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AN ALGEBRAIC APPROACH TO INTUITIONISTIC CONNECTIVES

XAVIER CAICEDO AND ROBERTO CIGNOLI

Abstract. It is shown that axiomatic extensions of intuitionistic propositional calculus defining univocally new connectives, including those proposed by Gabbay, are strongly complete with respect to valuations in Heyting algebras with additional operations. In all cases, the double negation of such a connective is equivalent to a formula of intuitionistic calculus. Thus, under the excluded third law it collapses to a classical formula, showing that this condition in Gabbay's definition is redundant. Moreover, such connectives can not be interpreted in all Heyting algebras, unless they are already equivalent to a formula of intuitionistic calculus. These facts relativize to connectives over intermediate logics. In particular, the intermediate logic with values in the chain of length n may be "completed" conservatively by adding a single unary connective, so that the expanded system does not allow further axiomatic extensions by new connectives.

§1. Introduction. If we consider intuitionistic and intermediate propositional calculi as logics with truth values in Heyting algebras, it is natural to consider new connectives for theses logics as operations in the algebras, univocally determined by their axioms, approach that we explore in this paper. Most of the proposed extensions of intuitionism by connectives have been introduced prima facie as deductive systems, before looking for a semantics for them. In particular, the proposal by Gabbay [6, 7] of a general definition of intuitionistic connective is given in deductive terms, and attempts to maintain the "intuitionistic character" of the corresponding axiomatic systems by asking that they be conservative over pure intuitionistic calculus, have the disjunction property, and collapse to classical calculus under the excluded third law. Among other results, we show that Gabbay's systems are strongly complete for their natural algebraic semantics.

There is a natural notion of intuitionistic connective for Kripke models, preserving the fundamental properties of Kripke semantics for intuitionism and yielding conservative extensions with the disjunction property (cf. [2]). These connectives correspond to certain operations in Heyting algebras of increasing sets of partial orders. More generally, we may consider the connectives of the logic of sheaves over a topological space X. That is, the morphisms of the *subobject classifier* Ω of the topos $\mathbb{S}h(X)$, which acts as the "object of truth values" for the inner logic of this topos in the same sense that $\{0, 1\}$ is the set of truth values for classical logic (cf. [5, 8]). As noticed in [3], these connectives are in correspondence with the operations f on the Heyting algebra of open subsets of X which satisfy the

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equation:

$$f(A_1, \dots, A_n) \wedge B = f(A_1 \wedge B, \dots, A_n \wedge B) \wedge B, \tag{1.1}$$

observation which extends to the logic of sheaves over any complete Heyting algebra.

We will see that this equation is related in arbitrary Heyting algebras to the notion of *affine completeness* studied in Universal Algebra (see [11], [12]). Moreover, in axiomatic extensions of intuitionistic calculus by a connective ∇, it corresponds to the validity of the axiom schema:

$$\bigwedge_{i=1}^{n} (\alpha_i \leftrightarrow \beta_i) \rightarrow (\nabla(\alpha_1, \dots, \alpha_n) \leftrightarrow \nabla(\beta_1, \dots, \beta_n)), \tag{1.2}$$

in turn, equivalent to strong completeness of the extensions with respect to their associated varieties of enriched Heyting algebras. This allows us to study intuitionistic connectives by algebraic means, since all axiomatic extensions determining univocally a connective, including those proposed by Gabbay, will be shown to contain the latter schema.[1]

In sections 2 and 3, we explore the algebraic meaning of equation (1.1), obtaining as a by-product simple proofs of the known affine completeness of boolean and finite Heyting algebras, and we consider the properties of operations implicitly defined by equations over a variety of Heyting algebras. In Section 4, we study axiomatically defined connectives for intuitionistic calculus and intermediate logics and show that they always satisfy schema (1.2). The double negation of any such connective must be equivalent to a formula of Heyting propositional calculus. Thus, under the excluded third law it collapses to a classical propositional formula, showing that this condition in Gabbay's definition is redundant. In the last two sections, we consider some examples, and show that the intermediate logic $\mathscr{L}_n$ with values in the chain of length n may be "completed" conservatively by adding a single unary connective S, so that any implicit connective of $\mathscr{L}_n + S$ is equivalent to a formula of this calculus.

§2. Compatible functions in Heyting algebras. We assume that the reader is familiar with the theory of Heyting algebras, also called pseudo-boolean algebras, and their relation with intuitionistic propositional calculus [4, 16, 17]. For more on Heyting algebras see also [1]. We will utilize $\rightarrow, \wedge, \vee, \neg, 0, 1$ for *relative pseudo-complement, meet, join, pseudo-complement, minimum*, and *maximum*, respectively; $x \leftrightarrow y$ will be used as an abbreviation for $(x \rightarrow y) \wedge (y \rightarrow x)$.

In general, H will denote a Heyting algebra. A term over the vocabulary $\tau = \{\neg, \wedge, \vee, \rightarrow, 0, 1\}$ will be called a *Heyting term*, and the function determined in H by a Heyting term t will be denoted t^H.

A n-ary *polynomial* of H is a function obtained by evaluating $m - n$ variables of t^H by fixed elements of H, for some m-ary term t $(m \geq n)$.

A function $f\colon H^n \rightarrow H$ is *compatible with a congruence relation* Θ of H if:

$$(x_i, y_i) \in \Theta \text{ for } i = 1, \dots, n \quad \text{implies} \quad (f(x_1, \dots, x_n), f(y_1, \dots, y_n)) \in \Theta.$$

[1]According to [21], this axiom appears in a definition of connective proposed by Novikov in the fifties.

f is a *compatible function of* H provided it is compatible with all congruence relations of H. This is equivalent to saying that the algebras H and $\langle H, f\rangle$ have the same congruences.

The simplest examples of compatible functions in a Heyting algebra H are the polynomial functions; in particular, all constant functions.

An algebra H is *affine complete* if any compatible function of H is given by a polynomial of H. It is *locally affine complete* provided that any compatible function is given by a polynomial on each finite subset of H. It is known that boolean algebras and finite Heyting algebras are affine complete [9], [15], [12, Cor. 3.6.1]. These facts appear as corollaries below.

Recall that the following relations hold in any Heyting algebra, due to the adjunction between $\wedge$ and $\to$:

$$x \wedge e = y \wedge e \text{ iff } e \leq x \leftrightarrow y \tag{2.1}$$

$$x \wedge (x \leftrightarrow y) = y \wedge (x \leftrightarrow y). \tag{2.2}$$

LEMMA 2.1. *The following are equivalent for any map* $f\colon H^n \to H$ *in a Heyting algebra* H.

a) $f(x_1, \dots, x_n) \wedge a = f(x_1 \wedge a, \dots, x_n \wedge a) \wedge a$, *for all* $x_i, a \in H$.
b) $\bigwedge_{i=1}^{n}(x_i \leftrightarrow y_i) \leq f(x_1, \dots, x_n) \leftrightarrow f(y_1, \dots, y_n)$, *for all* $x_i, y_i \in H$.
c) f *is a compatible function of* H.

PROOF. It is enough to consider unary functions. Assuming (a), then by (2.2): $f(x) \wedge (x \leftrightarrow y) = f(x \wedge (x \leftrightarrow y)) \wedge (x \leftrightarrow y) = f(y \wedge (x \leftrightarrow y)) \wedge (x \leftrightarrow y) = f(y) \wedge (x \leftrightarrow y)$. Therefore, from (2.1) follows (b): $x \leftrightarrow y \leq f(x) \leftrightarrow f(y)$. Since any congruence Θ on H is given by a filter F of H in the form: $x\Theta y$ iff $x \leftrightarrow y \in F$, the last inequality implies that f is compatible with Θ. Reciprocally, assuming (c), f must be compatible with the congruence associated to the principal filter $\langle (x \wedge a) \leftrightarrow x\rangle$. Hence, $a \leq (x \wedge a) \leftrightarrow x \leq f(x \wedge a) \leftrightarrow f(x)$, and so $f(x \wedge a) \wedge a = f(x) \wedge a$, by (2.1). ⊣

Condition (a) of the lemma is equivalent to the apparently stronger equation:

$$f(x_1, \dots, x_n) \wedge y_1 \wedge \dots \wedge y_n = f(x_1 \wedge y_1, \dots, x_n \wedge y_n) \wedge y_1 \wedge \dots \wedge y_n. \tag{2.3}$$

A compatible function which is not a polynomial is given in the next example (cf. [12, § 4.2]):

EXAMPLE 2.1. Let H be the totally-ordered Heyting algebra obtained by adding two new elements α, β to the set ω of natural numbers in such a way that $n < \alpha < \beta$ for each $n \in \omega$. Since the filter $\{\alpha, \beta\}$ is contained in every filter of H different from $\{\beta\}$, the following prescription

$$f(x) = \begin{cases} \alpha & \textit{if } x \textit{ is even or } x = \alpha, \\ \beta & \textit{if } x \textit{ is odd or } x = \beta, \end{cases}$$

defines a compatible function $f\colon H \to H$. On the other hand, given a $k+1$-variable Heyting term t and k elements $a_1, \dots, a_k$ of H, it is not hard to see that for sufficiently large $n \in \omega$, $t^H(n, a_1, \dots, a_k) = \alpha$ implies $t^H(n+1, a_1, \dots, a_k) = \alpha$. Hence, f can not coincide with a polynomial of H.

If $\{f_i\}_{i\in I}$ is a family of compatible functions of the same arity in H for which the join $f(x_1,\ldots,x_n) = \bigvee_i f_i(x_1,\ldots,x_n)$ exists for all $x_1,\ldots,x_n \in H$, then f satisfies condition (a) of the previous lemma because in a Heyting algebra $\wedge$ distributes over all existing joins. Hence, all existing joins of polynomials are compatible. The next theorem, which generalizes the *disjunctive normal form* of boolean algebras, shows that all compatible functions are joins of polynomials.

THEOREM 2.2. *Let $f : H^n \to H$ be a compatible function. Then for any subset $S \subseteq H$ and $x_1,\ldots,x_n \in S$:*

$$f(x_1,\ldots,x_n) = \bigvee_{(a_1,\ldots,a_n)\in S^n} f(a_1,\ldots,a_n) \wedge (x_1 \leftrightarrow a_1) \wedge \cdots \wedge (x_n \leftrightarrow a_n).$$

PROOF. Fix $(x_1,\ldots,x_n) \in S^n$. Then by (2.3) and (2.2), for all $a_1,\ldots,a_n \in S$:

$$\begin{aligned} &f(a_1,\ldots,a_n) \wedge (a_1 \leftrightarrow x_1) \wedge \cdots \wedge (a_n \leftrightarrow x_n) \\ &\qquad = f(x_1,\ldots,x_n) \wedge (a_1 \leftrightarrow x_1) \wedge \cdots \wedge (a_n \leftrightarrow x_n) \leq f(x_1,\ldots,x_n). \end{aligned}$$

Therefore $f(x_1,\ldots,x_n)$ is an upper bound of the set

$$T = \{f(a_1,\ldots,a_n) \wedge (a_1 \leftrightarrow x_1) \wedge \cdots \wedge (a_n \leftrightarrow x_n) \mid (a_1,\ldots,a_n) \in S^n\}.$$

But,

$$f(x_1,\ldots,x_n) = f(x_1,\ldots,x_n) \wedge (x_1 \leftrightarrow x_1) \wedge \cdots \wedge (x_n \leftrightarrow x_n) \in T.$$

Hence, $f(x_1,\ldots,x_n) = \max T$. ⊣

COROLLARY 2.3. *Any Heyting algebra is locally affine complete. Any finite Heyting algebra is affine complete.*

If, for a given a function $f : H^n \to H$, the join

$$\bar{f}(x_1,\ldots,x_n) = \bigvee_{(a_1,\ldots,a_n)\in H^n} f(a_1,\ldots,a_n) \wedge (x_1 \leftrightarrow a_1) \wedge \cdots \wedge (x_n \leftrightarrow a_n)$$

exists for all $x_1,\ldots,x_n$ in H, then $\bar{f}$ is the minimum compatible function above f. For example, let H_3 be the three-element chain $0 < a < 1$ endowed with its natural Heyting algebra structure and $f : H_3^2 \to H_3$ be the Łukasiewicz three-valued implication, which is not compatible, then $\bar{f}(x,y) = (x \to y) \vee a$. The functions f and $\bar{f}$ are shown in the following tables:

f	0	a	1
0	1	1	1
a	a	1	1
1	0	a	1

$\bar{f}$	0	a	1
0	1	1	1
a	a	1	1
1	a	a	1

A compatible function is always dense in a polynomial, as shown in the next theorem.

THEOREM 2.4. *For any n-ary compatible function f in a Heyting algebra H, there is a Heyting polynomial $p(x_1,\ldots,x_n)$ of H such that*

(2.4) $$p(x_1,\ldots,x_n) \leq f(x_1,\ldots,x_n) \leq \neg\neg p(x_1,\ldots,x_n).$$

More precisely, one may take:

$$p(x_1,\dots,x_n) = \bigvee_{(s_1,\dots,s_n)\in\{0,1\}^n} f(s_1,\dots,s_n)\wedge x_1^{s_1}\wedge\cdots\wedge x_n^{s_n},$$

where $x^1 = x$, $x^0 = \neg x$.

PROOF. By Theorem 2.2, $p(x_1,\dots,x_n) \leq f(x_1,\dots,x_n)$, since $x^1 = (x \leftrightarrow 1)$, $x^0 = (x \leftrightarrow 0)$. By (2.2) and (2.3),

$$\begin{aligned} p(x_1,\dots,x_n) &= \bigvee_{(s_1,\dots,s_n)\in\{0,1\}^n} f(x_1,\dots,x_n)\wedge x_1^{s_1}\wedge\cdots\wedge x_n^{s_n} \\ &= f(x_1,\dots,x_n)\wedge \bigvee_{(s_1,\dots,s_n)\in\{0,1\}^n} x_1^{s_1}\wedge\cdots\wedge x_n^{s_n}. \end{aligned}$$

Thus,

$$\begin{aligned} \neg\neg p(x_1,\dots,x_n) &= \neg\neg f(x_1,\dots,x_n)\wedge\neg\neg\Big(\bigvee_{(s_1,\dots,s_n)\in\{0,1\}^n} x_1^{s_1}\wedge\cdots\wedge x_n^{s_n}\Big) \\ &\geq f(x_1,\dots,x_n), \end{aligned}$$

considering that

$$\neg\neg\Big(\bigvee_{(s_1,\dots,s_n)\in\{0,1\}^n} x_1^{s_1}\wedge\cdots\wedge x_n^{s_n}\Big) = 1$$

in any Heyting algebra. ⊣

COROLLARY 2.5. [9] *Any boolean algebra is affine complete.*

The next two observations will be useful.

COROLLARY 2.6. *For any compatible function* f,

$$\neg\neg f(\neg\neg x_1,\dots,\neg\neg x_n) = \neg\neg f(x_1,\dots,x_n).$$

PROOF. From Theorem 2.4, $\neg\neg f = \neg\neg p$, and the statement holds for any Heyting polynomial, as may be shown by induction on complexity. ⊣

COROLLARY 2.7. *Let* $f\colon H \to H$ *be a compatible function such that* $f(1), f(0) \in \{1,0\}$. *Then either* $f(x) = \neg x$, $f(x) \equiv 0$, $f(x) \geq x\vee\neg x$, *or* $x \leq f(x) \leq \neg\neg x$.

PROOF. By Theorem 2.4 one of the next situations must hold:

$f(1)$	$f(0)$	
1	1	$x\vee\neg x \leq f(x) \leq \neg\neg(x\vee\neg x) = 1$
1	0	$x \leq f(x) \leq \neg\neg x$
0	1	$\neg x \leq f(x) \leq \neg\neg\neg x = \neg x$
0	0	$0 \leq f(x) \leq \neg\neg 0 = 0.$

⊣

EXAMPLE 2.2. Recall that an operator $\sim$ on a Heyting algebra is a *De Morgan negation* if it satisfies: $\sim\sim x = x$ and $\sim(x\vee y) = \sim x\wedge\sim y$ (cf. [1, 16]). It follows that $\sim 1 = 0$ and $\sim 0 = 1$. Thus, in case $\sim$ is compatible, Corollary 2.7 implies $\sim x = \neg x$. Hence,

> A *Heyting algebra admitting a compatible De Morgan negation* $\sim$ *must be a boolean algebra, where* $\sim x = \neg x$.

An operator $\Box$ is a *necessitation* if it satisfies $\Box x \leq x$ and $\Box 1 = 1$. If $\Box$ is compatible, the last equation and Corollary 2.7 imply $x \leq \Box x$. Hence,

> *The only compatible necessitation operator in a Heyting algebra is the identity.* (cf. [18, Prop. 4.1]).

§3. Equationally defined compatible operations on Heyting algebras. A set $E(f)$ of equations in the signature of Heyting algebras augmented with the n-ary function symbol f will be said to define an *implicit operation of Heyting algebras* if for any Heyting algebra H there is at most one function $f_H : H^n \to H$ such that $\langle H, f_H \rangle$ satisfies the universal closure of the equations in $E(f)$. f will be an *implicit compatible operation* provided all f_H are compatible. Beth's definability theorem guarantees that an implicit operation must be explicitly definable by a first order formula in the vocabulary of Heyting algebras. That does not mean that it has to be given by a Heyting term, even if it is compatible, as the following example illustrates.

EXAMPLE 3.1. The system $E(\gamma)$ consisting of the following three equations defines an implicit compatible operation $\gamma(x)$ of Heyting algebras, the smallest dense element above x.

C_1. $\neg\gamma(0) = 0$,
C_2. $\gamma(0) \to (x \vee \neg x) = 1$,
C_3. $\gamma(x) = x \vee \gamma(0)$.

Recall that an element x of a Heyting algebra H is *dense* if $\neg\neg x = 1$, and the dense elements form a filter of H. It should be clear that H has an element $\gamma(0)$ satisfying C_1 and C_2 if and only if the filter of dense elements of H is principal with generator $\gamma(0)$. This element exists in all finite Heyting algebras and, more generally, in atomic Heyting algebras where the supremum of the atoms exists. C_3 determines $\gamma(x)$ univocally as the smallest dense element above x, showing also that γ is a compatible function (being a polynomial). This operation is not expressible by a Heyting term, not even by an infinite combination of Heyting terms, because in the three-element chain $H_3 = \{0, a, 1\}$ we have $\gamma(0) = a$, while $t(0) \in \{0, 1\}$ for any Heyting term t.

The axioms of the operation $*$ on Heyting algebras introduced by Touraille in [20] imply compatibility and are satisfied by the operation γ. But Touraille's equations do not determine $*$ univocally since they are satisfied by the identity also.

EXAMPLE 3.2. The following set of equations $E(\rho)$ defines an implicit non compatible operation $\rho(x)$, the dual pseudo-complement (see [1, VIII.3]).

P_1. $x \vee \rho(x \vee y) = x \vee \rho(y)$,
P_2. $\rho(1) = 0$,
P_3. $\rho(0) = 1$.

Indeed, it is easy to check that any $\rho(x)$ satisfying $E(\rho)$ must be the least element of the set $\{y \in H : y \vee x = 1\}$. If ρ were compatible, we would have $\rho(x) = \neg x$ for each x by P_2, P_3 and Corollary 2.7. Then, by P_1, the equation $\neg x \vee x = 1$ would hold in any H where ρ exists, contradicting the fact that ρ is defined in all finite Heyting algebras. Not being compatible, ρ can not be expressible by a Heyting term.

The constant $\gamma(0)$ of Example 3.1 may be expressed as the infimum of the elements of the form $x \vee \neg x$. The next example shows that this kind of constant is not always determined by equations.

EXAMPLE 3.3. Let H be a Heyting algebra where $\delta^H = \bigwedge\{\neg x \vee \neg\neg x : x \in H\}$ exists. Then the stipulation $\Delta(x) := x \to \delta$ uniquely defines a polynomial function $\Delta^H : H \to H$. Despite its natural definition, Δ and δ can not be characterized by equations, because $\delta = \Delta(1)$ is not preserved by Δ-subalgebras. Indeed, if H is the finite Heyting algebra whose Hasse diagram is depicted in Figure 1, then $\delta^H = i$. Thus, $\Delta^H(x) = 1$ for $x \leq i$ and $\Delta^H(e) = \Delta^H(1) = i$. Hence, the Heyting subalgebra $S = \{0, i, 1\}$ is closed under Δ^H, but in this subalgebra $\delta^S = 1 > i$.

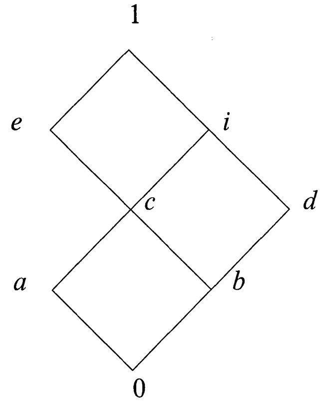

FIGURE 1

A compatible implicit operation does not need to exist in all Heyting algebras. The operation γ of Example 3.1 does not exists in the ordered real interval $[0, 1]$. This is a particular case of the next theorem, which shows that an implicit compatible operation can not exist in all Heyting algebras, unless it is explicitly definable by a Heyting term.

THEOREM 3.1. *If a system of equations $E(f)$ is satisfied by a unique n-ary compatible function f_H in each algebra H of a variety $\mathbb{V}$ of Heyting algebras, then there is a n-ary Heyting term $t(x_1, \ldots, x_n)$ such that $f_H = t^H$ in any $H \in \mathbb{V}$.*

PROOF. Let F be the free algebra of $\mathbb{V}$ on countable many generators $\overline{x}_n$, $n \in \omega$, and let $f_F(\overline{x}_1, \ldots, \overline{x}_n) = q^F(\overline{x}_1, \ldots, \overline{x}_n, \ldots, \overline{x}_{n+k})$, where $q(x_1, \ldots, x_n, \ldots, x_{n+k})$ is a Heyting term. Now, given a finite or countable Heyting algebra $H \in \mathbb{V}$ and $a_1, \ldots, a_n \in H$, let $h : F \to H$ be an onto homomorphism such that $h(\overline{x}_i) = a_i$ for $i = 1, \ldots, n$, and $h(\overline{x}_i) = a_n$ for $i = n, \ldots, k$. Since f_F is compatible with the kernel of h, the function $\widetilde{f} : H^n \to H$ given by

$$\widetilde{f}(h(t_1), \ldots, h(t_n)) = h(f_F(t_1, \ldots, t_n))$$

is well defined in H and, by construction, h is an homomorphism from $\langle F, f_F \rangle$ onto $\langle H, \widetilde{f} \rangle$.

Therefore, all positive universal sentences satisfied by the first algebra are satisfied by the second. In particular $\langle H, \widetilde{f}\rangle \models E(f)$. This shows that $\widetilde{f} = f_H$ does not depend on $a_1, \dots, a_n$ or h. Moreover, $f_H(a_1, \dots, a_n) = \widetilde{f}(h(\overline{x}_1), \dots, h(\overline{x}_n)) = h(f_F(\overline{x}_1, \dots, \overline{x}_n)) = h(q^F(\overline{x}_1, \dots, \overline{x}_n, \dots, \overline{x}_k)) = q^H(h(\overline{x}_1), \dots, h(\overline{x}_n), \dots, h(\overline{x}_{n+k})) = q^H(a_1, \dots, a_n, \dots, a_n)$. Hence, $f_H = t^H$, where $t(x_1, \dots, x_n) = q(x_1, \dots, x_n, \dots, x_n)$. Therefore t satisfies $E(f)$ in all finite or countable $H \in \mathbb{V}$. As $E(f)$ is at most countable this holds in any $H \in \mathbb{V}$ by the downward Löwenheim-Skolem theorem, and the identity $f_H = t^H$ follows by uniqueness. ⊣

If $E(f)$ defines implicitly a compatible operation, let $\mathbb{V}(E(f))$ be the variety of enriched Heyting algebras $\langle H, f_H\rangle$ satisfying $E(f)$. For each $A \in \mathbb{V}(E(f))$, let $\mathbf{H}(A)$ be the Heyting algebra reduct of A, and set

$$Red(\mathbb{V}(E(f))) = \{\mathbf{H}(A) \colon A \in \mathbb{V}(E(f))\}.$$

Clearly, this class is closed under products and, by compatibility of f, it is closed under homomorphic images. Moreover, Theorem 3.1 shows that if f is not given by a Heyting term, then $Red(\mathbb{V}(E(f)))$ can not be a variety and, by Birkhoff's theorem, it can not be closed under subalgebras. On the other hand, if $E(f) \models f = t$ for some Heyting term t then $Red(\mathbb{V}(E(f))) = \mathbb{V}(E(f/t))$. Thus:

Corollary 3.2. *An equationally defined implicit compatible operation of Heyting algebras is explicitly definable by a Heyting term if and only if the class of Heyting algebras where it exists is a variety (equivalently, it is closed under subalgebras).*

An algebra is *non trivial* if it has more than one element. A set of equations $E(f)$ is *non trivial* if $\mathbb{V}(E(f))$ contains non trivial algebras.

Theorem 3.3. *If $E(f)$ is non trivial and defines an implicit compatible operation f of Heyting algebras, then there is a Heyting term t such that $\neg\neg f = t$ in any algebra where f is defined. Moreover, f is defined in all boolean algebras.*

Proof. If $\langle H, f\rangle \in \mathbb{V}(E(f))$ is non trivial, with f compatible, a maximal congruence of H is also a congruence of $\langle H, f\rangle$ and yields a quotient $\langle H_2, \widetilde{f}\rangle$, where H_2 is the two elements boolean algebra and $\widetilde{f}$ satisfies $E(f)$. By functional completeness of this algebra, $\widetilde{f}$ coincides with a boolean term u in H_2. Since u satisfies the equations of $E(f)$ in H_2, it must satisfy them in all boolean algebras. On the other hand, $\neg\neg f_H$ satisfies $E(f)$ in the boolean algebra $Reg(H)$ of regular elements of any Heyting algebra H where it is defined, because the onto homomorphism of Heyting algebras $\neg\neg \colon H \to Reg(H)$ is also a homomorphism from $\langle H, f_H\rangle$ onto $(Reg(H), \neg\neg f_H)$ by Corollary 2.6. By uniqueness, $\neg\neg f_H \restriction Reg(H) = u^{Reg(H)}$. But $u^{Reg(H)} = \neg\neg u^H \restriction Reg(H)$, because of the interpretation of $\vee$ in $Reg(H)$ as $\neg\neg(a \vee b)$ in H. By regularity of $\neg\neg x$ and Corollary 2.6 again, $\neg\neg f_H(\mathbf{x}) = \neg\neg f_H(\neg\neg\mathbf{x}) = \neg\neg u^H(\neg\neg\mathbf{x}) = \neg\neg u^H(\mathbf{x})$ for any $\mathbf{x} \in H^n$. Take $t = \neg\neg u$. ⊣

In spite of Theorem 2.4, it is not possible to improve the previous theorem to have $t \leq f \leq \neg\neg t$. There is no Heyting term $t(x)$ such that $t(x) \leq \gamma(x) \leq \neg\neg t(x)$ in all algebras $\langle H, \gamma_H\rangle$, for the operation γ introduced in Example 3.1. Indeed, in the three-element chain H_3, we have $\gamma(0) = a$, while $t(0) = \neg\neg t(0) \in \{0, 1\}$ for any Heyting term t.

§4. Axiomatic extensions of intuitionistic calculus by implicit connectives. The language L of *formulas* of the intuitionistic propositional calculus is built in the usual way from the connective symbols $\rightarrow$, $\wedge$, $\vee$, $\neg$, corresponding to *implication, conjunction, disjunction and negation*, respectively, and the propositional variables π_i, $i = 0, 1, \ldots$ As in the language of Heyting algebras, $\varphi \leftrightarrow \psi$ will stand for an abbreviation of $(\varphi \rightarrow \psi) \wedge (\psi \rightarrow \varphi)$.

Given a set Φ of new connective symbols (of arbitrary arities), $L(\Phi)$ will denote the propositional language obtained by allowing the symbols of Φ in the formation rules of formulas. We write $L(\nabla)$ for $L(\{\nabla\})$.

To each set of formulas $\mathscr{A}(\Phi) \subseteq L(\Phi)$, associate the *axiomatic system* having $\mathscr{A}(\Phi) \cup Int$ for axiom schemas, where *Int* is a complete system of schemas for intuitionistic propositional calculus (as given for example in [16, 17]), and substitution in axiom schemas and *Modus Ponens* as only rules. Only this kind of systems will be considered. If Φ is empty and $\mathscr{A}$ is consistent, the system is an *intermediate logic*.

Given $\Gamma \cup \{\varphi\} \subseteq L(\Phi)$, the notation

$$\Gamma \vdash_{\mathscr{A}(\Phi)} \varphi$$

will indicate that φ is deducible from Γ in this calculus. We write $\vdash_{\mathscr{A}(\Phi)} \varphi$ if $\Gamma = \emptyset$, and $\Gamma \vdash \varphi$ for deducibility in pure intuitionistic calculus. It is immediate that the *deduction theorem* is satisfied:

$$\Gamma \cup \{\alpha\} \vdash_{\mathscr{A}(\Phi)} \varphi \text{ implies } \Gamma \vdash_{\mathscr{A}(\Phi)} \alpha \rightarrow \varphi,$$

Each formula $\varphi \in L(\Phi)$ may be seen as a term in the type $\tau \cup \Phi$ of Heyting algebras enlarged with the set Φ of operation symbols, in the variables π_i. Therefore, to each extension $\mathscr{A}(\Phi)$ of intuitionistic calculus we may associate the system of equations $E(\Phi) = \{\varphi = 1 : \varphi \in \mathscr{A}(\Phi) \cup Int\}$, and the corresponding variety of Heyting algebras

$$\mathbb{V}(\mathscr{A}(\Phi)) = \mathbb{V}(E(\Phi)),$$

Since $L(\Phi)$ with the syntactical operations is the absolute free algebra of type $\tau \cup \Phi$ on the set of propositional variables $\Pi = \{\pi_1, \pi_2, \ldots\}$, any function $v : \Pi \rightarrow$ Domain(A) with $A \in \mathbb{V}(\mathscr{A}(\Phi))$ (called an *A-valuation*) may be extended to a unique homomorphism $\overline{v} : L(\Phi) \rightarrow A$. Then we may define for any set $\Gamma \cup \{\varphi\} \subseteq L(\Phi)$ an *algebraic consequence relation* as follows.

DEFINITION 4.1. $\Gamma \Vdash_{\mathscr{A}(\Phi)} \varphi$ if and only if for any $A \in \mathbb{V}(\mathscr{A}(\Phi))$ and A-valuation v: $\overline{v}(\gamma) = 1$ *for all* $\gamma \in \Gamma$ *implies* $\overline{v}(\varphi) = 1$.

It is easy to check, by induction on the length of proofs, that $\vdash_{\mathscr{A}(\Phi)}$ is *sound* with respect to this semantics. That is,

$$\Gamma \vdash_{\mathscr{A}(\Phi)} \varphi \text{ implies } \Gamma \Vdash_{\mathscr{A}(\Phi)} \varphi. \tag{4.1}$$

In particular, $\vdash_{\mathscr{A}(\Phi)} \varphi$ implies that $\varphi = 1$ is an equation of the variety $\mathbb{V}(\mathscr{A}(\Phi))$. The reciprocal of (4.1), *strong algebraic completeness* of $\vdash_{\mathscr{A}(\Phi)}$, is not generally true. The following result characterizes those extensions for which it holds.

THEOREM 4.1. *The following conditions are equivalent for any* $\mathscr{A}(\Phi) \subseteq L(\Phi)$:

(1) $\vdash_{\mathscr{A}(\Phi)}$ *is strongly complete for the algebraic consequence relation. That is,* $\Gamma \Vdash_{\mathscr{A}(\Phi)} \varphi$ *implies* $\Gamma \vdash_{\mathscr{A}(\Phi)} \varphi$, *for any* $\Gamma \cup \{\varphi\} \subseteq L(\Phi)$.

(2) $\vdash_{\mathscr{A}(\Phi)} \bigwedge_{i=1}^{n}(\alpha_i \leftrightarrow \beta_i) \rightarrow (\nabla(\alpha_1,\dots,\alpha_n) \leftrightarrow \nabla(\beta_1,\dots,\beta_n))$, *for each* $\nabla \in \Phi$.

PROOF. ($1 \Rightarrow 2$) If $\overline{v}(\bigwedge_{i=1}^{n}(\alpha_i \leftrightarrow \beta_i)) = 1$ for some A-valuation v, then $\overline{v}(\alpha_i) = \overline{v}(\beta_i)$ for $i = 1,\dots,n$, and so $\overline{v}(\nabla(\alpha_1,\dots,\alpha_n) \leftrightarrow \nabla(\beta_1,\dots,\beta_n)) = \nabla^A(\overline{v}(\alpha_1),\dots,\overline{v}(\alpha_n)) \leftrightarrow \nabla^A(\overline{v}(\beta_1),\dots,\overline{v}(\beta_n)) = 1$, for each $\nabla \in \Phi$. Therefore,

$$\bigwedge_{i=1}^{n}(\alpha_i \leftrightarrow \beta_i) \Vdash_{\mathscr{A}(\Phi)} \nabla(\alpha_1,\dots,\alpha_n) \leftrightarrow \nabla(\beta_1,\dots,\beta_n).$$

From the strong completeness hypothesis and the deduction theorem we conclude (2).

($2 \Rightarrow 1$) Since $\vdash_{\mathscr{A}(\Phi)}$ includes the rules of the intuitionistic calculus, given $\Gamma \subseteq L(\Phi)$ and $\Theta = \{(\alpha,\beta)\colon \Gamma \vdash_{\mathscr{A}(\Phi)} \alpha \leftrightarrow \beta\}$, then $L(\Phi)/\Theta$ is a Heyting algebra. Denoting by $[\alpha]$ the equivalence class of $\alpha \in L(\Phi)$, (2) implies that the operation

$$f_\nabla([\alpha_1],\dots,[\alpha_n]) = [\nabla(\alpha_1,\dots,\alpha_n)]$$

is well defined for each $\nabla \in \Phi$, and thus $\langle L(\Phi)/\Theta, F\rangle$ becomes an algebra of $\mathbb{V}(\mathscr{A}(\Phi))$, where $F = \{f_\nabla\colon \nabla \in \Phi\}$. The $\langle L(\Phi)/\Theta, F\rangle$-valuation $v(\pi_i) = [\pi_i]$ extends to $\overline{v}(\alpha) = [\alpha]$ for any $\alpha \in L(\Phi)$. Therefore, $\overline{v}(\alpha) = 1 = [\pi_1 \rightarrow \pi_1]$ if and only if $\Gamma \vdash_{\mathscr{A}(\Phi)} \alpha \leftrightarrow (\pi_1 \rightarrow \pi_1)$, if and only if, $\Gamma \vdash_{\mathscr{A}(\Phi)} \alpha$. It follows that $\overline{v}(\psi) = 1$ for all $\psi \in \Gamma$. Hence, $\Gamma \Vdash_{\mathscr{A}(\Phi)} \varphi$ implies $\overline{v}(\varphi) = 1$; that is, $\Gamma \vdash_{\mathscr{A}(\Phi)} \varphi$. ⊣

DEFINITION 4.2. A set of formulas $\mathscr{A}(\nabla)$ will be said to define axiomatically a connective ∇ provided that

$$\vdash_{\mathscr{A}(\nabla)\cup\mathscr{A}(\nabla')} \nabla(\pi_1,\dots,\pi_n) \leftrightarrow \nabla'(\pi_1,\dots,\pi_n),$$

where ∇' is a new n ary connective symbol and $\mathscr{A}(\nabla') = \{\varphi(\nabla/\nabla')\colon \varphi \in \mathscr{A}(\nabla)\}$.

THEOREM 4.2. *If* $\mathscr{A}(\nabla)$ *defines axiomatically a connective, then it satisfies condition* (2) *in Theorem* 4.1.

PROOF. For notational simplicity we consider just the unary case. Given two fixed propositional variables $\mathbf{x}$ and $\mathbf{y}$ define by simultaneous induction two transformations $*$ and $+$ from $L(\nabla,\nabla')$ into $L(\nabla)$ as follows:

$\varphi^* = \mathbf{y}$ if $\varphi = \mathbf{x}$, and $\varphi^* = \varphi$ for other propositional variables,
$[\varphi \oplus \psi]^* = \varphi^* \oplus \psi^*$ for $\oplus = \wedge, \vee, \rightarrow$,
$[\neg\varphi]^* = \neg\varphi^*$,
$[\nabla\varphi]^* = \nabla\varphi^+$,
$[\nabla'\varphi]^* = \nabla\varphi^*$.

$\varphi^+ = \varphi$, for propositional variables,
$[\varphi \oplus \psi]^+ = \varphi^+ \oplus \psi^+$ for $\oplus = \wedge, \vee, \rightarrow$,
$[\neg\varphi]^+ = \neg\varphi^+$,
$[\nabla\varphi]^+ = \nabla\varphi^+$,
$[\nabla'\varphi]^+ = \nabla\varphi^*$.

Informally, to obtain φ^* change each occurrence of $\mathbf{x}$ in φ to $\mathbf{y}$, except those occurring under the *immediate* scope of ∇ (that is, occurring in a subformula $\nabla\theta$ of φ but not in a subformula $\nabla'\mu$ of $\nabla\theta$). Then, change all ∇' to ∇. In particular, the occurrences of $\nabla\mathbf{x}$ are not changed. But the occurrences of $\nabla'\mathbf{x}$ are changed to $\nabla\mathbf{y}$.

Claim 1: $\mathbf{x} \leftrightarrow \mathbf{y} \vdash \varphi^+ \leftrightarrow \varphi^*$.

By an easy induction on the complexity of $\varphi \in L(\nabla, \nabla')$. Indeed, for φ atomic we have two trivial cases: $\mathbf{x} \leftrightarrow \mathbf{y} \vdash \mathbf{x} \leftrightarrow \mathbf{y}$ and $\mathbf{x} \leftrightarrow \mathbf{y} \vdash \varphi \leftrightarrow \varphi$. For a Heyting connective the induction step follows from intuitionistic rules, and for ∇ and ∇' it is trivial since by definition $[\nabla'\varphi]^* = [\nabla'\varphi]^+$ and $[\nabla\varphi]^* = [\nabla\varphi]^+$.

Claim 2: A proof of $\vdash_{\mathscr{A}(\nabla)\cup\mathscr{A}(\nabla')} \varphi$ *may be transformed into a proof of* $\mathbf{x} \leftrightarrow \mathbf{y} \vdash_{\mathscr{A}(\nabla)} \varphi^*$.

By induction on the length of the proof. Let $\mathscr{A}(\nabla)_{\nabla,\nabla'}$ be the set of substitutions of formulas of $L(\nabla, \nabla')$ in schemas of $\mathscr{A}(\nabla)$, and $\mathscr{A}(\nabla)_\nabla$ the set of substitutions of formulas of $L(\nabla)$ in schemas of $\mathscr{A}(\nabla)$. Define similarly $\mathscr{A}(\nabla')_{\nabla,\nabla'}$. Now assume $\vdash_{\mathscr{A}(\nabla)\cup\mathscr{A}(\nabla')} \varphi$. If φ is a intuitionistic axiom, then φ^* is a intuitionistic axiom because $*$ respects the Heyting formula structure. If φ is an axiom in $\mathscr{A}(\nabla')_{\nabla,\nabla'}$, then $\varphi = \theta(\nabla', \pi_1, \dots, \pi_k)[\pi_i/\psi_i]$ with $\theta \in \mathscr{A}(\nabla')$ and $\psi_i \in L(\nabla, \nabla')$. So $\varphi^* = \theta(\nabla, \pi_1, \dots, \pi_k)[\pi_i/\psi_i^*] \in \mathscr{A}(\nabla)_\nabla$ because $*$ does not change the formula $\theta(\nabla', \pi_1, \dots, \pi_k)$, except for changing ∇' to ∇. Therefore, $\vdash_{\mathscr{A}(\nabla)} \varphi^*$. By a similar argument, if φ is an axiom in $\mathscr{A}(\nabla)_{\nabla,\nabla'}$, then φ^+ is an axiom in $\mathscr{A}(\nabla)_\nabla$. Therefore, $\vdash_{\mathscr{A}(\nabla)} \varphi^+$, and so, $\mathbf{x} \leftrightarrow \mathbf{y} \vdash_{\mathscr{A}(\nabla)} \varphi^*$ by Claim 1. If $\vdash_{\mathscr{A}(\nabla)\cup\mathscr{A}(\nabla')} \varphi$ follows by Modus Ponens from $\vdash_{\mathscr{A}(\nabla)\cup\mathscr{A}(\nabla')} \psi$ and $\vdash_{\mathscr{A}(\nabla)\cup\mathscr{A}(\nabla')} \psi \to \varphi$, we have by induction hypothesis $\mathbf{x} \leftrightarrow \mathbf{y} \vdash_{\mathscr{A}(\nabla)} \psi^*$ and $\mathbf{x} \leftrightarrow \mathbf{y} \vdash_{\mathscr{A}(\nabla)} (\psi \to \varphi)^* = \psi^* \to \varphi^*$, and so $\mathbf{x} \leftrightarrow \mathbf{y} \vdash_{\mathscr{A}(\nabla)} \varphi^*$.

Finally, let φ be the formula $\nabla\mathbf{x} \leftrightarrow \nabla'\mathbf{x}$. Then φ^* is: $(\nabla\mathbf{x})^* \leftrightarrow (\nabla'\mathbf{x})^* = \nabla(\mathbf{x})^+ \leftrightarrow \nabla(\mathbf{x})^* = \nabla\mathbf{x} \leftrightarrow \nabla\mathbf{y}$, and the theorem follows from Claim 2 and the hypothesis. The same idea works for the n-ary case if one defines:

$[\nabla(\varphi_1, \dots, \varphi_n)]^* = \nabla(\varphi_1^+, \dots, \varphi_n^+)$,

$[\nabla'(\varphi_1, \dots, \varphi_n)]^+ = \nabla(\varphi_1^*, \dots, \varphi_n^*)$. ⊣

From Lemma 2.1, and Theorems 4.1 and 4.2, we obtain:

COROLLARY 4.3. *If a set of formulas $\mathscr{A}(\nabla)$ defines axiomatically a connective then the corresponding set of equations defines an implicit compatible operation of Heyting algebras. Moreover, the system $\vdash_{\mathscr{A}(\nabla)}$ is strongly complete.*

By Theorem 3.3 we have, using completeness:

COROLLARY 4.4. *If $\mathscr{A}(\nabla)$ defines axiomatically a n-ary connective ∇, then there is a formula $\varphi(\pi_1, \dots, \pi_n) \in L$ such that*

$$\vdash_{\mathscr{A}(\nabla)} \neg\neg\nabla(\pi_1, \dots, \pi_n) \leftrightarrow \varphi(\pi_1, \dots, \pi_n).$$

Therefore, in the context of classical propositional calculus, ∇ collapses to a classical propositional formula. More precisely,

$$\vdash_{\mathscr{A}(\nabla)\cup\{\alpha\vee\neg\alpha\}} \nabla(\pi_1, \dots, \pi_n) \leftrightarrow \varphi(\pi_1, \dots, \pi_n). \tag{4.2}$$

Recall that $\mathscr{A}(\nabla)$ is a *conservative extension* of intuitionistic calculus or, more generally, of an intermediate logic **I**, if $\vdash_{\mathscr{A}(\nabla)}$ **I**, and $\vdash_{\mathscr{A}(\nabla)} \varphi$ implies $\vdash_{\mathbf{I}} \varphi$ for any $\varphi \in L$. This is not a restrictive condition because any consistent set of axiom schemas $\mathscr{A}(\nabla)$ is a conservative extension of a unique intermediate logic, namely the logic $\mathbf{I}(\mathscr{A}(\nabla)) = \{\phi \in L \colon \vdash_{\mathscr{A}(\nabla)} \phi\}$.

DEFINITION 4.3. If $\mathscr{A}(\nabla)$ defines axiomatically a connective ∇ and it is a conservative extension of an intermediate logic **I**, then we say that ∇ is an *implicit* *connective* of **I**. If, in addition,

$$\nvdash_{\mathscr{A}(\nabla)} \nabla(\pi_1, \dots, \pi_n) \leftrightarrow \varphi, \quad \text{for any } \varphi \in L,$$

then we say that ∇ is a *new* implicit connective of **I**.

Notice that Corollary 4.4 rules out the existence of new implicit connectives for classical propositional calculus.

Due to strong completeness, the conservativity condition in Definition 4.3 means that $Red(\mathbb{V}(\mathscr{A}(\nabla)))$ and $\mathbb{V}(\mathbf{I})$ satisfy the same equations and thus the class of reducts generates the variety. That is,

$$\mathbb{V}(\mathbf{I}) = HS(Red(\mathbb{V}(\mathscr{A}(\nabla)))),$$

because $Red(\mathbb{V}(\mathscr{A}(\nabla)))$ is closed under products. Due to compatibility of ∇ we may improve this to:

$$\mathbb{V}(\mathbf{I}) = S(Red(\mathbb{V}(\mathscr{A}(\nabla)))). \tag{4.3}$$

Indeed, if $H = H'/F$ for a filter F in $H' \leq H''$ with $\langle H'', f_\nabla\rangle \in \mathbb{V}(\mathscr{A}(\nabla))$, then F may be extended to F'' in H'' so that $F = F'' \cap H'$. Thus, $H'/F = H'/(F'' \cap H')$ is isomorphic to a subalgebra of H''/F''. But $\langle H'', f_\nabla\rangle/F'' \in \mathbb{V}(\mathscr{A}(\nabla))$ by compatibility of f_∇.

THEOREM 4.5. *If $\mathscr{A}(\nabla)$ defines an implicit connective of an intermediate logic **I**, then ∇ is new if and only if it is not defined in all algebras of $\mathbb{V}(\mathbf{I})$.*

PROOF. By Corollary 3.2 and strong completeness, ∇ is new if and only if $Red(\mathbb{V}(\mathscr{A}(\nabla)))$ is not a variety of Heyting algebras. After (4.3), this means $Red(\mathbb{V}(\mathscr{A}(\nabla))) \subsetneq \mathbb{V}(\mathbf{I})$. ⊣

In the case of pure intuitionistic propositional calculus, Definition 4.3 of a new implicit connective includes three of the conditions in Gabbay's definition of *intuitionistic connective* [7]; namely: uniqueness, conservativity, and being new. In addition, Gabbay requires condition (4.2) of Corollary 4.4, which is obviously redundant, and the *disjunction property*:

$$\vdash_{\mathscr{A}(\nabla)} \varphi \vee \psi \text{ implies } \vdash_{\mathscr{A}(\nabla)} \varphi \text{ or } \vdash_{\mathscr{A}(\nabla)} \psi.$$

This last property can not be required in general if we wish to consider connectives over arbitrary intermediate logics. We do not know if it is automatically inherited by the implicit connectives of pure intuitionistic calculus.

It should be clear that the disjunction property holds if and only if 1 is join-irreducible in the free algebras of the variety $\mathbb{V}(\mathscr{A}(\nabla))$. We use this fact in the next examples.

§5. Some examples. Corollary 4.3 shows that the dual pseudo-complement ρ considered in Example 3.2 can not be defined axiomatically, in spite of the fact that it is determined univocally by equations. No sound axiomatic system for this connective consisting of axiom schemas and Modus Ponens only may be complete, or prove uniqueness of ρ.

Other conservative axiomatic extensions of intuitionistic calculus by connectives found in the literature contain schema (1.2), and so they are strongly complete for algebraic semantics, but do not satisfy the uniqueness property of Definition 4.2.

For instance, the connective C introduced by Kaminski in [13] admits at least two different sound interpretations: the identity and double negation.

Let us consider more positive examples.

EXAMPLE 5.1. The following axiom system $\mathscr{A}(\gamma)$ defines a new implicit connective of intuitionistic calculus with the disjunction property. Hence, an intuitionistic connective in the sense of Gabbay.

C_1. $\neg\neg\gamma\alpha$

C_2. $\alpha \to \gamma\alpha$,

C_3. $\gamma\alpha \to (\alpha \vee \beta \vee \neg\beta)$,

C_4. $(\alpha \to \beta) \to (\gamma\alpha \to \gamma\beta)$.

We let the reader check that the corresponding equational system $E(\gamma)$ is equivalent to the one considered in Example 3.1. Then, by C_4 and Theorem 4.1, we know that γ is axiomatically defined by $\mathscr{A}(\gamma)$ and interpretable exactly in those algebras where the minimum dense exists. To show that this is a conservative extension of intuitionistic calculus, assume that $\vdash_{\mathscr{A}(\gamma)} \varphi$, where φ does not contain γ. Then φ holds in all finite Heyting algebras and thus $\vdash \varphi$, by the finite model property of intuitionistic propositional calculus. Since γ does not exist in all Heyting algebras, $\mathscr{A}(\gamma)$ defines a new implicit connective of intuitionistic calculus by Theorem 4.5.

The following algebraic argument shows that $\vdash_{\mathscr{A}(\gamma)}$ has the disjunction property. Given a Heyting algebra H, denote by H' the Heyting algebra obtained by adding a new greatest element $1'$ to H (see, for instance, [1]). It is clear that $\gamma(0)$ is defined in H' whenever $\gamma(0)$ is defined in H. Moreover, the prescription

$$f_H(x) = \begin{cases} x & \text{for } x \in H, \\ 1 & \text{for } x = 1', \end{cases}$$

defines a homomorphism from $\langle H', \gamma\rangle$ onto $\langle H, \gamma\rangle$. Let F be a free algebra in $\mathbb{V}(\mathscr{A}(\gamma))$. It is easy to check that the identity $id_F\colon F \to F$ can be lifted to a homomorphism $h\colon F \to F'$ in such a way that $f_F h = id_F$. If a, b are elements of F such that $a \vee b = 1$, then $h(a) \vee h(b) = 1'$. Since $1'$ is join-irreducible in F' we have $h(a) = 1'$ or $h(b) = 1'$. Hence $a = f_F(h(a)) = 1$ or $b = f_F(h(b)) = 1$. This shows that 1 is join-irreducible in F.

After completing the first draft of this paper, we learned that the constant $\gamma(0)$ has been already proposed by Smetanich as an example of a new intuitionistic connective in the sense of Novikov (see [21]).

EXAMPLE 5.2. The following set $\mathscr{A}(S)$ of schemas also defines a new implicit connective of intuitionistic calculus satisfying the disjunction property:

S_1. $\alpha \to S\alpha$

S_2. $S\alpha \to (\beta \vee (\beta \to \alpha))$

S_3. $(S\alpha \to \alpha) \to \alpha$.

Indeed, $S\alpha \vdash_{S_2} S'\alpha \vee (S'\alpha \to \alpha) \vdash_{S_3'} S'\alpha \vee \alpha \vdash_{S_1'} S'\alpha \vee S'\alpha \vdash S'\alpha$, and so $\vdash_{\mathscr{A}(S)\cup\mathscr{A}(S')} S\alpha \to S'\alpha$, proving uniqueness. The other properties may be verified by algebraic means. Taking into account that in a Heyting algebra $x \leq y$ iff $x \to y = 1$, the corresponding system of equations $E(S)$ can be expressed as follows:

E_1. $x \leq S(x)$
E_2. $S(x) \leq y \vee (y \to x)$,
E_3. $S(x) \to x = x$.

There is a (necessarily unique) operation satisfying E_1, E_2, E_3 in each complete well founded Heyting algebra H. For each $x \in H$, define $s(x) := \{p \in H : y < p$ implies $y \leq x\}$ and set $S(x) := \bigvee s(x)$. Since $x \in s(x)$, we have $x \leq S(x)$ and condition E_1 holds for S. To prove E_2, suppose $S(x) \not\leq y \vee (y \to x)$. Then $p \not\leq y \vee (y \to x)$ for some $p \in s(x)$. Therefore, $p \not\leq y$ and $p \not\leq y \to x$. The first inequality implies $p \wedge y < p$ and so $p \wedge y \leq x$ (since $p \in s(x)$). The second inequality implies $p \wedge y \not\leq x$, a contradiction. To prove E_3, suppose that for some $x \in H$, $S(x) \to x \not\leq x$. Since H is well founded, we may take $p \in H$ minimal such that $p \leq S(x) \to x$ and $p \not\leq x$; then $p \not\leq S(x)$. If $q < p$, then trivially $q \leq S(x) \to x$, and by minimality we must have $q \leq x$. Therefore $p \in s(x)$ and so $p \leq S(x)$, a contradiction.

It follows that in a finite chain H_n, endowed with its natural Heyting algebra structure,

$$S(x) = \begin{cases} x^+ & \text{if } 0 \leq x < 1 \\ 1 & \text{if } x = 1. \end{cases} \tag{5.1}$$

where x^+ denotes the *successor* of x.

As a matter of fact, S exists in a chain if and only if each element distinct from 1 has an immediate successor, in which case S is defined by (5.1). Indeed, if $0 \leq x < 1$, then $S(x) \to x = x < 1$. Therefore $x < S(x)$. Now, if $x < y$, then $S(x) \leq y \vee (y \to x) = y \vee x = y$. This shows that $S(x)$ is the immediate successor of x.

Since S exists in all finite Heyting algebras, conservativity over intuitionistic calculus will follow as in Example 5.1. Since S does not exists in [0, 1], we get from Theorem 4.5 that S is a new implicit connective of intuitionistic calculus.

To prove the disjunction property, suppose that S is defined on a Heyting algebra H and let H' be the Heyting algebra obtained by adding a new top element $1'$ to H. Then it is easy to check that the prescription $S(1') = 1'$ extends S to H' in such a way that conditions $E_1 - E_3$ are preserved. Then, argue as in Example 5.1.

EXAMPLE 5.3. Gabbay shows in [6, 7] that the following schemas satisfy his definition of an intuitionistic connective and have a complete semantics in finite Kripke models (it is proven in [21] that G_2 is a consequence of the other axioms).

G_1. $G\alpha \to (\beta \vee (\beta \to \alpha))$,
G_2. $(\alpha \to \beta) \to (G\alpha \to G\beta)$,
G_3. $\alpha \to G\alpha$,
G_4. $G\alpha \to \neg\neg\alpha$,
G_5. $(G\alpha \to \alpha) \to (\neg\neg\alpha \to \alpha)$.

This connective, as well as the connective γ of Example 5.1, are definable from S. The reader may verify easily that if we set

$$G\alpha := S\alpha \wedge \neg\neg\alpha$$

then we may deduce the axioms of G in $\mathscr{A}(S)$. Only G_5, that takes the form

$$((S\alpha \wedge \neg\neg\alpha) \to \alpha) \to (\neg\neg\alpha \to \alpha),$$

needs a little checking. Indeed, $(S\alpha \wedge \neg\neg\alpha) \to \alpha \vdash \neg\neg\alpha \to (S\alpha \to \alpha) \vdash_{S_3} \neg\neg\alpha \to \alpha$; then apply the deduction theorem. Similarly, the definition

$$\gamma\alpha := \alpha \vee S(\alpha \wedge \neg\alpha)$$

allows us to prove easily from $\mathscr{A}(S)$ the axioms of γ.

S is not definable from G or γ because the Heyting subalgebra $\{0, 1\}$ of the chain H_3 is closed under G but not under S, and the Heyting subalgebra $\{0, 0^+, 1\}$ of H_4 is closed under γ but not under S. Similarly, one may show that G and γ are not mutually definable. However, S is definable from G and γ together, since setting:

$$S\alpha := \gamma\alpha \vee G\alpha$$

allows us to deduce the axioms of S in the system in $\mathscr{A}(\gamma) \cup \mathscr{A}(G)$. For example, axiom S_3 becomes:

$$((\gamma\alpha \vee G\alpha) \to \alpha) \to \alpha.$$

By pure intuitionistic calculus: $(\gamma\alpha \vee G\alpha) \to \alpha \vdash (\gamma\alpha \to \alpha) \wedge (G\alpha \to \alpha)$. On the other hand, $\gamma\alpha \to \alpha \vdash \neg\neg\gamma\alpha \to \neg\neg\alpha \vdash_{C_1} \neg\neg\alpha$, and so $(\gamma\alpha \vee G\alpha) \to \alpha \vdash_{\mathscr{A}(\gamma)} \neg\neg\alpha \wedge (G\alpha \to \alpha) \vdash_{\mathscr{A}(G)} \alpha$.

From the algebraic point of view, this means that the varieties $\mathbb{V}(\mathscr{A}(S))$ and $\mathbb{V}(\mathscr{A}(\gamma) \cup \mathscr{A}(G))$ are mutually interpretable.

§6. The implicit connectives of intuitionistic n-valued logic. Let $\mathscr{L}_n$ be an axiomatization of the intermediate logic with values in H_n, the Heyting chain of length n, $n \geq 3$. For instance, we can add to *Int* the following axiom schemas ([10], see [19] for a different axiomatization):

$$(\phi \to \psi) \vee (\psi \to \phi),$$
$$(\phi_1 \to \phi_2) \vee \cdots \vee (\phi_n \to \phi_{n+1}).$$

Heyting three-valued logic, $\mathscr{L}_3$, may be axiomatized alternatively by adding the single axiom: $((x \to y) \to z) \to (((z \to x) \to z) \to z)$ (cf. [14]).

We show next that all implicit connectives of $\mathscr{L}_n$ are generated by the single connective S of Example 5.2. In fact, the logic $\mathscr{L}_n + S$, given by the union of $\mathscr{L}_n$ and the axiom system $\mathscr{A}(S)$ for the connective S, does not admit extensions by new implicit connectives, even if we allow S to appear in the new axioms.

S is new over $\mathscr{L}_n$ for $n \geq 3$ because the Heyting subalgebra $\{0, 1\}$ of H_n is not closed under the successor operation S_n defined on H_n by $\mathscr{A}(S)$.

THEOREM 6.1. *The system $\mathscr{L}_n + S$ is a conservative extension of $\mathscr{L}_n$, strongly complete for valuations in the algebra $\langle H_n, S_n \rangle$. Moreover, any implicit connective of $\mathscr{L}_n + S$ is equivalent in this system to a combination of $\wedge, \vee, \to, \neg$, and S.*

PROOF. Since the variety $\mathbb{V}(\mathscr{L}_n)$ is generated by H_n, then, by Jonsson's lemma, the subdirectly irreducible algebras of this variety are exactly the chains H_i, $i \leq n$. By compatibility, their respective expansions $\langle H_i, S_i \rangle$ are the subdirectly irreducible algebras of the variety $\mathbb{V}^* = \mathbb{V}(\mathscr{L}_n + S)$. Therefore, by the subdirect decomposition theorem, the algebras of $\mathbb{V}(\mathscr{L}_n)$ are embedded in reducts of algebras in $\mathbb{V}^*$, and thus the extension is conservative. To prove completeness with respect to valuations into

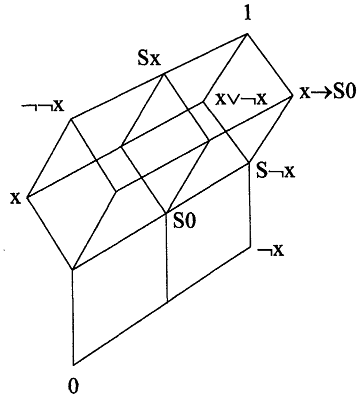

FIGURE 2

$\langle H_n, S_n\rangle$, it is enough to notice that this algebra generates $\mathbb{V}^*$. Indeed, by compatibility of S_n and uniqueness of S_i, the natural Heyting algebra homomorphism from H_n onto H_i, for $i \leq n$, is also a homomorphisms from $\langle H_n, S_n\rangle$ onto $\langle H_i, S_i\rangle$.

Now, if an axiom system $\mathscr{A}(S, \nabla)$ defines an implicit k-ary connective ∇ over $\mathscr{L}_n + S$, then the subdirectly irreducible algebras of $\mathbb{V}(\mathscr{A}(S, \nabla))$ have reducts among the subdirectly irreducible algebras of $\mathbb{V}^*$; that is, they are of the form $\langle H_i, S_i, \nabla_i\rangle$ for some $i \leq n$. Let m be the maximum such i. By affine completeness of finite Heyting algebras, $\nabla_m(x_1, \dots, x_k)$ is a Heyting polynomial $p(x_1, \dots, x_k, a_1, \dots, a_r)$ with $a_i \in H_m$. Since $S_m(x)$ is the successor function for $0 \leq x < 1$, by (5.1), it follows that each a_i is definable in $\langle H_m, S_m, \nabla_m\rangle$ by one of the closed terms 0, $S(0)$, $S(S(0))$, ..., $S^{m-1}(0), 1$. Therefore, $\nabla_m(x_1, \dots, x_k) = t(x_1, \dots, x_k)$, a term of type $\{\wedge, \vee, \rightarrow, \neg, S\}$. Again by compatibility of ∇_m and uniqueness of ∇_i, the other irreducible algebras (H_i, S_i, ∇_i) are homomorphic images of (H_m, S_m, ∇_m), and the last equation holds in all of them. Therefore, it holds in all algebras of $\mathbb{V}(\mathscr{A}(S, \nabla))$ and by completeness $\vdash_{\mathscr{A}(\nabla)} \nabla(\pi_1, \dots, \pi_k) \leftrightarrow t(\pi_1, \dots, \pi_k)$. ⊣

The proof of the theorem shows that the class of implicit connectives of $\mathscr{L}_n$ coincides with the set of Heyting polynomials of H_n; identical, by affine completeness, to the set of compatible functions of H_n. It may be shown (cf. [3]) that these are exactly the functions $f : H_n^k \rightarrow H_n$ satisfying, for some $a \in H_n$:

$$f(x_1, \dots, x_k) \begin{cases} \geq \min(x_1, \dots, x_k) & \text{if } \min(x_1, \dots, x_k) \leq a \\ = a & \text{if } \min(x_1, \dots, x_k) > a. \end{cases}$$

The unary implicit connectives of $\mathscr{L}_3$, which constitute also the free algebra in one generator of the variety $\mathbb{V}(\mathscr{L}_3 + S)$, are depicted in Figure 2, in terms of their generator S.

It would be interesting to have answers to the following general questions.

Does any implicit connective of pure intuitionistic calculus satisfies the disjunction property?

Does every intermediate logic have a unique completion by implicit connectives, as $\mathscr{L}_n$ does?

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DEPARTAMENTO DE MATEMÁTICAS
UNIVERSIDAD DE LOS ANDES
APARTADO AÉREO 4976
BOGOTÁ, D.C. — COLOMBIA
E-mail: xcaicedo@uniandes.edu.co

DEPARTAMENTO DE MATEMÁTICA
FACULTAD DE CIENCIAS EXACTAS Y NATURALES
UNIVERSIDAD DE BUENOS AIRES — CONICET
CIUDAD UNIVERSITARIA
1428 BUENOS AIRES - ARGENTINA
E-mail: cignoli@dm.uba.ar