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BACK-AND-FORTH SYSTEMS FOR ARBITRARY QUANTIFIERS

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ABSTRACT. $L_{\infty\omega}(K)$ is the logic obtained by adding a Lindström's quantifier $K\vec{x}_1 \ldots \vec{x}_k\,(\phi_1(\vec{x}_1)\ldots \phi_k(\vec{x}_k))$ to the logical operations of $L_{\infty\omega}$. The corresponding finitary logic is $L_{\omega\omega}(K)$, and $L_{\infty\omega}(K_i)_{i\in I}$ is obtained by adjoining a family of quantifiers.

In this paper, we give back-and-forth systems characterizing elementary equivalence in those logics and their fragments of bounded quantifier rank. This generalizes work of Fraïssé and Ehrenfeucht for $L_{\omega\omega}$, Karp for $L_{\infty\omega}$, Brown, Lipner, and Vinner for cardinal quantifiers, Badger for Magidor-Malitz quantifiers, and others. Our systems apply to higher order quantifiers also.

INTRODUCTION.

Ehrenfeucht 1961 and Fraïssé 1955 gave back-and-forth or game theoretical characterizations of elementary equivalence in first order logic, $L_{\omega\omega}$, later generalized by Karp 1965 to infinitary logic, $L_{\infty\omega}$. These characterizations were used to obtain results about definability of ordinals and preservation of elementary equivalence by operations on structures. Lindström 1969 used Fraissé-Ehrenfeucht games to characterize $L_{\omega\omega}$. Back-and-forth systems for logics with cardinal quantifiers are due to Vinner 1972 and others. Badger 1977 gives systems for logics with Magidor-Malitz quantifiers (Magidor and Malitz 1977), and shows the failure of interpolation and preservation of elementary equivalence by products in these logics. Krawczyk and Krynicki 1976 give systems for certain monotonic quantifiers, without any application. Makowsky 1977 a has similar systems and he studies monotonic quantifiers in detail. Back-and-forth systems for Stationary Logic, $L(aa)$ (Barwise, Kaufmann and Makkai 1977), were given independently by Kaufmann 1978, Makowsky 1977 b and the author (Caicedo 1977 b).

In our doctoral dissertation, we presented back-and-forth systems characterizing elementary equivalence in logics obtained by adding to first order logic quantifiers of the form $Q\vec{x}\,\phi(\vec{x})$, this means binding one or several variables in a single formula, and gave various applications, particularly to $L_{\omega\omega}(Q_1)$ and $L(aa)$. In this paper, we introduce back-and-forth systems appropriate for quantifiers binding several formulas:

$$Q\vec{x}_1 \ldots \vec{x}_n(\phi_1(\vec{x}_1), \ldots, \phi_n(\vec{x}_n))$$

Although the methods may be applied successfully to second and higher order quantifiers, as they were applied to $L(aa)$ in Caicedo 1978, the corresponding results will be published elsewhere.

We assume as known the notions of an abstract logic, as well as the extension relation between logics, and the notion of a generalized quantifier (Lindström 1966, Barwise 1974, Makowski, Shelah, and Stavi 1976). $\mathfrak{A}, \mathfrak{B}, \ldots$ denote classical structures, and $A, B, \ldots$ denote their universes.

In Section 1, we introduce quantifier symbols and their interpretations. Instead of considering quantifiers in the sense of Lindström 1966 and Mostowski 1957 only, we deal with the more general case of so called "weak models" where the quantifier interpretation forms part of the structure. Lindström-Mostowski quantifiers are recovered as families of weak models where the interpretations of the quantifiers are determined, up to isomorphism, by the domain of the structure.

In Sections 2 and 3, we define the back-and-forth systems and prove the characterization of elementary equivalence.

In Section 4, we consider *monadic* quantifiers, those where the quantifier binds a single variable in each formula, and extend a result of Friedman 1973 about the failure of Beth's definability theorem in cardinality logics to logics with these quantifiers. Also we show that any extension of $L_{\omega\omega}(Q_0)$ by monadic quantifiers satisfying interpolation must satisfy the downward Löwenheim-Skolem theorem.

In Sections 5 and 6, we give a simpler version of back-and-forth for *cofilter* quantifiers, which becomes PC definable. The main applications deal with (infinitary) extensions by monadic quantifiers of $L_{\omega\omega}(Q_1)$, logic with the quantifier "there are uncountably many". These include an analogue of Lindström's theorem for $L_{\omega\omega}(Q_1)$, a relative interpolation theorem in $L_{\omega\omega}(Q_1)$ with respect to such extensions, and the existence of models satisfying few types in those extensions. Makowsky and Stavi discovered independently the relative interpolation theorem in $L_{\omega\omega}(Q_1)$ and $L(aa)$, with respect to their infinitary extensions.

Finally, in Section 7 we show that elementary equivalence is preserved by cartesian products in a natural extension of logic with Magidor-Malitz quantifiers.

§1 GENERALIZED QUANTIFIERS.

A *quantifier symbol* is a symbol Q together with a sequence of positive integers $\langle n_1, \ldots, n_k \rangle$ called the *type* of the quantifier symbol. Given a set of relation, function and constant symbols, the language $L_{\omega\omega}(Q_j)_{j\in J}$ is obtained by adding to the usual formation rules of $L_{\omega\omega}$ for atomic formulas, $\neg$, $\wedge$, and $\exists$ the new rule:

> If Q_j is a quantifier symbol of type $\langle n_1, \ldots, n_k \rangle$, $\phi_1, \ldots, \phi_k$ are formulas, and $\vec{x}_1, \ldots, \vec{x}_k$ are lists of $n_1, \ldots, n_k$ variables, respectively, then $Q_j \vec{x}_1, \ldots, \vec{x}_k(\phi_1, \ldots, \phi_k)$ is a formula.

It is understood that only those free variables of ϕ_i which appear in the list $\vec{x}_i$ are bound by the quantifier.

If $\mathfrak{A}$ is a structure in the ordinary sense and Q is a quantifier symbol of type $\langle n_1, \ldots, n_k \rangle$, then an *interpretation for* Q *in* $\mathfrak{A}$ is a family

$q \subseteq \mathcal{P}(A^{n_1}) \times \ldots \times \mathcal{P}(A^{n_k})$. An $L_{\omega\omega}(Q_j)_{j\in J}$-*structure* has the form $(\mathfrak{A};q_j)_{j\in J}$ where q_j is an interpretation of Q_j in $\mathfrak{A}$. The semantics of $L_{\omega\omega}(Q_j)_{j\in J}$ is defined in the usual way, except for the additional clause:

$$(\mathfrak{A};\, q_j)_{j\in J} \vDash Q_j \vec{x}_1,\ldots,\vec{x}_n(\phi_1,\ldots,\phi_k) \text{ if and only if}$$

$$(\pi\vec{x}_1\,\phi_1,\ldots,\pi\,\vec{x}_k\,\phi_k)\in q_j, \quad \text{where}$$

$$\pi\,\vec{x}\,\phi = \{\vec{a}\,|\,(\mathfrak{A}\,;\,q_j)_{j\in J} \vDash \phi(\vec{x}/\vec{a})\}.$$

The language $L_{\infty\omega}(Q_j)_{j\in J}$ and its semantics are defined similarly, allowing infinitary conjunctions. The quantifier rank of a formula of this language is the ordinal defined inductively by:

$$\mathrm{qr}(\phi) = 0, \quad \text{if } \phi \text{ is atomic}$$
$$\mathrm{qr}(\neg\phi) = \mathrm{qr}(\phi)$$
$$\mathrm{qr}(\wedge\theta) = \sup\{\mathrm{qr}(\phi)\,|\,\phi\in\theta\}$$
$$\mathrm{qr}(\exists x\,\phi) = \mathrm{qr}(\phi)+1$$
$$\mathrm{qr}(Q_j\vec{x}_1,\ldots,\vec{x}_k(\phi_1,\ldots,\phi_k)) = \max_i \mathrm{qr}(\phi_i) + \max_i n_i,$$

where $\langle n_1,\ldots,n_k\rangle$ is the type of Q_j.

Let α be an ordinal, or the symbol ∞ considered greater than all the ordinals, then $L^{\alpha}_{\infty\omega}(Q_j)_{j\in J}$ consists of the formulas of rank less than α. Two quantifier structures are α-elementarily equivalent, $(\mathfrak{A};\,q_j)_{j\in J} \overset{\alpha}{\equiv} (\mathfrak{B};\,r_j)_{j\in J}$, if they satisfy the same sentences of this language. The proof of the following lemmas is similar to that of Lemma 3.2.1 in Caicedo 1978.

LEMMA 1.1 *If the number of relation, function, constant, and quantifier symbols is finite, then* $L^{\omega}_{\infty\omega}(Q_j)_{j\in J}$ *is equivalent to* $L_{\omega\omega}(Q_j)_{j\in J}$. *Moreover, for each finite* n *and* k *there is a finite number of non-equivalent formulas of quantifier rank equal to* n *with at most* k *variables.*

LEMMA 1.2 *Given* $(\mathfrak{A};\,q_j)_{j\in J}$, *there is at most a set of formulas of* $L_{\infty\omega}(Q_j)_{j\in J}$ *which are non-equivalent in this structure.*

A *Lindström-Mostowski quantifier of type* $\langle n_1,\ldots,n_k\rangle$ is a function $\hat{Q}$ which assigns to each set A a quantifier interpretation $\hat{Q}(A)\subseteq \mathcal{P}(A^{n_1})\times\ldots\times\mathcal{P}(A^{n_r})$, with the property that if $f : A\to B$ is a bijection, then for all $(S_1,\ldots,S_k)$, $(S_1,\ldots,S_k)\in\hat{Q}(A)$ iff $(f'(S_1),\ldots,f'(S_k))\in\hat{Q}(B)$. It is readily seen that this corresponds to the definition of a quantifier as a class $\mathbf{K}$ of structures of the same type, closed under isomorphism (Lindström 1966), by defining

$$\hat{Q}(A) = \{(S_1,\ldots,S_k)\,|\,(A, S_1,\ldots,S_k)\in\mathbf{K}\}.$$

A Lindström-Mostowski family of quantifiers $\{\hat{Q}_j\,|\,j\in J\}$ defines an extension L^{+} of $L_{\omega\omega}$ in the sense of Lindstrom 1969 and Barwise 1974 by taking the sentences of $L_{\infty\omega}(Q_j)_{j\in J}$ as the *syntax*, and defining for an ordinary structure :

$$\mathfrak{A} \models_{L^+} \phi \quad \text{iff} \quad (\mathfrak{A}; \hat{Q}_j(A))_{j\in J} \models \phi .$$

When the quantifier $\hat{Q}_j$ interpreting the symbol Q_j is understood, we will abuse the language writing $L_{\infty\omega}(Q_j)_{j\in J}$ for L^+.

The existential quantifier is the Lindström-Mostowski quantifier of type $\langle 1 \rangle$ defined by the function $\hat{\exists}(A) = \{S \subseteq A \mid S \neq \emptyset\}$. The following quantifiers are well know.

Cardinal quantifiers, type $\langle 1 \rangle$, for each ordinal $\alpha : Q_\alpha(A) = \{S \subseteq A \mid |S| \geq \omega_\alpha\}$.

Magidor-Malitz quantifiers, for each ordinal α and finite n, the quantifier of type $\langle n \rangle$: $Q^n_\alpha(A) = \{S \subseteq A^n \mid \exists I \subseteq A$ such that $I^n \subseteq S$ and $|I| \geq \omega_\alpha\}$.

Chang quantifier, type $\langle 1 \rangle$: $Q(A) = \{S \subseteq A \mid |S| = |A|\}$.

Hartig quantifier, type $\langle 1,1 \rangle$: $H(A) = \{(S,T) \mid S \subseteq A,\ T \subseteq A,\ |S| = |T|\}$.

Henkin quantifier, type $\langle 4 \rangle$: $Hen(A) = \{S \subseteq A^4 \mid \exists f, g : A \to A$ such that $f \times g \subseteq S\}$

Note that the logics obtained from these quantifiers do not include those where the meaning of quantifier is not determined by the domain of the structure, like Sgro's topological logic (Sgro 1977). However, if we consider only logics for classical structures with a finite number of relations, functions, and constants, then any logic is a sublogic of some $L_{\omega\omega}(Q_j)_{j\in J}$. Even more, if the logic L is closed under substitution of relation symbols for formulas then L is equivalent to some $L_{\omega\omega}(Q_j)_{j\in J}$.

§ 2. BACK-AND-FORTH SYSTEMS.

Through this section $(\mathfrak{A}, q)$ and $(\mathfrak{B}, r)$ will be quantifier structure where q and r are interpretations for a quantifier symbol of type $\langle n_1, \ldots, n_k \rangle$. A and B are assumed to be disjoint. Sequences in $A^n \cup B^n$ $(n \in \omega)$ will be denoted $\sigma, \sigma', \vec{a}, \vec{a}', \tau, \tau', \vec{b}, \vec{b}'$; the value of n will be clear from the context. Concatenation is denoted by juxtaposition.

DEFINITION 2.1.- A *back-and-forth between* $(\mathfrak{A}; q)$ *and* $(\mathfrak{B}; r)$ consists of a linearly ordered set $\mathbf{P} = (P, <)$, called the set of *parameters*, and a family $\{E^p_n \mid p \in P\}$ of equivalence relations in $A^n \cup B^n$ for each $n \in \omega$, subject to properties (i) to (iv) below.

Before stating the properties, we introduce some convenient notation. $\sigma \overset{p}{\sim} \sigma'$ will denote $(\sigma, \sigma') \in E^p_n$; the value of n will be clear from the context. For each k, n and $\sigma \in A^k$, we can restrict the equivalence relation E^p_{k+n} to elements of A^n in the following way: $\vec{a} \underset{\sigma}{\overset{p}{\sim}} \vec{a}'$ iff $\sigma\vec{a} \overset{p}{\sim} \sigma\vec{a}'$ $(\vec{a}, \vec{a}' \in A^n)$. For $k = 0$ and $\sigma = \emptyset$ we get E^p_n restricted to A^n. Obviously, $\underset{\sigma}{\overset{p}{\sim}}$ is an equivalence relation.

$$[\vec{a}]^p_\sigma = \{\vec{x} \in A^n \mid \sigma\, \vec{x} \overset{p}{\sim} \sigma\, \vec{a}\}$$

is the corresponding equivalence class of $\vec{a}$ in A^n. If $X \subseteq A^n$ we write, abusing the language, $[X]^p_\sigma = \cup\{[\vec{a}]^p_\sigma \mid \vec{a} \in X\}$.

Analogous notation and observations are valid with respect to sequences in B.

PROPERTIES.

(i) $\emptyset \overset{p}{\sim} \emptyset$

(ii) (Extension property). Let $s = \max\{n_1, \ldots, n_k\}$ and $p < p_1 < \ldots < p_s = p'$, then for all sequences σ in A and τ in B such that $\sigma \overset{p'}{\sim} \tau$ there exist functions $f_i : A^{n_i} \to B^{n_i}$, $1 \leqslant i \leqslant k$, such that:

(A) $\sigma\, \vec{a} \overset{p}{\sim} \tau\, f_i(\vec{a})$ for all $\vec{a} \in A^{n_i}$, $1 \leqslant i \leqslant k$.

(B) If $X_i \subseteq A^{n_i}$, $1 \leqslant i \leqslant k$, then $([X_1]^p_\sigma, \ldots, [X_k]^p_\sigma) \in q$

implies $([f'_1(X_1)]^p_\tau, \ldots, [f'_k(X_k)]^p_\tau) \in r$.

(iii) As (ii), interchanging the roles of A and B.

(iv) (Isomorphism property). If $(a_1, \ldots, a_n) \overset{p}{\sim} (b_1, \ldots, b_n)$ then the assignment $a_i \rightsquigarrow b_i$ is a partial isomorphism from $\mathfrak{A}$ to $\mathfrak{B}$.

DEFINITION 2.2.- $(P, \{E^p_n \mid p \in P,\ n \in \omega\})$ is a *back-and-forth from* $(\mathfrak{A}; q_j)_{j \in J}$ *to* $(\mathfrak{B}; r_j)_{j \in J}$ if it is one from $(\mathfrak{A}; q_j)$ to $(\mathfrak{B}; r_j)$ for each $j \in J$. The existence of such a relation is denoted by $(\mathfrak{A}; q_j)_{j \in J} \overset{P}{\sim} (\mathfrak{B}; r_j)_{j \in J}$

REMARK. We assume that any back-and-forth satisfies the extension property for the existential quantifier. It is enough to postulate in (ii), for each $\sigma \overset{p'}{\sim} \tau$ and $p < p'$ the existence of a function $f : A \to B$ such that $\sigma a \overset{p}{\sim} \tau\, f(a)$. Property (B) holds automatically.

DEFINITION 2.3.- $\{E_n \mid n \in \omega\}$ is a *back-and-forth without parameters from* $(\mathfrak{A}; q_j)_{j \in J}$ *to* $(\mathfrak{B}; r_j)_{j \in J}$ if E_n is an equivalence relation in $A^n \cup B^n$ for each n, and properties (i) to (iv) of Def. 2.1 hold, dropping the parameter conditions. Such relation is denoted by $(\mathfrak{A}; q_j)_{j \in J} \sim (\mathfrak{B}; r_j)_{j \in J}$.

§ 3 CHARACTERIZATION OF ELEMENTARY EQUIVALENCE.

Let $\mathfrak{A}$ and $\mathfrak{B}$ be classical structures. $C = \{q_j \mid j \in J\}$ and $D = \{r_j \mid j \in J\}$ are interpretations in $\mathfrak{A}$ and $\mathfrak{B}$, respectively, of the quantifier symbols

$\{Q_j \mid j \in J\}$.

THEOREM 3.1.- *If* $(\mathfrak{A};\, C) \overset{\alpha}{\equiv} (\mathfrak{B};\, D)$ *then* $(\mathfrak{A};\, C) \overset{(\alpha,<)}{\sim} (\mathfrak{B};\, D)$.

PROOF. Suppose $(\mathfrak{A};C) \overset{\alpha}{\equiv} (\mathfrak{B};D)$. For every $\beta < \alpha$ and $\sigma, \sigma' \in A^n$; $\tau, \tau' \in B^n$ define:

$$\sigma \overset{\beta}{\sim} \sigma' \quad \text{iff} \quad (\mathfrak{A}, \sigma; C) \overset{\beta+1}{\equiv} (\mathfrak{A}, \sigma', C).$$

$$\tau \overset{\beta}{\sim} \tau' \quad \text{iff} \quad (\mathfrak{B}, \tau; D) \overset{\beta+1}{\equiv} (\mathfrak{B}, \tau'; D).$$

$$\sigma \overset{\beta}{\sim} \tau \ (\tau \overset{\beta}{\sim} \sigma) \quad \text{iff} \quad (\mathfrak{A}, \sigma; C) \overset{\beta+1}{\equiv} (\mathfrak{B}, \tau; D).$$

We show that this gives a back-and-forth with parameters $(\alpha,<)$. It is clear that $\overset{\beta}{\sim}$ is an equivalence relation and properties (i) and (iv) of Def. 2.1 hold. Since (ii) and (iii) are symmetric, it is enough to show (ii). Given $\sigma \in A^n, \vec{a} \in A^k, \beta < \alpha$ let

$$t^{\beta}_{\vec{a}} = \{\phi(\vec{x}) \mid \mathrm{qr}(\phi) \leqslant \beta \quad \text{and} \quad (\mathfrak{A}, \sigma; C) \models \phi(\vec{a})\}.$$

By Lemma 1.2, the conjunction $\wedge t_{\vec{a}}$ may be considered a formula of $L_{\infty\omega}(Q_j)_{j\in J}$ Since the logic is closed under negations:

$$[\vec{a}]^{\beta}_{\sigma} = \{\vec{a}' \in A^k \mid (\mathfrak{A}, \sigma; C) \models \wedge t_{\vec{a}}(\vec{a}')\}. \tag{1}$$

Now we are ready to check the extension property. Let Q_j be a quantifier symbol of type $\langle n_1, \ldots, n_k \rangle$ and $s = \max_i n_i$. Suppose that $\sigma \overset{\beta'}{\sim} \tau$, with $\tau \in B^n$ and $\beta < \beta_1 < \ldots < \beta_s = \beta'$. For each $\vec{a} \in A^{n_i}$, $(\mathfrak{A}, \sigma; C) \models \wedge t_{\vec{a}}(\vec{a})$, and so $(\mathfrak{A}, \sigma; C) \models \exists \vec{x}_i \wedge t_{\vec{a}}(\vec{x}_i)$. This last formula has $\mathrm{qr} = \mathrm{qr}(\wedge t_{\vec{a}}) + n_i = \beta + n_i \leqslant \beta + s \leqslant \beta'$. Since $\sigma \overset{\beta'}{\sim} \tau$ then:

$$(\mathfrak{A}, \sigma; C) \overset{\beta'+1}{\equiv} (\mathfrak{B}, \tau; D) \tag{2}$$

and so $(\mathfrak{B}, \tau; D) \models \exists \vec{x}_i \wedge t_{\vec{a}}(\vec{x}_i)$. Choose $\vec{b} \in B^{n_i}$ such that $(\mathfrak{B}, \tau; C) \models \wedge t_{\vec{a}}(\vec{b})$, and define $f_i(\vec{a}) = \vec{b}$. Since $t_{\vec{a}}$ is a complete set of formulas of rank $\leqslant \beta$ we have $(\mathfrak{A}, \sigma, \vec{a}; C) \overset{\beta+1}{\equiv} (\mathfrak{B}, \tau, f_i(\vec{a}); D)$ and so $\sigma\vec{a} \overset{\beta}{\sim} \tau f(\vec{a})$, the first condition of the extension property.

Now let $X_i \subseteq A^{n_i}$, $i = 1, \ldots, k$. For each i let $T_i(x)$ be the formula $\bigvee_{\vec{a} \in X_i} \wedge t_{\vec{a}}(\vec{x})$. By (1):

$$[X_i]^{\beta}_{\sigma} = \{\vec{x} \in A^{n_i} \mid (\mathfrak{A}, \sigma; C) \models T_i(\vec{x})\}.$$

Similarly, in $\mathfrak{B}$:

$$[f'_i(X_i)]^{\beta'}_\tau = \{\vec{x}\in B^{n_i} \mid (\mathfrak{B}, \tau; D) \models T_i(\vec{x})\}.$$

Moreover, the formula $Q_j\vec{x}_1,\ldots,\vec{x}_k(T_1(\vec{x}_1),\ldots,T_k(\vec{x}_k))$ has $\mathrm{qr} = \beta + s \leqslant \beta'$. Therefore, by (2) and the definition of quantifier, $([X_1]^\beta_\sigma,\ldots,[X_k]^\beta_\sigma) \in q_j$ implies succesively:

$$(\mathfrak{A}, \sigma; C) \models Q_j\vec{x}_1,\ldots,\vec{x}_k(T_1(\vec{x}_1),\ldots,T_r(\vec{x}_r)) ,$$

$$(\mathfrak{B}, \tau; D) \models Q_j\vec{x}_1,\ldots,\vec{x}_k(T_1(\vec{x}_1),\ldots,T_r(\vec{x}_r)) ,$$

$$([f'_1(\vec{x}_1)]^\beta_\tau ,\ldots, [f'_k(\vec{x}_k)]^\beta_\tau) \in r_j . \ \blacksquare$$

REMARK 3.2.- It is clear from the demonstration of the last theorem that the functions $f_1,\ldots,f_k$ in the extension property may be chosen homogeneously, so they work for all quantifiers of type $\langle n_1,\ldots,n_k\rangle$. Even more, it is possible to choose for each p', σ, and τ, in advance, a family $\{f^n : A^n \to B^n \mid n\in\omega\}$ which works for all quantifiers.

THEOREM 3.3.- *If* $(\mathfrak{A}; C) \overset{(\alpha,<)}{\sim} (\mathfrak{B}; D)$ *then* $(\mathfrak{A}; C) \overset{\alpha}{\equiv} (\mathfrak{B}; D)$.

PROOF. Assume $(\mathfrak{A}; C) \overset{(\alpha,<)}{\sim} (\mathfrak{B}; D)$. We prove by induction on the complexity of the formula $\phi(y)$, that if $\mathrm{qr}(\phi) \leqslant \beta < \alpha$, then for all σ and τ, $\sigma \overset{\beta}{\sim} \tau$ implies:

$$(\mathfrak{A}; C) \models \phi(\sigma) \quad \text{iff} \quad (\mathfrak{B}; D) \models \phi(\tau). \tag{1}$$

The result follows from property (i) taking $\sigma = \tau = \emptyset$. For atomic formulas, (1) follows from the isomorphism property (iv). The inductive step for negations and conjunctions is trivial. Let $\phi = Q_j\vec{x}_1,\ldots,\vec{x}_k(\phi_1,\ldots,\phi_k)$ be of $\mathrm{qr} \leqslant \beta$ and suppose that (1) holds for $\phi_1,\ldots,\phi_k$. Let $p = \max_i \mathrm{qr}(\phi_i)$, $s = \max_i n_i$, then $p + s \leqslant \beta$. Define for each i:

$$X_i = \{\vec{a}\in A^{n_i} \mid (\mathfrak{A}, \sigma; C) \models \phi_i(\vec{a})\}$$

$$Y_i = \{\vec{b}\in B^{n_i} \mid (\mathfrak{B}, \tau; D) \models \phi_i(\vec{b})\} .$$

Since $p < p+1 < \ldots < p+s \leqslant \beta$, then by the extension properties there are functions $f_i : A^{n_i} \to B^{n_i}$, $g_i : B^{n_i} \to A^{n_i}$ such that

$$\sigma\,\vec{a} \overset{p}{\sim} \tau\, f_i(\vec{a}) \quad \text{for all } \vec{a}\in A^{n_i}. \tag{I}$$

$$\tau\,\vec{b} \overset{p}{\sim} \sigma\, g_i(\vec{b}) \quad \text{for all } \vec{b}\in B^{n_i}. \tag{II}$$

If $([X_1]^p_\sigma,\ldots,[X_k]^p_\sigma) \in q_j$ then $([f'_i(X_1)]^p_\tau,\ldots,[f'_i(X_k)]^p_\tau) \in r_j$. (III)

If $([y_1]^p_\tau, \ldots, [y_k]^p_\tau) \in r_j$ then $([g'_i(y_1)]^p_\sigma, \ldots, [g'_i(y_k)]^p_\sigma) \in q_j$. (IV)

CLAIM 1. $X_i = [X_i]^p_\sigma$, $Y_i = [Y_i]^p_\tau$.

If $\sigma \vec{a}' \overset{p}{\sim} \sigma \vec{a}$ where $(\mathfrak{A}, \sigma; C) \models \phi_i(\vec{a})$, then $(\mathfrak{B}, \tau; D) \models \phi_i(f_i(\vec{a}))$ because of (I), the fact that $qr(\phi_i) \leq p$, and the induction hypothesis. By transitivity: $\sigma \vec{a}' \overset{p}{\sim} \tau f_i(\vec{a})$, and so we have again $(\mathfrak{A}, \sigma, C) \models \phi_i(\vec{a}')$. This shows $[X_i]^p_\sigma \subseteq X_i$. The other direction is trivial, and the case of Y_i is similar.

CLAIM 2. $f'_i(X_i) \subseteq Y_i$, $g'_i(Y_i) \subseteq X_i$.

This follows from (I), (II), and the induction hypothesis for ϕ_i.

CLAIM 3. $[f'_i(X_i)]^p_\tau = Y_i$, $[g'_i(Y_i)]^p_\sigma = X_i$.

The inclusion from left to right follows from Claims 1 and 2. Suppose $\vec{b} \in Y_i$ then $\tau \vec{b} \overset{p}{\sim} \sigma g_i(\vec{b}) \overset{p}{\sim} \tau f_i g_i(\vec{b})$ and so $\vec{b} \underset{\tau}{\overset{p}{\sim}} f_i g_i(\vec{b})$. But $g_i(\vec{b}) \in g'_i(Y_i) \subseteq X_i$ and we have $\vec{b} \in [f'_i(X_i)]^p_\tau$.

By (III), (IV), and Claims 1 and 2, we have that $(\mathfrak{A}, \sigma; C) \models Q_j \vec{x}_1, \ldots, \vec{x}_k (\phi_1, \ldots, \phi_k)$ iff $(X_1, \ldots, X_k) \in q_j$ iff $(Y_1, \ldots, Y_k) \in r_j$ iff $(\mathfrak{B}, \tau; D) \models Q_j \vec{x}_1, \ldots, \vec{x}_k (\phi_1, \ldots, \phi_k)$. ∎

COROLLARY 3.4.- (*a*) $(\mathfrak{A}; C) \overset{\alpha}{\equiv} (\mathfrak{B}; D)$ *iff* $(\mathfrak{A}; C) \overset{(\alpha, <)}{\sim} (\mathfrak{B}; D)$.
(*b*) $(\mathfrak{A}; C) \overset{\infty}{\equiv} (\mathfrak{B}; D)$ *iff* $(\mathfrak{A}; C) \overset{(\alpha, <)}{\sim} (\mathfrak{B}; D)$ *for all ordinals* α.

COROLLARY 3.5.- *If the families* C, D *and the similarity types of* $\mathfrak{A}$ *and* $\mathfrak{B}$ *are finite, then the following conditions are equivalent:*

(*i*) $(\mathfrak{A}; C) \equiv (\mathfrak{B}; D)$ (*equivalent for finitary formulas*),

(*ii*) $(\mathfrak{A}; C) \overset{(\omega, <)}{\sim} (\mathfrak{B}; D)$,

(*iii*) $(\mathfrak{A}; C) \overset{(n, <)}{\sim} (\mathfrak{B}; D)$ *for all* $n \in \omega$.

PROOF. (*ii*) ⇒ (*iii*) and (*iii*) ⇒ (*i*) are trivial; (*i*) ⇒ (*ii*) follows from Lemma 1.1 and Theorem 3.1. ∎

There are more convenient characterizations of ∞-elementary equivalency than Corollary 3.4 (*b*).

THEOREM 3.6.- *The following are equivalent:*

(*i*) $(\mathfrak{A}; C) \overset{\infty}{\equiv} (\mathfrak{B}; D)$.

(ii) $(\mathfrak{A};\ C) \overset{\mathrm{P}}{\sim} (\mathfrak{B};\ D)$ *where* P *is non-well ordered.*
(iii) $(\mathfrak{A};\ C) \sim (\mathfrak{B};\ D)$ *(back-and-forth without parameters).*

PROOF. *(i)* ⇒ *(iii)*. The same as the proof of Theorem 3.1, but simplifying the definition of back-and-forth to: $\sigma \sim \sigma'$ iff $(\mathfrak{A}, \sigma;\ C) \overset{\infty}{\equiv} (\mathfrak{A}, \sigma';\ D)$, etc. Instead of $t^{\beta}_{\vec{a}}$ one defines $t_{\vec{a}} = \{\phi(\vec{y}) \mid (\mathfrak{A}, \sigma;\ C) \models \phi(\vec{a})\}$. By Lemma 1.2, $\wedge t_{\vec{a}}$ is a sentence of $L_{\infty\omega}(Q_j)_{j \in J}$. The rest of the proof is simpler because we do not need parameter conditions.

(iii) ⇒ *(ii)*. Choose any non-well ordered set P and define $\sigma \overset{p}{\sim} \tau$ iff $\sigma \sim \tau$.

(ii) ⇒ *(i)*. Let $p_1 > p_2 > \ldots$ be an infinite descending sequence of P. One shows, as in the proof of Theorem 3.3, by induction on the complexity of $\phi(\vec{y})$, that for all n:

$$\sigma \overset{p_n}{\sim} \tau \ \text{ implies: } \ (\mathfrak{A};\ C) \models \phi(\sigma) \ \text{ iff } \ (\mathfrak{B};\ D) \models \phi(\tau).$$

To use the extension property in the inductive step for $Q_j\ \vec{x}_1,\ldots,\vec{x}_m(\phi_1,\ldots,\phi_m)$ one chooses $p_{n+s} < p_{n+s-1} < \ldots < p_n$. ■

§ 4 INTERPOLATION AND MONADIC QUANTIFIERS.

A quantifier symbol of type $\langle n_1,\ldots, n_k\rangle$ is *monadic* if $n_i = 1$ for all i. Sintactically, it binds a single variable in each formula: $Q x_1 \ldots x_n (\phi_1(x_1),\ldots, \phi_n(x_n))$. This includes the cardinal quantifiers Q_α: "there are at least ω_α elements", as well as Chang's and Hartig's quantifier, but not the Magidor - Malitz quantifiers. Throughout this section L_M will denote the logic $L_{\infty\omega}(Q_j)_{j \in J}$, where Q_j runs through all the possible Lindström-Mostowski monadic quantifiers.

LEMMA 4.1.- *If* $\mathfrak{A}$ *and* $\mathfrak{B}$ *are structures of power at most* ω_1, *then* $\mathfrak{A} \equiv_{L_{\infty\omega}(Q_1)} \mathfrak{B}$ *implies* $\mathfrak{A} \equiv_{L_M} \mathfrak{B}$.

PROOF. Let $2^n = \{\delta \mid \delta : n \to 2\}$, for $S \subseteq A$ let $S^0 = S$ and $S^1 = A - S$. For each $n \in \omega$ and $F : 2^n \to \{\kappa \mid \kappa \text{ cardinal},\ \kappa \leqslant \omega_1\}$ define the quantifier: $Q_F(A) = \{(S_1,\ldots, S_n) \mid \forall \delta \in 2^n\ |\bigcap_{i<n} S^{\delta(i)}_{i+1}| = F(\delta)\}$. Since a monadic structure $(A, S_1,\ldots, S_n)$ is completely determined by the above set of cardinals, there corresponds to it a unique F such that $(T_1,\ldots,T_n) \in Q_F(B)$ iff $(B, T_1,\ldots, T_n) \approx (A, S_1,\ldots, S_n)$. Let $\exists !^{\kappa} x\, \phi(x)$ mean that the truth set of $\phi(x)$ has exaclty κ elements ($\kappa \leqslant \omega_1$). Clearly, for structures of power at most ω_1, $\mathfrak{A} \equiv_{L_{\infty\omega}(Q_1)} \mathfrak{B}$ implies $\mathfrak{A} \equiv_{L_{\infty\omega}(Q_1,\ \exists !^{\kappa})_\kappa} \mathfrak{B}$, and so $\mathfrak{A} \equiv_{L_{\infty\omega}(Q_F)_F} \mathfrak{B}$, because the Q_F's are definable from the former quantifiers. If ϕ^0 is ϕ and ϕ^1 is $\neg\phi$:

$$\mathcal{Q}_F x_1 \dots x_n(\phi_1(x_1)\dots \phi_n(x_n)) \Leftrightarrow \bigwedge_{\delta\in 2^n} [\exists !^{F(\delta)} x \bigwedge_{i<n} \phi_{i+1}(x)^{\delta(i)}] .$$

By Therorem 3.6: $(\mathfrak{A}; \mathcal{Q}_F(A))_F \sim (\mathfrak{B}; \mathcal{Q}_F(B))_F$. To conclude the proof, we show that the given back-and-forth has the extension property with respect to each monadic quantifier, and apply Theorem 3.3.

Let $\sigma \overset{\beta'}{\sim} \tau$, $\beta < \beta'$, and let $f : A \to B$ be the function given homogeneously by the extension property for all the $\mathcal{Q}_F$'s (see Remark 3.2).
If $M = ([X_1]^{\beta}_{\sigma},\dots,[X_k]^{\beta}_{\sigma}) \in \mathcal{Q}_j(A)$, choose F such that $M \in \mathcal{Q}_F(A)$, then $M' = ([f'(X_1)]^{\beta}_{\tau},\dots,[f'(X_k)]^{\beta}_{\tau}) \in \mathcal{Q}_F(B)$ by the extension property. Hence, $(A,M) \approx (B, M')$ and so $M' \in \mathcal{Q}_j(B)$ by definition of Lindström-Mostowski quantifier. ∎

The proof of the following lemma is analogous. Let $L_C = L_{\infty\omega}(\mathcal{Q}_\alpha)_{\alpha\in Or}$, where $\mathcal{Q}_\alpha$ runs over all the cardinal quantifiers.

LEMMA 4.2.- *$\mathfrak{A} \equiv_{L_C} \mathfrak{B}$ implies $\mathfrak{A} \equiv_{L_M} \mathfrak{B}$.*

In Friedman 1973, Friedman gives a sentence ϕ of $L_{\omega\omega}(\mathcal{Q}_\alpha)$ (respectively, $L_{\omega\omega}(\mathcal{Q})$, where $\mathcal{Q}$ is Chang's quantifier) containing a relation symbol P, such that for every $\mathfrak{A}$ there is at most an interpretation P for which $(\mathfrak{A},P) \vDash \phi$. However, $\mathbf{K} = \{\mathfrak{A} \mid (\mathfrak{A}, P) \vDash \phi$ for some $P\}$ is not elementary in L_C. This is shown giving a pair of structures $\mathfrak{A} \in \mathbf{K}$, $\mathfrak{B} \notin \mathbf{K}$ which are elementarily equivalent in L_C. It is observed that this is enough to show the failure of Beth's definability theorem for any logic between $L_{\omega\omega}(\mathcal{Q}_\alpha)$ (respectively $L_{\omega\omega}(\mathcal{Q})$) and L_C. After Lemma 4.2, we have:

THEOREM 4.3.- Beth $(L_{\omega\omega}(\mathcal{Q}_\alpha), L_M)$ *fails for any* $\alpha > 0$. *Also,* Beth $(L_{\omega\omega}(\mathcal{Q}), L_M)$ *fails.*

For example, Beth's theorem does not hold in logic with the Hartig quantifier since it is between $L_{\omega\omega}(\mathcal{Q})$ and L_M. The first part of the last theorem is not true for $\alpha = 0$ because $L_{\omega_1\omega}$ is a sublogic of L_M extending $L_{\omega\omega}(\mathcal{Q}_0)$ and satisfying interpolation; hence, Beth's theorem. The same is true of $\Delta(L_{\omega\omega}(\mathcal{Q}_0))$ (see Barwise 1974). The next theorem imposes a restriction in such logics. A logic L^+ has *relativizations* if for every monadic predicate V and sentence ϕ of L^+ there is a sentence ϕ^V such that $(\mathfrak{A}, \overline{V}) \vDash \phi^V$ iff $\mathfrak{A} \upharpoonright \overline{V} \vDash \phi$. Here, $\overline{V}$ is an interpretation of V and $\mathfrak{A} \upharpoonright \overline{V}$ is the restriction of the universe and relations of $\mathfrak{A}$ to the set $\overline{V}$. Almost all interesting logics have this property. An exception is logic with Chang's quantifier. The *Löwenheim number* of a logic L^+ is the smallest cardinal κ such that if a sentence of L^+ has a model it has one of power at most κ. It may not exist. L^+ satisfies the downward Löwenheim-Skolem theorem if its Löwenheim number is ω.

THEOREM 4.4.- *Let L^+ be a logic between $L_{\omega\omega}$ and L_M having relativization and satisfying interpolation. If L^+ has Löwenheim number κ, then for any sentence ϕ having infinite models* $\mathrm{Sup}\{|\mathfrak{A}| \leqslant \kappa | \mathfrak{A} \models \phi\} = \kappa$. *If L^+ does not have Löwenheim number such sentence has models of arbitrarily large cardinality.*

PROOF. Suppose that there is ϕ with infinite models of size $\lambda, \lambda < \kappa$, but not of any size between λ^+ and κ (included). By definition of Löwenheim number, there is a sentence ψ with all its models of power greater than λ. Let θ be a sentence whose models are the equivalence relations. The classes $\mathbf{K}_1$ (respectively $\mathbf{K}_2$) of models of θ having as many equivalence classes as a model of ϕ (respectively, a model of ψ) are PC classes of L^+, defined by the projection of the sentences:

$$\theta \wedge \text{"}f \text{ is a function onto } V\text{"} \wedge \forall x \forall y\ (xEy \leftrightarrow f(x) = f(y)) \wedge \phi^V \ \ (\text{resp. } \psi^V).$$

Since these sentences do not have common models of power less or equal than κ, $\mathbf{K}_1$ and $\mathbf{K}_2$ are disjoint PC classes of L^+. However, they are inseparable in L^+.If $(A,E) \in \mathbf{K}_1$ has λ has equivalence classes of power λ', and $(A', E') \in \mathbf{K}_2$ has λ' equivalence classes of power λ', an easy back-and-forth argument shows that $(A, E) \equiv_{L_C} (A', E')$, and so $(A, E) \equiv_{L_M} (A', E')$. From this we conclude that interpolation fails in L^+. ■

COROLLARY 4.5.- *Let L^+ be a logic between $L_{\omega\omega}(Q_0)$ and L_M having relativization and satisfying interpolation, then L^+ satisfies the downward Löwenheim-Skolem theorem.*

PROOF. There is a sentence in $L_{\omega\omega}(Q_0)$, and therefore in L^+, which has models of power ω but not larger. By Theorem 4.4 it must have Löwenheim number ω. ■

§ 5 COFILTER QUANTIFIERS, A SIMPLER BACK-AND-FORTH.

Let Q be a quantifier symbol of type $\langle n \rangle$, $q \subseteq \mathcal{P}(A^n)$ is a *cofilter interpretation of* Q if it satisfies *Monotonicity:* $S \in q$ and $S \subset S'$ imply $S' \in q$, and *Distributivity:* $S \cup S' \in q$ implies $S \in q$ or $S' \in q$. Obviously, q is a cofilter interpretation iff the "dual" interpretation $\tilde{q} = \{S \mid A - S \notin q\}$ is a filter over A. In terms of the language, $(\mathfrak{A}; q)$ must satisfy the schemata: $\forall \vec{x}(\phi \to \psi) \to (Q\vec{x}\phi \to Q\vec{x}\psi)$, and $Q\vec{x}(\phi \vee \psi) \to (Q\vec{x}\phi \vee Q\vec{x}\psi)$. A cofilter quantifier is *ω-complete* if $\tilde{q}$ is an ω-complete filter. Equivalently, if $\cup\{S_n \mid n \in \omega\} \in q$ implies $S_n \in q$ for some n.

The cardinal quantifiers are cofilter, Q_α is ω-complete if its cofinality is greater than ω, as is the case of Q_1. Magidor-Malitz quantifiers are not cofilter, however the quantifier Q_0^n is equivalent to the cofilter quantifier H_0^n, where $H_0^n x_1 \ldots x_n\, \phi(x_1, \ldots, x_n)$ means "there is an infinite set I such that for all distinct $a_1, \ldots, a_n \in I$ there is a permutation π for which $\phi(a_{\pi 1}, \ldots, a_{\pi n})$ holds".

The back-and-forth characterization of elementary equivalence may be simplified in the case of logics with cofilter quantifiers:

DEFINITION 5.1.- A *simple back-and-forth from* $(\mathfrak{A}; q)$ to $(\mathfrak{B}, r)$ is defined as in Def. 2.1 (respectively Def. 2.3), except parts (A) and (B) of the extension property which are changed now to:

(ii^+) For all $\vec{a} \in A^n$ there is $\vec{b} \in B^n$ such that:

(A^+) $\sigma\,\vec{a} \overset{p}{\sim} \tau\,\vec{b}$.

(B^+) $[\vec{a}\,]^p_\sigma \in q$ implies $[\vec{b}\,]^p_\tau \in r$.

(iii^+) The analogue in the other direction.

In the following theorem, the subscript $S(Fin)$ indicates that there is a simple back-and-forth from $(\mathfrak{A}; q)$ to $(\mathfrak{B}; r)$ where the number of equivalence classes of each E^p_k is finite. Similarly, $S(\omega)$ indicates a simple back-and-forth where each E^p_k has at most countably many equivalence classes.

THEOREM 5.1.- *Let* C *and* D *be finite families of cofilter quantifier interpretations and assume the number of relations in each structure is finite, then:*

(*a*) $(\mathfrak{A}; C) \overset{n}{\equiv} (\mathfrak{B}; D)$ *iff* $(\mathfrak{A}; C) \underset{S(Fin)}{\overset{(n,<)}{\sim}} (\mathfrak{B}; D)$.

If in addition the quantifiers are ω*-complete:*

(*b*) $(\mathfrak{A}; C) \overset{n}{\equiv} (\mathfrak{B}; D)$ *iff* $(\mathfrak{A}; C) \underset{S(\omega)}{\overset{(n,<)}{\sim}} (\mathfrak{B}; D)$.

(*c*) *If* $(\mathfrak{A}; C) \underset{S(\omega)}{\overset{P}{\sim}} (\mathfrak{B}, D)$ *with* P *non well ordered, then*

$(\mathfrak{A}; C) \overset{\infty}{\equiv} (\mathfrak{B}, D)$.

PROOF. (*a*) Since a back-and-forth is a simple back-and-forth, from left to right follows from Theorem 3.1 and Lemma 1.1, plus the observation that the equivalence classes of E^k_n result definable by formulas of qr at most k. For the other direction one shows that if q and r are cofilter interpretations, then properties (ii^+-iii^+) imply the original extension property. Given $\sigma \overset{k}{\sim} \tau$ choose for each $\vec{a} \in A^n$ some $f(\vec{a}) = \vec{b}$ such that (A^+) and (B^+) hold. Assume as in ii-B of Def. 2.1 that $X \subseteq A^n$ and $[X]^k_\sigma \in q$. Since $[X]^k_\sigma = \cup\{[\vec{a}]^k \mid \vec{a} \in X\}$, and the number of equivalence classes of the relation $\overset{k}{\sim}$ must be finite, the union is of a finite number of classes. By the distributivity of q, $[\vec{a}]^k_\sigma \in q$ for some $\vec{a} \in X$. By (B^+), $[f(\vec{a})]^k_\tau \in r$. By monotonicity of r, $[f'(X)]^k_\tau \in r$.

(*b*) and (*c*) are similar; the converse of (*c*) fails because the number of definable subsets may be too large. ■

§ 6 APPLICATIONS TO $L_{\omega\omega}(Q_1)$.

Let $\mathfrak{A}_i$ be a structure of type τ_i, then $[\mathfrak{A}_0, \ldots, \mathfrak{A}_n]$ is the structure that has the disjoint union $\cup \{A_i \mid i \leq n\}$ as universe and whose relations are the corresponding copies of the relations of each $\mathfrak{A}_i$. A relation $\mathbf{R}$ between structures is PC in L^+ if there is a sentence in L^+ such that

$$\mathbf{R}(\mathfrak{A}_1, \ldots, \mathfrak{A}_n) \text{ iff } [\mathfrak{A}_1, \ldots, \mathfrak{A}_n, \mathfrak{B}] \models \phi.$$

Let $t : \tau \to \tau'$ be an *interpretation* between similarity types (languages) as in Barwise 1974. If $\mathfrak{A}$ is a τ'-structure, then $\mathfrak{A} \upharpoonright t$ denotes the restriction (projection) of $\mathfrak{A}$ to a τ-structure, via t. In the case of a *simple* interpretation, we write $\mathfrak{A} \upharpoonright \tau$.

LEMMA 6.1.- *Let C be a finite family of Lindström-Mostowski quantifiers such that $L_{\omega\omega}(Q_1, C)$ has relativizations. Let $t_i : \tau \to \tau_i$ be interpretations where τ is finite. Then the following relation between structures $\mathfrak{A}$, $\mathfrak{B}$, and $\mathbf{P}$ is PC in the given logic:* $(\mathfrak{A} \upharpoonright t_1; Q_1, C) \overset{\mathbf{P}}{\underset{S(\omega)}{\sim}} (\mathfrak{B} \upharpoonright t_2; Q_1, C)$.

PROOF. It is a routine exercise to write down sentences stating that some $\mathbf{P}$ and $\{E_n^p \mid p \in P, n \in \omega\}$ form a simple back-and-forth from $\mathfrak{A} \upharpoonright t_1$ to $\mathfrak{B} \upharpoonright t_2$. The only second order statement in the original definition, the extension property, has become first order. Q_1 and the quantifiers of C are needed to state the extension property. Q_1 is also needed to put a countable bound on the number of equivalence classes. The infinite family of relations $E_n(p, x_1, \ldots, x_n, y_1, \ldots, y_n)$ expressing the E_n^p's may be reduced to a single one by a suitable encoding of the finite sequences, see Caicedo 1978 for details. Relativization is needed to express the meaning of each quantifier in $\mathfrak{A} \upharpoonright t_1$ and $\mathfrak{B} \upharpoonright t_2$ instead of $[\mathfrak{A}, \mathfrak{B}, \mathbf{P}, \ldots]$. ∎

Let $\theta(U, V, P, <)$ be the sentence given by the lemma where U, V, P, and $<$ denote respectively the universe of $\mathfrak{A}$, the universe of $\mathfrak{B}$ and the ordered set of parameters P. If we drop the finiteness conditions, θ becomes a set of sentences.

In the next lemma, a class $\mathbf{K}$ is PC if there exists a *countable* set of sentences T and an interpretation t such that $\mathbf{K} = \{\mathfrak{A} \upharpoonright t \mid \mathfrak{A} \models T\}$. L^+ satisfies $LS(\omega_1)$ if every sentence (theory) having a model has one of power at most ω_1.

LEMMA 6.2.- *Let $\mathbf{K}_1$ and $\mathbf{K}_2$ be PC classes of a countably compact logic L^+ extending $L_{\omega\omega}(Q_1)$ and having relativizations. If they are inseparable by a sentence of $L_{\omega\omega}(Q_1)$, then there are $\mathfrak{A} \in \mathbf{K}_1$, $\mathfrak{B} \in \mathbf{K}_2$ such that $(\mathfrak{A}; Q_1) \overset{\mathbf{P}}{\underset{S(\omega)}{\sim}} (\mathfrak{B}; Q_1)$ with $\mathbf{P}$ non-well ordered. In case L^+ satisfies $LS(\omega_1)$ for countable theories $\mathfrak{A}$ and $\mathfrak{B}$ may be chosen of power at most ω_1.*

PROOF. Let $t_i : \tau \to \tau_i$ and $\mathbf{K}_i = \{\mathfrak{A} \upharpoonright t_i \mid \mathfrak{A} \models T_i\}$, with $T_i \subseteq L^+$. By countable compactness, we may assume the number of symbols in τ and τ_i to be finite (see Flum 1975). Suppose K_1 and K_2 are inseparable in $L_{\omega\omega}(Q_1)$. Since there are finitely many non equivalent sentences of rank at most n in this logic, there exist $\mathfrak{A}_n \in K_1$, $\mathfrak{B}_n \in K_2$ such that $\mathfrak{A}_n \overset{n}{\equiv}_{L_{\omega\omega}(Q_1)} \mathfrak{B}_n$. Otherwise, we could separate the classes by a sentence of rank n. By Theorem 5.1 $(\mathfrak{A}_n; Q_1) \underset{S(\omega)}{\overset{(n,<)}{\sim}} (\mathfrak{B}_n; Q_1)$.

In this way, each finite part of the following theory of L^+ has a model:

$$\theta(U, V, P, <) \cup \{T_1^U, T_2^V\} \cup \{P(c_n), c_{n+1} < c_n \mid n \in \omega\}.$$

Here, $T^U = \{\phi^U \mid \phi \in T\}$. By compactness, there is a model $\mathfrak{M} = [\mathfrak{A}', \mathfrak{B}', P, \ldots]$ of the full set. Making $\mathfrak{A} = \mathfrak{A}' \upharpoonright t_1$ and $\mathfrak{B} = \mathfrak{B}' \upharpoonright t_2$, we have the result. ∎

The next theorem is analogous to Lindström's theorem for $L_{\omega\omega}$. It is the best possible result in the sense that it is possible to have proper countably compact extensions of $L_{\omega\omega}(Q_1)$ by non-monadic quantifiers satisfying $LS(\omega_1)$, for example logic with the Magidor-Malitz quantifiers Q_1^n.

THEOREM 6.3.- *Let L^+ be a logic between $L_{\omega\omega}(Q_1)$ and L_M having relativizations. If L^+ satisfies compactness and $LS(\omega_1)$ for countable theories, then $L^+ \equiv L_{\omega\omega}(Q_1)$.*

PROOF. Suppose $\phi \in L^+ - L_{\omega\omega}(Q_1)$, then the classes $K_1 = Mod(\phi)$, $K_2 = Mod(\neg\phi)$ are inseparable in $L_{\omega\omega}(Q_1)$. By Lemma 5.3 and Theorem 3.6, there exists $\mathfrak{A} \models \phi$ and $\mathfrak{B} \models \neg\phi$ such that $\mathfrak{A} \equiv_{L_M} \mathfrak{B}$, a contradiction. ∎

The next theorem shows that interpolation fails strongly in L_M. It implies Theorem 4.3 for $\alpha = 1$.

THEOREM 6.4.- *Let ϕ and ψ be sentences of $L_{\omega\omega}(Q_1)$ such that $\phi \models \psi$. If they have an interpolant in L_M, they have one in $L_{\omega\omega}(Q_1)$.*

PROOF. Let τ be the common language of ϕ and ψ; if they do not have an interpolant in $L_{\omega\omega}(Q_1)$, then the PC classes $K_1 = \{\mathfrak{A} \upharpoonright \tau \mid \mathfrak{A} \models \phi\}$ and $K_2 = \{\mathfrak{B} \upharpoonright \tau \mid \mathfrak{B} \models \psi\}$ are inseparable. By Lemma 5.3, there are structures $\mathfrak{A} \models \phi$ and $\mathfrak{B} \models \psi$ such that $\mathfrak{A} \upharpoonright \tau \equiv_{L_M} \mathfrak{B} \upharpoonright \tau$. Therefore, ϕ and ψ do not have interpolants in L_M. ∎

THEOREM 6.5.- *Let T be a countable theory in $L_{\omega\omega}(Q_1)$. If T has a (uncountable) model, it has a (uncountable) model satisfying at most countably many n-types in L_M, for each $n \in \omega$.*

PROOF. Take $K_1 = K_2 = Mod(T)$ in Lemma 5.3. They are obviously inseparable; then we have $(\mathfrak{A}; Q_1) \underset{S(\omega)}{\overset{P}{\sim}} (\mathfrak{B}; Q_1)$ with $\mathbf{P}$ non-well ordered, $\mathfrak{A}$ and $\mathfrak{B}$ of power at most ω_1. If $\vec{a} \sim \vec{a}'$ in A^n, find $\vec{b} \in B^n$ such that $\vec{a} \sim \vec{b}$; then $\vec{a} \sim \vec{b} \sim \vec{a}'$. Hence, $(\mathfrak{A}, \vec{a}) \equiv_{L_{\infty\omega}(Q_1)} (\mathfrak{B}, \vec{b}) \equiv_{L_{\infty\omega}(Q_1)} (\mathfrak{A}, \vec{a}')$; see Theorem 3.6. By Lemma 4.1, $(\mathfrak{A}, \vec{a}) \equiv_{L_M} (\mathfrak{A}, \vec{a}')$. Therefore, $\vec{a}$ and $\vec{a}'$ satisfy the same type in L_M. Since the number of equivalence classes of $\sim$ is countable, the same is true of the number of types. ■

In $L_{\omega\omega}$ we have that a theory with infinite models has an uncountable model satisfying at most countably many types *over each countable subset*. This is not true here as shown by the counterexample:

$$Q_1 x P(x), \quad \neg Q_1 x R(x), \quad \text{" } < \textit{is a linear order"},$$
$$\forall x \, \forall y (P(x) \wedge P(y) \wedge x \neq y \rightarrow \exists z (R(z) \wedge x < z \wedge z < y)).$$

Any model satisfies uncountably many types over the interpretation of R.

The last three theorems hold for any countable compact logic $L_{\omega\omega}(Q_1, C)$ having relativizations and satisfying $LS(\omega_1)$, where C is a finite family of Linds.-Most. cofilter quantifiers. In Theorems 5.5 and 5.6 one must change L_M for the result obtained by adding to $L_{\omega\omega}(Q_1, C)$ all monadic quantifiers. However, we do not know of any concrete example. On the other hand, one can show analogues of these theorems for Stationary logic $L(aa)$; see Caicedo 1977 b. So, any countable theory of $L(aa)$ has a model satisfying at most countably many types in the infinitary logic $L(aa)_M$. $L(aa)$ is maximal in $L(aa)_M$, with respect to compactness and $LS(\omega_1)$, and any pair of sentences of $L(aa)$ with an interpolant in $L(aa)_M$ have one in $L(aa)$.

§ 7 AN EXTENSION OF $L_{\omega\omega}(Q_0^{<\omega})$ WHERE ELEMENTARY EQUIVALENCE IS PRESERVED BY PRODUCTS.

In Badger 1977, L.Badger shows that elementary equivalence of structures is not preserved by products in the logic $L_{\omega\omega}(Q_0^{<\omega})$ obtained by adding the Magidor-Malitz quantifiers Q_0^n under the infinite interpretation to first order logic (cf. Magidor and Malitz 1977). His proof is based on the undefinability of well order in the logic. Actually, his proof gives the stronger result: no extension of this logic where well order is undefinable satisfies preservation of elementary equivalence by products. In this section, we define a natural extension of the above logic where elementary equivalence is preserved by finite products. Badger's counterexample does not work there because well order is definable.

For each n define an n-variable quantifier symbol R^n where $R^n x_1, \ldots, x_n \, \phi(x_1, \ldots, x_n)$ has the interpretation: "*there is an infinite linearly ordered set* $(I, <)$ *without last element such that for all* $x_1, \ldots, x_n$ *in* I, $x_1 < \ldots < x_n$ *implies* $\phi(x_1, \ldots, x_n)$".

The same quantifier is obtained if one asks the set $(I,<)$ to be well ordered or just of type $(\omega,<)$. By Ramsey's Theorem this is a cofilter quantifier. Let $L_{\omega\omega}(R^{<\omega})$ result by adding these quantifiers to first order logic; it extends logic with Magidor-Malitz quantifiers because:

$$Q_0^n x_1,\dots,x_n\phi(x_1,\dots,x_n) \Leftrightarrow R^n x_1,\dots,x_n \bigwedge_\pi \phi(x_{\pi 1},\dots,x_{\pi n}).$$

where π runs over all permutations. The extension is proper because well order is definable there by the sentence $\neg R^2 xy(y<x)$, and it is not definable in $L_{\omega\omega}(Q_0^{<\omega})$ by Badger's Theorem 4.10 in Badger 1977. Using our back-and-forth for cofilter quantifiers we show:

THEOREM 7.1.- *$L_{\omega\omega}(R^{<\omega})$-elementary equivalence is preserved by finite products of structures.*

PROOF. Let $\mathfrak{A}$ and $\mathfrak{B}$ (respectively $\mathfrak{A}'$ and $\mathfrak{B}'$) be $L_{\omega\omega}(R^{<\omega})$-elementary equivalent. Since every sentence has finitely many relation and quantifier symbols, there is no loss of generality in assuming that the similarity type is finite and restricting our considerations to $C = \{R^1,\dots,R^m\}$. By Theorem 5.1: $(\mathfrak{A};C) \overset{\omega}{\underset{S(Fin)}{\sim}} (\mathfrak{B};C)$, and the same is true for $\mathfrak{A}'$ and $\mathfrak{B}'$. Let $B_i = (\omega,<,\{E^k_{i,n} \mid n,k\in\omega\})$, $i = 1,2$, be the corresponding back-and-forths. For each pair of sequences $\vec{x} = (x_1,\dots,x_n)$, $\vec{x}' = (x'_1,\dots,x'_n)$ define $\vec{x}+\vec{x}' = ((x_1x'_1),\dots,(x_nx'_n))$. Define E^k_n in $(A\times A')^n \cup (B\times B')^n$ by:

$$\vec{x}+\vec{x}' \overset{k}{\leftrightsquigarrow} \vec{y}+\vec{y}' \text{ iff } \vec{x}\,E^k_{1,n}\,\vec{y} \text{ and } \vec{x}'E^k_{2,n}\vec{y}'.$$

We claim that the system $B^+ = (\omega,<,\{E^k_n \mid k,n\in\omega\})$ is a simple back-and-forth from $\mathfrak{A}\times\mathfrak{A}'$ to $\mathfrak{B}\times\mathfrak{B}'$, with finitely many equivalence classes for each E^k_n.

The fact that E^k_n is an equivalence relation, and properties (i) and (iv) of Definition 2.1 follow easily because they are expressed by universal Horn sentences, that are preserved by cartesian products, and we defined the new relations as the "product relations" are ordinarily defined in the cartesian product. E^k_n has finitley many equivalence classes because for any $\vec{x}+\vec{y}$ the equivalence class of this sequence with respect to E^k_n has the form:

$$[\vec{x}+\vec{x}']^k = \{\vec{u}+\vec{u}' \mid \vec{u}\in[\vec{x}]^k \text{ and } \vec{u}'\in[\vec{x}']^k\}.$$

Therefore, if $E^k_{1,n}$ has M classes and $E^k_{2,n}$ has N classes, E^k_n has MN classes. It remains to show the extension property $(ii^+ - iii^+)$ for each R^r. Let $\sigma = \vec{x}+\vec{x}' \in (A\times A')^n$ and $\tau = \vec{y}+\vec{y}' \in (B\times B')^n$ be such that $\sigma \overset{k'}{\leftrightsquigarrow} \tau$, and suppose that $k+r<k'$. By definition:

$$\vec{x} \overset{k'}{\sim} \vec{y} \quad\text{and}\quad \vec{x}' \overset{k'}{\sim} \vec{y}'. \tag{1}$$

Let $\vec{a} + \vec{a}' \in (A \times A')^n$. By the extension property in B_1 and B_2, there exist $\vec{b} = (b_1, \ldots, b_n)$ and $\vec{b}' = (b'_1, \ldots, b'_n) \in B'^n$ such that:

$$\vec{x}\vec{a} \overset{k}{\sim} \vec{y}\vec{b} \quad \text{and} \quad \vec{x}'\vec{a}' \overset{k}{\sim} \vec{y}'\vec{b}' \tag{2}$$

$$\begin{aligned} &[\vec{a}]^k_{\vec{x}} \in R^n(A) \text{ implies } [\vec{b}]^k_y \in R^n(B) \\ &[\vec{a}']^k_{\vec{x}'} \in R^n(A') \text{ implies } [\vec{b}']^k_{y'} \in R^n(B'). \end{aligned} \tag{3}$$

From (2): $(\sigma, \vec{a} + \vec{a}') \overset{k}{\approx} (\tau, \vec{b} + \vec{b}')$.

It remains to show that (4) below implies (5):

$$S = [\vec{a} + \vec{a}']^k_\sigma \in R^n(A \times A') \tag{4}$$

$$[\vec{b} + \vec{b}']^k_\tau \in R^n(B \times B') \tag{5}$$

Assuming (4), we get a linear order $(I, <)$ of type $(\omega, <)$ with $I \subseteq A \times A'$, such that for all $(u_1 w_1) < \ldots < (u_n w_n)$ in I,

$$(\vec{x}, u_1, \ldots, u_n) \overset{k}{\sim} (\vec{x}, \vec{a})$$

$$(\vec{x}', w_1, \ldots, w_n) \overset{k}{\sim} (\vec{x}', \vec{a}').$$

If J and J' are the projections of I in A and A' respectively, one of them, say J, must be infinite. Choose a function $g : J \to J'$ such that $(u, g(u)) \in I$ for all $u \in J$, and define a linear order in J by: $u_1 < u_2$ iff $(u_1, g(u_1)) < (u_2, g(u_2))$. It must have type $(\omega, <)$. Moreover $u_1 < \ldots < u_n$ in J implies $(u_1, g(u_1)) < \ldots < (u_n, g(u_n))$ in I, and so $(u_1, \ldots, u_n) \in [\vec{a}]^k_{\vec{x}}$. Hence, $[\vec{a}]^k_{\vec{x}} \in R^n(A)$ and $[\vec{b}]^k_{\vec{y}} \in R^n(B)$ by (3). Let $(L, <)$ of type $(\omega, <)$ with $L \subseteq B$ be such that $u_1 < \ldots < u_n$ in L implies $(u_1, \ldots, u_n) \in [\vec{b}]^k_{\vec{y}}$. Now we consider two cases with respect to J'.

CASE 1. J' is finite.

Then there is $w \in J'$ such that $\{u \in J \mid (u, w) \in I\}$ is infinite and there exist $(u_1 w) < \ldots < (u_n w)$ in I. Hence, $((u_1 w) \ldots (u_n w)) \in S$ and so $(\vec{x}', w, \ldots, w) \overset{k}{\sim} (\vec{x}', a'_1, \ldots, a'_n) \overset{k}{\sim} (\vec{y}', b'_1, \ldots, b'_n)$. By the isomorphism property of back-and-forth, this implies $b'_1 = \ldots = b'_n = b'$. Define the infinite set $I^+ = L \times \{b'\} \subseteq B \times B'$, and order it by: $(u_1 b') < (u_2 b')$ iff $u_1 < u_2$. Then

$$(u_1 b') < \ldots < (u_n b') \text{ implies } (u_1 \ldots u_n) \in [\vec{b}]^k_{\vec{y}}, \text{ and so}$$

$$((u_1 b') \ldots (u_n b')) \in [\vec{b} + \vec{b}']^k_\tau. \text{ This shows (5).}$$

CASE 2. Both J, J' are infinite.

Then we also have $[\vec{b}']^k_{\vec{y}'} \in R^n(B')$, by (3). Let $(L', <)$ be a linear order of type $(\omega, <)$ for which $u_1' < \ldots < u_n'$ implies $(u_1', \ldots, u_n') \in [\vec{b}']^k_{\vec{y}}$. Let $f: L \to L'$ be a one to one order preserving function and define $I^+ = \{(u, f(u)) \mid u \in L\}$ with the order $(u_1, f(u_1)) < (u_2, f(u_2))$ iff $u_1 < u_2$ iff $f(u_1) < f(u_2)$. If $(u_1, f(u_1)) < \ldots < (u_n, f(u_n))$ then $(u_1, \ldots, u_n) \in [\vec{b}]^k_{\vec{y}}$ and $(f(u_1), \ldots; f(u_n)) \in [\vec{b}']^k_{\vec{y}}$, which proves $((u_1, f(u_1)), \ldots (u_n, f(u_n)) \in [\vec{b} + \vec{b}']^k_{\tau}$ and shows (5) again. With this we finish the proof that

$$(\mathfrak{A} \times \mathfrak{A}'; C) \underset{S(Fin)}{\overset{\omega}{\sim}} (\mathfrak{B} \times \mathfrak{B}'; C)$$

Now apply Theorem 5.1. ■

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