriminator term operation on $\mathcal{V}_S$ is a majority term operation for $\mathcal{V}$) and (ii) principal congruences are complemented (cf. [7, Theorem 3.8]), so all of the above applies to dual discriminator varieties. From this point we restrict our focus to dual discriminator varieties. One essential consequence of this strengthened assumption is that dual discriminator varieties have a majority term.

Let $\mathbf{A}$ be a structure and $f\colon A^n \to A$ be a function. We say that a first-order formula $\Phi(\bar{x},y)$ *defines* the function f in $\mathbf{A}$ provided that $\mathbf{A} \models \Phi(\bar{a},b)$ if and only if $f(\bar{a}) = b$. For functions $f\colon A^n \to A$ and $g\colon B^n \to B$ let $f \times g\colon (A\times B)^n \to A \times B$ be defined by $(f\times g)((a_1,b_1),\dots,(a_n,b_n)) := (f(\bar{a}), g(\bar{b}))$.

Lemma 3.8 (From [10, Corollary 4.4]**).** *Let $\mathcal{K}$ be a first-order axiomatizable class of algebras with a majority term and suppose that for each $\mathbf{A} \in \mathcal{K}$, the first-order formula $\Phi(\bar{x},y)$ defines a function $F_{\mathbf{A}}\colon A^n \to A$ in $\mathbf{A}$. The following are equivalent:*

(1) *There is a term $t(\bar{x})$ such that for all $\mathbf{A} \in \mathcal{K}$ and all $\bar{a} \in A^n$ we have $F_{\mathbf{A}}(\bar{a}) = t^{\mathbf{A}}(\bar{a})$.*
(2) *For all $\mathbf{A},\mathbf{B} \in \mathcal{K}$, all $\mathbf{S} \le \mathbf{A}\times\mathbf{B}$ and all $\bar{s} \in S^n$ we have that $(F_{\mathbf{A}} \times F_{\mathbf{B}})(\bar{s}) \in S$.*

We shall need the characterization of the subalgebras of twofold products under the presence of the dual discriminator.

Lemma 3.9 ([7, Theorem 2.4]**).** *Let $\mathbf{A}$ and $\mathbf{B}$ be algebras, and suppose there is a term $q(x,y,z)$ that interprets as the dual discriminator in both $\mathbf{A}$ and $\mathbf{B}$. Let $\mathbf{C} \le \mathbf{A}\times\mathbf{B}$, and let A_0 and B_0 be the projections of C into A and B respectively. One of the following holds:*

(1) *$C = A_0 \times B_0$.*
(2) *C is the graph of an isomorphism between $\mathbf{A}_0$ and $\mathbf{B}_0$.*
(3) *$C = (\{a\}\times B_0) \cup (A_0 \times \{b\})$ for some $a \in A_0$ and $b \in B_0$, and with $|A_0|, |B_0| > 1$.*

We are now ready to prove the main result in this section.

Proof of the non-idempotent case of Theorem 1.1. We argue that a minimal non-idempotent dual discriminator variety is minimal as a quasivariety.

By Proposition 3.5 we may assume that:

(∗) there do not exist $t(x)$ and $\mathbf{S} \in \mathcal{V}_S$ satisfying (1) and (2) of Proposition 3.5. (Expressed in a positive form, we are assuming that for any $t(x)$ and $\mathbf{S} \in \mathcal{V}_S$, if $t^{\mathbf{S}}(x)$ is a constant function, then its range must be a singleton subuniverse.)

By Lemma 3.7, it suffices to prove that if ρ is any quasi-identity in one variable such that $\mathcal{V} \not\models \rho$, then there are no nontrivial models of ρ in $\mathcal{V}$. So, fix ρ equal to $\forall x(\alpha(x) \to \beta(x))$ such that $\mathcal{V} \not\models \rho$. From the fact that $\mathcal{V} \not\models \rho$ it follows that $\mathcal{V}_S \not\models \rho$, and thus, by Lemma 3.4, we have

$$\mathcal{V}_S \models \exists x(\alpha(x) \wedge \neg\beta(x)). \tag{A}$$

Let

$$\gamma(x) := \alpha(x) \wedge \beta(x) \quad \text{and} \quad \varphi(x) := \alpha(x) \wedge \neg\beta(x).$$

Our next step is to prove the following:

$$\mathcal{V}_S \models \gamma(x) \vee \bigvee_{t \in T(x)} \varphi(t(x)). \tag{B}$$

To see this, fix $\mathbf{S} \in \mathcal{V}_S$ and $s \in S$. Let $\mathbf{S}_0$ be the subalgebra of $\mathbf{S}$ generated by s; there are two cases. If $\mathbf{S}_0$ is trivial, then clearly $\mathbf{S} \models \gamma(s)$. Otherwise, $\mathbf{S}_0$ is a simple algebra, and thus, by (A), we have that $\mathbf{S}_0 \models \exists x\, \varphi(x)$. Hence, there is a term $t \in T(x)$ such that $\mathbf{S}_0 \models \varphi(t(s))$, which in turn implies that $\mathbf{S} \models \varphi(t(s))$, since $\varphi(t(x))$ is quantifier-free. This completes the proof of (B). Next, observe that by compactness there are terms $t_1(x), \ldots, t_n(x)$ such that

$$\mathcal{V}_S \models \gamma(x) \vee \varphi(t_1(x)) \vee \cdots \vee \varphi(t_n(x)). \tag{3.1}$$

For each $\mathbf{S}$ in $\mathcal{V}_S$ define a function $F_{\mathbf{S}} \colon S \to S$ by

$$F_{\mathbf{S}}(x) = \begin{cases} x & \text{if } \gamma(x), \\ t_1(x) & \text{if } \varphi(t_1(x)) \wedge \neg\gamma(x), \\ \vdots & \vdots \\ t_n(x) & \text{if } \varphi(t_n(x)) \wedge \neg\varphi(t_{n-1}(x)) \wedge \cdots \wedge \neg\varphi(t_1(x)) \wedge \neg\gamma(x). \end{cases}$$

We aim to apply Lemma 3.8 to obtain a term $r(x)$ that interprets as $F_{\mathbf{S}}$ in each simple algebra $\mathbf{S}$ in $\mathcal{V}$. It is not hard to write the previous displayed line in the form of a first-order formula $\Phi(x, y)$ which defines $F_{\mathbf{S}}$ for each $\mathbf{S} \in \mathcal{V}_S$, hence we need to check that Lemma 3.8 (2) holds with $\mathcal{K} = \mathcal{V}_S$. Assume $\mathbf{C} \leq \mathbf{A} \times \mathbf{B}$ with $\mathbf{A}, \mathbf{B} \in \mathcal{V}_S$. By Lemma 3.9, we know that either C is direct product, a graph of an isomorphism, or $C = (\{a\} \times B_0) \cup (A_0 \times \{b\})$ with A_0 and B_0 nontrivial and $\mathbf{A}_0, \mathbf{B}_0 \in \mathcal{V}_S$. It is easy to see that Lemma 3.8 (2) holds in the first two cases. We prove that it also holds in the third case. By symmetry, we only need to check that $(F_{\mathbf{A}}(a), F_{\mathbf{B}}(b')) \in C$ for each $b' \in B_0$.

With the intention of deriving a contradiction, assume that $F_{\mathbf{A}}(a) \neq a$. There must exist $i \in \{1, \ldots, n\}$ such that $F_{\mathbf{A}}(a) = t_i^{\mathbf{A}}(a)$ and $\mathbf{A} \models \varphi(t_i^{\mathbf{A}}(a))$ (hence $\mathbf{A} \models \neg\beta(t_i^{\mathbf{A}}(a))$). We prove now that $t_i^{\mathbf{B}_0}$ is the constant function with range $\{b\}$. Choose any $b'' \in B_0$. Since $t_i^{\mathbf{A}_0}(a) = F_{\mathbf{A}}(a) \neq a$, and $t_i^{\mathbf{C}}(a, b'') = (t_i^{\mathbf{A}_0}(a), t_i^{\mathbf{B}_0}(b''))$ must be in C, it follows that $t_i^{\mathbf{B}_0}(b'') = b$, and therefore that $t_i^{\mathbf{B}_0}(x)$ is constant with range $\{b\}$. From assumption $(*)$ from the second paragraph of this proof we derive that $\{b\}$ must be a singleton subuniverse of the simple algebra $\mathbf{B}_0$. However, the fact that the range of $t_i(x)$ is a singleton subuniverse is expressible by a family of universal sentences, so, by Lemma 3.4, if this is true of $\mathbf{B}_0$ it must be true for any algebra in $\mathcal{V}_S$, such as $\mathbf{A}$. But it is not true for $\mathbf{A}$, since $\mathbf{A} \models \neg\beta(t_i^{\mathbf{A}}(a))$ and β is an equation between terms. This contradiction implies that $F_{\mathbf{A}}(a) = a$, and thus it is clear that $(F_{\mathbf{A}}(a), F_{\mathbf{B}}(b')) \in C$.

We have established the necessary information to conclude that there exists a term $r(x)$ satisfying

$$r^{\mathbf{S}}(x) = F_{\mathbf{S}}(x) \quad \text{for all} \quad \mathbf{S} \in \mathcal{V}_S.$$

By the definition of the functions $F_{\mathbf{S}}$, and the fact that $F_{\mathbf{S}}(x) = x$ if $\gamma(x)$ holds, we have

$$\mathcal{V}_S \models \gamma(x) \vee \varphi(r(x)), \quad \text{and} \tag{C}$$

$$\mathcal{V}_S \models \gamma(r(x)) \vee \varphi(r(x)). \tag{C'}$$

Since $\gamma(r(x))$ is the formula $\alpha(r(x)) \wedge \beta(r(x))$ and $\varphi(r(x))$ is the formula $\alpha(r(x)) \wedge \neg\beta(r(x))$, we get from (C') that $\mathcal{V}_S \models \alpha(r(x))$. Since $\alpha(r(x))$ is a conjunction of equations we get

$$\mathcal{V} \models \alpha(r(x)). \tag{D}$$

To conclude the proof we show that $\mathcal{V} \setminus \{\text{trivial algebras}\} \models \neg\rho$. Let $\mathbf{E} \in \mathcal{V}$ be nontrivial and let $\{\mathbf{A}_i : i \in I\} \subseteq \mathcal{V}_S$ be such that $\mathbf{E} \leq_{sd} \prod_{i\in I} \mathbf{A}_i$. If it were the case that $\mathbf{E} \models \forall x\, \gamma(x)$, then since $\gamma(x)$ is a conjunction of equations and the representation $\mathbf{E} \leq_{sd} \prod_{i\in I} \mathbf{A}_i$ is subdirect, we would have $\{\mathbf{A}_i : i \in I\} \models \forall x\, \gamma(x)$, which contradicts (A). Therefore, there must exist $e \in E$ such that $\mathbf{E} \models \neg\gamma(e)$, which in turn produces $i \in I$ such that $\mathbf{A}_i \models \neg\gamma(e_i)$. From (C) we derive that $\mathbf{A}_i \models \varphi(r(e_i))$, and in particular we have $\mathbf{A}_i \models \neg\beta(r(e_i))$. It follows that $\mathbf{E} \models \neg\beta(r(e))$, which in combination with (D) yields that $e \in E$ is a witness to

$$\mathbf{E} \models \exists x(\alpha(r(x)) \wedge \neg\beta(r(x))),$$

hence that $r(e) \in E$ is a witness to

$$\mathbf{E} \models \exists x(\alpha(x) \wedge \neg\beta(x)).$$

This shows that $\mathbf{E} \models \neg(\forall x(\alpha(x) \to \beta(x)))$, so $\mathbf{E}$ fails ρ. □

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